Pegging the Interest Rate on Bank Reserves: A Resolution of New Keynesian Puzzles and Paradoxes

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October 14, 2017

Abstract

We develop a model of monetary policy with a simple departure from the basic New Keynesian (NK) model. In this model, the central bank sets independently the interest rate on bank reserves and the nominal stock of bank reserves. Because reserves reduce the costs of banking, the model delivers local-equilibrium determinacy under a permanent interest-rate peg. As a result, it does not share the puzzling and paradoxical implications of the basic NK model under a temporary interest-rate peg (e.g., in the context of a liquidity trap). More specifically, it offers a resolution of the "forward-guidance puzzle," a related puzzle about fiscal multipliers, and the "paradox of flexibility," even for an *arbitrarily* small departure from the basic NK model (i.e., arbitrarily small banking costs). It still solves or attenuates these puzzles and paradox for a *vanishingly* small departure, and also solves the "paradox of toil" in that case. This limit result provides an equilibrium-selection device in the basic NK model, and brings this canonical sticky-price model at par with its sticky-information counterpart in terms of their ability to solve or attenuate the four puzzles and paradoxes.

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1 Introduction

The Great Recession has led central banks to peg their policy rates near zero and provide forward guidance about low future policy rates. It has also rekindled interest in the use of discretionary fiscal policy as a stabilization tool, and, in Europe, sparked debate about implementing structural reforms. The mainstream New Keynesian (NK) literature, however, is of limited help for economists and policymakers in this context, because standard NK models have puzzling and paradoxical implications about the consequences of forward guidance, fiscal policy, and structural reforms under a temporary interest-rate peg − e.g., in the context of a liquidity trap during which the interest rate is pegged to its zero lower bound. In this paper, we show that a simple and possibly minimal departure from the basic NK model, involving the central bank pegging the interest rate on bank reserves (as central banks did in reality), offers a resolution of these puzzles and paradoxes.

In standard NK models, and in particular in the basic NK model studied in Woodford (2003) and Galí (2015), the effects of a temporary interest-rate peg on current inflation and output become unboundedly large as the duration of the peg goes to infinity (the so-called "forward-guidance puzzle") or as prices become perfectly flexible (the so-called "paradox of flexibility"). Moreover, the effects on current inflation and output of a given fiscal expansion at the end of the peg also grow explosively as the duration of the peg goes to infinity (what we henceforth call the "fiscal-multiplier puzzle").[1] These implications are perplexing because inflation, output, and fiscal multipliers all take finite values in the limit case of a permanent peg or perfectly flexible prices.[2] There is, thus, a stark discontinuity in the limit as the duration of the peg goes to infinity, or as the degree of price stickiness goes to zero. In addition to these limit implications, the basic NK model has another perplexing implication known as the "paradox of toil:" positive supply shocks − such as downward shifts in the labor-disutility function, labor-tax cuts, technology improvements, and reductions in market power − are contractionary under a temporary interest-rate peg.[3]

Most of these puzzles and paradoxes find their source in NK models' property of exhibiting equilibrium indeterminacy under a permanent interest-rate peg. The basic NK model, in particular, has a stable eigenvalue under a (permanent or temporary) peg, but no predetermined variable. Under a temporary peg, when the model is iterated backward in time, this excess

[1]These results can be found in, e.g., Werning (2012), Carlstrom, Fuerst, and Paustian (2015), Farhi and Werning (2016), and Cochrane (2017a). The phrases "forward-guidance puzzle" and "paradox of flexibility" were coined by, respectively, Del Negro, Giannoni, and Patterson (2015), and Eggertsson and Krugman (2012).

[2]A permanent interest-rate peg generates multiple local equilibria in the basic NK model. What we mean is that inflation, output, and fiscal multipliers take finite values in each of these equilibria. Similarly, inflation is not uniquely pinned down under perfectly flexible prices, but takes a finite value.

[3]These results can be found in, e.g., Eggertsson (2010, 2011, 2012), Eggertsson and Krugman (2012), Eggertsson, Ferrero, and Raffo (2014), Kiley (2016), and Wieland (2016). Available empirical evidence, presented in Wieland (2016), Cohen-Setton, Hausman, and Wieland (2017), and Garín, Lester, and Sims (2017), does not seem to support this implication of the model. The phrase "paradox of toil" was coined by Eggertsson (2010).

stable eigenvalue magnifies the effects of future terminal conditions (at the end of the peg) on initial outcomes (at the start of the peg), so that these effects grow explosively as the duration of the peg goes to infinity − thus giving rise to the forward-guidance and fiscal-multiplier puzzles, as highlighted in Carlstrom, Fuerst, and Paustian (2015) and Cochrane (2017a). Moreover, the indeterminacy property of the basic NK model is also behind the paradox of flexibility, as we highlight in the text: as prices become more and more flexible, the stable eigenvalue converges towards zero, so that initial outcomes explode even for a peg of given short duration.

In turn, NK models' property of exhibiting indeterminacy under a permanent interest-rate peg comes from the fact that in these models, following Sargent and Wallace (1975), the central bank is assumed to set the nominal interest rate on a bond that serves only as a store of value (i.e., has no non-pecuniary "convenience yield").[4] Once the central bank pegs this interest rate, it commits to buy or sell the bond at the implied price. This makes the money supply endogenous, and it makes any arbitrary price level (with the associated nominal money stock) consistent with an equilibrium.

Our starting point is to note that central banks, during the Great Recession, did not peg "the" interest rate that appears in the IS equation of NK models. In reality, the lower bound on nominal interest rates forced them to peg the interest rate on bank reserves (IOR rate), which is the interest rate that they directly control. In our model, the central bank can set independently the IOR rate and the nominal stock of bank reserves, because bank reserves are the unit of account (and, essentially, a means of payment). And, because bank reserves reduce the costs of banking, setting exogenously these two instruments delivers determinacy.[5] Under flexible prices, setting the IOR rate determines the demand for real reserves; and, given the outstanding nominal stock of reserves, this pins down the price level. Under sticky prices à la Calvo (1983), setting exogenously the IOR rate and the stock of reserves amounts to following a "shadow Wicksellian rule" for the interest rate on a bond that serves only as a store of value: it is as if the central bank directly controlled this interest rate and set it as an increasing function of output and the price level.[6] Wicksellian rules are well known to ensure determinacy in the basic NK model (as shown in Woodford, 2003, Chapter 4). We show that this specific

[4]The assumption that the bond in question has absolutely no convenience yield (such as eligibility as collateral) is important for the determinacy properties of standard models, as Canzoneri and Diba (2005) illustrate.

[5]In the basic NK model with money entering the utility function in a separable way, setting exogenously the money supply and the interest rate on money would also ensure determinacy, as follows from Woodford's (2003, Chapter 4) analysis. The structure that we add to the basic NK model enables us to interpret money more specifically as bank reserves, on which interest can be paid, rather than household cash. The resulting reduced form is not isomorphic to the reduced form of the basic NK model with money in the utility function (whether money enters the utility function in a separable or non-separable way), except in the specific case of a corridor system (which we consider in Subsection 6.4). Other contributions (Canzoneri, Henderson, and Rogoff, 1983; Adão, Correia, and Teles, 2003) also note that letting the central bank somehow set both the interest rate and the money supply would deliver determinacy in their models.

[6]This rule is a "shadow rule" in our setup in the sense that what it actually describes is the private sector's behavior, not the central bank's. Since our central bank sets its policy instruments exogenously, it does not react to deviations from equilibrium; we do not need, therefore, to worry about the *feasibility* of its off-equilibrium reaction (Bassetto, 2005; Loisel, 2016). The *credibility* of its off-equilibrium-behavior threat may, however, still be an issue (Cochrane, 2011), even though this off-equilibrium behavior is passive.

Wicksellian rule, given its implied coefficients, also ensures determinacy in our extended NK model with banking costs, for all functional forms of the utility and production functions and all values of the structural and (steady-state) policy parameters.

Because it delivers determinacy under a permanent (IOR-rate) peg, our model solves the forward-guidance and fiscal-multiplier puzzles, as well as the paradox of flexibility. More specifically, in our model, the effects of a temporary (IOR-rate) peg do not grow explosively as its duration becomes longer and longer, nor as prices become more and more flexible; instead, they converge towards the finite effects of a permanent peg or the finite flexible-price effects. And fiscal interventions in the vanishingly distant future have vanishingly small effects, instead of unboundedly large effects, on current outcomes.

The key element behind our resolution of these "limit" puzzles and paradox (through determinacy) is the exogeneity of the IOR rate, instead of the interest rate on bonds as in the literature. In our model, if the central bank did set exogenously one of its two instruments − the IOR rate and the stock of reserves − and followed a rule for the other instrument that made the interest rate on bonds equal to an exogenous target both in and out of equilibrium, then indeterminacy would ensue, and all three limit puzzles and paradox would reemerge. Central banks, however, did not follow such a rule for the stock of reserves, which they set for other (quantitative-easing) purposes; and they could not follow such a rule for the IOR rate anyway, as this rate was already at its lower bound.

Our model solves these limit puzzles and paradox even with an *arbitrarily* small departure from the basic NK model, i.e. with arbitrarily small banking costs and convenience yield of bank reserves. In the limit, as we make the departure from the basic NK model *vanishingly* small, our model serves to select uniquely a particular equilibrium of the basic NK model under a temporary interest-rate peg followed by a permanent one.[7,8] We show that this particular equilibrium exhibits neither the fiscal-multiplier puzzle nor the paradox of flexibility, and exhibits an attenuated form of the forward-guidance puzzle (which asymptotically makes inflation and output grow linearly, not exponentially, with the duration of the interest-rate peg).[9] We also show that our selected equilibrium does not exhibit the paradox of toil, and relate this feature to inflation inertia arising from the presence of a state variable in our model.[10] Overall, this

[7]Our equilibrium-selection device, based on a limit uniqueness result, is reminiscent of the one developed in the global-games literature pioneered by Carlsson and van Damme (1993) and Morris and Shin (1998, 2000).

[8]The basic NK model has an infinity of local equilibrium paths under a temporary interest-rate peg followed by a permanent one. Cochrane (2017a) shows that any of these paths can be obtained as the unique local equilibrium under a temporary interest-rate peg followed by a suitably designed interest-rate rule. He describes two such equilibria (the "backward-stable" and "no-inflation-jump" equilibria) that do not exhibit any of the puzzles and paradoxes. We briefly compare these two equilibria with our selected equilibrium in the text.

[9]Since it does not exhibit the paradox of flexibility, it satisfies Cochrane's (2017a) "local-to-frictionless" equilibrium-selection criterion, which requires that equilibrium outcomes converge towards flexible-price equilibrium outcomes as prices become more and more flexible. This criterion does not select a unique equilibrium, but rules out some equilibria.

[10]For positive banking costs, this state variable − the stock of reserves − both delivers determinacy under a permanent IOR-rate peg (by matching the stable eigenvalue) and generates inflation inertia. As we shrink the

equilibrium brings the basic NK model at par with Mankiw and Reis's (2002) sticky-information model in terms of their ability to solve or attenuate the four NK puzzles and paradoxes.[11]

Most of our analysis is conducted in the context of a benchmark model of a cashless economy − an extension of the basic NK model with banking costs − in which the nominal stock of bank reserves is set exogenously by the central bank. As a robustness check, we consider two departures from this benchmark setting. One departure recognizes that the central bank may follow a quantitative-easing rule that makes the quantity of reserves react to the price and output levels. We find that our results are robust to what we consider plausible specifications of such a rule. The second departure recognizes that in reality, what the central bank controls is the monetary base, and cash held outside banks makes the quantity of bank reserves endogenous. We show that our results are essentially robust to the introduction of household cash into the model.[12]

In our benchmark model, the demand for reserves cannot be satiated. Satiation of the demand for reserves in our model would raise the spectre of indeterminacy issues highlighted in Sargent and Wallace (1985): it would make the marginal convenience yield of bank reserves equal to zero, and the demand for real reserve balances indeterminate. This would undo the main mechanism that delivers determinacy and solves the puzzles and paradoxes in our model. Some observers of the U.S. situation (e.g., Cochrane, 2014) assert that the demand for bank reserves is currently satiated. And Reis (2016) presents empirical evidence that post-QE1 data fail to reject the satiation hypothesis.

We will make some arguments in the text that the marginal convenience yield of reserves (albeit very small) may not be zero. In the end, however, we cannot make a persuasive argument either way. It seems hard to discriminate between the view that the marginal convenience yield of reserves is exactly zero and our preferred view that it may be small and fairly flat, but still positive and inversely related to the amount of reserves. The latter view substitutes a narrative in which small shocks may lead to large changes in the demand for reserves for a narrative in which banks are truly indifferent across a range of values for their reserve balances; or, equivalently, a narrative in which quantitative easing has, on the margin, some (possibly very small) effects on the economy, for a narrative in which it has no effects at all.

Some remarks may serve to put our results in the context of the recent and fast-growing literature on NK puzzles and paradoxes. First, our model offers a full resolution of these puzzles

costs to zero, our model converges to the purely forward-looking basic NK model, but the equilibrium we select in this way retains the inflation inertia.

[11]The ability of Mankiw and Reis's (2002) model to solve or attenuate the puzzles and paradoxes is studied by Carlstrom, Fuerst, and Paustian (2015) and Kiley (2016). Hagedorn (2017) also finds that the puzzles and paradoxes may not arise in a sticky-information model; his model, in addition, delivers determinacy of the steady-state price level, because it assumes that government expenditures are, to some extent, specified in nominal terms.

[12]We think our results would also hold up if we added other realistic features (associated with central-bank operating procedures) that lead to limited endogenous variation in reserves. For example, the Federal Reserve's reverse-repo facility allows financial entities with no access to the IOR rate to affect the monetary base. But the resulting endogeneity is limited because the Federal Reserve sets the reverse-repo rate below the IOR rate.

and paradoxes. This is in contrast to contributions that obtain more credible quantitative results by introducing realistic features into the basic NK model to attenuate the effects of an interest-rate peg − or the effects of fiscal policy under an interest-rate peg − for a *given* duration of the peg and a *given* degree of price stickiness. These features can be perpetual youth (Del Negro, Giannoni, and Patterson, 2015), information frictions (Wiederholt, 2015; Angeletos and Lian, 2016), incomplete markets (McKay, Nakamura, and Steinsson, 2016, 2017; Kaplan, Moll, and Violante, 2016), bounded rationality (Farhi and Werning, 2017), heterogenous beliefs (Andrade, Gaballo, Mengus, and Mojon, 2017), or adaptive expectations (Gertler, 2017). In our view, these attenuations are very welcome but do not substitute for a full resolution, as a model's quantitative predictions for a given duration of the peg and a given degree of price stickiness can still be questioned if its implications in the limit as the duration of the peg goes to infinity or as the degree of price stickiness goes to zero seem puzzling or paradoxical.[13]

Second, the literature has proposed models that can solve the forward-guidance puzzle by "discounting" the IS equation or the Phillips curve of the basic NK model, i.e. by scaling down the coefficients of their expectational terms. The source of this discounting can be bounded rationality (Gabaix, 2016), information frictions (Angeletos and Lian, 2016), or incomplete markets (Bilbiie, 2017).[14] In the text, we highlight four differences between these models and ours. First, these models do not solve the paradox of flexibility. Second, they require a discrete departure from the basic NK model to solve the forward-guidance puzzle. Third and fourth, they cannot solve this puzzle without generating a *negative* long-term relationship between the inflation rate and the nominal interest rate on bonds, nor without having non-standard and far-reaching implications for equilibrium determinacy in "normal times," i.e. away from the zero lower bound on nominal interest rates.[15] By contrast, our model preserves the standard Fisher effect of the basic NK model, i.e. the *one-to-one* long-term relationship between the inflation rate and the nominal interest rate on bonds; and, under a corridor system, it inherits all the standard implications of the basic NK model for equilibrium determinacy in normal times.[16]

Finally, three other modifications to the basic NK model have been proposed to solve (at least some of) the puzzles and paradoxes. Cochrane (2017a, 2017b) shows that non-Ricardian fiscal

[13] García-Schmidt and Woodford (2015) express a similar point of view.

[14] Angeletos and Lian's (2016) resolution result (as opposed to their attenuation results) is stated in the last sentence of Proposition 3 of their paper. Bilbiie (2017) considers a somewhat different definition of the forward-guidance puzzle; but the necessary and sufficient condition (stated in Proposition 2 of his paper) for his model to solve the puzzle in his sense is identical to the necessary and sufficient condition for his model to deliver determinacy under a permanent interest-rate peg and thus solve the puzzle in our sense. Other contributions (e.g. Del Negro, Giannoni, and Patterson, 2015; McKay, Nakamura, and Steinsson, 2016, 2017) also involve some discounting of the IS equation or the Phillips curve of the basic NK model, but pursue only an attenuation, not a full resolution, of the forward-guidance puzzle.

[15] Cochrane (2016) has already noted these four properties. Our contribution in this respect is to show that his numerical results, obtained in the context of a calibrated version of Gabaix's (2016) benchmark model, apply more generally to any calibration of any discounting model of the class we consider − but not to our model.

[16] More specifically, if the central bank maintains a fixed spread between the IOR rate and the interbank rate (as in a typical corridor system), then the reduced form of our model becomes isomorphic to the reduced form of the basic NK model for any given interest-rate rule.

policy solves all the NK puzzles and paradoxes. His resolution rests on the fiscal theory of the price level, and requires a discrete departure from the basic NK model; ours is purely monetary, and works for an arbitrarily small departure from the basic NK model. García-Schmidt and Woodford (2015) solve the forward-guidance puzzle by replacing rational expectations with reflective expectations. Their resolution does not require a discrete departure from the basic NK model, as it works for any degree of reflection, in particular for degrees of reflection that are arbitrarily large and hence arbitrarily close to perfect foresight.[17] Our resolution of the forward-guidance puzzle rests on a different mechanism, which preserves the basic NK model's analytical tractability and which we show also enables us to solve the fiscal-multiplier puzzle, the paradox of flexibility, and the paradox of toil. Ravn and Sterk (2017) show that incomplete markets can solve the paradox of toil − again, through a different mechanism: our resolution of this paradox rests on inflation inertia after the supply shock, unlike theirs.

The rest of the paper is organized as follows. Section 2 presents the benchmark model that is used in the next four sections. Section 3 shows that, when the central bank sets exogenously the IOR rate and the stock of reserves, the model has a unique steady state and a unique equilibrium in the neighborhood of this steady state. Section 4 shows that, as a consequence, the model solves the three limit puzzles and paradox. Section 5 establishes that the model still solves or attenuates these limit puzzles and paradox for a vanishingly small departure from the basic NK model, and also solves the paradox of toil in this case. Section 6 highlights four differences between the implications of our model and those of the "discounting" models proposed in the literature to solve the forward-guidance puzzle. Section 7 shows that our resolution of the limit puzzles and paradox is essentially robust to the relaxation of two assumptions: the exogeneity of nominal reserves, and the absence of cash. Section 8 shows that our results require the demand for reserves not to be (fully) satiated in equilibrium, and discusses whether or not this has been the case in the U.S. over the past few years. We then conclude and provide a technical appendix.

2 Benchmark Model

In our benchmark model, monopolistic firms use labor to produce goods. They need to pay wages before they can produce and sell their output. They borrow the wage bill from banks. Banks use labor and reserves to make loans. The central bank sets both the interest rate on bank reserves and the quantity of bank reserves. The model is essentially non-parametric, as we do not specify any functional form for the utility and production functions, in order to broaden the scope of our results. For simplicity, we assume that households do not hold cash and that there is no finite satiation level in the demand for reserves; these assumptions will be relaxed

[17] In the limit as the distance between their model and the basic NK model goes to zero, however, the forward-guidance puzzle reemerges in the same strong form as in the basic NK model.

in Subsection 7.2 and Section 8 respectively.

2.1 Households

Each household consists of workers and bankers. Households get utility from consumption (c) and disutility from labor (h for workers, h^b for bankers). Their intertemporal utility function is

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - v^b\left(h_{t+k}^b\right) \right] \right\}$$

where $\beta \in (0,1)$. The consumption-utility function u, defined over the set of positive real numbers $\mathbb{R}_{>0}$, is twice differentiable, strictly increasing ($u' > 0$), strictly concave ($u'' < 0$), and satisfies the usual Inada conditions

$$\lim_{c \to 0} u'(c) = +\infty, \tag{1}$$

$$\lim_{c \to +\infty} u'(c) = 0. \tag{2}$$

The labor-disutility functions v and v^b, defined over the set of non-negative real numbers $\mathbb{R}_{\geq 0}$, are twice differentiable, strictly increasing ($v' > 0$ and $v^{b\prime} > 0$), and weakly convex ($v'' \geq 0$ and $v^{b\prime\prime} \geq 0$).

Bankers use their own labor h_t^b and (real) reserves at the central bank m_t to produce (real) loans ℓ_t according to the technology

$$\ell_t = f^b\left(h_t^b, m_t\right).$$

The production function f^b, defined over $\left(\mathbb{R}_{\geq 0}\right)^2$, is twice differentiable, strictly increasing ($f_h^b > 0$ and $f_m^b > 0$), homogeneous of degree $d \in (0,1]$, and such that $f_{hh}^b < 0$, $f_{mm}^b < 0$, $f_{hm}^b \geq 0$,

$$\forall h_t^b \in \mathbb{R}_{\geq 0}, \lim_{m_t \to +\infty} f_m^b\left(h_t^b, m_t\right) = 0, \tag{3}$$

$$\forall h_t^b \in \mathbb{R}_{\geq 0}, \lim_{m_t \to 0} f_h^b\left(h_t^b, m_t\right) = 0. \tag{4}$$

Assumption (3) is a standard Inada condition, while Assumption (4) articulates a sense in which holding reserves is essential for banking. The assumption of decreasing or constant returns to scale ($d \leq 1$) is not necessary for our results, but simplifies our non-parametric analysis.[18] As we show in Appendix A.1, it implies that f^b is concave ($f_{hh}^b f_{mm}^b - \left(f_{hm}^b\right)^2 \geq 0$). Similarly, the assumption that labor and reserves are complements ($f_{hm}^b \geq 0$) could be relaxed to some extent without affecting our results. The set of functions f^b satisfying all these assumptions is broad enough to include, for instance, any constant-elasticity-of-substitution (CES) function, as well as any CES function raised to a power d such that $(s-1)/s \leq d < 1$, where s denotes the elasticity of substitution.

[18] In a previous version of this paper (Diba and Loisel, 2017), we allow for increasing returns to scale ($d > 1$) in the context of a parametric model.

The function f^b is, of course, a convenient short cut to capture the role of bank reserves − which in reality may come, for example, from a maturity mismatch between banks' assets and liabilities. Our results will not depend on the quantitative importance of this role: the elasticity of loans to reserves, $m_t f^b_m(h^b_t, m_t)/f^b(h^b_t, m_t)$, may be arbitrarily small for any (h^b_t, m_t) in $(\mathbb{R}_{\geq 0})^2$. What we need for our results, however, is that this elasticity is not zero in equilibrium. This condition is necessarily met in our benchmark model, because we assume that there is no finite satiation level in the demand for reserves. We will relax this assumption in Section 8.

Given the properties of f^b, we can invert it and get

$$h^b_t = g^b(\ell_t, m_t),$$

where the function g^b is implicitly and uniquely defined over $(\mathbb{R}_{\geq 0})^2$ by

$$\ell_t = f^b[g^b(\ell_t, m_t), m_t].$$

The utility cost of banking, as a function of loans and reserves, is therefore defined over $(\mathbb{R}_{\geq 0})^2$ by

$$\Gamma(\ell_t, m_t) \equiv v^b\left[g^b(\ell_t, m_t)\right].$$

We derive some properties of the functions g^b and Γ in Appendices A.2, A.3, and A.4. In particular, we establish the following lemma in Appendix A.3:

Lemma 1 (Properties of Function Γ): *The banking-cost function Γ is strictly increasing in loans ($\Gamma_\ell > 0$); strictly decreasing in reserves ($\Gamma_m < 0$); convex ($\Gamma_{\ell\ell} > 0$, $\Gamma_{mm} > 0$, $\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 \geq 0$); and such that $\Gamma_{\ell m} < 0$,*

$$\forall \ell_t \in \mathbb{R}_{>0}, \lim_{m_t \to +\infty} \Gamma_m(\ell_t, m_t) = 0, \tag{5}$$

$$\forall \ell_t \in \mathbb{R}_{>0}, \lim_{m_t \to 0} \Gamma_\ell(\ell_t, m_t) = +\infty. \tag{6}$$

The property that Γ_m is not zero (except asymptotically, as $m_t \to +\infty$) reflects our assumption that there is no finite satiation point in the demand for reserves. The negative-cross-derivative property ($\Gamma_{\ell m} < 0$) says that a marginal increase in reserves decreases costs by more the larger are loans, while property (6) reflects our assumption that holding reserves is essential for banking.[19]

In addition to making loans ℓ_t and holding reserve balances m_t at the central bank, households hold bonds b_t (or issue bonds when $b_t < 0$), which serve only as stores of value.[20] Loans,

[19]These properties of Γ are quite similar to the banking-cost assumptions made by Cúrdia and Woodford (2011). Three differences between the two models are worth mentioning: (i) our benchmark model has no finite satiation level in the demand for reserves, unlike their model; (ii) we link banking costs to time spent on banking activities, in order to make our steady-state analysis tractable, while they link them to goods consumed in banking activities; and (iii) the borrowers in our model are firms (borrowing the wage bill), while they are impatient households in their model.

[20]Bonds issued by households can also be thought of as deposits issued by bankers.

reserves, and bonds are one-period non-contingent assets. We let I_t^ℓ, I_t^m, and I_t^b denote the corresponding gross nominal interest rates. We let P_t denote the price level, and $\Pi_t \equiv P_t/P_{t-1}$ the gross inflation rate. The household budget constraint, expressed in real terms, is then

$$c_t + b_t + \ell_t + m_t \leq \frac{I_{t-1}^b}{\Pi_t} b_{t-1} + \frac{I_{t-1}^\ell}{\Pi_t} \ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \omega_t, \tag{7}$$

where w_t represents the real wage and ω_t captures firm profits and the government's lump-sum taxes or transfers.

Households choose b_t, c_t, h_t, ℓ_t, and m_t to maximize their utility function, rewritten as

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - \Gamma\left(\ell_{t+k}, m_{t+k}\right) \right] \right\},$$

subject to their budget constraint (7), taking all prices (I_t^b, I_t^ℓ, I_t^m, P_t, and w_t) as given. Letting λ_t denote the Lagrange multiplier on the period-t budget constraint, the first-order conditions of households' optimization problem are

$$\lambda_t = u'\left(c_t\right), \tag{8}$$

$$\frac{1}{I_t^b} = \beta \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\lambda_t \Pi_{t+1}} \right\}, \tag{9}$$

$$\lambda_t w_t = v'\left(h_t\right), \tag{10}$$

$$\Gamma_\ell\left(\ell_t, m_t\right) + \lambda_t = \beta I_t^\ell \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\},$$

$$\Gamma_m\left(\ell_t, m_t\right) + \lambda_t = \beta I_t^m \mathbb{E}_t \left\{ \frac{\lambda_{t+1}}{\Pi_{t+1}} \right\}.$$

Using (9), we can rewrite the last two conditions as

$$\frac{I_t^\ell}{I_t^b} = 1 + \frac{\Gamma_\ell\left(\ell_t, m_t\right)}{\lambda_t}, \tag{11}$$

$$\frac{I_t^m}{I_t^b} = 1 + \frac{\Gamma_m\left(\ell_t, m_t\right)}{\lambda_t}. \tag{12}$$

Condition (11) implies that loans pay more interest than bonds, because the marginal banking cost is positive ($\Gamma_\ell > 0$). Condition (12) implies that reserves pay less interest than bonds, because they serve to reduce banking costs ($\Gamma_m < 0$). Assuming that households' optimization problem is also subject to a standard no-Ponzi-game condition, the transversality condition is

$$\lim_{k \to +\infty} \mathbb{E}_t \left\{ \beta^{t+k} \lambda_{t+k} a_{t+k} \right\} = 0, \tag{13}$$

where $a_t \equiv b_t + \ell_t + m_t$ denotes households' total assets. The second-order conditions of households' optimization problem are met because of the convexity of the banking-cost function Γ.

2.2 Firms

There is a continuum of monopolistically competitive firms owned by households and indexed by $i \in [0,1]$. Each firm i uses $h_t(i)$ units of labor to produce

$$y_t(i) = f[h_t(i)] \tag{14}$$

units of output. The production function f, defined over $\mathbb{R}_{\geq 0}$, is twice differentiable, strictly increasing ($f' > 0$), weakly concave ($f'' \leq 0$), and such that $f(0) = 0$. To generate a demand for bank loans, we assume that firm i has to borrow its nominal wage bill $W_t h_t(i)$ from banks, at the gross nominal interest rate I_t^{ℓ}, before it can produce and sell its output. Thus, the nominal value of firm i's loan $L_t(i)$ must satisfy

$$W_t h_t(i) \leq L_t(i). \tag{15}$$

Following Calvo (1983), we assume that each firm, whatever its history, has the probability $\theta \in [0,1)$ not to be allowed to reset its price in any period. We postpone the analysis of the sticky-price case ($\theta > 0$) to the next section. In the limit case where prices are perfectly flexible ($\theta = 0$), firm i chooses its price $P_t(i)$ at date t to maximize

$$\mathbb{E}_t \left\{ P_t(i) y_t(i) - \frac{\beta \lambda_{t+1} I_t^{\ell} L_t(i)}{\lambda_t \Pi_{t+1}} \right\}$$

subject to the production function (14), the borrowing constraint (15), and the demand schedule

$$y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\varepsilon} y_t, \tag{16}$$

where $\varepsilon > 0$ denotes the elasticity of substitution between differentiated goods and $y_t \equiv [\int_0^1 y_t(i)^{(\varepsilon-1)/\varepsilon} di]^{\varepsilon/(\varepsilon-1)}$. The first-order condition of this optimization problem implies

$$P_t(i) = \frac{\varepsilon}{\varepsilon - 1} \mathbb{E}_t \left\{ \frac{\beta \lambda_{t+1} I_t^{\ell} W_t}{\lambda_t \Pi_{t+1} f'[h_t(i)]} \right\}.$$

Using the Euler equation (9), we can rewrite this pricing equation as

$$P_t(i) = \frac{\varepsilon}{\varepsilon - 1} \frac{I_t^{\ell} W_t}{I_t^b f'[h_t(i)]}.$$

In a symmetric (flexible-price) equilibrium, all firms set the same price:

$$P_t = \frac{\varepsilon}{\varepsilon - 1} \frac{I_t^{\ell} W_t}{I_t^b f'(h_t)}. \tag{17}$$

2.3 Government

The government consists of a monetary authority and a fiscal authority. The monetary authority has two independent instruments: the (gross) nominal interest rate on reserves $I_t^m \geq 0$, and the monetary base, which in our benchmark model is made only of nominal reserves $M_t > 0$.[21] Changes in reserve balances are matched by changes in the monetary authority's holdings of bonds issued by households or the fiscal authority. The fiscal authority consumes an exogenous quantity $g_t \geq 0$ of goods, sets lump-sum transfers T_t (or taxes when $T_t < 0$) to households, and satisfies its present-value budget constraint at any prevailing price path (making fiscal policy Ricardian). We assume for simplicity that government expenditures g_t are wasted, but the results would be unchanged if they entered households' utility function in a separable way.

The consolidated budget constraint of the government is

$$M_t + B_t = I^m_{t-1} M_{t-1} + I^b_{t-1} B_{t-1} + P_t g_t + T_t.$$

The monetary authority may inject reserve balances either through open-market operations, which increase M_t holding $M_t + B_t$ constant, or through helicopter drops (monetized fiscal transfers), which increase M_t holding B_t constant. Which way it injects reserve balances does not matter for our steady-state and local-equilibrium analyses below.[22]

Our analysis below does not rely on the existence of an effective lower bound for the IOR rate (other than $I^m_t \geq 0$). A more realistic model in which vault cash is substitutable to some extent for deposits at the central bank could imply a positive effective lower bound for I^m_t, although there is no particular reason to think that this lower bound would be unity. To capture the lower bound in a simple and stark way, we will assume that vault cash (with no interest payments) is a perfect substitute for deposits at the central bank in terms of reducing banking costs. This introduces a zero lower bound (ZLB) for the net nominal IOR rate in our model. In an equilibrium with $I^m_t > 1$, banks will hold no cash. In an equilibrium with $I^m_t = 1$, the composition of reserve balances will be indeterminate, but also inconsequential; so, we will assume that banks hold no cash in equilibrium − and we refer to this setup as our model without cash to distinguish it from the model in Subsection 7.2 where we introduce household cash via a cash-in-advance constraint.

2.4 Market Clearing

The bond-market-clearing condition is

$$b_t = \frac{B_t}{P_t},$$

the reserve-market-clearing condition is

$$m_t = \frac{M_t}{P_t}, \tag{18}$$

and the goods-market-clearing condition is

$$c_t + g_t = y_t. \tag{19}$$

[21] In Subsection 7.2, the monetary base will be made of bank reserves and household cash.

[22] It does matter, though, for the global-equilibrium analysis that we conduct in a previous version of this paper (Diba and Loisel, 2017).

3 Determinacy Under Exogenous Monetary Policy

This section shows that when the central bank sets exogenously the IOR rate and the stock of reserves, our benchmark model has a unique steady state and a unique equilibrium in the neighborhood of this steady state. This determinacy result will be key to understand, in the following sections, how our model solves the puzzles and paradoxes that arise in standard NK models under a temporary interest-rate peg.

We start by showing that global equilibrium dynamics under flexible prices can be summarized by a single equation relating h_t to h_{t+1}. We then use this dynamic equation to establish the existence and uniqueness of a steady state under sticky prices. We next log-linearize the model around this unique steady state and prove local-equilibrium determinacy. Finally, we interpret this determinacy result with the help of a "shadow Wicksellian rule" for the interest rate on bonds.

3.1 Dynamic Equation Under Flexible Prices

In this subsection, we consider the case in which prices are flexible ($\theta = 0$). To derive the key equation summarizing global equilibrium dynamics in this case, we first use (8), (10), (14), (15) holding with equality, and (19), to express loans ℓ_t as a function of employment h_t:

$$\ell_t = \mathcal{L}\left(h_t\right) \equiv \frac{h_t v'\left(h_t\right)}{u'\left[f\left(h_t\right) - g_t\right]}. \tag{20}$$

The function $\mathcal{L}$ is defined over $(\underline{h}_t, +\infty)$, where $\underline{h}_t \geq 0$ is implicitly and uniquely defined by $f(\underline{h}_t) = g_t$. It is strictly increasing ($\mathcal{L}' > 0$) with

$$\lim_{h_t \to \underline{h}_t} \mathcal{L}\left(h_t\right) = 0, \tag{21}$$

$$\lim_{h_t \to +\infty} \mathcal{L}\left(h_t\right) = +\infty. \tag{22}$$

Next, we note that under flexible prices, the pricing equation (17) gives the real wage

$$w_t = \frac{\varepsilon - 1}{\varepsilon} f'\left(h_t\right) \frac{I_t^b}{I_t^\ell}. \tag{23}$$

We then use households' first-order condition (11), together with (8), (10), (14), (19), (20), and (23), to get a relationship between reserves m_t and employment h_t:

$$\Gamma_\ell\left[\mathcal{L}\left(h_t\right), m_t\right] = \mathcal{A}\left(h_t\right) \equiv u'\left[f\left(h_t\right) - g_t\right] \left\{ \frac{\varepsilon - 1}{\varepsilon} \frac{u'\left[f\left(h_t\right) - g_t\right] f'\left(h_t\right)}{v'\left(h_t\right)} - 1 \right\}. \tag{24}$$

Because $\Gamma_\ell > 0$, we restrict the domain of definition of $\mathcal{A}$ to $(\underline{h}_t, h_t^\star)$, where, given the Inada conditions (1) and (2), $h_t^\star > \underline{h}_t$ is implicitly and uniquely defined by

$$\frac{u'\left[f\left(h_t^\star\right) - g_t\right] f'\left(h_t^\star\right)}{v'\left(h_t^\star\right)} = \frac{\varepsilon}{\varepsilon - 1}.$$

The function $\mathcal{A}$ is strictly decreasing ($\mathcal{A}' < 0$) with

$$\lim_{h_t \to \underline{h}_t} \mathcal{A}(h_t) = +\infty, \tag{25}$$

$$\lim_{h_t \to h_t^\star} \mathcal{A}(h_t) = 0. \tag{26}$$

Note that $h_t^\star$ represents the value that h_t would take in the absence of financial frictions, i.e. if the marginal banking cost Γ_ℓ were zero.

Since $\Gamma_{\ell\ell} > 0$, $\Gamma_{\ell m} < 0$, $\mathcal{L}' > 0$, and $\mathcal{A}' < 0$, Equation (24) implicitly and uniquely defines a function $\mathcal{M}$ such that

$$m_t = \mathcal{M}(h_t). \tag{27}$$

The function $\mathcal{M}$ is strictly increasing ($\mathcal{M}' > 0$). The reason is that under flexible prices, firms' profit maximization makes their real marginal cost equal to the inverse of their mark-up ($(\varepsilon - 1)/\varepsilon$), which is constant over time; since real marginal cost depends positively on employment and negatively on reserves (through borrowing costs), real reserves need to react positively to employment to keep real marginal cost constant. Moreover, given (6), $\mathcal{M}$ is defined over $(\underline{h}_t, \overline{h}_t)$, where $\overline{h}_t \in (\underline{h}_t, h_t^\star]$ is implicitly and uniquely defined by

$$\lim_{m_t \to +\infty} \Gamma_\ell \left[\mathcal{L}\left(\overline{h}_t\right), m_t\right] = \mathcal{A}\left(\overline{h}_t\right). \tag{28}$$

The uniqueness of $\overline{h}_t$ follows from $\mathcal{A}' < 0$, $\mathcal{L}' > 0$, and $\Gamma_{\ell\ell} > 0$, while its existence is ensured by (25) and (26). Finally, given (6) and (28), we have

$$\lim_{h_t \to \underline{h}_t} \mathcal{M}(h_t) = 0, \tag{29}$$

$$\lim_{h_t \to \overline{h}_t} \mathcal{M}(h_t) = +\infty. \tag{30}$$

Thus, in our benchmark model with no satiation point in the demand for reserves, real reserve balances grow without bound as employment rises towards its upper bound $\overline{h}_t$. This upper bound coincides with the frictionless employment level $h_t^\star$ in the case where the marginal banking cost Γ_ℓ converges to zero as real reserves tend to infinity. In general, however, we allow the marginal banking cost to converge to a positive value – in which case we have $\overline{h}_t < h_t^\star$, and our economy with the financial friction cannot attain the employment level of the frictionless economy.

Finally, we use households' first-order conditions (9) and (12), together with (8), (14), (18), (19), (20), and (27), to get the dynamic equation in employment:

$$1 + \frac{\Gamma_m \left[\mathcal{L}(h_t), \mathcal{M}(h_t)\right]}{u'\left[f(h_t) - g_t\right]} = \beta I_t^m \mathbb{E}_t \left\{ \frac{u'\left[f(h_{t+1}) - g_{t+1}\right] \mathcal{M}(h_{t+1})}{\mu_{t+1} u'\left[f(h_t) - g_t\right] \mathcal{M}(h_t)} \right\}, \tag{31}$$

where $\mu_t \equiv M_t / M_{t-1}$ denotes the (gross) growth rate of nominal reserves. In the next subsection, we use this dynamic (flexible-price) equation to establish the existence and uniqueness of a steady state in our benchmark (sticky-price) model.[23]

[23] In a previous version of this paper (Diba and Loisel, 2017), we also use this dynamic equation to characterize

3.2 Steady-State Existence and Uniqueness

We now allow prices to be sticky ($\theta \geq 0$) and assume that I_t^m can vary exogenously around a given value $I^m \geq 1$, μ_t around the value $\mu = 1$, and g_t around a given value $g \geq 0$ (variables without time subscript denote steady-state values). In any steady state, both nominal reserves (because $\mu = 1$) and real reserves (by definition of a steady state) are constant over time. Therefore, prices are also constant over time, and the set of steady states is the same under sticky prices ($\theta > 0$) as under flexible prices ($\theta = 0$). To characterize this set, we can therefore use the flexible-price dynamic equation (31). When $h_{t+1} = h_t$ and $(I_t^m, \mu_t, g_t) = (I^m, 1, g)$, this dynamic equation boils down to the static equation

$$\mathcal{F}(h_t) \equiv \frac{\Gamma_m [\mathcal{L}(h_t), \mathcal{M}(h_t)]}{u'[f(h_t) - g]} = \beta I^m - 1, \tag{32}$$

where $\mathcal{L}(h_t)$ and $\mathcal{M}(h_t)$ are evaluated at $g_t = g$. The function $\mathcal{F}$ is defined over $(\underline{h}, \overline{h})$, where $\underline{h}$ and $\overline{h}$ denote the values of $\underline{h}_t$ and $\overline{h}_t$ respectively when $g_t = g$. We prove the following lemma in Appendix B.1:

Lemma 2 (Properties of Function $\mathcal{F}$): *The function $\mathcal{F}$ is strictly increasing ($\mathcal{F}' > 0$), with*

$$\begin{aligned} \lim_{h_t \to \underline{h}} \mathcal{F}(h_t) &= -\infty, \\ \lim_{h_t \to \overline{h}} \mathcal{F}(h_t) &= 0. \end{aligned} \tag{33}$$

The static equation (32) and Lemma 2 straightforwardly imply the following proposition:

Proposition 1 (Steady-State Existence and Uniqueness): *In the benchmark model, when I_t^m, μ_t, and g_t vary exogenously around the values $I^m \geq 1$, $\mu = 1$, and $g \geq 0$, there is a unique steady state if and only if $I^m < \beta^{-1}$.*

Since bonds serve only as a store of value, we have $I^b = \beta^{-1}$, so that the condition for steady-state existence and uniqueness stated in Proposition 1 is that the steady-state IOR rate I^m be set strictly below the steady-state interest rate on bonds I^b. When $I^m \geq I^b$, there is no steady state because banks would be tempted to issue infinite amounts of debt and deposit the proceeds at the central bank. When $I^m < I^b$, the first-order condition (12) implies that the convenience yield of bank reserves is positive (i.e., we have $\Gamma_m < 0$ in equilibrium), and

the set of all *global* perfect-foresight equilibria under flexible prices and permanent pegs (which includes, but is not limited to, the steady state of our model). We show that the steady state is the unique "determinate" equilibrium among them, which makes it a natural "focal point" on which private agents could coordinate, and we argue that the other equilibria seem unlikely to be of practical relevance in the context of the policy questions we want to address in this paper. This result provides some justification for our focus, following standard practice, on *local* rational-expectations equilibria in the rest of the paper (i.e. rational-expectations equilibria in the neighborhood of the steady state).

this basically pins down the demand for real reserves. Since the nominal stock of reserves is exogenous, pinning down the demand for real reserves also pins down the price level.

Proposition 1 thus implies that the type of indeterminacy discussed in Sargent and Wallace (1975) does not arise in our model. This type of indeterminacy associates any value in a continuum of initial price levels with the same constant values for real variables (and inflation). In our setup, the initial price level is uniquely pinned down at the steady state.

Given Lemma 2, the steady-state employment level $h = \mathcal{F}^{-1}(\beta I^m - 1)$ is strictly increasing in the steady-state IOR rate I^m. This is because an increase in I^m reduces the opportunity cost of holding reserves $I^b/I^m = (\beta I^m)^{-1}$. The lower opportunity cost, in turn, decreases the banking cost Γ and the banking spread I^ℓ/I^b. The lower spread (borrowing cost) increases the real wage, which stimulates employment and output.

3.3 Local-Equilibrium Determinacy

We now log-linearize the model around its unique steady state and show that there is a unique rational-expectations equilibrium in the neighborhood of this steady state.

In Appendix C.1, we show that the aggregate price-setting behavior of firms is described by the following log-linearized Phillips curve:

$$\pi_t = \beta \mathbb{E}_t \left\{ \pi_{t+1} \right\} + \kappa_y \widehat{y}_t - \kappa_m \widehat{m}_t - \kappa_g \widetilde{g}_t, \tag{34}$$

where variables with hats denote log deviations from steady-state values, $\pi_t \equiv \log(\Pi_t)$, $\widetilde{g}_t \equiv (g/y)\widehat{g}_t$, and $\kappa_y > 0$, $\kappa_m > 0$, and $\kappa_g > 0$ are three reduced-form parameters defined in Appendix C.1. This Phillips curve departs from the standard NK Phillips curve in two respects. First, the parameters κ_y and κ_g now depend (positively) on $\Gamma_{\ell\ell}$, as an increase in output (for given government expenditures) or a decrease in government expenditures (for a given output) raises firms' marginal cost of production also through the resulting increase in loans and banking costs. Second, and more importantly, a new term appears on the right-hand side, $-\kappa_m \widehat{m}_t$, which reflects a cost channel of monetary policy: an increase in real reserve balances reduces firms' marginal cost of production through the resulting decrease in banking costs. The parameter κ_m thus depends (positively) on $|\Gamma_{\ell m}|$.

Log-linearizing the Euler equation (9), and using the goods-market-clearing condition (19), gives the standard IS equation

$$\widehat{y}_t = \mathbb{E}_t \left\{ \widehat{y}_{t+1} \right\} - \frac{1}{\sigma} \mathbb{E}_t \left\{ i_t^b - \pi_{t+1} \right\} + \widetilde{g}_t - \mathbb{E}_t \left\{ \widetilde{g}_{t+1} \right\}, \tag{35}$$

where $i_t^b \equiv \widehat{I}_t^b$ and $\sigma \equiv -u''(c)y/u'(c) > 0$. This IS equation involves the interest rate on bonds, i_t^b, which is not directly controlled by the central bank. To relate this interest rate to the monetary-policy instruments, we log-linearize the first-order condition (12) in Appendix

C.1 and get

$$i_t^b - i_t^m = \sigma\delta_y\widehat{y}_t - \sigma\delta_m\widehat{m}_t - \sigma\delta_g\widetilde{g}_t, \tag{36}$$

where $i_t^m \equiv \widehat{I}_t^m$ and $\delta_y > 0$, $\delta_m > 0$, and $\delta_g > 0$ are three reduced-form parameters defined in Appendix C.1. Thus, the spread between the interest rates on bonds and on reserves depends positively on output, and negatively on real reserve balances and government expenditures. The reason is that this spread represents the marginal opportunity cost of holding reserves (rather than bonds serving only as stores of value). It has to be equal to the marginal benefit of holding reserves, i.e. the marginal effect of reserves on banking costs, which depends positively on loans − and hence positively on output (for given government expenditures) and negatively on government expenditures (for a given output) − and negatively on reserves. The parameters δ_y and δ_g thus depend (positively) on $|\Gamma_{\ell m}|$, and the parameter δ_m (positively) on Γ_{mm}.

Using the Phillips curve (34), the IS equation (35), the spread equation (36), and the (first difference of the) log-linearized reserve-market-clearing condition

$$\pi_t = -\Delta\widehat{m}_t + \widehat{\mu}_t \tag{37}$$

(where Δ denotes the first-difference operator), we then get the following dynamic equation in $\widehat{m}_t$:

$$\mathbb{E}_t\left\{L\mathcal{P}\left(L^{-1}\right)\widehat{m}_t\right\} = \frac{\kappa_y}{\beta\sigma}i_t^m + \mathbb{E}_t\left\{\mathcal{Q}_\mu\left(L^{-1}\right)\widehat{\mu}_t\right\} + \mathbb{E}_t\left\{\mathcal{Q}_g\left(L^{-1}\right)\widetilde{g}_t\right\}, \tag{38}$$

where L denotes the lag operator,

$$\begin{aligned}\mathcal{P}(X) \equiv\ & X^3 - \left[(1+\delta_y) + \frac{1+\beta-\kappa_m}{\beta} + \frac{\kappa_y}{\beta\sigma}\right]X^2 + ... \\ & \left[(1+\delta_y)\frac{1+\beta-\kappa_m}{\beta} + \frac{1}{\beta} + \left(\frac{1}{\sigma}+\delta_m\right)\frac{\kappa_y}{\beta}\right]X - \left(\frac{1+\delta_y}{\beta}\right),\end{aligned}$$

and $\mathcal{Q}_\mu(X)$ and $\mathcal{Q}_g(X)$ are two polynomials defined in Appendix C.1. The right-hand side of the dynamic equation (38) is exogenous, as it involves only monetary-policy instruments and government expenditures, so that $\mathcal{P}(X)$ is the (monic) characteristic polynomial of this dynamic equation. In Appendix C.2, we establish the following lemma − which holds whatever the functional forms of the (dis)utility and production functions u, v, v^b, f, and f^b, the values of the structural parameters $\beta \in (0,1)$, $\varepsilon > 0$, and $\theta \in (0,1)$, and the steady-state values of the IOR rate and government expenditures $I^m \in [1, \beta^{-1})$ and $g \geq 0$:

Lemma 3 (Roots of Polynomial $\mathcal{P}$): *The roots of $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that*

$$0 < \rho < 1 < \omega_1 < \omega_2.$$

This lemma straightforwardly implies that Blanchard and Kahn's (1980) conditions are met, so that we get the following proposition:

Proposition 2 (Local-Equilibrium Determinacy): *In the benchmark model, when I_t^m, μ_t, and g_t vary exogenously around the values $I^m \in [1, \beta^{-1})$, $\mu = 1$, and $g \geq 0$, there is a unique rational-expectations equilibrium in the neighborhood of the unique steady state.*

This proposition stands in contrast to a well known result of the NK literature. In the basic NK model, there is only one interest rate, namely the interest rate on bonds, and setting it exogenously leads to local-equilibrium *indeterminacy* for all structural-parameter values, as shown in Woodford (2003, Chapter 4) and Galí (2015, Chapter 4). In most extensions of the basic NK model, there is also only one interest rate, and setting it exogenously typically leads to local-equilibrium indeterminacy as well.[24] In our model, by contrast, there are two interest rates, one on bonds and the other on reserves, and setting exogenously the interest rate on reserves (together with the stock of reserves) leads to local-equilibrium *determinacy* for all functional forms of the (dis)utility and production functions u, v, v^b, f, and f^b, and all values of the structural and (steady-state) policy parameters β, ε, θ, I^m, and g. In the next subsection, we interpret this determinacy result with the help of a "shadow Wicksellian rule."

3.4 A Shadow Wicksellian Rule

The key element at the source of last subsection's determinacy result is the equilibrium relationship (36) between the interest rate on bonds, the interest rate on reserves, output, and real reserve balances. This relationship can be viewed as a "shadow rule" for the interest rate on bonds i_t^b, as if the central bank directly controlled this interest rate. Since the IOR rate and nominal reserve balances are set exogenously, this shadow rule is "Wicksellian" in the terminology of Woodford (2003): it makes i_t^b react positively to output, the price level (through $\widehat{m}_t$), and no other endogenous variable. It is well known that Wicksellian rules always ensure local-equilibrium determinacy in the basic NK model − as Woodford (2003, Chapter 4) shows. Our model, however, differs from the basic NK model in that it features a cost channel of monetary policy (i.e. $\kappa_m > 0$). We would not always get local-equilibrium determinacy if the parameters κ_y and κ_m of the Phillips curve and the coefficients $\sigma\delta_y$ and $\sigma\delta_m$ of the shadow Wicksellian rule for i_t^b were allowed to take any independent positive values. We always get local-equilibrium determinacy only because these reduced-form parameters and coefficients are related to each other through inequality constraints, as they come from the same primitive functions (most notably the production function f^b and the disutility function v^b).

More specifically, the two key inequality constraints that are proved and used in Appendix C.2 to establish local-equilibrium determinacy are

$$\delta_y \kappa_m < \delta_m \kappa_y, \tag{39}$$

$$\sigma \delta_m < \delta_y. \tag{40}$$

[24] Woodford (2003, Chapter 8) calls this result the "Sargent-Wallace property" of NK models.

The inequality (39), in particular, corresponds to the well known "Taylor principle" discussed by Woodford (2003, Chapter 4). This principle, which is a necessary condition for local-equilibrium determinacy in the basic NK model under a variety of interest-rate rules, states that the nominal interest rate should react more than one-to-one to the inflation rate in the long run. In our model, the relationship between the nominal interest rate on bonds and the inflation rate in the long run can be easily derived from (34), (36), and (37) as $\Delta i^b = (\delta_m \kappa_y - \delta_y \kappa_m)\sigma\pi/\kappa_y$, where Δi^b and π denote the long-run values of Δi_t^b and π_t respectively. Thus, the inequality (39) implies that the nominal interest rate on bonds reacts infinitely more than one-to-one to the inflation rate in the long run; and, therefore, that the shadow rule for i_t^b implied by the exogenous setting of i_t^m and $\widehat{\mu}_t$ satisfies the Taylor principle.

Another way to see the key role played by this shadow rule in Proposition 2's determinacy result is to consider for a moment the alternative case in which the central bank sets the reserves-growth rate $\widehat{\mu}_t$ exogenously, adopts an exogenous target i_t^{b*} for the interest rate on bonds i_t^b, and sets endogenously the interest rate on reserves i_t^m according to the feedback rule

$$i_t^m = i_t^{b*} - \sigma\delta_y \widehat{y}_t + \sigma\delta_m \widehat{m}_t + \sigma\delta_g \widetilde{g}_t, \tag{41}$$

which corresponds to (36) in which i_t^b is replaced by i_t^{b*}.[25] In this case, the central bank hits its target i_t^{b*} both in and out of equilibrium, as (36) and (41) imply $i_t^b = i_t^{b*}$. Then, the dynamics of $\widehat{y}_t$, π_t, and $\widehat{m}_t$ are governed by the three-equation system made of the Phillips curve (34), the IS equation (35) in which i_t^b is replaced by i_t^{b*}, and the reserve-market-clearing condition (37), while i_t^m is residually determined by the feedback rule (41). Thus, the relationship (36) plays no role in local-equilibrium (in)determinacy in this case. Using (34), (35) with $i_t^b = i_t^{b*}$, and (37), we then get the following dynamic equation in π_t:

$$\beta \mathbb{E}_t\{\pi_{t+2}\} - \left[1 + \beta + \left(\frac{\kappa_y}{\sigma} - \kappa_m\right)\right]\mathbb{E}_t\{\pi_{t+1}\} + \pi_t = -\frac{\kappa_y}{\sigma} i_t^{b*} + \kappa_m \mathbb{E}_t\{\widehat{\mu}_{t+1}\} - (\kappa_y - \kappa_g)\,\mathbb{E}_t\{\Delta\widetilde{g}_{t+1}\}.$$

Now, the inequalities (39) and (40) together imply the inequality $\kappa_y/\sigma - \kappa_m > 0$. Using this last inequality, we easily show that the characteristic polynomial of this dynamic equation has one root inside the unit circle and one root outside.[26] Thus, when the central bank "sets" exogenously i_t^b and $\widehat{\mu}_t$, we always get local-equilibrium *indeterminacy*. What matters for Proposition 2's determinacy result is not the exogeneity of *either* interest rate and the reserves-growth rate: it is the exogeneity of the *IOR rate* and the reserves-growth rate.

In this sense, our model does not disagree with standard NK models about the consequences of pegging the interest rate on a bond that serves only as a store of value (even when the central

[25]The corresponding global feedback rule for I_t^m is $I_t^m/I_t^{b*} = 1 + \Gamma_m(\ell_t, m_t)/\lambda_t$, where I_t^{b*} is the exogenous target for I_t^b. Since $\mu = 1$, we need the steady-state value of I_t^{b*} to be equal to β^{-1} for a steady state to exist – and there is then an infinity of steady states. We are here log-linearizing the model around anyone of these steady states, keeping for simplicity the same notations as previously.

[26]This characteristic polynomial is isomorphic to its counterpart in the basic NK model under an interest-rate peg, as one moves from the latter to the former simply by replacing $\kappa/\sigma > 0$ by $\kappa_y/\sigma - \kappa_m > 0$, where κ denotes the slope of the standard NK Phillips curve.

bank jointly sets the reserves-growth rate exogenously). Our point, instead, is that the interest rate that central banks have to peg – and did peg – at the zero lower bound is the IOR rate, which is the interest rate that they directly control. In our model, pegging the IOR rate (jointly with setting the reserves-growth rate exogenously) delivers determinacy because reserves do *not* serve only as a store of value.[27]

4 Resolution of the Limit Puzzles and Paradox

In the basic NK model and its usual extensions, pegging the interest rate temporarily at its zero lower bound has (at least) three puzzling and paradoxical implications: the forward-guidance puzzle, the fiscal-multiplier puzzle, and the paradox of flexibility – which we call the "limit" puzzles and paradox because they are about the behavior of equilibrium outcomes at the limit as the duration of the peg goes to infinity, or as the degree of price stickiness goes to zero. In this section, we show that our model does not share any of these implications. The reason is that these implications are mechanically connected to NK models' property of exhibiting indeterminacy under a permanent interest-rate peg. In our model, once interest rates get close to their zero lower bound, the central bank has to suspend whatever rule it would pursue under more normal circumstances and instead peg temporarily the IOR rate, which is the interest rate that it directly controls. Because a permanent peg of the IOR rate (with exogenous nominal reserves) delivers determinacy in our model, a temporary peg of the IOR rate (with exogenous nominal reserves) does not give rise to any of the limit puzzles and paradox.

For simplicity, we maintain in this section the assumption that the stock of nominal reserves is exogenous. In our view, this assumption is not necessarily a bad approximation of reality. The way central banks have conducted balance-sheet policies over the past few years – while pegging their policy rates close to zero – seems to us more like discretionary changes in policy than rules: we cannot detect any systematic pattern of balance-sheet policies reacting to, say, quarterly data, and the Federal Reserve's policy in the last years of its policy-rate peg was to keep the size of its balance sheet constant (replacing assets as they mature). We will show in Subsection 7.1 that our determinacy results – and hence our resolution of the limit puzzles and paradox – are essentially robust to the relaxation of this assumption.

4.1 The Forward-Guidance Puzzle

The first puzzle that we consider is the so-called "forward-guidance puzzle," which can be summarized as follows: in the basic NK model, the effects of a temporary interest-rate peg

[27] Similarly, if the central bank did "set" exogenously i_t^b and i_t^m, we would always get local-equilibrium *indeterminacy* in our model, as can be easily inferred from Subsection 6.4's results. But central banks did not seem to adjust the stock of reserves to hit a specific target for the interest rate on bonds. In fact, in the U.S., the effective federal-funds rate was consistently below the official target in the last months of 2008, and the Federal Reserve moved from a target point to a target range when reaching the zero lower bound in December 2008.

on current inflation and output become unboundedly large as the duration of this peg goes to infinity, if the economy is back to steady state after the peg. This result, shown in Werning (2012), Carlstrom, Fuerst, and Paustian (2015), and Cochrane (2017a), is a puzzle notably because a *permanent* interest-rate peg has *finite* effects on inflation and output.[28] There is, thus, a stark discontinuity in the limit as the duration of the peg goes to infinity.

The temporary interest-rate peg in question may be due to a situation in which (i) a liquidity trap compels the central bank to keep the interest rate at its lower bound during T_1 periods, and (ii) the central bank promises to keep the interest rate at its lower bound during T_2 periods after the end of the trap (hence the name of "forward-guidance puzzle"). The puzzle in this case has two manifestations: the log-deviations of inflation and output from their steady-state values, at the start of the trap, become infinitely negative as $T_1 \rightarrow +\infty$ for a given T_2, and infinitely positive as $T_2 \rightarrow +\infty$ for a given T_1. These two manifestations are, of course, mirror images of each other.

To see how our model solves this puzzle, consider a temporary peg of the IOR rate: assume that the economy is at the steady state at date 0 (so that $\widehat{m}_0 = 0$), and that, unexpectedly, i_t^m (the log-deviation of the IOR rate from its steady-state value) takes the value i^* from date 1 to date T, and the value zero from date $T+1$ onwards.[29] As in Werning (2012), Carlstrom, Fuerst, and Paustian (2015), and Cochrane (2017a), we assume for simplicity that T is deterministic and known at date 1. In the NK literature, liquidity traps are typically obtained as the result of a large negative discount-factor shock. For simplicity again, following Carlstrom, Fuerst, and Paustian (2015) and Cochrane (2017a), we do not explicitly introduce such a shock into our model. Thus, in our setup, a positive value of i^* can represent a liquidity-trap situation in which setting the interest rate at its lower bound is not enough to offset the negative discount-factor shock; and a negative value of i^* can represent a situation in which, in accordance with some earlier forward guidance, the central bank keeps the interest rate at its lower bound even though the negative discount-factor shock has ceased to affect the economy.

In the absence of reserves-growth-rate and government-expenditures shocks (i.e., when $\widehat{\mu}_t = \widetilde{g}_t = 0$), the dynamic equation (38) can be rewritten as

$$\mathbb{E}_t \left\{ (1 - \omega_1 L)(1 - \omega_2 L) q_{t+2} \right\} = \frac{\kappa_y}{\beta \sigma} i_t^m, \tag{42}$$

where $q_t \equiv \widehat{m}_t - \rho \widehat{m}_{t-1}$. The unique stationary solution to this dynamic equation from date $T+1$ onwards is $q_t = 0$ for $t \geq T+1$. Using this dynamic equation from date 1 to date T, together with the terminal conditions $q_{T+1} = q_{T+2} = 0$, we then get

$$q_t = \left(1 - d_1 \omega_1^{t-T-1} + d_2 \omega_2^{t-T-1}\right) q^*$$

[28] A permanent interest-rate peg generates multiple local equilibria in the basic NK model. What we mean is that inflation and output take finite values in each of these equilibria.

[29] As we explain at the end of the subsection, our assumption that the central bank reverts to a permanent peg of i_t^m at date $T+1$ (rather than to a rule for i_t^b as in the literature) is made for simplicity and does not matter for our resolution of the forward-guidance puzzle − nor will it matter for our resolution of the fiscal-multiplier puzzle and the paradox of flexibility in the following subsections.

for $1 \leq t \leq T+2$, where

$$d_1 \equiv \frac{\omega_2 - 1}{\omega_2 - \omega_1}, \quad d_2 \equiv \frac{\omega_1 - 1}{\omega_2 - \omega_1}, \quad \text{and} \quad q^* \equiv \frac{\kappa_y i^*}{\beta\sigma\left(\omega_1 - 1\right)\left(\omega_2 - 1\right)}.$$

Finally, using the initial condition $\widehat{m}_0 = 0$, the Phillips curve (34), and the reserve-market-clearing condition (37), we get

$$\pi_1 = -\left(1 - d_1\omega_1^{-T} + d_2\omega_2^{-T}\right) q^*, \tag{43}$$

$$\widehat{y}_1 = \Big\{-\left(1 - \beta\rho - \kappa_m\right) - d_1\omega_1^{-T}\left[\beta\left(\omega_1 + \rho - 1\right) + \kappa_m - 1\right] + ... \\ d_2\omega_2^{-T}\left[\beta\left(\omega_2 + \rho - 1\right) + \kappa_m - 1\right]\Big\} \frac{q^*}{\kappa_y}. \tag{44}$$

As the duration of the peg goes to infinity, π_1 and $\widehat{y}_1$ converge towards some finite values:

$$\lim_{T\to+\infty} \pi_1 = -q^* \quad \text{and} \quad \lim_{T\to+\infty} \widehat{y}_1 = -\left(\frac{1 - \beta\rho - \kappa_m}{\kappa_y}\right) q^*.$$

These finite limit values coincide with the values that π_1 and $\widehat{y}_1$ would take under a permanent peg. Indeed, if i_t^m took the value i^* at all dates $t \geq 1$, then the unique stationary solution of the dynamic equation (42) would be $q_t = q^*$ for $t \geq 1$. Using $\widehat{m}_0 = 0$, the Phillips curve (34), and the reserve-market-clearing condition (37), we would then get $\pi_1 = -q^*$ and $\widehat{y}_1 = -\left(1 - \beta\rho - \kappa_m\right) q^*/\kappa_y$. As a consequence, we obtain the following result:

Proposition 3 (Resolution of the Forward-Guidance Puzzle): *In the benchmark model, the responses of π_1 and $\widehat{y}_1$ to a temporary IOR-rate peg of expected duration T ($i_t^m = i^*$ for $1 \leq t \leq T$, $i_t^m = 0$ for $t \geq T+1$) converge, as T goes to $+\infty$, towards their (finite) responses to the corresponding permanent IOR-rate peg ($i_t^m = i^*$ for $t \geq 1$).*

In the basic NK model, as pointed out by Carlstrom, Fuerst, and Paustian (2015) and Cochrane (2017a), the source of the forward-guidance puzzle lies in the model's property of exhibiting indeterminacy under a permanent interest-rate peg. Indeed, this property implies that, under a temporary interest-rate peg, the dynamic system has a "stable eigenvalue" (i.e. an eigenvalue whose modulus is lower than one) that is not matched by any predetermined variable. As a consequence, starting from some terminal condition (at the end of the peg), the economy explodes backward in time. In our model, by contrast, the stable eigenvalue (ρ) is matched by a predetermined variable, namely the lagged stock of reserves ($\widehat{m}_{t-1}$). This is the feature that delivers determinacy under a permanent IOR-rate peg and, therefore, also solves the forward-guidance puzzle.

Our resolution of the forward-guidance puzzle does not rest on our simplifying assumption that the central bank reverts to a permanent peg of i_t^m at date $T+1$. In the alternative case in which the central bank would set its policy instruments i_t^m and μ_t from date $T+1$ onwards to, say, follow a rule for i_t^b ($i_t^b = \phi\pi_t$ with $\phi > 1$) under a corridor system ($i_t^b - i_t^m = 0$), the computations would be more complex, but the logic of the argument would not change: there would still be two terminal conditions (at the end of the peg) to match the two unstable eigenvalues (ω_1 and ω_2), and one initial condition ($\widehat{m}_0 = 0$) to match the stable eigenvalue (ρ), so that the economy would not explode as the duration of the peg goes to infinity.[30]

4.2 The Fiscal-Multiplier Puzzle

We now turn to what we call the "fiscal-multiplier puzzle." Consider an interest-rate peg of known duration, and assume that the government credibly announces that it will increase government expenditures by a given amount at the end of the peg. In the basic NK model, as shown in Farhi and Werning (2016) and Cochrane (2017a), the effect of this expected future fiscal expansion on inflation and output at the start of the peg grows exponentially with the duration of the peg, if the economy is back to steady state after the peg.[31] In fact, this effect still grows exponentially even when the amount of government expenditures declines towards zero as the duration of the peg goes to infinity, provided that it does not decline too fast. Thus, news about government expenditures that are both vanishingly distant and vanishingly small can have exploding effects today.

As pointed out by Cochrane (2017a), the culprit is, again, the basic NK model's property of exhibiting indeterminacy under a permanent interest-rate peg. Indeed, this property implies that, under a temporary interest-rate peg, the dynamic system has a stable eigenvalue but no predetermined variable. As a consequence, when we iterate the model backward in time, this eigenvalue magnifies the effects of the fiscal expansion (at the end of the peg) on initial outcomes (at the start of the peg), so that these effects grow explosively as the duration of the peg goes to infinity, giving rise to the fiscal-multiplier puzzle. Our model, by contrast, delivers determinacy under a permanent IOR-rate peg, as we have shown in the previous section. As a consequence, it solves the fiscal-multiplier puzzle, as we now show.

As in the previous subsection, we shut down reserves-growth-rate shocks ($\widehat{\mu}_t = 0$) and consider a temporary IOR-rate peg followed by a permanent one ($i_t^m = i^*$ for $1 \leq t \leq T$, $i_t^m = 0$ for $t \geq T+1$).[32] For simplicity, we assume further that the IOR rate is pegged at its steady-state value not only under the permanent peg, but also under the temporary one (i.e., $i^* = 0$).[33] In

[30] We omit the derivations to save space.

[31] The analyses of Christiano, Eichenbaum, and Rebelo (2011), Eggertsson (2011), and Woodford (2011) are related but distinct in two main ways. First, they assume that the length of the liquidity-trap period is stochastic, not deterministic. Second, they consider a fiscal expansion that lasts over the whole liquidity-trap period. Woodford (2011) shows that the resulting fiscal multiplier becomes unboundedly large as the probability to remain in the liquidity trap approaches a certain value (lower than one).

[32] For the same reason as previously, our assumption that the central bank reverts to a permanent peg of i_t^m at date $T+1$ (rather than to a rule for i_t^b as in the literature) is made for simplicity and does not matter for our resolution of the fiscal-multiplier puzzle.

[33] This assumption does not affect our results since the effect of $\widetilde{g}_t$ is independent of the effect of i^* in our log-linearized setup.

this case, the dynamic equation (38) can be rewritten as

$$\mathbb{E}_t\left\{(1-\omega_1 L)(1-\omega_2 L)\, q_{t+2}\right\} = \left[\frac{(1+\delta_y)\,\kappa_g - (1+\delta_g)\,\kappa_y}{\beta}\right]\widetilde{g}_t + \left(\frac{\kappa_y-\kappa_g}{\beta}\right)\mathbb{E}_t\left\{\widetilde{g}_{t+1}\right\}. \quad (45)$$

Now assume that the economy is at the steady state at date 0 (so that $\widehat{m}_0 = 0$), and that the government unexpectedly announces at date 1 that (i) $\widetilde{g}_T = \widetilde{g}^* \neq 0$ for some date $T \geq 2$, and (ii) $\widetilde{g}_t = 0$ for all dates $t \geq 1$ such that $t \neq T$. Using (45) from date $T+1$ onwards, we first get $q_t = 0$ for $t \geq T+1$. Then, using (45) at date T and $q_{T+1} = q_{T+2} = 0$, we get

$$q_T = \left[\frac{(1+\delta_y)\,\kappa_g - (1+\delta_g)\,\kappa_y}{\beta\omega_1\omega_2}\right]\widetilde{g}^*.$$

Similarly, using (45) at date $T-1$, $q_{T+1} = 0$, and the above expression for q_T, we get

$$q_{T-1} = \left\{\frac{(\kappa_y-\kappa_g)\,\omega_1\omega_2 + \left[(1+\delta_y)\,\kappa_g - (1+\delta_g)\,\kappa_y\right](\omega_1+\omega_2)}{\beta\omega_1^2\omega_2^2}\right\}\widetilde{g}^*.$$

Next, using (45) from date 1 to date $T-2$, together with the above terminal conditions on q_{T-1} and q_T, we get

$$q_t = \left[\frac{e_1\omega_1^{t-T-1} - e_2\omega_2^{t-T-1}}{\beta\,(\omega_2-\omega_1)}\right]\widetilde{g}^*$$

for $1 \leq t \leq T$, where $e_j \equiv (\kappa_y-\kappa_g)\omega_j + [(1+\delta_y)\kappa_g - (1+\delta_g)\kappa_y]$ for $j \in \{1,2\}$. Finally, using the initial condition $\widehat{m}_0 = 0$, the Phillips curve (34), and the reserve-market-clearing condition (37), we get

$$\pi_1 = \left[\frac{e_2\omega_2^{-T} - e_1\omega_1^{-T}}{\beta\,(\omega_2-\omega_1)}\right]\widetilde{g}^*, \quad (46)$$

$$\widehat{y}_1 = \left\{\frac{\left[\beta\,(\omega_1+\rho-1)+\kappa_m-1\right]e_1\omega_1^{-T} - \left[\beta\,(\omega_2+\rho-1)+\kappa_m-1\right]e_2\omega_2^{-T}}{\beta\kappa_y\,(\omega_2-\omega_1)}\right\}\widetilde{g}^*, \quad (47)$$

and therefore

$$\lim_{T\to+\infty}\pi_1 = 0 \quad \text{and} \quad \lim_{T\to+\infty}\widehat{y}_1 = 0.$$

This result can be stated in the following way:

Proposition 4 (Resolution of the Fiscal-Multiplier Puzzle): *In the benchmark model, the responses of π_1 and $\widehat{y}_1$ to a given expected fiscal expansion at date T converge towards zero as T goes to $+\infty$.*

Again, the resolution of the puzzle reflects the fact that our model economy does not explode as we go backward in time under a temporary interest-rate peg. And this is because the stable eigenvalue (ρ) is matched by a predetermined variable ($\widehat{m}_{t-1}$).

4.3 The Paradox of Flexibility

We finally consider the so-called "paradox of flexibility." In the basic NK model, the effects on inflation and output of an interest-rate peg of given finite duration become unboundedly large as prices become perfectly flexible, if the economy is back to steady state after the peg.[34] Similarly, the effect on inflation and output of a given fiscal expansion at the end of the peg also grows explosively as prices become perfectly flexible. These results, shown in Werning (2012), Farhi and Werning (2016), and Cochrane (2017a), are puzzling because output, expected inflation, and fiscal multipliers take finite values in the limit case of perfectly flexible prices (while realized inflation is indeterminate).[35] There is, thus, a stark discontinuity in the limit as the degree of price stickiness goes to zero.

Unlike the previous puzzles, the paradox of flexibility is about the limit not as the duration of the peg goes to infinity, but instead as prices become perfectly flexible. Like them, however, it is related to the basic NK model's property of exhibiting indeterminacy under a permanent interest-rate peg. Indeed, under a temporary peg, the stable eigenvalue of the dynamic system, which is not matched by any predetermined variable, converges towards zero as price stickiness vanishes, making endogenous variables explode.[36] In our model, by contrast, endogenous variables converge towards their finite flexible-price values as price stickiness vanishes, because there is no such excess stable eigenvalue.

To prove that our model does indeed solve the paradox of flexibility, we first establish the following lemma in Appendix C.3:

Lemma 4 (Limits of Some Parameters as $\theta \to 0$): *As $\theta \to 0$, we have*

$$\rho \to 0, \qquad \omega_1 \to \omega_1^n \equiv 1 + \frac{\sigma(\delta_m - \delta_y \psi)}{1 - \sigma\psi}, \qquad \omega_2 \to +\infty,$$
$$\kappa_m \rho \to \frac{\sigma\psi(1+\delta_y)}{(1-\sigma\psi) + \sigma(\delta_m - \delta_y\psi)}, \quad \text{and} \quad \frac{\omega_2}{\kappa_y} \to \frac{1-\sigma\psi}{\beta\sigma},$$

where $\psi \equiv \kappa_m/\kappa_y$ is independent of θ and such that $0 < \psi < \min(\sigma^{-1}, \delta_m/\delta_y)$.

Using this lemma, we can easily determine the limits of π_1 in (43) and $\widehat{y}_1$ in (44) as $\theta \to 0$:

$$\lim_{\theta \to 0} \pi_1 = -\left[\frac{1-(\omega_1^n)^{-T}}{\delta_m - \delta_y\psi}\right]\frac{i^*}{\sigma} \quad \text{and} \quad \lim_{\theta \to 0} \widehat{y}_1 = \left[\frac{1-(\omega_1^n)^{-T}}{\delta_m - \delta_y\psi}\right]\frac{\psi i^*}{\sigma}, \tag{48}$$

[34] Again, the sign of these unboundedly large effects depend on whether the temporary interest-rate peg in question is due to a liquidity trap ($i^* > 0$) or a forward-guidance policy ($i^* < 0$).

[35] Christiano, Eichenbaum, and Rebelo (2011) find numerically that inflation and output decrease and fiscal multipliers increase with price flexibility in a liquidity trap, but do not obtain limit results.

[36] The analytical expression for this stable eigenvalue, noted $\overline{\rho}$, is given in Appendix C.5. Using this expression, it is easy to check that $\lim_{\theta \to 0} \overline{\rho} = \lim_{\overline{\kappa}_y \to +\infty} \overline{\rho} = 0$.

as well as the limits of π_1 in (46) and $\widehat{y}_1$ in (47) as $\theta \to 0$ (for $T \geq 2$):

$$\lim_{\theta\to 0} \pi_1 = -\left[(1-\vartheta)\,\omega_1^n + (1+\delta_y)\,\vartheta - (1+\delta_g)\right]\left[\frac{\sigma\,(\omega_1^n)^{-T}}{1-\sigma\psi}\right]\widetilde{g}^*, \tag{49}$$

$$\lim_{\theta\to 0} \widehat{y}_1 = \left[(1-\vartheta)\,\omega_1^n + (1+\delta_y)\,\vartheta - (1+\delta_g)\right]\left[\frac{\sigma\psi\,(\omega_1^n)^{-T}}{1-\sigma\psi}\right]\widetilde{g}^*, \tag{50}$$

where $\vartheta \equiv \kappa_g/\kappa_y$ is independent of θ. These limits are finite, unlike their counterparts in the basic NK model. In Appendix B.2, we log-linearize the flexible-price version of our benchmark model (studied in Subsection 3.1) and show that the values taken by π_1 and $\widehat{y}_1$ under flexible prices coincide with the above limits. We summarize these results as follows:

Proposition 5 (Resolution of the Paradox of Flexibility): *In the benchmark model, the responses of π_1 and $\widehat{y}_1$ to a temporary IOR-rate peg of expected duration T and to a given expected fiscal expansion at date T converge, as θ goes to 0, towards the (finite) corresponding responses under flexible prices.*

Again, the resolution of the paradox reflects the fact that the stable eigenvalue (ρ) under an interest-rate peg is matched by a predetermined variable ($\widehat{m}_{t-1}$) in our model. So the convergence of this eigenvalue towards zero as prices become perfectly flexible does not translate into explosive equilibrium outcomes under a temporary peg, unlike in the basic NK model.

5 The Basic-NK-Model Limit

We have so far shown that our model solves the three limit puzzles and paradox for any given (dis)utility and production functions u, v, v^b, f, and f^b, any given values of the structural parameters $\beta \in (0,1)$, $\varepsilon > 0$, and $\theta \in (0,1)$, and any given steady-state values of the IOR rate and government expenditures $I^m \in [1, \beta^{-1})$ and $g \geq 0$. In this section, we first show that the disutility function v^b and the steady-state value I^m can be chosen so as to make our model arbitrarily close, in terms of steady state and reduced form, to the basic NK model characterized by the same (dis)utility and production functions u, v, and f, the same values of the structural parameters β, ε, and θ, and the same steady-state value of government expenditures g. Thus, our model solves these limit puzzles and paradox even for an *arbitrarily* small departure from the basic NK model.

In the limit, as we make the departure from the basic NK model *vanishingly* small, our model serves to select uniquely a particular equilibrium of the basic NK model under a temporary interest-rate peg followed by a permanent one. We show that this particular equilibrium (i) exhibits neither the fiscal-multiplier puzzle nor the paradox of flexibility, (ii) exhibits an attenuated form of the forward-guidance puzzle, and (iii) does not exhibit the paradox of toil. As we discuss, our selected equilibrium thus closes the gap between the basic NK model and Mankiw and Reis's (2002) sticky-information model in terms of their ability to solve or attenuate the four puzzles and paradoxes. We also discuss how our selected equilibrium compares to the basic NK model's equilibria considered by Cochrane (2017a).

5.1 Convergence Towards the Basic NK Model

To show that our model can involve an arbitrarily small departure from the basic NK model in a quantitatively measurable sense, we replace the disutility function v^b by γv^b (and hence the banking-cost function Γ by $\gamma\Gamma$), where $\gamma > 0$ is a scale parameter, and we establish the following proposition in Appendix C.4:

Proposition 6 (Convergence Towards the Steady State and Reduced Form of the Basic NK Model): *As $(I^m, \gamma) \to (\beta^{-1}, 0)$ with $(\beta^{-1} - I^m)/\gamma$ bounded away from zero and infinity, the steady state and reduced form of the benchmark model converge towards the steady state and reduced form of the basic NK model, i.e. $h \to h^\star$ and $(\kappa_y, \kappa_m, \kappa_g, \delta_y, \delta_m, \delta_g) \to (\overline{\kappa}_y, 0, \overline{\kappa}_g, 0, 0, 0)$, where $\overline{\kappa}_y$ and $\overline{\kappa}_g$ denote the coefficients of $\widehat{y}_t$ and $-\widetilde{g}_t$ in the standard NK Phillips curve.*

Making the steady-state IOR rate I^m go to the steady-state interest rate on bonds $I^b = \beta^{-1}$ asymptotically removes the steady-state opportunity cost of holding reserves. Making the banking-cost-scale parameter γ go to zero asymptotically removes: (i) the steady-state marginal banking cost Γ_ℓ, provided that the steady-state value of real reserve balances m is bounded away from zero, and (ii) the steady-state marginal benefit of holding reserves Γ_m, even when m is bounded from above. Imposing that $(\beta^{-1} - I^m)/\gamma$ be bounded away from zero and infinity ensures that the steady-state opportunity cost and marginal benefit of holding reserves go hand in hand to zero, so that m is itself bounded away from zero and infinity. Asymptotically, given that all steady-state costs related to banking and reserve holding are removed, the steady-state employment level takes its frictionless value ($h = h^\star$), the marginal cost of production becomes insensitive to the volume of loans ($\kappa_y = \overline{\kappa}_y$ and $\kappa_g = \overline{\kappa}_g$), the cost channel of monetary policy is shut down ($\kappa_m = 0$), and the interest-rate spread becomes insensitive to output, reserves, and government expenditures ($\delta_y = \delta_m = \delta_g = 0$).

Proposition 6 enables us to consider: (i) a sequence of models converging towards the basic NK model, each of them solving the limit puzzles and paradox; and (ii) the corresponding sequence of unique equilibrium outcomes under a temporary IOR-rate peg followed by a permanent one. The limit of this sequence of equilibrium outcomes coincides with a particular equilibrium of the basic NK model − out of an infinity of equilibria − under a temporary interest-rate peg followed by a permanent one. Thus, our approach provides an equilibrium-selection device in the basic NK model, based on a limit uniqueness result, which is reminiscent of the equilibrium-selection device developed in the global-games literature pioneered by Carlsson and van Damme (1993) and Morris and Shin (1998, 2000). In the rest of the section, we study the properties of our selected equilibrium.

5.2 Resolution or Attenuation of the Limit Puzzles and Paradox

We first show that our selected equilibrium exhibits neither the fiscal-multiplier puzzle nor the paradox of flexibility, and exhibits an attenuated form of the forward-guidance puzzle. More specifically, we proceed as follows in Appendix C.5: (i) we start from our benchmark model's unique equilibrium outcomes when $i_t^m = i^*$ for $1 \leq t \leq T$, $i_t^m = 0$ for $t \geq T+1$, $\widetilde{g}_T = \widetilde{g}^*$ for $t = T$, $\widetilde{g}_T = 0$ for $t \neq T$, and $\widehat{\mu}_t = 0$ for $t \geq 1$, for a given duration of the IOR-rate peg T and a given degree of price stickiness θ; (ii) we determine the limit of these equilibrium outcomes as $(I^m, \gamma) \to (\beta^{-1}, 0)$ with $(\beta^{-1} - I^m)/\gamma$ bounded away from zero and infinity; and (iii) we compute the limit of these limit equilibrium outcomes as $T \to +\infty$ and $\theta \to 0$. We obtain the following results:

Proposition 7 (Resolution of the Fiscal-Multiplier Puzzle and the Paradox of Flexibility and Attenuation of the Forward-Guidance Puzzle in the Basic-NK-Model Limit): *In our selected equilibrium of the basic NK model,*

(i) the responses of π_1 and $\widehat{y}_1$ to a temporary IOR-rate peg of expected duration T grow (asymptotically) linearly in T as $T \to +\infty$;

(ii) the responses of π_1 and $\widehat{y}_1$ to a given expected fiscal expansion at date T converge towards zero as $T \to +\infty$;

(iii) the responses of π_1 to a temporary IOR-rate peg of expected duration T and to a given expected fiscal expansion at date T converge towards some finite values as $\theta \to 0$;

(iv) the responses of $\widehat{y}_1$ to a temporary IOR-rate peg of expected duration T and to a given expected fiscal expansion at date T converge, as $\theta \to 0$, towards the (finite) corresponding responses in the flexible-price version of the basic NK model.

This proposition highlights three properties of our selected equilibrium. First, this equilibrium is immune from the fiscal-multiplier puzzle: the reaction of initial output and inflation to a given fiscal expansion at the end of the interest-rate peg does not grow explosively as the duration of the peg goes to infinity, but instead converges towards zero. Second, it is also immune from the paradox of flexibility: initial output and inflation do not explode as prices become perfectly flexible; instead, output converges towards its (finite) value in the flexible-price version of the basic NK model, and inflation towards some finite value.[37] And third, it

[37]The limit value of inflation cannot be related to its value in the flexible-price version of the basic NK model, since the latter value is indeterminate.

suffers from an attenuated form of the forward-guidance puzzle: initial output and inflation grow (asymptotically) linearly with the duration of the peg, not exponentially.

The reason why our selected equilibrium exhibits (an attenuated form of) the forward-guidance puzzle is that our benchmark model implies price-level stationarity in response to a temporary IOR-rate peg ($p_\infty = p_0$ when $i_t^m = i^*$ for $1 \leq t \leq T$ and $i_t^m = 0$ for $t \geq T+1$). Therefore, in our selected equilibrium of the basic NK model, the price level is also stationary in response to a temporary interest-rate peg ($p_\infty = p_0$ when $i_t^b = i^*$ for $1 \leq t \leq T$ and $i_t^b = 0$ for $t \geq T+1$). Now, iterating the IS equation (35) forward under this temporary interest-rate peg, using price-level stationarity, and using the terminal condition $\widehat{y}_\infty = 0$, leads to

$$\widehat{y}_1 = \frac{-Ti^*}{\sigma} - \frac{\pi_1}{\sigma}.$$

This relationship is consistent with $\widehat{y}_1$ and π_1 growing linearly in T (puzzle attenuation), but inconsistent with $\widehat{y}_1$ and π_1 remaining finite as $T \to +\infty$ (puzzle resolution). Thus, any equilibrium of the basic NK model in which the price level is stationary in response to a temporary interest-rate peg may attenuate but cannot fully solve the forward-guidance puzzle.[38]

5.3 Resolution of the Paradox of Toil

We now show that our selected equilibrium does not exhibit the so-called "paradox of toil."[39] This paradox can be summarized as follows: positive supply shocks − such as downward shifts in the labor-disutility function, labor-tax cuts, technology improvements, and reductions in market power − are contractionary under a temporary interest-rate peg in the "standard equilibrium" of the basic NK model (i.e. the equilibrium in which the economy is back to its steady state immediately after the peg).

This implication of the basic NK model under a temporary interest-rate peg is not puzzling in the same way as the three implications that we have addressed so far: it is about sign issues for a given duration of the peg and a given degree of price stickiness, not about size issues in the limit as the duration of the peg goes to infinity, or as the degree of price stickiness goes to zero. Nonetheless, we show that the presence of a state variable in our benchmark model − the stock of reserves − serves to solve not only the three limit puzzles and paradox, but also the paradox of toil. While it solves the former by matching the stable eigenvalue and thus delivering determinacy under a permanent IOR-rate peg, as we showed in the previous section, it solves the latter by generating inflation inertia (which does not vanish in the basic-NK-model limit), as we now show.

[38] If the temporary IOR-rate peg that we consider in our benchmark model were accompanied by a temporary change in the reserves-growth rate of the same amount ($i_t^m = \widehat{\mu}_t = i^*$ for $1 \leq t \leq T$ and $i_t^m = \widehat{\mu}_t = 0$ for $t \geq T+1$), then the price level would not be stationary, and our selected equilibrium would not exhibit the forward-guidance puzzle.

[39] We have no particular reason to doubt that our benchmark model also solves this paradox for any calibration outside its basic-NK-model limit (based on our experimentations with the relevant restrictions on parameters), but we have not been able to prove it.

For simplicity, we focus on the effects of cost-push shocks, i.e. exogenous variations in the elasticity of substitution between differentiated goods.[40] The introduction of cost-push shocks into our model leaves the IS equation (35), the spread equation (36), and the reserve-market-clearing condition (37) unchanged, and transforms the Phillips curve (34) into

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa_y \widehat{y}_t - \kappa_m \widehat{m}_t - \kappa_g \widetilde{g}_t + \kappa_\varphi \widehat{\varphi}_t, \tag{51}$$

where $\widehat{\varphi}_t$ denotes the log-deviation of desired markups (i.e. flexible-price markups $\varepsilon_t/(\varepsilon_t - 1)$) from their steady-state value, and $\kappa_\varphi > 0$ is a reduced-form parameter defined in Appendix C.6. We assume that: (i) the economy is at the steady state at date 0 (so that $\widehat{m}_0 = 0$); (ii) agents learn at date 1 that $\widehat{\varphi}_t$ will take the value $\widehat{\varphi}^* > 0$ from date 1 to date T, and the value zero from date $T+1$ onwards; and (iii) no other shock hits the economy ($i_t^m = \widehat{\mu}_t = \widetilde{g}_t = 0$). In Appendix C.6, we derive the unique equilibrium outcomes of our benchmark model in this case, and we determine the limit of these equilibrium outcomes as $(I^m, \gamma) \to (\beta^{-1}, 0)$ with $(\beta^{-1} - I^m)/\gamma$ bounded away from zero and infinity. We obtain the following results:

Proposition 8 (Resolution of the Paradox of Toil in the Basic-NK-Model Limit): *In our selected equilibrium of the basic NK model, a positive cost-push shock lasting from date* 1 *to date* T *is contractionary from date* 1 *to date* T *(i.e. such that* $\widehat{y}_t < 0$ *for* $1 \leq t \leq T$*).*

Thus, our benchmark model solves the paradox of toil for a vanishingly small departure from the basic NK model − as well as, by continuity, for a sufficiently small departure. To understand how it does so, it is useful to understand first how this paradox arises in the standard equilibrium of the basic NK model. So consider the basic NK model, with its IS equation $\widehat{y}_t = \mathbb{E}_t\{\widehat{y}_{t+1}\} - (i_t^b - \mathbb{E}_t\{\pi_{t+1}\})/\sigma$ and its Phillips curve $\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \kappa\widehat{y}_t + \kappa_\varphi\widehat{\varphi}_t$, where $\kappa > 0$. Assume again that $\widehat{\varphi}_t = \widehat{\varphi}^* > 0$ for $1 \leq t \leq T$ and $\widehat{\varphi}_t = 0$ for $t \geq T+1$. Assume moreover that the central bank pegs i_t^b to, say, zero from date 1 to date T, and that the economy is back to its steady state from date $T+1$ onwards. Since $\pi_{T+1} = \widehat{y}_{T+1} = 0$, the IS equation at date T implies $\widehat{y}_T = 0$, and the Phillips curve at date T then implies $\pi_T > 0$. In turn, since $\pi_T > 0$ and $\widehat{y}_T = 0$, the IS equation at date $T-1$ implies $\widehat{y}_{T-1} > 0$, and the Phillips curve at date $T-1$ then implies $\pi_{T-1} > 0$. Iterating this reasoning backward in time eventually leads to $\pi_t > 0$ for $1 \leq t \leq T$, $\widehat{y}_t > 0$ for $1 \leq t \leq T-1$, and $\widehat{y}_T = 0$: output is positive or zero during the whole higher-desired-markup period.

The key difference is that, in our selected equilibrium, $\pi_{T+1} < 0$ (as we show in Appendix C.6). That inflation is *non-zero* at $T+1$ comes from the fact that our benchmark model has a state variable, the stock of reserves, which introduces inertia into the dynamics − and this inertia is crucially preserved in the basic-NK-model limit. That inflation is *negative* at $T+1$ seems

[40] Cost-push shocks are considered in, e.g., Eggertsson (2010, 2012) and Eggertsson, Ferrero, and Raffo (2014). Our model, in its basic-NK-model limit, also solves the paradox of toil in response to other supply shocks like productivity shocks, labor-disutility shocks, and labor-tax shocks (which show up not only in the Phillips curve, but also in the spread equation, unlike cost-push shocks). We omit the derivations to save space.

natural, as desired markups fall from date T to date $T+1$. Now, iterating the basic NK model's IS equation and Phillips curve forward from the initial condition $\pi_{T+1} < 0$ leads to $\pi_t < 0$ and $\widehat{y}_t < 0$ for all $t \geq T+1$. Then, $\pi_{T+1} < 0$ and $\widehat{y}_{T+1} < 0$ imply $\widehat{y}_T < 0$ (through the IS equation at date T): output is negative at the last date of the higher-desired-markup period.

To understand why output is also negative from date 1 to date $T-1$, we show in Appendix C.6 that $\pi_1 > 0$ and that π_t is decreasing in t for $1 \leq t \leq T+1$. These results seem sensible too: inflation rises from zero at the start of the higher-desired-markup period, and then decreases as the remaining duration of the higher-desired-markup period shortens. Call $T_\pi \in \{2, ..., T+1\}$ the first date when inflation turns negative. Using the IS equation, we get that if $T_\pi = 2$, then output increases from date 1 to date T ($\Delta\widehat{y}_t > 0$ for $2 \leq t \leq T$); since $\widehat{y}_T < 0$, this implies that $\widehat{y}_t < 0$ for $1 \leq t \leq T$. Alternatively, if $3 \leq T_\pi \leq T$, then output first decreases from date 1 to date $T_\pi - 1$ ($\Delta\widehat{y}_t < 0$ for $2 \leq t \leq T_\pi - 1$), and then increases from date $T_\pi - 1$ to date T ($\Delta\widehat{y}_t > 0$ for $T_\pi \leq t \leq T$); and if $T_\pi = T+1$, then output decreases from date 1 to date T ($\Delta\widehat{y}_t < 0$ for $2 \leq t \leq T$). But in these last two cases, the terminal condition makes $\widehat{y}_T$ sufficiently negative for output to remain negative from 1 to $T-1$.

We have so far considered the effects of a T-period-long increase in market power. Our results, however, also have implications for the timing of structural reforms aimed at *permanently* reducing market power. Consider an economy at the start of a liquidity trap: to boost output, should it implement structural reforms now, or rather wait until the trap is over? The answer depends on which equilibrium of the basic NK model is considered. According to the standard equilibrium, the answer is "wait." According to our selected equilibrium, on the contrary, "now" is preferable: delaying the reforms to date $T+1$, compared to implementing them at date 1, reduces output by the same amount at each date as does our positive cost-push shock lasting from date 1 to date T.

5.4 Comparison With Other Equilibria

We end this section with a brief comparison between our selected equilibrium and some equilibria studied in the literature.

Cochrane (2017a) characterizes the set of all equilibrium paths, in the basic NK model, with $i_t^b = i^*$ for $1 \leq t \leq T$ and $i_t^b = 0$ for $t \geq T+1$. He shows that any of these paths can be obtained as the unique local equilibrium under a temporary interest-rate peg followed by a suitably designed interest-rate rule. He describes, among them, three particular equilibria. The first equilibrium is the "standard equilibrium," characterized by zero inflation immediately after the temporary peg ($\pi_{T+1} = 0$), and exhibiting all four puzzles and paradoxes.[41] The other

[41] We suspect that we would have obtained this equilibrium of the basic NK model (as the limit of our benchmark model's equilibrium outcomes) if we had instead assumed that the central bank of our benchmark model reverts at date $T+1$ to a rule ensuring $\pi_{T+1} = 0$, rather than to the permanent peg $i_t^m = 0$ (since we would then have had $\pi_{T+1} = 0$ along all the sequence of equilibrium outcomes and, therefore, also at the limit of this sequence).

two equilibria do not exhibit any of the puzzles and paradoxes. In particular, since they do not exhibit the paradox of flexibility, like our selected equilibrium, they satisfy Cochrane's (2017a) "local-to-frictionless" equilibrium-selection criterion, which requires that equilibrium outcomes converge towards flexible-price equilibrium outcomes as prices become more and more flexible. This criterion does not select a unique equilibrium, but rules out some equilibria.

More specifically, the two equilibria studied by Cochrane (2017a) and not exhibiting any of the puzzles and paradoxes are: (i) the "backward-stable" equilibrium, in which inflation goes to zero backward in time ($\lim_{t\to-\infty}\pi_t = 0$) when the interest-rate peg between 1 and T is announced at date $-\infty$; and (ii) the "no-inflation-jump" equilibrium, in which inflation is zero at the start of the peg ($\pi_1 = 0$). Two differences between these equilibria and ours are worth emphasizing. First, unlike these equilibria, our equilibrium still exhibits (a weak form of) the forward-guidance puzzle. Second, at the start of a liquidity trap, inflation is negative in our equilibrium ($\partial\pi_1/\partial i^* < 0$), while it is positive in the backward-stable equilibrium ($\partial\pi_1/\partial i^* > 0$) and, by construction, zero in the no-inflation-jump equilibrium ($\partial\pi_1/\partial i^* = 0$).[42]

Finally, Carlstrom, Fuerst, and Paustian (2015) and Kiley (2016) show that Mankiw and Reis's (2002) sticky-information model solves the fiscal-multiplier puzzle, the paradox of flexibility, and the paradox of toil, and attenuates the forward-guidance puzzle − exactly like the basic NK model with our selected equilibrium.[43] Thus, our selected equilibrium brings the canonical sticky-price model at par with its sticky-information cousin in terms of their ability to solve or attenuate all four NK puzzles and paradoxes. Kiley (2016) also points out that Mankiw and Reis's (2002) model implies price-level stationarity when the central bank follows a non-inertial interest-rate rule after the temporary interest-rate peg; in this case, we can conduct the same reasoning as at the end of Subsection 5.2, and attribute the inability of that model to (fully) solve the forward-guidance puzzle to price-level stationarity, for the same reason as in the basic NK model with our selected equilibrium.

6 Comparison With Discounting Models

In this section, we generalize Cochrane's (2016) comments on Gabaix (2016) to highlight four properties of the "discounting models" proposed in the literature to solve the forward-guidance puzzle (Gabaix, 2016; Angeletos and Lian, 2016; and Bilbiie, 2017), and we show how our

Thus, under this alternative assumption, our benchmark model would still solve the limit puzzles and paradox (as discussed in the previous section), but would not solve them asymptotically in the basic-NK-model limit.

[42]The result $\partial\pi_1/\partial i^* > 0$ in the backward-stable equilibrium is straightforwardly obtained from Cochrane's (2017a) Equation (34) by setting $C = 0$ and $t = T_l$; the result $\partial\pi_1/\partial i^* < 0$ in our equilibrium is straightforwardly obtained from the expression of π_1 in Appendix C.5 by noting that $T - (1-\overline{\omega}_2^{-T})/(\overline{\omega}_2 - 1) > T - T/\overline{\omega}_2 > 0$.

[43]Of course, the paradox of flexibility solved by Mankiw and Reis's (2002) model is about the effects of information flexibility, not price flexibility.

model is different.[44] First, these models do not solve the paradox of flexibility.[45] Second, they require a sufficiently large departure from the basic NK model to solve the forward-guidance puzzle. Third, they cannot solve this puzzle without generating a *negative* long-term relationship between the inflation rate and the interest rate on bonds; by contrast, our model generates the standard Fisher effect, i.e. a *one-to-one* long-term relationship between these two variables.[46] And fourth, these models cannot solve the forward-guidance puzzle without having non-standard and far-reaching implications for equilibrium determinacy in "normal times," i.e. away from the zero lower bound on nominal interest rates. By contrast, our model does not necessarily question the implications of the basic NK model about how monetary policy works during normal times; for example, under a corridor system, our model inherits the familiar conditions for equilibrium determinacy.

By "discounting model," we mean more specifically in this section any model whose reduced form, in the absence of shocks other than interest-rate shocks, is made of an IS equation and a Phillips curve of type

$$\widehat{y}_t = \xi_1 \mathbb{E}_t \{\widehat{y}_{t+1}\} - \frac{\xi_2}{\sigma} \mathbb{E}_t \left\{ i_t^b - \pi_{t+1} \right\}, \tag{52}$$

$$\pi_t = \beta \xi_3 (\theta) \mathbb{E}_t \{\pi_{t+1}\} + \kappa (\theta) [\widehat{y}_t - \xi_4 (\theta) \mathbb{E}_t \{\widehat{y}_{t+1}\}], \tag{53}$$

where $\beta \in (0,1)$, $\sigma > 0$, $\xi_1 > 0$, $\xi_2 > 0$, and, for all $\theta \in (0,1)$, $\xi_3(\theta) \geq 0$, $\xi_4(\theta) \in [0,1)$, and $\kappa(\theta) > 0$, with $\lim_{\theta \to 0} \xi_3(\theta) < +\infty$ and $\lim_{\theta \to 0} \kappa(\theta) = +\infty$. This class of reduced forms nests the reduced form of the basic NK model as a special case in which $\xi_1 = \xi_2 = \xi_3(\theta) = 1$ and $\xi_4(\theta) = 0$. More generally, it allows the coefficients of $\mathbb{E}_t\{\widehat{y}_{t+1}\}$ and $\mathbb{E}_t\{\pi_{t+1}\}$ to be smaller ("positive discounting") or larger ("negative discounting") than in the basic NK model, and also allows for a $\mathbb{E}_t\{\widehat{y}_{t+1}\}$ term in the Phillips curve. In particular, this class encompasses the reduced forms of three models that have been shown to be able to fully solve the forward-guidance puzzle: (i) Gabaix's (2016) benchmark model, in which $(\xi_1, \xi_3(\theta)) \in (0,1)^2$ and $\xi_4(\theta) = 0$; (ii) Angeletos and Lian's (2016) model, in which $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta)) \in (0,1)^4$; and (iii) Bilbiie's (2017) model with myopic firms, in which $(\xi_1, \xi_2) \in (0,1)^2$ and $\xi_3(\theta) = \xi_4(\theta) = 0$. In addition, it also encompasses the reduced forms of: (iv) Bilbiie's (2017) model with non-myopic firms, in which $(\xi_1, \xi_2) \in (0,1)^2$, $\xi_3(\theta) = 1$, and $\xi_4(\theta) = 0$; (v) McKay, Nakamura, and Steinsson's (2017) model, in which also $(\xi_1, \xi_2) \in (0,1)^2$, $\xi_3(\theta) = 1$, and $\xi_4(\theta) = 0$; and (vi) Ravn and Sterk's (2017) model with risk-neutral equity investors, in which $\xi_3(\theta) = 1$ and $\xi_4(\theta) \in (0,1)$.[47]

[44] None of these four properties is related to the fiscal-multiplier puzzle or the paradox of toil, which are not addressed in Angeletos and Lian (2016), Gabaix (2016), and Bilbiie (2017). Exploring the implications of discounting models for these puzzle and paradox would be interesting, but is beyond the scope of this paper.

[45] They may *attenuate* this paradox, though, as Angeletos and Lian (2016) show in the context of their model.

[46] Gabaix (2016), however, introduces price indexation and "inflation guidance" into his benchmark model and shows that the resulting model can both solve the forward-guidance puzzle and make inflation respond positively to the nominal interest rate in the long term.

[47] However, it does not encompass the reduced forms of McKay, Nakamura, and Steinsson's (2016) and Del Negro, Giannoni, and Patterson's (2015) models, which involve some discounting too but are more complex.

6.1 Paradox of Flexibility

Unlike our model, discounting models do not solve the paradox of flexibility:

Proposition 9 (Paradox of Flexibility in Discounting Models): *In models whose reduced form is made of an IS equation of type (52) and a Phillips curve of type (53), the responses of* $|\pi_1|$ *and* $|\widehat{y}_1|$ *to a temporary interest-rate peg starting at date* 1 *go to infinity as* $\theta \to 0$.

This result, proved in Appendix C.7, comes from two facts: (i) discounting models generate indeterminacy under a permanent peg of i_t^b when prices are sufficiently flexible, as their dynamic system then has one stable eigenvalue not matched by any predetermined variable, and (ii) this eigenvalue converges to zero as prices become more and more flexible. Under a temporary peg of i_t^b, this eigenvalue magnifies the effects of future terminal conditions (at the end of the peg) on initial outcomes (at the start of the peg), so that these effects grow explosively as prices become more and more flexible − thus giving rise to the paradox of flexibility.

In turn, indeterminacy under sufficiently flexible prices follows, by continuity, from indeterminacy under perfectly flexible prices. Under perfectly flexible prices, the Phillips curve (53) collapses to the dynamic equation $\widehat{y}_t = [\lim_{\theta \to 0} \xi_4(\theta)]\mathbb{E}_t\{\widehat{y}_{t+1}\}$, which pins down $\widehat{y}_t$ uniquely if $\lim_{\theta \to 0} \xi_4(\theta) \neq 1$. Under a permanent peg of i_t^b, the IS equation (52) then pins down expected future inflation $\mathbb{E}_t\{\pi_{t+1}\}$, but not current inflation π_t. Thus, discounting the basic NK model may deliver determinacy under a permanent interest-rate peg for some degrees of price stickiness, but cannot do it for sufficiently small degrees of price stickiness.

In our model, by contrast, the interest rate pegged is the IOR rate, not the interest rate on bonds. In the limit of perfectly flexible prices (and, for simplicity, absent government-expenditures shocks), the Phillips curve (34) expresses output as a function of real reserves ($\widehat{y}_t = \psi\widehat{m}_t$), in such a way that the spread equation (36), under a permanent IOR-rate peg ($i_t^m = i^*$), expresses the interest rate on bonds as a decreasing function of real reserves and, hence, an increasing function of the price level (the shadow Wicksellian rule $i_t^b = i^* - \sigma(\delta_m - \delta_y\psi)\widehat{m}_t$). The IS equation (35) then leads to the dynamic equation $[(\sigma^{-1} - \psi) + (\delta_m - \delta_y\psi)]\widehat{m}_t = (\sigma^{-1} - \psi)\mathbb{E}_t\{\widehat{m}_{t+1}\} + \sigma^{-1}i^*$, which pins down $\widehat{m}_t$ uniquely.

6.2 Distance From the Basic NK Model

Unlike our model, discounting models require a discrete departure from the basic NK model to deliver determinacy under a permanent interest-rate peg and, therefore, to solve the forward-guidance puzzle. Indeed, for a sufficiently small departure, i.e. for $(\xi_1, \xi_2, \xi_3(\theta), \xi_4(\theta))$ sufficiently close to $(1, 1, 1, 0)$, indeterminacy obtains, by continuity with the basic NK model, so that the responses of π_1 and $\widehat{y}_1$ to a temporary interest-rate peg lasting from date 1 to date T go to infinity as $T \to +\infty$. Thus, a sufficiently high degree of bounded rationality (in Gabaix, 2016), information frictions (in Angeletos and Lian, 2016), or market incompleteness (in Bilbiie, 2017), is needed to solve the forward-guidance puzzle.

By contrast, our model solves the forward-guidance puzzle even for an arbitrarily small departure from the basic NK model, i.e. even for arbitrarily small banking costs and convenience yield of bank reserves. Again, the key element at the source of this result is that the interest rate pegged is the IOR rate, not the interest rate on bonds. Under a permanent IOR-rate peg, the interest rate on bonds evolves according to a shadow Wicksellian rule. The smaller the departure from the basic NK model, the smaller the coefficients of output and the price level in this shadow rule. If these coefficients were exactly zero, then indeterminacy would ensue; but as long as the coefficient of the price level is positive, however tiny it is, determinacy prevails.

This discontinuity comes from the fact that reacting to the price level provides an infinite leverage to satisfy the Taylor principle, as discussed by Woodford (2003, Chapter 4) in the context of the basic NK model: in response to a permanent increase in inflation, which leads to an infinite increase in the price level in the long run, a Wicksellian rule prescribes an infinite increase in the interest rate in the long run, so that the interest rate reacts (infinitely) more than one-to-one to inflation in the long run. Our resolution of the forward-guidance puzzle (as well as the fiscal-multiplier puzzle and the paradox of flexibility) for an arbitrarily small departure from the basic NK model exploits this discontinuity.

6.3 Fisher Effect

Because it involves the standard IS equation (35), our model trivially implies the standard Fisher effect, i.e. a *one-to-one* long-term relationship between the inflation rate and the interest rate on bonds. Thus, a permanent rise in the nominal-reserves-growth rate will raise the inflation rate and the interest rate on bonds by the same amount in the long term.

By contrast, discounting models cannot deliver determinacy under a permanent interest-rate peg without making the inflation rate respond *negatively* to the interest rate in the long term; therefore, they cannot both solve the forward-guidance puzzle and imply a long-term relationship consistent in sign (let alone in size) with the standard Fisher effect. This result is stated in the following proposition, which is proved in Appendix C.8:

Proposition 10 (No Fisher Effect in Discounting Models): *In models whose reduced form is made of an IS equation of type (52) and a Phillips curve of type (53), if a permanent peg of* i_t^b *ensures local-equilibrium determinacy, then* π_t *responds negatively to* i_t^b *in the long term.*

The proof in Appendix C.8 is simple, but mechanical. In what follows, we offer an interpretation of Proposition 10 that involves, again, a shadow interest-rate rule and the Taylor principle. The question (negatively) answered by Proposition 10 is whether the system made of the modified IS equation (52), the modified Phillips curve (53), and the permanent peg $i_t^b = i^{b*}$ can have a unique stationary solution and make inflation, in this unique stationary solution, depend positively on i^{b*}. This question will receive exactly the same answer if that system is replaced by the system made of the standard IS equation (35), the modified Phillips curve (53), and the shadow interest-rate rule

$$i_t^b = \xi_2 i^{b*} + \sigma (1 - \xi_1) \mathbb{E}_t \{\widehat{y}_{t+1}\} + (1 - \xi_2) \mathbb{E}_t \{\pi_{t+1}\}. \tag{54}$$

Indeed, the two systems have exactly the same implications for local-equilibrium determinacy and the dynamics of inflation and output (they differ only in terms of the implied dynamics for i_t^b). So consider the latter system. The Taylor principle, shown in Appendix C.8 to be valid in this context, states that a necessary condition for local-equilibrium determinacy is that the modified Phillips curve (53) and the shadow interest-rate rule (54) should make the interest rate react more than one-to-one to the inflation rate in the long term, that is to say

$$\zeta \equiv \frac{\sigma (1 - \xi_1) [1 - \beta \xi_3 (\theta)]}{\kappa (\theta) [1 - \xi_4 (\theta)]} + (1 - \xi_2) > 1. \tag{55}$$

In the unique local equilibrium, the (constant) interest rate i^b and the (constant) inflation rate π are therefore linked to each other by the relationship $i^b = \xi_2 i^{b*} + \zeta \pi$, where $\zeta > 1$. Now, the standard IS equation (35) implies that they should be equal to each other: $i^b = \pi$. As a consequence, we get

$$\pi = \frac{-\xi_2 i^{b*}}{\zeta - 1}.$$

Thus, the necessary condition for local-equilibrium determinacy (55) imposed by the Taylor principle requires that π be *negatively* related to i^{b*}.

In our model, the previous reasoning does not apply. Despite the standard nature of its IS equation (which trivially implies the Fisher effect), our model delivers determinacy under a permanent interest-rate peg essentially because the interest rate pegged is the IOR rate, not the interest rate on bonds as in the basic NK model and in discounting models. Under a permanent IOR-rate peg, the interest rate on bonds evolves according to the shadow Wicksellian rule (36), which always ensures determinacy.

6.4 Normal Times

Discounting models have only one interest rate, which is the policy instrument. To solve the forward-guidance puzzle, they need to deliver determinacy when this interest rate is permanently pegged at its zero lower bound. The parameter restrictions that deliver determinacy under a peg at the ZLB, however, also deliver determinacy under a peg at a higher interest rate; and, by continuity, policy rules that make the interest rate react sufficiently *weakly* to inflation will also deliver determinacy. Thus, discounting models do not support the conventional wisdom that interest-rate rules have to be active during normal times.

By contrast, our model has two interest rates. At the zero lower bound on nominal interest rates, the central bank has no other possibility than pegging the IOR rate, which is the interest rate it directly controls. Away from this lower bound, however, it has various possibilities. One of these possibilities, which has been commonly chosen by central banks, is to operate under a corridor system, i.e. to maintain a fixed spread between the IOR rate and the interbank rate and set an interbank-rate target depending on the state of the economy. The interbank rate, in our model, coincides with the interest rate on bonds I_t^b. It is easy to show, along the same lines as in Subsections 3.1 and 3.2, that our model has a unique zero-inflation steady state under a corridor system.[48] We log-linearize the model around this steady state and, for simplicity, keep the same notations as previously. Under a corridor system, the spread equation (36) becomes

$$\widehat{m}_t = \frac{\delta_y}{\delta_m}\widehat{y}_t - \frac{\delta_g}{\delta_m}\widetilde{g}_t,$$

so that the Phillips curve (34) can be rewritten as

$$\pi_t = \beta \mathbb{E}_t \left\{\pi_{t+1}\right\} + \left(\frac{\delta_m \kappa_y - \delta_y \kappa_m}{\delta_m}\right)\widehat{y}_t - \left(\frac{\delta_m \kappa_g - \delta_g \kappa_m}{\delta_m}\right)\widetilde{g}_t.$$

The inequality (39) ensures that this Phillips curve has a positive slope (i.e. the coefficient of $\widehat{y}_t$ is positive). In addition, the inequality $\delta_g \kappa_m < \delta_m \kappa_g$, established in Appendix C.9, ensures that the coefficient of $\widetilde{g}_t$ in this Phillips curve is negative. Therefore, the reduced form of our model, made of the IS equation (35) and this Phillips curve, is then isomorphic to the basic NK model's reduced form for any given rule for the interest rate on bonds.[49] As a consequence, our model then inherits all the standard implications of the basic NK model for equilibrium determinacy and dynamics away from the zero lower bound on the nominal IOR rate.

7 Robustness Analysis

In Section 4, we traced back our benchmark model's ability to solve the limit puzzles and paradox to its ability to deliver equilibrium determinacy under a permanent IOR-rate peg. In this section, we show that this determinacy result is essentially robust to the relaxation of two assumptions in turn: the exogeneity of nominal reserves, and the absence of cash.

[48] This steady state does not depend on the particular rule considered for the interbank-rate target I_t^{b*}. In particular, the employment level at this steady state is equal to $\mathcal{F}(-S)$, where $S \equiv (I^b - I^m)/I^b) > 0$ denotes the constant value of the spread. For the steady state to exist, though, the rule has to prescribe the value β^{-1} for I_t^{b*} at the steady state. Any given rule $I_t^{b*} = R(\Theta_t)$ for the interbank-rate target I_t^{b*}, where Θ_t is a vector of variables observable at date t, can be "implemented" by the rule $\mu_t = \beta R(\Theta_{t-1})\lambda_t m_t/(\lambda_{t-1}m_{t-1})$ for the policy instrument μ_t, in the spirit of Adão, Correia, and Teles (2011) and Loisel (2009).

[49] Woodford (2003, Chapter 4) obtains a similar isomorphism result in the context of the basic NK model with money entering the utility function in a non-separable way, when the central bank maintains a fixed spread between the interest rate on money and the interest rate on bonds.

7.1 Endogenous Nominal Reserves

We have so far assumed, for simplicity, that the stock of nominal reserves is exogenous. As we explained, this assumption does not seem to us like a bad approximation of reality. Nonetheless, the alternative assumption of endogenous nominal reserves may also seem relevant, perhaps especially so for long IOR-rate-peg durations. In this subsection, we assume that the central bank sets the stock of nominal reserves according to the rule

$$M_t = P_t \mathcal{R}\left(P_t, y_t\right), \tag{56}$$

where the function $\mathcal{R}$, from $\mathbb{R}^2_{>0}$ to $\mathbb{R}_{>0}$, is differentiable, decreasing in P_t ($\mathcal{R}_P < 0$), and non-increasing in y_t ($\mathcal{R}_y \leq 0$). This assumption ensures that real reserve balances respond negatively to the price level for a given output level, and non-positively to the output level for a given price level. This specification nests, in particular, the previous case of (constant) exogenous nominal reserves, when $\mathcal{R}\left(P_t, y_t\right) = M/P_t$ with $M > 0$.

The endogenization of nominal reserves does not change any of the equilibrium conditions stated in Section 2 and Subsection 3.1. In particular, the dynamic equation (31) still holds under flexible prices; simply, the factor $\mu_{t+1} \equiv M_{t+1}/M_t$ in this equation is now endogenous. Inverting the rule (56) leads to $P_t = \mathcal{S}(m_t, y_t)$, where the function $\mathcal{S}$, defined from $\mathbb{R}^2_{>0}$ to $\mathbb{R}_{>0}$, is differentiable, decreasing in m_t ($\mathcal{S}_m < 0$), and non-increasing in y_t ($\mathcal{S}_y \leq 0$). The dynamic equation (31) can therefore be rewritten as

$$1 + \mathcal{F}\left(h_t\right) = \beta I_t^m \mathbb{E}_t \left\{ \frac{u'\left[f\left(h_{t+1}\right) - g_{t+1}\right] \mathcal{S}\left[\mathcal{M}\left(h_t\right), f\left(h_t\right)\right]}{u'\left[f\left(h_t\right) - g_t\right] \mathcal{S}\left[\mathcal{M}\left(h_{t+1}\right), f\left(h_{t+1}\right)\right]} \right\},$$

where I_t^m and g_t vary exogenously around some given values $I^m \geq 1$ and $g \geq 0$. In any steady state, this dynamic equation boils down to the same static equation (32) as previously. Therefore, the necessary and sufficient condition on I^m for existence and uniqueness of a steady state is again $I^m < \beta^{-1}$. Log-linearizing the model around this unique steady state, we get the same Phillips curve (34), IS equation (35), and spread equation (36) as previously. Using the reserve-market-clearing condition (18), we can log-linearize the rule (56) as

$$\widehat{m}_t = -\nu_P \widehat{P}_t - \nu_y \widehat{y}_t, \tag{57}$$

where

$$\nu_P \equiv \frac{-P\mathcal{R}_P(P,y)}{\mathcal{R}(P,y)} > 0 \quad \text{and} \quad \nu_y \equiv \frac{-y\mathcal{R}_y(P,y)}{\mathcal{R}(P,y)} \geq 0.$$

Using (34), (35), (36), and (57), we easily get the following dynamic equation in $\widehat{m}_t$ and $\widehat{y}_t$:

$$\mathbb{E}_t \left\{ \begin{bmatrix} \widehat{m}_{t+1} \\ \widehat{m}_t \\ \widehat{y}_{t+1} \\ \widehat{y}_t \end{bmatrix} \right\} = \mathbf{A}^r \begin{bmatrix} \widehat{m}_t \\ \widehat{m}_{t-1} \\ \widehat{y}_t \\ \widehat{y}_{t-1} \end{bmatrix} + \mathbb{E}_t \left\{ \mathbf{B}^r \begin{bmatrix} i_t^m \\ \widetilde{g}_{t+1} \\ \widetilde{g}_t \end{bmatrix} \right\}, \tag{58}$$

where the 4×4 matrix $\mathbf{A}^r$ and 4×3 matrix $\mathbf{B}^r$ are defined in Appendix D.1 (the superscript "r" standing for "rule"). In Appendix D.2, we establish the following lemma:

Lemma 5 (Eigenvalues of Matrix $\mathbf{A}^r$): *The matrix $\mathbf{A}^r$ has two eigenvalues inside the unit circle and two eigenvalues outside if*

$$\nu_P \leq \left\{ \left(\frac{I^\ell - I^b}{I^\ell} \right) \left[\frac{-m\Gamma_{\ell m}(\ell, m)}{\Gamma_\ell(\ell, m)} \right] \right\}^{-1}. \tag{59}$$

This lemma straightforwardly implies, through Blanchard and Kahn's (1980) conditions, the following proposition:

Proposition 11 (Local-Equilibrium Determinacy Under Endogenous Nominal Reserves): *In the benchmark model, when I_t^m and g_t vary exogenously around the values $I^m \in [1, \beta^{-1})$ and $g \geq 0$, while M_t is set according to the rule (56), there is a unique rational-expectations equilibrium in the neighborhood of the unique steady state if Condition (59) is met.*

This proposition is the counterpart, under endogenous nominal reserves, of Proposition 2. It can be interpreted in the same way as Proposition 2, with the help of a "shadow rule" for the interest rate on bonds. This shadow rule is still Wicksellian, i.e. it still makes the interest rate (on bonds) react positively to output, the price level, and no other endogenous variable. Indeed, the spread equation (36) makes i_t^b depend positively on $\widehat{y}_t$ and negatively on $\widehat{m}_t$; in turn, $\widehat{m}_t$ now depends, through the rule (57), negatively on $\widehat{y}_t$ and $\widehat{P}_t$. Again, it is well known that Wicksellian rules always ensure local-equilibrium determinacy in the basic NK model. Because of its cost channel of monetary policy (i.e. $\kappa_m > 0$), however, our model differs from the basic NK model, and the shadow Wicksellian rule for i_t^b does not always ensure local-equilibrium determinacy. Proposition 11 provides a simple sufficient condition for this rule to ensure determinacy − namely, Condition (59).[50]

We view Condition (59) as likely to be met. To support this view, we make standard or conservative assumptions about parameter values. We set the steady-state spread between loan and bond rates $(I^\ell - I^b)/I^\ell$ to the value 3%, which is the average difference, over the period from January 1997 to March 2017, between the bank prime loan rate and the 3-month AA financial commercial paper rate in the U.S.[51] We assume for simplicity that the production function f^b is isoelastic: $f^b(h_t^b, m_t) = (h_t^b)^{\chi_h}(m_t)^{\chi_m}$, where $(\chi_h, \chi_m) \in (0,1)^2$ with $\chi_h + \chi_m \leq 1$. The elasticity of marginal banking costs to reserves can then be rewritten, after some algebra, as

$$\frac{-m\Gamma_{\ell m}(\ell, m)}{\Gamma_\ell(\ell, m)} = \left[1 + \frac{v^{b\prime\prime}(h^b)\, h^b}{v^{b\prime}(h^b)} \right] \frac{\chi_m}{\chi_h}.$$

[50] The necessary and sufficient condition for determinacy, stated in Appendix D.2, is substantially more complex. We focus on the sufficient condition (59) because (i) it involves fewer parameters, (ii) how it is affected by each parameter is clearer, and (iii) it is, as we argue, likely to be met anyway.

[51] The period considered does not matter, as this spread has been remarkably stable over time.

We consider the value 0.02 for the ratio χ_m/χ_h, which is the ratio of the cost of bank reserves to the compensation of employees in banking in 2015.[52] We assume conservatively that the inverse of the steady-state Frisch elasticity of bankers' labor supply is $h^b v^{b\prime\prime}/v^{b\prime} = 5$.[53] The right-hand side of (59) is then still a comfortably high 278. To calibrate the left-hand side of (59), we assume conservatively that $\nu_y = 0$, and compute ν_P as 1 minus the elasticity of the change in nominal reserves to the deviation of inflation from target between December 2008 and December 2015 (the period during which the net IOR rate was pegged at 0.25%), which gives $\nu_P = 51$.[54] We thus find that Condition (59) is met by a large margin even under our conservative assumptions, and conclude that setting exogenously the IOR rate and endogenously nominal reserves still delivers local-equilibrium determinacy, except for implausible calibrations.

To solve the forward-guidance and fiscal-multiplier puzzles (as in Subsections 4.1 and 4.2), we need local-equilibrium determinacy for all empirically relevant calibrations. To solve the paradox of flexibility (as in Subsection 4.3), we also need local-equilibrium determinacy when the price-stickiness parameter θ is small enough. This additional need is satisfied because Condition (59) does not involve θ. We conclude that endogenizing nominal reserves does not affect the ability of our model to solve these limit puzzles and paradox, except for implausible calibrations.[55]

7.2 Household Cash

Our benchmark model is specific in that households hold money only in the form of reserves, in their capacity as bankers. This makes our point stark because banks cannot collectively change the aggregate nominal quantity of reserves outstanding. In reality, bank reserves can fall if households demand more cash. We now illustrate that our results do not unravel when we allow for such leakages out of reserve balances. We introduce household cash into our benchmark model through a cash-in-advance constraint, and we study the consequences for local-equilibrium determinacy of setting exogenously the IOR rate I_t^m, the growth rate μ_t of

[52] More specifically, we compute this ratio as $\chi_m/\chi_h = [(I^b - I^m)/I^b]M/C$, where the spread $(I^b - I^m)/I^b$ is measured by the difference between the 3-month AA financial commercial paper rate and the interest rate on excess reserves, M by total reserves of depository institutions, and C by the compensation of employees in Federal Reserve banks, credit intermediation, and related activities. All data are U.S. data. We choose the year 2015 as it is the most recent year for which data on C are available, and we conservatively use the maximum value reached by the spread $(I^b - I^m)/I^b$ over that year. The same computation for the year 2007 (before the crisis) would lead to an even lower value for χ_m/χ_h, namely 0.01, and considering this value would only strengthen our point.

[53] The value 5 for the inverse of a Frisch elasticity of labor supply lies at the upper end of the range of microeconomic estimates, and is much higher than values commonly considered in macroeconomics. Lower values thus seem arguably more likely, and considering such values would only strengthen our point.

[54] More specifically, we compute ν_P as $1 - [log(M_{2015:12}) - log(M_{2008:12})/[log(P_{2015:12}) - log(P_{2008:12}) - 0.14]$, where M_t and P_t are measured by respectively the total reserves of depository institutions and the consumer price index, while 0.14 corresponds to the Federal Reserve's 2% annual-inflation target over seven years.

[55] To solve or attenuate these limit puzzles and paradox in the basic-NK-model limit (as in Subsection 5.2), we also need local-equilibrium determinacy when the scale parameter γ and the steady-state interest-rate spread $I^m - \beta^{-1}$ are small enough. This additional need is satisfied because Condition (59) is necessarily met when γ and $I^m - \beta^{-1}$ go to zero, since the spread $(I^\ell - I^b)/I^\ell$ then goes to zero.

the monetary base (made of bank reserves and household cash), and government expenditures g_t.

More specifically, we assume that each period is made of a financial exchange followed by a goods exchange. Households acquire cash in the financial exchange and use it to buy goods in the goods exchange; firms receive this cash in the goods exchange and have to wait until the next period's financial exchange to get rid of it. Thus, households choose bonds b_t, consumption c_t, work hours h_t, loans ℓ_t, reserves m_t, and (now) cash m_t^c to maximize the same intertemporal utility function as previously, rewritten as

$$U_t = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \beta^k \left[u\left(c_{t+k}\right) - v\left(h_{t+k}\right) - \Gamma\left(\ell_{t+k}, m_{t+k}\right) \right] \right\}$$

subject to the budget constraint

$$m_t^c + b_t + \ell_t + m_t \leq \frac{m_{t-1}^c - c_{t-1}}{\Pi_t} + \frac{I_{t-1}^b}{\Pi_t} b_{t-1} + \frac{I_{t-1}^\ell}{\Pi_t} \ell_{t-1} + \frac{I_{t-1}^m}{\Pi_t} m_{t-1} + w_t h_t + \omega_t$$

and the cash-in-advance constraint

$$m_t^c \geq c_t, \tag{60}$$

taking all prices (I_t^b, I_t^ℓ, I_t^m, P_t, and w_t) as given. Letting λ_t and λ_t^c denote the Lagrange multipliers on these two constraints respectively, the first-order conditions of households' optimization problem are again (8), (9), (10), (11), (12), and now

$$\lambda_t^c + \frac{\beta \lambda_{t+1}}{\Pi_{t+1}} - \lambda_t = 0.$$

None of the other equilibrium conditions stated in Section 2 is changed, except the reserve-market-clearing condition (18), which becomes the money-market-clearing condition

$$m_t + m_t^c = \frac{M_t}{P_t}, \tag{61}$$

as the monetary base controlled by the central bank is now made not only of bank reserves, but also of household cash.

Under flexible prices, the relationship (27) still holds. Using households' first-order conditions (9) and (12), together with (8), (14), (19), (20), (27), (60) holding with equality, and (61), we obtain the following dynamic equation in h_t under flexible prices:

$$1 + \mathcal{F}\left(h_t\right) = \beta I_t^m \mathbb{E}_t \left\{ \frac{u'\left[f\left(h_{t+1}\right) - g_{t+1}\right]\left[\mathcal{M}\left(h_{t+1}\right) + f\left(h_{t+1}\right) - g_{t+1}\right]}{\mu_{t+1} u'\left[f\left(h_t\right) - g_t\right]\left[\mathcal{M}\left(h_t\right) + f\left(h_t\right) - g_t\right]} \right\},$$

where I_t^m, μ_t, and g_t vary exogenously around the values $I^m \geq 1$, 1, and $g \geq 0$. In any steady state, this dynamic equation boils down to the same static equation (32) as previously. Therefore, the necessary and sufficient condition on I^m for existence and uniqueness of a steady state is again $I^m < \beta^{-1}$. Log-linearizing the model around this unique steady state, we get the same IS equation (35) and spread equation (36) as previously, since the money-market-clearing condition plays no role in their derivation. As we show in Appendix D.3, the Philips curve becomes

$$\pi_t = \beta \mathbb{E}_t \{\pi_{t+1}\} + \kappa_y \widehat{y}_t - \kappa_m \widehat{m}_t + \kappa_b i_t^b - \kappa_g \widetilde{g}_t, \tag{62}$$

where $\kappa_b > 0$ is a reduced-form parameter defined in Appendix D.3. The only difference with respect to the previous Phillips curve (34) lies in the presence of the new term $\kappa_b i_t^b$, which captures the opportunity cost for firms of holding their cash from one period to the next. Finally, using the goods-market-clearing condition (19) and the binding cash-in-advance constraint (60), we can log-linearize the new money-market-clearing condition (61) as

$$\widehat{\left(\frac{M_t}{P_t}\right)} = (1-\alpha_c)\, \widehat{m}_t + \frac{\alpha_c}{1-\alpha_g} \widehat{y}_t - \frac{\alpha_c}{1-\alpha_g} \widetilde{g}_t, \tag{63}$$

or equivalently, in first difference,

$$\pi_t = -(1-\alpha_c)\, \Delta \widehat{m}_t - \frac{\alpha_c}{1-\alpha_g} \Delta \widehat{y}_t + \widehat{\mu}_t + \frac{\alpha_c}{1-\alpha_g} \Delta \widetilde{g}_t, \tag{64}$$

where

$$\alpha_c \equiv \frac{f(h) - g}{f(h) - g + \mathcal{M}(h)} \in (0,1) \quad \text{and} \quad \alpha_g \equiv \frac{g}{y} \in (0,1)$$

denote respectively the steady-state share of household cash in the monetary base and the steady-state share of government consumption in total consumption. Using (35), (36), (62), and (64), we easily get the following dynamic equation in $\widehat{m}_t$ and $\widehat{y}_t$:

$$\mathbb{E}_t \left\{ \begin{bmatrix} \widehat{m}_{t+1} \\ \widehat{m}_t \\ \widehat{y}_{t+1} \\ \widehat{y}_t \end{bmatrix} \right\} = \mathbf{A}^c \begin{bmatrix} \widehat{m}_t \\ \widehat{m}_{t-1} \\ \widehat{y}_t \\ \widehat{y}_{t-1} \end{bmatrix} + \mathbb{E}_t \left\{ \mathbf{B}^c \begin{bmatrix} i_t^m \\ \widehat{\mu}_{t+1} \\ \widehat{\mu}_t \\ \widetilde{g}_{t+1} \\ \widetilde{g}_t \\ \widetilde{g}_{t-1} \end{bmatrix} \right\}, \tag{65}$$

where the 4×4 matrix $\mathbf{A}^c$ and 4×6 matrix $\mathbf{B}^c$ are defined in Appendix D.4 (the superscript "c" standing for "cash"). In Appendix D.5, we establish the following lemma:

Lemma 6 (Eigenvalues of Matrix $\mathbf{A}^c$): *The matrix $\mathbf{A}^c$ has two eigenvalues inside the unit circle and two eigenvalues outside if*

$$\left(\frac{1-\beta I^m}{1-\alpha_c}\right) \left\{ \left(\frac{\varepsilon}{\varepsilon - 1}\right) \left[\frac{f(h)}{h f'(h)}\right] \left(\frac{m}{y}\right) + \frac{1}{\beta I^m} \right\} \left[\frac{\ell \Gamma_{\ell m}(\ell, m)}{\Gamma_m(\ell, m)}\right] < 1. \tag{66}$$

This lemma straightforwardly implies, through Blanchard and Kahn's (1980) conditions, the following proposition:

Proposition 12 (Local-Equilibrium Determinacy in the Model With Cash): *In the model with cash, when I_t^m, μ_t, and g_t vary exogenously around the values $I^m \in [1, \beta^{-1})$, $\mu = 1$,* *and $g \geq 0$, there is a unique rational-expectations equilibrium in the neighborhood of the unique steady state if Condition (66) is met.*

This proposition is the counterpart, in the model with cash and under exogenous nominal reserves, of Propositions 2 and 11. It can be interpreted in the same way as these propositions, with the help of a "shadow rule" for the interest rate on bonds. This shadow rule is still Wicksellian, i.e. it still makes the interest rate (on bonds) react positively to output, the price level, and no other endogenous variable. Indeed, the spread equation (36) makes i_t^b depend positively on $\widehat{y}_t$ and negatively on $\widehat{m}_t$; in turn, $\widehat{m}_t$ now depends, through the new money-market-clearing condition (63), negatively on $\widehat{y}_t$ and $\widehat{P}_t$.

Proposition 12 provides a simple sufficient condition for this shadow Wicksellian rule to ensure determinacy – namely, Condition (66).[56] We view this condition as likely to be met. To support this view, we make standard or conservative assumptions about parameter values. More specifically, we set the steady-state elasticity of output to labor $hf'(h)/f(h)$ to 0.66; the elasticity of substitution between goods ε to 6 (implying a 20% markup); the ratio of reserves to (quarterly) output m/y to 0.60, and the share of cash in the monetary base α_c to 0.40, based on the most recent available data; and the spread $1 - \beta I^m = (I^b - I^m)/I^b$ to 0.2%, as in the previous subsection. We assume for simplicity, as in the previous subsection, that the production function f^b is isoelastic: $f^b(h_t^b, m_t) = (h_t^b)^{\chi_h}(m_t)^{\chi_m}$, where $(\chi_h, \chi_m) \in (0,1)^2$ with $\chi_h + \chi_m \leq 1$. The elasticity $\ell\Gamma_{\ell m}/\Gamma_m$ can then be rewritten, after some algebra, as

$$\frac{\ell\Gamma_{\ell m}(\ell, m)}{\Gamma_m(\ell, m)} = \left[1 + \frac{v^{b\prime\prime}(h^b)\, h^b}{v^{b\prime}(h^b)}\right] \frac{1}{\chi_h}.$$

We consider the value 0.47 for χ_h, which is the ratio of the compensation of employees to the value added in Federal Reserve banks, credit intermediation, and related activities, in 2015. Finally, we assume conservatively that the inverse of the steady-state Frisch elasticity of bankers' labor supply is $h^b v^{b\prime\prime}/v^{b\prime} = 5$, as in the previous subsection. The left-hand side of (66) is then still a comfortably low 0.09, one order of magnitude below its right-hand side. We thus find that Condition (66) is met by a large margin even under our conservative assumptions. Therefore, setting exogenously the IOR rate and the monetary base delivers local-equilibrium determinacy except for implausible calibrations. Since Condition (66) does not involve θ, like Condition (59), we conclude that the introduction of household cash into the monetary base does not affect the ability of our model to solve the three limit puzzles and paradox, except for implausible calibrations.[57]

[56] Again, the necessary and sufficient condition for determinacy, stated in Appendix D.5, is substantially more complex. We focus on the sufficient condition (66) because it is more transparent and, as we argue, likely to be met anyway.

[57] In addition, like Condition (59), Condition (66) is necessarily met when γ and $I^m - \beta^{-1}$ are small enough. However, unlike our benchmark model (with exogenous or endogenous nominal reserves), our model with cash does not converge towards the basic NK model as γ and $I^m - \beta^{-1}$ go to zero, because of the term $\kappa_b i_t^b$ in its Phillips curve (62); instead, it converges towards the cost-channel model of Ravenna and Walsh (2006).

8 Finite Satiation Level

Our benchmark model assumes that there is no finite satiation level of demand for reserves – i.e., that the marginal convenience yield of reserves remains positive at any finite level of reserve balances. This assumption is analytically convenient but cannot be literally true. For example, if reserves reduce banking costs only by helping banks manage the liquidity risk associated with short-term deposits, then the marginal convenience yield must be zero once reserves are as large as deposits.

In this section, we first summarize (relegating formal statements and details to Appendix E) the consequences of introducing a finite satiation level of demand for reserves into our benchmark model, as in Cúrdia and Woodford (2011). Not surprisingly, we find that our main results remain essentially intact if and only if reserves are below the satiation level in equilibrium. We then discuss arguments that seem relevant for gauging whether or not the demand for reserves is currently satiated in the United States.

8.1 Summary of the Results

To introduce a finite satiation level of demand for reserves into our benchmark model, we remove the assumption that $f_m^b > 0$ and $f_{mm}^b < 0$ for all $\left(h_t^b, m_t\right) \in (\mathbb{R}_{\geq 0})^2$, and replace it by the following assumption: for any $h_t^b \in \mathbb{R}_{>0}$, there exists $\underline{m}^f\left(h_t^b\right) \in \mathbb{R}_{>0}$ such that (i) $f_m^b = f_{mm}^b = 0$ if $m_t \geq \underline{m}^f\left(h_t^b\right)$, and (ii) $f_m^b > 0$ and $f_{mm}^b < 0$ if $m_t < \underline{m}^f\left(h_t^b\right)$. This change affects the properties of the banking-cost function Γ stated in Lemma 1. We no longer have $\Gamma_m < 0$, $\Gamma_{mm} > 0$, and $\Gamma_{\ell m} < 0$ for all $(\ell_t, m_t) \in (\mathbb{R}_{\geq 0})^2$. Instead, we get that for any $\ell_t \in \mathbb{R}_{>0}$, there exists $\underline{m}(\ell_t) \in \mathbb{R}_{>0}$ such that (i) $\Gamma_m = \Gamma_{mm} = \Gamma_{\ell m} = 0$ if $m_t \geq \underline{m}(\ell_t)$, and (ii) $\Gamma_m < 0$, $\Gamma_{mm} > 0$, and $\Gamma_{\ell m} < 0$ if $m_t < \underline{m}(\ell_t)$.

As we show in Appendix E, our results for the steady state(s) do not change dramatically. In particular, setting $I^m < \beta^{-1}$ still leads to a unique steady state, which lies outside the satiation region. The novelty is that setting $I^m = \beta^{-1}$ is now consistent with existence of steady state – and the resulting steady states exhibit the kind of indeterminacy discussed in Sargent and Wallace (1985). More precisely, because we have a representative-agent setup (in contrast to Sargent and Wallace's overlapping-generations setup), the indeterminacy in our model implies that any level of real reserve balances in the satiation range can be an equilibrium, but this indeterminacy does not permeate to equilibrium values of other real variables. This policy setting I^m exactly at a critical value (β^{-1}), however, represents a knife-edge case in our model.[58]

[58] In a previous version of this paper (Diba and Loisel, 2017), we also characterize the set of all global perfect-foresight equilibria under flexible prices and permanent pegs – which includes, but is not limited to, the steady state(s) of our model. The novelty here is that the equilibria other than the steady state(s) now involve real reserve balances that converge to the satiation range (while in the benchmark model, real reserve balances would grow without bound). But the steady state, when $I^m < \beta^{-1}$, remains the unique "determinate" equilibrium

Turning to our local-equilibrium analysis, we obtain exactly the same results as in the benchmark model if we log-linearize the model around a steady state outside the satiation region. By contrast, a log-linear approximation around a steady state inside the satiation region leads to a reduced form isomorphic to the reduced form of the basic NK model: compared to our benchmark model, we then have $\kappa_m = \delta_y = \delta_m = \delta_g = 0$.[59] Thus, to apply our proposed resolution of the NK puzzles and paradoxes to the current U.S. situation, we must assume that the demand for reserves is not currently satiated in the U.S.

8.2 Gauging the Satiation Threshold

The fact that the stock of bank reserves is large (currently about $2 trillion) does not necessarily mean that their marginal convenience yield is zero. The daily flow of transactions on Fedwire is currently about the same size as the stock of reserves. For some small banks, the marginal convenience yield of reserves may still be linked to the risk of being forced to borrow from the Discount Window, or to borrow federal funds at rates exceeding the IOR rate, in order to satisfy reserve requirements. For large banks, the risk of hitting the required reserve threshold is negligible, but the risk of hitting the lower bound on a capital requirement may be an important consideration. Reserves have emerged as their preferred short-term asset in their liquidity portfolios; the convenience yield of reserves is thus linked to the need for liquidity.[60]

If we could associate an observable interest rate with the rate I_t^b in our model, then satiation would correspond to $I_t^m = I_t^b$, and our benchmark model would involve $I_t^m < I_t^b$. The problem with this approach is that we have not developed a structural model of how banking costs arise and how holding reserves reduces these costs. If we associate these costs with liquidity management, then other money-market instruments, like Treasury bills, also seem likely to have a convenience yield and therefore cannot serve as our proxy for I_t^b.

In the context of our stylized model, it seems natural to think of I_t^b as the interbank rate. We could take this idea to the data if some banks (in reality) were active lenders in the federal-funds market. If so, we could measure the convenience yield of reserves as the spread between the federal-funds and IOR rates, since the lender in the federal-funds market gives up the convenience yield in exchange for this spread. But banks are not currently active lenders in the federal-funds market, and the reasons have to do with institutional features that our model does not capture. Indeed, the effective federal-funds rate has remained below the IOR rate in the aftermath of the crisis, contradicting both versions of our model (with and without satiation threshold). One likely reason (noted, for example, in Williamson, 2015) is that government-

among them, which makes it a natural "focal point" on which private agents could coordinate.

[59]In addition, if marginal banking costs Γ_ℓ are zero under satiation, then κ_y and κ_g take the same values as in the basic NK model, and the reduced form of our model becomes exactly identical to the reduced form of the basic NK model.

[60]Osborne and Sim (2015) also emphasize the relevance of new liquidity standards in their commentary on demand for reserves in the U.K.

sponsored enterprises (GSEs) are not eligible to receive interest on deposits at the Federal Reserve. The GSEs have become large lenders in the federal-funds market; and money-market mutual funds are large lenders in the Eurodollar market. Banks borrowing these funds pay the federal-funds rate and receive the IOR rate on their deposits at the Federal Reserve. If the marginal bank is a large domestic institution, it incurs balance-sheet costs such as deposit-insurance fees (which depend on total assets) and capital costs from increased leverage. These costs can create a wedge, keeping the effective federal-funds rate below the IOR rate. Foreign banking organizations (FBOs) have a cost advantage in these transactions because they do not pay the Federal Deposit Insurance Corporation insurance premium (and, more recently, also because of cross-country differences in implementation of the Basel III liquidity-coverage regulations). And FBOs have emerged as disproportionately large borrowers in the federal-funds and Eurodollar markets.[61]

Against this background, it does not seem farfetched to us to assume that the marginal dollar of reserves provides some liquidity services to a bank − or, stretching beyond our model, generates transaction services indirectly for an individual whose money-market fund lends overnight funds to a bank. Envisioning an equilibrium in which FBOs have a higher effective pecuniary return on reserves and hold disproportionately large reserve balances seems easier in the context of our benchmark model; the higher non-pecuniary return associated with the smaller reserve balances of domestic banks would compensate for the difference in effective pecuniary returns. We find it harder to envision an equilibrium under satiation in which FBOs face a higher effective IOR rate and yet do not eliminate the spread with the federal-funds rate.

Our interpretation of some commentary about satiation of demand for reserves is that many policymakers and economists ask whether other liquid assets like Treasury bills are now very close substitutes for bank reserves.[62] We do not think that the case for our benchmark model necessarily rests on the answer to this question. We could, for example, assume that T-Bills and reserves are perfect substitutes in our specification of banking costs, as long as they *both* have a convenience yield and their supply is essentially determined by policy. A number of empirical contributions (e.g., Friedman and Kuttner, 1998, Greenwood and Vayanos, 2014, and Krishnamurthy and Vissing-Jorgensen, 2012) support the view that government debt has a convenience yield that is inversely related to its outstanding stock.

Nor does our benchmark model necessarily contradict the view that the demand for reserves may be very flat, and that small shocks may lead to large changes in the demand for reserves.

[61] Ennis and Wolman (2015), for example, note that in 2011 the assets of FBOs were about 10 to 20 percent of the assets of domestic institutions, but the two groups had roughly the same amount of reserve balances.

[62] For example, Osborne and Sim (2015) state that reserves lost some of their "specialness" after the Bank of England started remunerating them at the policy rate, and under these circumstances reserves "become much more like part of the broader spectrum of liquid assets, assessed against their liquidity value and expected return." As another example, Ennis and Wolman (2015) argue that reserves were still special in their data set on the basis of a negative correlation between the T-Bill-IOR spread and the reserve-to-deposit ratio in weekly data ending in 2011; they interpret the negative correlation as evidence in favor of Ireland's (2014) model in which reserves play a special role, and the demand for reserves is determinate even when the IOR rate is equal to a market rate.

Our model only assumes that banks are not truly indifferent across a range of values for their reserve balances. As long as this is granted, we can allow for a convenience yield that is very small and very flat. While empirical evidence (e.g., Reis, 2016) rules out the possibility that the convenience yield of bank reserves remains large, commentary by market analysts (e.g., Pozsar, 2017) still suggests that the convenience yield is non-zero and inversely related to the stock of reserves.

9 Conclusion

In this paper, we have proposed a resolution of three limit puzzles and paradox that arise in the basic NK model under a temporary interest-rate peg (e.g., in the context of a liquidity trap): the forward-guidance puzzle, the fiscal-multiplier puzzle, and the paradox of flexibility. This resolution rests on the ability of our model to deliver equilibrium determinacy under a permanent interest-rate peg. In turn, this ability comes from the fact that the central bank can set the interest rate on bank reserves and the supply of bank reserves independently, and that these reserves reduce the costs of banking. The endogenization of bank reserves and the introduction of household cash alongside bank reserves into the monetary base leave all these results essentially unaffected.

Moreover, our model still solves these limit puzzles and paradox for an *arbitrarily* small departure from the basic NK model, i.e. for arbitrarily small banking costs. In fact, even a *vanishingly* small departure from the basic NK model is enough to solve the fiscal-multiplier puzzle and the paradox of flexibility, attenuate the forward-guidance puzzle, and, in addition, solve the paradox of toil. This limit result provides an equilibrium-selection device in the basic NK model under a temporary interest-rate peg followed by a permanent one − and this device uniquely selects one of Cochrane's (2017a) local-to-frictionless equilibria. Overall, our selected equilibrium brings the basic NK model at par with Mankiw and Reis's (2002) sticky-information model in terms of their ability to solve or attenuate the four NK puzzles and paradoxes.

We identify two main avenues for future research. First, we could investigate the implications of our model for other interesting issues raised by NK liquidity traps. Other puzzling implications of NK models under a temporary interest-rate peg are that capital-tax cuts can reduce investment (the so-called "paradox of thrift," obtained by Eggertsson, 2011) and that capital-destruction shocks and negative oil-supply shocks can be expansionary (as shown by Wieland, 2016). Considering these implications would require adding capital and oil to our model.

Second, we plan to explore further the implications of our model for central banks' exit strategies. In a previous version of this paper (Diba and Loisel, 2017), we show that a permanent increase in the interbank rate can be inflationary in the short term. Our model can thus provide some conditional support for the "neo-Fisherian" view about the short-term inflationary effects of "normalizing" interest rates, which has been at the center of recent work and debate.[63] Whether neo-Fisherian effects arise or not in our model depends on how the two monetary-policy instruments (the IOR rate and the growth rate of reserves) are used to generate the permanent increase in the interbank rate.[64] This issue deserves further scrutiny in future work because central banks' exit strategies can involve several interesting combinations of when and how interest rates are normalized and balance-sheet adjustments occur.

Appendix A: Benchmark Model

In this appendix and the following ones, we omit time subscripts and function arguments whenever no ambiguity results.

A.1 Concavity of Function f^b

Since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $f^b\left(xh_t^b, xm_t\right) = x^d f^b\left(h_t^b, m_t\right)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$df^b = h^b f_h^b + m f_m^b. \tag{A.1}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to h^b and m leads to

$$\begin{aligned} -(1-d) f_h^b &= h^b f_{hh}^b + m f_{hm}^b, \\ -(1-d) f_m^b &= h^b f_{hm}^b + m f_{mm}^b, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} f_{hh}^b &= -\frac{(1-d) f_h^b + m f_{hm}^b}{h^b}, \\ f_{mm}^b &= -\frac{(1-d) f_m^b + h^b f_{hm}^b}{m}. \end{aligned}$$

Using these expression for f_{hh}^b and f_{hm}^b, as well as (A.1), we get

$$\begin{aligned} f_{hh}^b f_{mm}^b - \left(f_{hm}^b\right)^2 &= \frac{1-d}{h^b m}\left[(1-d) f_h^b f_m^b + f_{hm}^b \left(h^b f_h^b + m f_m^b\right)\right] \\ &= \frac{1-d}{h^b m}\left[(1-d) f_h^b f_m^b + d f^b f_{hm}^b\right] \\ &\geq 0, \end{aligned}$$

which implies (together with $f_{hh}^b \leq 0$ and $f_{mm}^b \leq 0$) that the function f^b is (weakly) concave.

[63] Cochrane (2017b) and Schmitt-Grohé and Uribe (2014, 2017) provide examples of models that do produce neo-Fisherian effects, while Cochrane (2017c), García-Schmidt and Woodford (2015), and Kocherlakota (2016) provide examples of models that do not.

[64] We find in particular that the specific combination of stepwise changes in these instruments that generates a stepwise increase in the interbank rate does produce a neo-Fisherian effect.

A.2 Properties of Function g^b

Computing the first and second derivatives of the left- and right-hand sides of $\ell_t = f^b[g^b(\ell_t, m_t), m_t]$ with respect to ℓ_t and m_t gives

$$\begin{aligned}
1 &= f_h^b g_\ell^b, \\
0 &= f_h^b g_m^b + f_m^b, \\
0 &= f_{hh}^b \left(g_\ell^b\right)^2 + f_h^b g_{\ell\ell}^b, \\
0 &= f_{hh}^b g_\ell^b g_m^b + f_{hm}^b g_\ell^b + f_h^b g_{\ell m}^b, \\
0 &= f_{hh}^b \left(g_m^b\right)^2 + 2 f_{hm}^b g_m^b + f_h^b g_{mm}^b + f_{mm}^b.
\end{aligned}$$

Using these equations and $f_h^b > 0$, $f_m^b > 0$, $f_{hh}^b < 0$, $f_{hm}^b \geq 0$, and $f_{mm}^b < 0$, we sequentially get

$$\begin{aligned}
g_\ell^b &= \frac{1}{f_h^b} > 0, \\
g_m^b &= \frac{-f_m^b}{f_h^b} < 0, \\
g_{\ell\ell}^b &= \frac{-f_{hh}^b}{\left(f_h^b\right)^3} > 0, \\
g_{\ell m}^b &= \frac{f_m^b f_{hh}^b}{\left(f_h^b\right)^3} - \frac{f_{hm}^b}{\left(f_h^b\right)^2} < 0, \\
g_{mm}^b &= \frac{-f_{hh}^b \left(f_m^b\right)^2}{\left(f_h^b\right)^3} + 2\frac{f_m^b f_{hm}^b}{\left(f_h^b\right)^2} - \frac{f_{mm}^b}{f_h^b} > 0.
\end{aligned}$$

Then, using these expressions for $g_{\ell\ell}^b$, g_{mm}^b, $g_{\ell m}^b$, and the concavity of f^b established in Appendix A.1, we easily get

$$g_{\ell\ell}^b g_{mm}^b - \left(g_{\ell m}^b\right)^2 = \frac{f_{hh}^b f_{mm}^b - \left(f_{hm}^b\right)^2}{\left(f_h^b\right)^4} \geq 0,$$

so that the function g^b is (weakly) convex.

Moreover, since f^b is homogeneous of degree d, we have $\forall x \in \mathbb{R}_{\geq 0}$, $g^b\left(x^d \ell_t, x m_t\right) = x g^b\left(\ell_t, m_t\right)$. Computing the first derivative of the left- and right-hand sides of this equation with respect to x at $x = 1$ leads to

$$g^b = d\ell g_\ell^b + m g_m^b. \tag{A.2}$$

In turn, computing the first derivative of the left- and right-hand sides of the last equation with respect to ℓ and m leads to

$$(1-d)\, g_\ell^b = d\ell g_{\ell\ell}^b + m g_{\ell m}^b, \tag{A.3}$$

$$0 = d\ell g_{\ell m}^b + m g_{mm}^b, \tag{A.4}$$

which can be rewritten as

$$g_{\ell\ell}^b = \frac{(1-d)\, g_\ell^b - m g_{\ell m}^b}{d\ell}, \tag{A.5}$$

$$g_{mm}^b = \frac{-d\ell g_{\ell m}^b}{m}. \tag{A.6}$$

Finally, as a consequence of (3) and (4), we have

$$\forall \ell_t \in \mathbb{R}_{\geq 0}, \lim_{m_t \to +\infty} g_m^b(\ell_t, m_t) = 0, \tag{A.7}$$

$$\forall \ell_t \in \mathbb{R}_{\geq 0}, \lim_{m_t \to 0} g_\ell^b(\ell_t, m_t) = +\infty. \tag{A.8}$$

A.3 Proof of Lemma 1

Computing the first and second derivatives of the left- and right-hand sides of $\Gamma(\ell_t, m_t) \equiv v^b[g^b(\ell_t, m_t)]$ with respect to ℓ_t and m_t gives

$$\begin{aligned}
\Gamma_\ell &= v^{b\prime} g_\ell^b > 0, \\
\Gamma_m &= v^{b\prime} g_m^b < 0, \\
\Gamma_{\ell\ell} &= v^{b\prime\prime} \left(g_\ell^b\right)^2 + v^{b\prime} g_{\ell\ell}^b > 0, \\
\Gamma_{mm} &= v^{b\prime\prime} \left(g_m^b\right)^2 + v^{b\prime} g_{mm}^b > 0, \\
\Gamma_{\ell m} &= v^{b\prime\prime} g_\ell^b g_m^b + v^{b\prime} g_{\ell m}^b < 0,
\end{aligned}$$

where the inequalities follow from $v^{b\prime} > 0$, $v^{b\prime\prime} \geq 0$, $g_\ell^b > 0$, $g_m^b < 0$, $g_{\ell\ell}^b > 0$, $g_{mm}^b > 0$, and $g_{\ell m}^b < 0$. In addition, using first (A.5) and (A.6) and then (A.2), we easily get

$$\begin{aligned}
\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2 &= \left(v^{b\prime}\right)^2 \left[g_{\ell\ell}^b g_{mm}^b - \left(g_{\ell m}^b\right)^2\right] + ... \\
&\quad v^{b\prime} v^{b\prime\prime} \left[\left(g_\ell^b\right)^2 g_{mm}^b + \left(g_m^b\right)^2 g_{\ell\ell}^b - 2 g_\ell^b g_m^b g_{\ell m}^b\right] \\
&= \frac{-(1-d)\left(v^{b\prime}\right)^2 g_\ell^b g_{\ell m}^b}{m} + ... \\
&\quad \frac{v^{b\prime} v^{b\prime\prime}}{d\ell m} \left[-g_{\ell m}^b \left(d\ell g_\ell^b + m g_m^b\right)^2 + (1-d)\, m g_\ell^b \left(g_m^b\right)^2\right] \\
&= \frac{-(1-d)\left(v^{b\prime}\right)^2 g_\ell^b g_{\ell m}^b}{m} + ... \\
&\quad \frac{v^{b\prime} v^{b\prime\prime}}{d\ell m} \left[-\left(g^b\right)^2 g_{\ell m}^b + (1-d)\, m g_\ell^b \left(g_m^b\right)^2\right] \\
&\geq 0,
\end{aligned} \tag{A.9}$$

so that the function Γ is (weakly) convex. Finally, using (A.7) and (A.8), we straightforwardly get (5) and (6).

A.4 Other Properties of Function Γ

Using (A.2) and (A.5), we get

$$\begin{aligned} d\ell\Gamma_{\ell\ell} + m\Gamma_{\ell m} &= d\ell\left[v^{b\prime\prime}\left(g_\ell^b\right)^2 + v^{b\prime}g_{\ell\ell}^b\right] + m\left(v^{b\prime\prime}g_\ell^b g_m^b + v^{b\prime}g_{\ell m}^b\right) \\ &= v^{b\prime\prime}g_\ell^b\left(d\ell g_\ell^b + m g_m^b\right) + v^{b\prime}\left(d\ell g_{\ell\ell}^b + m g_{\ell m}^b\right) \\ &= v^{b\prime\prime}g^b g_\ell^b + (1-d)\,v^{b\prime}g_\ell^b \\ &\geq 0. \end{aligned} \tag{A.10}$$

Similarly, using (A.2) and (A.4), we also get

$$\begin{aligned} \ell\Gamma_{\ell m} + m\Gamma_{mm} &= (1-d)\,\ell\Gamma_{\ell m} + d\ell\Gamma_{\ell m} + m\Gamma_{mm} \\ &= (1-d)\,\ell\Gamma_{\ell m} + d\ell\left[v^{b\prime\prime}g_\ell^b g_m^b + v^{b\prime}g_{\ell m}^b\right] + m\left[v^{b\prime\prime}\left(g_m^b\right)^2 + v^{b\prime}g_{mm}^b\right] \\ &= (1-d)\,\ell\Gamma_{\ell m} + v^{b\prime\prime}g_m^b\left(d\ell g_\ell^b + m g_m^b\right) + v^{b\prime}\left(d\ell g_{\ell m}^b + m g_{mm}^b\right) \\ &= (1-d)\,\ell\Gamma_{\ell m} + v^{b\prime\prime}g^b g_m^b \\ &\leq 0. \end{aligned} \tag{A.11}$$

Finally, we have

$$\frac{\Gamma_{\ell m}}{\Gamma_\ell} - \frac{\Gamma_{mm}}{\Gamma_m} = \frac{\left(v^{b\prime}\right)^2\left(g_m^b g_{\ell m}^b - g_\ell^b g_{mm}^b\right)}{\Gamma_\ell\Gamma_m} = \frac{\left(v^{b\prime}\right)^2\left(f_h^b f_{mm}^b - f_m^b f_{hm}^b\right)}{\Gamma_\ell\Gamma_m\left(f_h^b\right)^3} > 0. \tag{A.12}$$

Appendix B: Benchmark Model Under Flexible Prices

B.1 Proof of Lemma 2

Using (24), we can rewrite $\mathcal{F}(h_t)$ as

$$\mathcal{F}(h_t) = \mathcal{F}_1(h_t)\,\mathcal{F}_2(h_t)\,,$$

where the functions $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined over $(\underline{h}, \overline{h})$ by

$$\begin{aligned} \mathcal{F}_1(h_t) &\equiv \frac{\Gamma_m\left[\mathcal{L}(h_t)\,,\mathcal{M}(h_t)\right]}{\Gamma_\ell\left[\mathcal{L}(h_t)\,,\mathcal{M}(h_t)\right]}, \\ \mathcal{F}_2(h_t) &\equiv \frac{\varepsilon-1}{\varepsilon}\frac{u'\left[f(h_t)-g\right]f'(h_t)}{v'(h_t)} - 1. \end{aligned}$$

Since

$$\mathcal{F}_1(h_t) \equiv \frac{\Gamma_m\left[\mathcal{L}(h_t)\,,\mathcal{M}(h_t)\right]}{\Gamma_\ell\left[\mathcal{L}(h_t)\,,\mathcal{M}(h_t)\right]} = \frac{g_m^b\left[\mathcal{L}(h_t)\,,\mathcal{M}(h_t)\right]}{g_\ell^b\left[\mathcal{L}(h_t)\,,\mathcal{M}(h_t)\right]},$$

we have

$$\begin{aligned} \left(g_\ell^b\right)^2\mathcal{F}_1' &= g_\ell^b\left(g_{\ell m}^b\mathcal{L}' + g_{mm}^b\mathcal{M}'\right) - g_m^b\left(g_{\ell\ell}^b\mathcal{L}' + g_{\ell m}^b\mathcal{M}'\right) \\ &= -g_{\ell m}^b\left(d\mathcal{L}g_\ell^b + \mathcal{M}g_m^b\right)\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b\frac{\mathcal{L}'}{d\mathcal{L}} \\ &= -g^b g_{\ell m}^b\left(\frac{\mathcal{M}'}{\mathcal{M}} - \frac{\mathcal{L}'}{d\mathcal{L}}\right) - (1-d)\,g_\ell^b g_m^b\frac{\mathcal{L}'}{d\mathcal{L}}, \end{aligned}$$

where the second equality is obtained by using (A.5) and (A.6), and the third equality by using (A.2). Now, deriving the left- and right-hand sides of (24) with respect to h_t gives

$$\Gamma_{\ell m}\mathcal{M}'(h_t) + \Gamma_{\ell\ell}\mathcal{L}'(h_t) = \mathcal{A}'(h_t) < 0.$$

This inequality, together with (A.10), implies

$$\frac{\mathcal{M}'(h_t)}{\mathcal{M}(h_t)} > \frac{\mathcal{L}'(h_t)}{d\mathcal{L}(h_t)}, \tag{B.1}$$

from which we conclude that $\mathcal{F}_1' > 0$. Then, using $\mathcal{F}_1' > 0$, $\mathcal{F}_1 < 0$, $\mathcal{F}_2' < 0$, and $\mathcal{F}_2 > 0$, we get that the function $\mathcal{F}$ is strictly increasing ($\mathcal{F}' > 0$). Moreover, $\mathcal{F}_1' > 0$ and $\mathcal{F}_1 < 0$ imply that $\lim_{h_t \to \underline{h}} \mathcal{F}_1(h_t) < 0$, while the Inada condition (1) implies that $\lim_{h_t \to \underline{h}} \mathcal{F}_2(h_t) = +\infty$, so that

$$\lim_{h_t \to \underline{h}} \mathcal{F}(h_t) = -\infty.$$

Finally, both $\lim_{h_t \to \overline{h}} \mathcal{F}_1(h_t)$ and $\lim_{h_t \to \overline{h}} \mathcal{F}_2(h_t)$ are finite, since $\mathcal{F}_1$ is increasing and negative, and $\mathcal{F}_2$ decreasing and positive. If $\overline{h} < h^\star$, then the Inada condition (5) implies $\lim_{h_t \to \overline{h}} \mathcal{F}_1(h_t) = 0$. Alternatively, if $\overline{h} = h^\star$, then $\lim_{h_t \to \overline{h}} \mathcal{F}_2(h_t) = 0$. We conclude that, in both cases,

$$\lim_{h_t \to \overline{h}} \mathcal{F}(h_t) = 0.$$

B.2 Log-Linearization

Log-linearizing the pricing equation (23), and using (C.2), gives

$$\widehat{w}_t = \frac{ff''}{(f')^2}\widehat{y}_t - \left(i_t^\ell - i_t^b\right). \tag{B.2}$$

Using (B.2), (C.5), (C.6), and (C.7), we get

$$\widehat{y}_t = \psi\widehat{m}_t + \vartheta\widetilde{g}_t, \tag{B.3}$$

where ψ and ϑ are defined in Subsection 4.3. Using the IS equation (35), the spread equation (36), the reserve-market-clearing condition (37), and (B.3), we get

$$\begin{aligned} \omega_1^n\widehat{y}_t \;\; = \;\; & \mathbb{E}_t\{\widehat{y}_{t+1}\} + \frac{\psi}{1-\sigma\psi}\left(i_t^m - \mathbb{E}_t\{\widehat{\mu}_{t+1}\}\right) + ... \\ & \left[\frac{(\vartheta-\sigma\psi)+\sigma(\delta_m\vartheta-\delta_g\psi)}{1-\sigma\psi}\right]\widetilde{g}_t - \left(\frac{\vartheta-\sigma\psi}{1-\sigma\psi}\right)\mathbb{E}_t\{\widetilde{g}_{t+1}\}, \end{aligned}$$

where ω_1^n is also defined in Subsection 4.3. Since $\omega_1^n > 1$, this dynamic equation meets Blanchard and Kahn's (1980) conditions and therefore has a unique stationary solution.

When $i_t^m = i^*$ for $1 \leq t \leq T$, $i_t^m = 0$ for $t \geq T+1$, and $\widehat{\mu}_t = \widetilde{g}_t = 0$ for $t \geq 1$, this solution, for $t = 1$, is

$$\widehat{y}_1 = \left[\frac{1-(\omega_1^n)^{-T}}{\delta_m - \delta_y\psi}\right]\frac{\psi i^*}{\sigma}. \tag{B.4}$$

When in addition $\widehat{m}_0 = 0$, using (37), (B.3), and (B.4), we get

$$\pi_1 = -\left[\frac{1-(\omega_1^n)^{-T}}{\delta_m - \delta_y \psi}\right] \frac{i^*}{\sigma}.$$

These values of $\widehat{y}_1$ and π_1 coincide with the values reported in (48).

Alternatively, when $i_t^m = \widehat{\mu}_t = 0$ for $t \geq 1$, $\widetilde{g}_T = \widetilde{g}^* \neq 0$ for some date $T \geq 2$, and $\widetilde{g}_t = 0$ for all dates $t \geq 1$ such that $t \neq T$, this solution, for $t = 1$, is

$$\widehat{y}_1 = \left[(1-\vartheta)\,\omega_1^n + (1+\delta_y)\,\vartheta - (1+\delta_g)\right] \left[\frac{\sigma\psi\,(\omega_1^n)^{-T}}{1-\sigma\psi}\right] \widetilde{g}^*. \tag{B.5}$$

When in addition $\widehat{m}_0 = 0$, using (37), (B.3), and (B.5), we get

$$\pi_1 = -\left[(1-\vartheta)\,\omega_1^n + (1+\delta_y)\,\vartheta - (1+\delta_g)\right] \left[\frac{\sigma\,(\omega_1^n)^{-T}}{1-\sigma\psi}\right] \widetilde{g}^*.$$

These values of $\widehat{y}_1$ and π_1 coincide with the values reported in (49) and (50).

Appendix C: Benchmark Model Under Sticky Prices

C.1 Log-Linearization

Firm i chooses $\widetilde{P}_t(i)$ to maximize

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \left[\frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \widetilde{P}_t\,(i)\, y_{t+k}\,(i) - \beta \frac{\lambda_{t+k+1}}{\lambda_t \Pi_{t,t+k+1}} I_{t+k}^{\ell} W_{t+k} f^{-1}\left[y_{t+k}\,(i)\right] \right] \right\}$$

subject to

$$y_{t+k}\,(i) = \left[\frac{\widetilde{P}_t\,(i)}{P_{t+k}}\right]^{-\varepsilon} y_{t+k},$$

where $\Pi_{t,t+k} \equiv P_{t+k}/P_t$ for any $k \in \mathbb{N}$. The first-order condition is

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \left[\frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \widetilde{P}_t\,(i) - \frac{\beta\varepsilon}{\varepsilon-1} \frac{\lambda_{t+k+1}}{\lambda_t \Pi_{t,t+k+1}} \frac{I_{t+k}^{\ell} W_{t+k}}{f'\left[h_{t+k}\,(i)\right]} \right] y_{t+k}\,(i) \right\} = 0.$$

Using the law of iterated expectations and the Euler equation (9), we can rewrite this first-order condition as

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \left[\widetilde{P}_t\,(i) - \frac{\varepsilon}{\varepsilon-1} \frac{I_{t+k}^{\ell}}{I_{t+k}^{b}} \frac{W_{t+k}}{f'\left[h_{t+k}\,(i)\right]} \right] y_{t+k}\,(i) \right\} = 0,$$

or equivalently

$$\mathbb{E}_t \left\{ \sum_{k=0}^{+\infty} (\beta\theta)^k \frac{\lambda_{t+k}}{\lambda_t \Pi_{t,t+k}} \left[\frac{\widetilde{P}_t\,(i)}{P_t} - \frac{\varepsilon}{\varepsilon-1} \frac{I_{t+k}^{\ell}}{I_{t+k}^{b}} \frac{w_{t+k}\Pi_{t,t+k}}{f'\left[h_{t+k}\,(i)\right]} \right] y_{t+k}\,(i) \right\} = 0.$$

Log-linearizing this equation around the unique zero-inflation steady state leads to

$$\widetilde{p}_t - p_t = (1-\beta\theta)\,\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left(i^\ell_{t+k} - i^b_{t+k} + \widehat{w}_{t+k} + p_{t+k} - p_t - \widehat{mp}_{t+k|t}\right)\right\}, \tag{C.1}$$

where $\widetilde{p}_t \equiv \log(\widetilde{P}_t)$, $p_t \equiv \log(P_t)$, variables with hats denote log deviations from steady-state values, $i^\ell_t \equiv \widehat{I}^\ell_t$, $i^b_t \equiv \widehat{I}^b_t$, and $mp_{t+k|t}$ denotes the marginal productivity in period $t+k$ for a firm whose price was last set in period t. Now, log-linearizing the production function (14) gives

$$\widehat{h}_t = \frac{f}{f'h}\widehat{y}_t, \tag{C.2}$$

so that $\widehat{mp}_{t+k|t}$ can be rewritten as

$$\begin{aligned}\widehat{mp}_{t+k|t} &= \frac{f''h}{f'}\widehat{h}_{t+k|t} = \widehat{mp}_{t+k} + \frac{f''h}{f'}\left(\widehat{h}_{t+k|t} - \widehat{h}_{t+k}\right) = \widehat{mp}_{t+k} + \frac{ff''}{(f')^2}\left(\widehat{y}_{t+k|t} - \widehat{y}_{t+k}\right)\\ &= \widehat{mp}_{t+k} - \frac{\varepsilon ff''}{(f')^2}\left(\widetilde{p}_t - p_{t+k}\right),\end{aligned}$$

where mp_{t+k} denotes the average marginal productivity in period $t+k$. Using this result and

$$\pi_t \equiv \log(\Pi_t) = (1-\theta)\left(\widetilde{p}_t - p_{t-1}\right),$$

and following the same steps as in, e.g., Galí (2015, Chapter 3), we can rewrite (C.1) as

$$\pi_t = \beta\mathbb{E}_t\{\pi_{t+1}\} + \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon ff''}{(f')^2}\right]}\left(i^\ell_t - i^b_t + \widehat{w}_t - \widehat{mp}_t\right). \tag{C.3}$$

Now, log-linearizing the goods-market-clearing condition (19) gives

$$\widetilde{c}_t + \widetilde{g}_t = \widehat{y}_t. \tag{C.4}$$

Log-linearizing the first-order condition (11), and using (C.4), gives

$$i^\ell_t - i^b_t = \alpha_\ell\frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\widehat{\ell}_t + \alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell}\widehat{m}_t - \alpha_\ell\frac{u''y}{u'}\widehat{y}_t + \alpha_\ell\frac{u''y}{u'}\widetilde{g}_t, \tag{C.5}$$

where

$$\alpha_\ell \equiv \frac{I^\ell - I^b}{I^\ell} \in (0,1).$$

Log-linearizing the first-order condition (10), and using (C.2) and (C.4), gives

$$\widehat{w}_t = \left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h}\right)\widehat{y}_t + \frac{u''y}{u'}\widetilde{g}_t. \tag{C.6}$$

Log-linearizing the constraint (15) holding with equality, and using (C.2) and (C.6), gives

$$\widehat{\ell}_t = \left(-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} + \frac{f}{f'h}\right)\widehat{y}_t + \frac{u''y}{u'}\widetilde{g}_t. \tag{C.7}$$

Using (C.5), (C.6), (C.7), and

$$\widehat{mp}_t = \frac{ff''}{(f')^2}\widehat{y}_t,$$

we can then rewrite (C.3) as the Phillips curve

$$\pi_t = \beta \mathbb{E}_t \left\{ \pi_{t+1} \right\} + \kappa_y \widehat{y}_t - \kappa_m \widehat{m}_t - \kappa_g \widetilde{g}_t,$$

where

$$\begin{aligned}
\kappa_y &\equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon f f''}{(f')^2}\right]}\left[\alpha_\ell \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right) - ... \right. \\
&\left. (1+\alpha_\ell)\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right] > 0, \\
\kappa_m &\equiv -\frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon f f''}{(f')^2}\right]}\alpha_\ell \frac{\Gamma_{\ell m} m}{\Gamma_\ell} > 0, \\
\kappa_g &\equiv -\frac{(1-\theta)(1-\beta\theta)}{\theta\left[1-\frac{\varepsilon f f''}{(f')^2}\right]}\frac{u''y}{u'}\left[1+\alpha_\ell\left(1+\frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\right)\right] > 0.
\end{aligned}$$

Log-linearizing the first-order condition (12), and using (C.4), gives

$$i_t^b - i_t^m = -\alpha_m \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\widehat{\ell}_t - \alpha_m \frac{\Gamma_{mm} m}{\Gamma_m}\widehat{m}_t + \alpha_m \frac{u''y}{u'}\widehat{y}_t - \alpha_m \frac{u''y}{u'}\widetilde{g}_t, \tag{C.8}$$

where $i_t^m \equiv \widehat{I}_t^m$ and

$$\alpha_m \equiv \frac{I^m - I^b}{I^m} < 0.$$

Using (C.7), we can rewrite (C.8) as

$$i_t^b - i_t^m = \sigma \delta_y \widehat{y}_t - \sigma \delta_m \widehat{m}_t - \sigma \delta_g \widetilde{g}_t,$$

where

$$\begin{aligned}
\delta_y &\equiv -\alpha_m + \alpha_m \frac{u'}{u''y}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right) > 0, \\
\delta_m &\equiv -\alpha_m \frac{u'}{u''y}\frac{\Gamma_{mm} m}{\Gamma_m} > 0, \\
\delta_g &\equiv -\alpha_m \left(1+\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right) > 0.
\end{aligned}$$

The dynamic equation in $\widehat{m}_t$ resulting from equations (34) to (37) is easily obtained as (38) with

$$\begin{aligned}
\mathcal{Q}_\mu(X) &\equiv X^2 - \left[(1+\delta_y)+\frac{1}{\beta}+\frac{\kappa_y}{\beta\sigma}\right]X + \left(\frac{1+\delta_y}{\beta}\right), \\
\mathcal{Q}_g(X) &\equiv \left(\frac{\kappa_y-\kappa_g}{\beta}\right)X + \left[\frac{(1+\delta_y)\kappa_g-(1+\delta_g)\kappa_y}{\beta}\right].
\end{aligned}$$

C.2 Proof of Lemma 3

We can rewrite $\mathcal{P}(X)$ as

$$\begin{aligned}
\mathcal{P}(X) &= X^3 - \left(\frac{1+2\beta}{\beta}+K_1+\frac{K_2}{\beta}\right)X^2 + ... \\
&\left(\frac{2+\beta}{\beta}+\frac{1+\beta}{\beta}K_1+\frac{K_2}{\beta}+\frac{K_3}{\beta}\right)X - \frac{1+K_1}{\beta},
\end{aligned}$$

where

$$\begin{aligned}
K_1 &\equiv \delta_y > 0, \\
K_2 &\equiv \frac{\kappa_y}{\sigma} - \kappa_m > 0, && \text{(C.9)} \\
K_3 &\equiv \delta_m \kappa_y - \delta_y \kappa_m > 0. && \text{(C.10)}
\end{aligned}$$

The inequality $K_3 > 0$ follows from

$$\begin{aligned}
\frac{-\theta\left[1-\frac{\varepsilon f f''}{(f')^2}\right]}{\alpha_m (1-\theta)(1-\beta\theta)} K_3 &= \frac{u'}{u''y}\frac{\Gamma_{mm} m}{\Gamma_m}\left[\alpha_\ell \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right) - \ldots\right. \\
&\left.(1+\alpha_\ell)\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right] + \ldots \\
&\alpha_\ell \frac{\Gamma_{\ell m} m}{\Gamma_\ell}\left[1-\frac{u'}{u''y}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right)\right] \\
&= \frac{\alpha_\ell \ell m}{\Gamma_\ell \Gamma_m}\frac{u'}{u''y}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right)\left[\Gamma_{\ell\ell}\Gamma_{mm}-(\Gamma_{\ell m})^2\right] + \ldots \\
&\frac{u'}{u''y}\frac{\Gamma_{mm} m}{\Gamma_m}\left[-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}-\frac{ff''}{(f')^2}\right] + \ldots \\
&\alpha_\ell m\left(\frac{\Gamma_{\ell m}}{\Gamma_\ell}-\frac{\Gamma_{mm}}{\Gamma_m}\right) \\
&> 0,
\end{aligned}$$

where the last inequality is obtained using (A.9) and (A.12).

The inequality $K_2 > 0$ follows from $K_3 > 0$ and

$$\begin{aligned}
\sigma\delta_m - \delta_y &= \alpha_m \frac{\Gamma_{mm} m}{\Gamma_m} + \alpha_m - \alpha_m \frac{u'}{u''y}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(-\frac{u''y}{u'}+\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right) \\
&= \frac{\alpha_m}{\Gamma_m}\left(\ell\Gamma_{\ell m} + m\Gamma_{mm}\right) + \alpha_m - \alpha_m \frac{u'}{u''y}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\left(\frac{v''h}{v'}\frac{f}{f'h}+\frac{f}{f'h}\right) \\
&< 0, && \text{(C.11)}
\end{aligned}$$

where the last inequality is obtained using (A.11).

We have

$$\begin{aligned}
\mathcal{P}(0) &= -\left(\frac{1+K_1}{\beta}\right) < 0, && \text{(C.12)} \\
\mathcal{P}(1) &= \frac{K_3}{\beta} > 0. && \text{(C.13)}
\end{aligned}$$

By rewriting $\mathcal{P}(X)$ as

$$\mathcal{P}(X) = (X-1-K_1)\left[X^2 - \left(\frac{1+\beta+K_2}{\beta}\right)X + \frac{1}{\beta}\right] - \left(\frac{K_1 K_2 - K_3}{\beta}\right)X$$

and noting that (C.11) implies

$$K_3 < K_1 K_2, \tag{C.14}$$

we also get

$$\mathcal{P}(1+K_1) = \frac{-(K_1K_2-K_3)(1+K_1)}{\beta} < 0,$$
$$\mathcal{P}(1+K_4) = \frac{-(K_1K_2-K_3)(1+K_4)}{\beta} < 0,$$

where

$$K_4 \equiv \frac{1+\beta+K_2+\sqrt{(1+\beta+K_2)^2-4\beta}}{2\beta} - 1 > 0.$$

Therefore, the roots of $\mathcal{P}(X)$ are three real numbers ρ, ω_1, and ω_2 such that

$$0 < \rho < 1 < \omega_1 < 1+\min(K_1,K_4) \leq 1+\max(K_1,K_4) < \omega_2.$$

C.3 Proof of Lemma 4

As prices become perfectly flexible ($\theta \to 0$), we have $\kappa_y \to +\infty$ and $\kappa_m \to +\infty$. For any $X \in \mathbb{R}$, we have

$$\lim_{\theta\to 0} \frac{\mathcal{P}(X)}{\kappa_y} = \frac{-(1-\sigma\psi)}{\beta\sigma}X^2 + \frac{(1-\sigma\psi)+\sigma(\delta_m-\delta_y\psi)}{\beta\sigma}X, \tag{C.15}$$

where $\psi \equiv \kappa_m/\kappa_y$ is independent of θ. The two roots of the polynomial on the right-hand side of (C.15) are 0 and $1+\sigma(\delta_m-\delta_y\psi)/(1-\sigma\psi)$. The latter root is higher than 1, because of (C.9) and (C.10), and lower than $1+\delta_y$, because of (C.14). Since ρ is the unique root of $\mathcal{P}(X)$ inside $(-1,1)$, and ω_1 its unique root inside $(1,1+\delta_y)$, we conclude that

$$\lim_{\theta\to 0}\rho = 0, \tag{C.16}$$
$$\lim_{\theta\to 0}\omega_1 = 1+\frac{\sigma(\delta_m-\delta_y\psi)}{1-\sigma\psi}. \tag{C.17}$$

Moreover, using

$$\rho\omega_1\omega_2 = -\mathcal{P}(0) = \frac{1+\delta_y}{\beta}, \tag{C.18}$$

$\rho > 0$, (C.16), and (C.17), we get

$$\lim_{\theta\to 0}\omega_2 = +\infty. \tag{C.19}$$

Next, using (C.16) and

$$\begin{aligned} 0 = \mathcal{P}(\rho) = {} & \rho^3 - \left[\frac{1+\beta-\kappa_m}{\beta} + \frac{\kappa_y}{\beta\sigma} + (1+\delta_y)\right]\rho^2 + \ldots \\ & \left[(1+\delta_y)\frac{1+\beta-\kappa_m}{\beta} + \frac{1}{\beta} + \left(\frac{1}{\sigma}+\delta_m\right)\frac{\kappa_y}{\beta}\right]\rho - \left(\frac{1+\delta_y}{\beta}\right), \end{aligned}$$

we get

$$\lim_{\theta\to 0}\kappa_m\rho = \frac{\sigma\psi(1+\delta_y)}{(1-\sigma\psi)+\sigma(\delta_m-\delta_y\psi)}. \tag{C.20}$$

Finally, using (C.17), (C.18), and (C.20), we get

$$\lim_{\theta\to 0}\frac{\omega_2}{\kappa_y} = \frac{1-\sigma\psi}{\beta\sigma}.$$

C.4 Proof of Proposition 6

The proof of steady-state convergence largely rests on reasonings similar to the ones conducted in Subsections 3.1 and 3.2 and in Appendix B.1. With the introduction of parameter γ, Equation (24) becomes, at the steady state,

$$\gamma\Gamma_\ell\left[\mathcal{L}\left(h\right),m\right]=\mathcal{A}\left(h\right). \tag{C.21}$$

This equation implicitly and uniquely defines a function $\widetilde{\mathcal{M}}$ such that

$$m=\widetilde{\mathcal{M}}\left(h,\gamma\right).$$

The function $\widetilde{\mathcal{M}}$ is strictly increasing in each of its two arguments ($\widetilde{\mathcal{M}}_h>0$ and $\widetilde{\mathcal{M}}_\gamma>0$). For any given $\gamma>0$, the function $h\mapsto\widetilde{\mathcal{M}}(h,\gamma)$ is defined over $(\underline{h},\overline{h})$, where $\overline{h}\in(\underline{h},h^\star]$ is implicitly and uniquely defined by

$$\lim_{m_t\to+\infty}\gamma\Gamma_\ell\left[\mathcal{L}\left(\overline{h}\right),m_t\right]=\mathcal{A}\left(\overline{h}\right).$$

Note that $\overline{h}$ depends on γ and satisfies

$$\lim_{\gamma\to0}\overline{h}=h^\star. \tag{C.22}$$

Now, with the introduction of parameter γ, Equation (32) becomes

$$\widetilde{\mathcal{F}}\left(h,\gamma\right)\equiv\frac{\gamma\Gamma_m\left[\mathcal{L}\left(h\right),\widetilde{\mathcal{M}}\left(h,\gamma\right)\right]}{u'\left[f\left(h\right)-g\right]}=\beta I^m-1. \tag{C.23}$$

Lemma 2 implies that, for any given $\gamma>0$,

$$\widetilde{\mathcal{F}}_h>0\quad\text{and}\quad\lim_{h_t\to\overline{h}}\widetilde{\mathcal{F}}(h_t,\gamma)=0. \tag{C.24}$$

We can rewrite $\widetilde{\mathcal{F}}\left(h,\gamma\right)$ as

$$\widetilde{\mathcal{F}}\left(h,\gamma\right)=\widetilde{\mathcal{F}}_1\left(h,\gamma\right)\mathcal{F}_2\left(h\right),$$

where, for any given $\gamma>0$, the function $h\mapsto\widetilde{\mathcal{F}}_1(h,\gamma)$ is defined over $(\underline{h},\overline{h})$ by

$$\widetilde{\mathcal{F}}_1\left(h,\gamma\right)\equiv\frac{\Gamma_m\left[\mathcal{L}\left(h\right),\widetilde{\mathcal{M}}\left(h,\gamma\right)\right]}{\Gamma_\ell\left[\mathcal{L}\left(h\right),\widetilde{\mathcal{M}}\left(h,\gamma\right)\right]}=\frac{g_m^b\left[\mathcal{L}\left(h\right),\widetilde{\mathcal{M}}\left(h,\gamma\right)\right]}{g_\ell^b\left[\mathcal{L}\left(h\right),\widetilde{\mathcal{M}}\left(h,\gamma\right)\right]},$$

while $\mathcal{F}_2$ is defined in Appendix B.1. We have

$$\left(g_\ell^b\right)^2\widetilde{\mathcal{F}}_{1,\gamma}=\left(g_\ell^bg_{mm}^b-g_m^bg_{\ell m}^b\right)\widetilde{\mathcal{M}}_\gamma=-g_{\ell m}^b\left(d\mathcal{L}g_\ell^b+\mathcal{M}g_m^b\right)\frac{\widetilde{\mathcal{M}}_\gamma}{\widetilde{\mathcal{M}}}=-g^bg_{\ell m}^b\frac{\widetilde{\mathcal{M}}_\gamma}{\widetilde{\mathcal{M}}}>0,$$

where the second equality is obtained by using (A.6), and the third equality by using (A.2). Therefore, we get $\widetilde{\mathcal{F}}_{1,\gamma}>0$ and hence, using $\mathcal{F}_2>0$,

$$\widetilde{\mathcal{F}}_\gamma>0. \tag{C.25}$$

Using (C.22), (C.23), (C.24), and (C.25), we conclude that

$$\lim_{(I^m,\gamma)\to(\beta^{-1},0)} h = h^\star. \tag{C.26}$$

As a consequence, the steady-state values of all endogenous variables converge, as $(I^m, \gamma) \to (\beta^{-1}, 0)$, towards their counterparts in the corresponding basic NK model $-$ with the exception of the steady-state value of real reserves m, which does not exist in the basic NK model.

We now show that m is bounded away from zero and infinity as $(I^m, \gamma) \to (\beta^{-1}, 0)$ when $(\beta^{-1} - I^m)/\gamma$ is bounded away from zero and infinity. Rewrite (C.23) as

$$\frac{-\Gamma_m\left[\mathcal{L}(h), m\right]}{u'\left[f(h) - g\right]} = \frac{1 - \beta I^m}{\gamma}. \tag{C.27}$$

Since the right-hand side of this equation is bounded away from zero, (5), (C.26), and (C.27) imply that m is bounded from above. Moreover, (C.25) and $\widetilde{\mathcal{F}} < 0$ imply that, for any given h_t,

$$\lim_{\gamma\to 0} \frac{-\widetilde{\mathcal{F}}(h_t, \gamma)}{\gamma} = +\infty. \tag{C.28}$$

Now, using (6) and (C.21), we get, for any given h_t,

$$\lim_{\gamma\to 0} \widetilde{\mathcal{M}}(h_t, \gamma) = 0$$

and therefore

$$\lim_{\gamma\to 0} \frac{-\widetilde{\mathcal{F}}(h_t, \gamma)}{\gamma} = \lim_{\gamma\to 0} \frac{-\Gamma_m\left[\mathcal{L}(h_t), \widetilde{\mathcal{M}}(h_t, \gamma)\right]}{u'\left[f(h_t) - g\right]} = \lim_{m_t\to 0} \frac{-\Gamma_m\left[\mathcal{L}(h_t), m_t\right]}{u'\left[f(h_t) - g\right]}. \tag{C.29}$$

Using (C.28) and (C.29), we then get, for any given h_t,

$$\lim_{m_t\to 0} \frac{-\Gamma_m\left[\mathcal{L}(h_t), m_t\right]}{u'\left[f(h_t) - g\right]} = +\infty.$$

Using this result, (C.26), (C.27), and the fact that the right-hand side of (C.27) is bounded from above, we conclude that m is bounded away from zero.

Finally, (C.26) and the boundedness of m away from zero and infinity imply that, as $(I^m, \gamma) \to (\beta^{-1}, 0)$: (i) the elasticities $\Gamma_{\ell\ell}\ell/\Gamma_\ell$, $\Gamma_{mm}m/\Gamma_m$, $\Gamma_{\ell m}\ell/\Gamma_m$, and $\Gamma_{\ell m}m/\Gamma_\ell$ are themselves bounded away from zero and infinity; and (ii) the parameter $\alpha_\ell \equiv (I^\ell - I^b)/I^\ell$, which can be rewritten as

$$\alpha_\ell = \frac{\gamma\Gamma_\ell(\ell, m)}{u'\left[f(h) - g\right] + \gamma\Gamma_\ell(\ell, m)}$$

by using the first-order condition (11) amended to take into account the introduction of parameter γ, converges towards zero. Since $\alpha_m \equiv (I^m - I^b)/I^m$ also converges towards zero as $(I^m, \gamma) \to (\beta^{-1}, 0)$, using the definitions of κ_y, κ_m, κ_g, δ_y, δ_m, and δ_g, we conclude that

$$\kappa_y \to \overline{\kappa}_y \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon f f''}{(f')^2}\right]}\left[-\frac{u''y}{u'} + \frac{v''h}{v'}\frac{f}{f'h} - \frac{ff''}{(f')^2}\right], \quad \kappa_g \to \overline{\kappa}_g \equiv -\frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon f f''}{(f')^2}\right]}\frac{u''y}{u'},$$

and $(\kappa_m, \delta_y, \delta_m, \delta_g) \to 0$ as $(I^m, \gamma) \to (\beta^{-1}, 0)$, where the functions in $\overline{\kappa}_y$ and $\overline{\kappa}_g$ are evaluated at $h = h^\star$ and $c = f(h^\star) - g$.

C.5 Proof of Proposition 7

In this appendix, for the sake of brevity, we replace "$(I^m, \gamma) \to (\beta^{-1}, 0)$ with $(\beta^{-1} - I^m)/\gamma$ bounded away from zero and infinity" by "$D \to 0$," where D stands for "Distance between the basic NK model and our model." Using Proposition 6, we easily get that, as $D \to 0$,

$$\mathcal{P}(X) \to \frac{1}{\beta}(X-1)\left[\beta X^2 - \left(1+\beta+\frac{\overline{\kappa}_y}{\overline{\sigma}}\right)X + 1\right]$$

for any $X \in \mathbb{R}$, where $\overline{\sigma} \equiv -u''[f(h^\star)-g]f(h^\star)/u'[f(h^\star)-g] > 0$, from which we conclude that $\omega_1 \to 1$ and

$$\rho \quad \to \quad \overline{\rho} \equiv \frac{1}{2\beta}\left[1+\beta+\frac{\overline{\kappa}_y}{\overline{\sigma}} - \sqrt{\left(1+\beta+\frac{\overline{\kappa}_y}{\overline{\sigma}}\right)^2 - 4\beta}\right] \in (0,1),$$

$$\omega_2 \quad \to \quad \overline{\omega}_2 \equiv \frac{1}{2\beta}\left[1+\beta+\frac{\overline{\kappa}_y}{\overline{\sigma}} + \sqrt{\left(1+\beta+\frac{\overline{\kappa}_y}{\overline{\sigma}}\right)^2 - 4\beta}\right] > 1.$$

Using these results and L'Hospital's rule, we can easily determine the limits of π_1 in (43) and $\widehat{y}_1$ in (44) as $D \to 0$:

$$\lim_{D\to 0}\pi_1 = \frac{-\overline{\kappa}_y i^*}{\beta\overline{\sigma}(\overline{\omega}_2-1)}\left[T - \frac{1-\overline{\omega}_2^{-T}}{\overline{\omega}_2 - 1}\right] \quad \text{and} \quad \lim_{D\to 0}\widehat{y}_1 = \frac{-i^*}{\beta\overline{\sigma}}\left[\frac{T}{\overline{\omega}_2} + \frac{\overline{\kappa}_y\left(1-\overline{\omega}_2^{-T}\right)}{\overline{\sigma}(\overline{\omega}_2-1)^2}\right],$$

from which we get

$$\lim_{T\to+\infty}\lim_{D\to 0}\frac{\pi_1}{T} = \frac{-\overline{\kappa}_y i^*}{\beta\overline{\sigma}(\overline{\omega}_2-1)}, \quad \lim_{T\to+\infty}\lim_{D\to 0}\frac{\widehat{y}_1}{T} = \frac{-i^*}{\beta\overline{\sigma}\overline{\omega}_2},$$
$$\lim_{\theta\to 0}\lim_{D\to 0}\pi_1 = -i^*T, \quad \lim_{\theta\to 0}\lim_{D\to 0}\widehat{y}_1 = 0,$$

which proves Point (i) and half of Points (iii) and (iv) of the proposition. We can also easily determine the limits of π_1 in (46) and $\widehat{y}_1$ in (47) as $D \to 0$:

$$\lim_{D\to 0}\pi_1 = \frac{(\overline{\kappa}_y - \overline{\kappa}_g)\,\overline{\omega}_2^{-T}}{\beta}\widetilde{g}^* \quad \text{and} \quad \lim_{D\to 0}\widehat{y}_1 = \frac{-(\overline{\kappa}_y - \overline{\kappa}_g)\,\overline{\omega}_2^{-T}}{\beta\overline{\sigma}}\widetilde{g}^*,$$

from which we get

$$\lim_{T\to+\infty}\lim_{D\to 0}\pi_1 = \lim_{T\to+\infty}\lim_{D\to 0}\widehat{y}_1 = 0$$

and, for $T \geq 2$,

$$\lim_{\theta\to 0}\lim_{D\to 0}\pi_1 = \lim_{\theta\to 0}\lim_{D\to 0}\widehat{y}_1 = 0,$$

which proves Point (ii) and the other half of Points (iii) and (iv) of the proposition.

C.6 Proof of Proposition 8

It is easy to show, along the same lines as in Woodford (2003) or Galí (2015), that the introduction of cost-push shocks into our model transforms the Phillips curve (34) into (51) with

$$\kappa_\varphi \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon f f''}{(f')^2}\right]} > 0.$$

Using the IS equation (35), the spread equation (36), the reserve-market-clearing condition (37), and the Phillips curve (51), in the absence of IOR-rate, reserves-growth-rate, and government-expenditures shocks (i.e., when $i_t^m = \widehat{\mu}_t = \widetilde{g}_t = 0$), we get the following dynamic equation in $\widehat{m}_t$:

$$\mathbb{E}_t \left\{ L\mathcal{P}\left(L^{-1}\right) \widehat{m}_t \right\} = \mathbb{E}_t \left\{ \mathcal{Q}_{\varphi}\left(L^{-1}\right) \widehat{\varphi}_t \right\},$$

which can be rewritten as

$$\mathbb{E}_t \left\{ (1-\omega_1 L)(1-\omega_2 L) q_{t+2} \right\} = \mathbb{E}_t \left\{ \mathcal{Q}_{\varphi}\left(L^{-1}\right) \widehat{\varphi}_t \right\},$$

where $q_t \equiv \widehat{m}_t - \rho \widehat{m}_{t-1}$ and

$$\mathcal{Q}_{\varphi}(X) \equiv \frac{\kappa_{\varphi}}{\beta} X - \frac{(1+\delta_y)\kappa_{\varphi}}{\beta}.$$

Using this dynamic equation from date $T+1$ onwards, we get $q_t = 0$ for $t \geq T+1$. Using it at date T, together with $q_{T+1} = q_{T+2} = 0$, we get

$$q_T = \frac{-(1+\delta_y)}{\beta \omega_1 \omega_2} \kappa_{\varphi} \widehat{\varphi}^*.$$

Finally, using the dynamic equation at dates 1 to $T-1$, together with the above terminal conditions on q_T and q_{T+1}, we get

$$\begin{aligned} q_t \;\; = \;\; & \left\{ \left[(1+\delta_y) - \omega_1\right] \left(\frac{\omega_2 - 1}{\omega_2 - \omega_1}\right) \omega_1^{t-T-1} + \left[\omega_2 - (1+\delta_y)\right] \left(\frac{\omega_1 - 1}{\omega_2 - \omega_1}\right) \omega_2^{t-T-1} - \delta_y \right\} \ldots \\ & \frac{\kappa_{\varphi} \widehat{\varphi}^*}{\beta(\omega_1 - 1)(\omega_2 - 1)} \end{aligned}$$

for $1 \leq t \leq T+1$. As we converge towards the basic NK model, we have $\rho \to \overline{\rho} \in (0,1)$, $\omega_1 \to 1$, $\omega_2 \to \overline{\omega}_2 > 1$, and $\delta_y \to 0$. Using L'Hospital's rule, we then easily get that, in the basic-NK-model limit,

$$q_t = -\left(1 - \overline{\omega}_2^{t-T-1}\right) \frac{\kappa_{\varphi} \widehat{\varphi}^*}{\beta(\overline{\omega}_2 - 1)}$$

for $1 \leq t \leq T+1$. Using the initial condition $\widehat{m}_0 = 0$, we then get

$$\widehat{m}_t = -\left\{ \left(\frac{1-\overline{\rho}^t}{1-\overline{\rho}}\right) - \overline{\omega}_2^{t-T-1} \left[\frac{1 - \left(\frac{\overline{\rho}}{\overline{\omega}_2}\right)^t}{1 - \left(\frac{\overline{\rho}}{\overline{\omega}_2}\right)}\right] \right\} \frac{\kappa_{\varphi} \widehat{\varphi}^*}{\beta(\overline{\omega}_2 - 1)}$$

for $0 \leq t \leq T+1$. Next, using $\pi_t = -\Delta \widehat{m}_t$, we get

$$\pi_t = \left\{ \left[1 - \left(\frac{1-\overline{\rho}}{\overline{\omega}_2 - \overline{\rho}}\right) \overline{\omega}_2^{-T}\right] \overline{\rho}^{t-1} - \left(\frac{\overline{\omega}_2 - 1}{\overline{\omega}_2 - \overline{\rho}}\right) \overline{\omega}_2^{t-T-1} \right\} \frac{\kappa_{\varphi} \widehat{\varphi}^*}{\beta(\overline{\omega}_2 - 1)} \tag{C.30}$$

for $1 \leq t \leq T+1$. Finally, using $\overline{\kappa}_y \widehat{y}_t = \pi_t - \beta \mathbb{E}_t \{\pi_{t+1}\} - \kappa_{\varphi} \widehat{\varphi}_t$ and $\beta \overline{\rho} \overline{\omega}_2 = 1$, we get

$$\begin{aligned} \widehat{y}_t \;\; & = \;\; -\left\{ \left(\frac{1-\overline{\rho}^t}{1-\overline{\rho}}\right) - \overline{\omega}_2^{t-T-1} \left[\frac{1 - \left(\frac{\overline{\rho}}{\overline{\omega}_2}\right)^t}{1 - \left(\frac{\overline{\rho}}{\overline{\omega}_2}\right)}\right] \right\} \frac{(1-\overline{\rho})\kappa_{\varphi} \widehat{\varphi}^*}{\overline{\kappa}_y} \\ & < \;\; -\left\{ \left(\frac{1-\overline{\rho}^t}{1-\overline{\rho}}\right) - \left[\frac{1 - \left(\frac{\overline{\rho}}{\overline{\omega}_2}\right)^t}{1 - \left(\frac{\overline{\rho}}{\overline{\omega}_2}\right)}\right] \right\} \frac{(1-\overline{\rho})\kappa_{\varphi} \widehat{\varphi}^*}{\overline{\kappa}_y} \\ & < \;\; 0 \end{aligned}$$

for $1 \leq t \leq T$, which proves Proposition 8. The interpretation of Proposition 8 in the text uses three additional results: (i) π_t is decreasing in t for $1 \leq t \leq T+1$, as apparent from (C.30); (ii)

$$\pi_1 = \left(1 - \overline{\omega}_2^{-T}\right) \frac{\kappa_\varphi \widehat{\varphi}^*}{\beta\left(\overline{\omega}_2 - 1\right)} > 0;$$

and (iii) $\pi_{T+1} < 0$. This last result is shown by: (i) writing $\pi_{T+1} = A(T)\kappa_\varphi \widehat{\varphi}^* / [\beta(\overline{\omega}_2 - 1)]$, where

$$A(T) \equiv \left[1 - \left(\frac{1-\overline{\rho}}{\overline{\omega}_2 - \overline{\rho}}\right) \overline{\omega}_2^{-T}\right] \overline{\rho}^{t-1} - \left(\frac{\overline{\omega}_2 - 1}{\overline{\omega}_2 - \overline{\rho}}\right) \overline{\omega}_2^{t-T-1};$$

(ii) noting that $A(T)$ is decreasing in T for $T \geq 1$:

$$A\left(T+1\right) - A\left(T\right) = -\left(1 - \overline{\rho}\right)\left(1 - \overline{\omega}_2^{-T-1}\right) \overline{\rho}^T < 0;$$

and (iii) noting that

$$A\left(1\right) = \frac{-\left(1-\overline{\rho}\right)\left(\overline{\omega}_2 - 1\right)}{\overline{\omega}_2} < 0.$$

C.7 Proof of Proposition 9

Consider a temporary interest-rate peg: assume that i_t^b takes the value $i^{b*} \neq 0$ from date 1 to date $T \geq 1$, and that the economy is at its steady state from date $T+1$ onwards (i.e. $i_t^b = \widehat{y}_t = \pi_t = 0$ for $t \geq T+1$). We start with the case in which $\xi_3(\theta) > 0$ or $\xi_4(\theta) > 0$. In this case, for $1 \leq t \leq T$, the system made of the IS equation (52) and the Phillips curve (53) can easily be rewritten as

$$\mathbb{E}_t \left\{ \begin{bmatrix} \widehat{y}_{t+1} \\ \pi_{t+1} \end{bmatrix} \right\} = \mathbf{C} \begin{bmatrix} \widehat{y}_t \\ \pi_t \end{bmatrix} + \mathbf{D} i^{b*} \tag{C.31}$$

with

$$\mathbf{C} \equiv \frac{1}{\varphi\left(\theta\right)} \begin{bmatrix} \kappa\left(\theta\right)\xi_2 + \beta\sigma\xi_3\left(\theta\right) & -\xi_2 \\ \kappa\left(\theta\right)\sigma\left[\xi_4\left(\theta\right) - \xi_1\right] & \sigma\xi_1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} \equiv \frac{\xi_2}{\varphi\left(\theta\right)} \begin{bmatrix} \beta\xi_3\left(\theta\right) \\ \kappa\left(\theta\right)\xi_4\left(\theta\right) \end{bmatrix},$$

where $\varphi(\theta) \equiv \beta\sigma\xi_1\xi_3(\theta) + \kappa(\theta)\xi_2\xi_4(\theta) > 0$. The characteristic polynomial of $\mathbf{C}$ is

$$\mathcal{C}\left(X\right) \equiv X^2 - \frac{\sigma\xi_1 + \kappa\left(\theta\right)\xi_2 + \beta\sigma\xi_3\left(\theta\right)}{\varphi\left(\theta\right)} X + \frac{\sigma}{\varphi\left(\theta\right)}.$$

Since $\mathcal{C}(0) \neq 0$, $\mathbf{C}$ is invertible. Iterating the dynamic equation (C.31) between 1 and T, and using the terminal condition $\widehat{y}_{T+1} = \pi_{T+1} = 0$ and the invertibility of $\mathbf{C}$, we get

$$\begin{bmatrix} \widehat{y}_1 \\ \pi_1 \end{bmatrix} = -\left(\sum_{k=1}^{T} \mathbf{C}^{-k}\right) \mathbf{D} i^{b*}.$$

For any $X \in \mathbb{R}$, we have

$$\lim_{\theta \to 0} \frac{\varphi\left(\theta\right)\mathcal{C}\left(X\right)}{\kappa\left(\theta\right)\xi_2} = \left[\lim_{\theta \to 0} \xi_4\left(\theta\right)\right] X^2 - X.$$

One root of the polynomial on the right-hand side of this equation is zero. Therefore, one root of $\mathcal{C}(X)$ converges towards zero as $\theta \to 0$, which implies in turn that $\lim_{\theta \to 0} ||\mathbf{C}^{-1}|| = +\infty$. Since $||\mathbf{D}||$ is bounded away from zero as $\theta \to 0$, we eventually get $\lim_{\theta \to 0} |\widehat{y}_1| = \lim_{\theta \to 0} |\pi_1| = +\infty$.

In the alternative case in which $\xi_3(\theta) = \xi_4(\theta) = 0$, the system made of the IS equation (52) and the Phillips curve (53) implies the following dynamic equation in inflation for $1 \leq t \leq T$:

$$\left[\xi_1 + \frac{\kappa(\theta)\,\xi_2}{\sigma}\right] \mathbb{E}_t\{\pi_{t+1}\} = \pi_t + \frac{\kappa(\theta)\,\xi_2 i^{b*}}{\sigma}. \tag{C.32}$$

Iterating this dynamic equation between 1 and T, and using the terminal condition $\pi_{T+1} = 0$, we get

$$\pi_1 = \left\{\left[\xi_1 + \frac{\kappa(\theta)\,\xi_2}{\sigma}\right]^T - 1\right\} \frac{\kappa(\theta)\,\xi_2 i^{b*}}{\sigma(1-\xi_1) - \kappa(\theta)\,\xi_2},$$

so that $\lim_{\theta\to 0} |\pi_1| = +\infty$. Using the Phillips curve (53), we then get $\lim_{\theta\to 0} |\widehat{y}_1| = +\infty$.

C.8 Proof of Proposition 10

Consider first the case in which $\xi_3(\theta) > 0$ or $\xi_4(\theta) > 0$. In this case, under a permanent peg $i_t^b = i^{b*}$, the system made of the IS equation (52) and the Phillips curve (53) can easily be rewritten as (C.31). If the peg ensures local-equilibrium determinacy, then $\mathcal{C}(X)$, the characteristic polynomial of $\mathbf{C}$ (derived in Appendix C.7), must have no root inside the unit circle, because the system has two non-predetermined variables. In particular, $\mathcal{C}(X)$ must have no root inside the real-number interval $[0, 1]$, which requires that $\mathcal{C}(0)\,\mathcal{C}(1) > 0$, i.e. equivalently

$$\sigma(1-\xi_1)[1-\beta\xi_3(\theta)] - \kappa(\theta)\,\xi_2[1-\xi_4(\theta)] > 0. \tag{C.33}$$

In the unique local equilibrium, the (constant) inflation rate is easily obtained as

$$\pi_t = \pi^* \equiv \frac{-\kappa(\theta)\,\xi_2[1-\xi_4(\theta)]\,i^{b*}}{\sigma(1-\xi_1)[1-\beta\xi_3(\theta)] - \kappa(\theta)\,\xi_2[1-\xi_4(\theta)]}.$$

Given (C.33), π^* is negatively related to i^{b*}.

In the alternative case in which $\xi_3(\theta) = \xi_4(\theta) = 0$, under a permanent peg $i_t^b = i^{b*}$, the system made of the IS equation (52) and the Phillips curve (53) implies the dynamic equation (C.32). Therefore, for the peg to ensure determinacy, we need

$$\sigma(1-\xi_1) - \kappa(\theta)\,\xi_2 > 0. \tag{C.34}$$

In the unique local equilibrium, the (constant) inflation rate is easily obtained as

$$\pi_t = \pi^* \equiv \frac{-\kappa(\theta)\,\xi_2 i^{b*}}{\sigma(1-\xi_1) - \kappa(\theta)\,\xi_2}.$$

Given (C.34), π^* is negatively related to i^{b*}.

C.9 Proof of Inequality $\delta_g \kappa_m < \delta_m \kappa_g$

The inequality $\delta_g \kappa_m < \delta_m \kappa_g$ follows from

$$
\begin{aligned}
\left\{ \frac{-\theta \left[1 - \frac{\varepsilon f f''}{(f')^2}\right]}{\alpha_m (1-\theta)(1-\beta\theta)} \right\} (\delta_m \kappa_g - \delta_g \kappa_m) &= -\left[1 + \alpha_\ell \left(1 + \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\right)\right] \frac{\Gamma_{mm} m}{\Gamma_m} + ... \\
& \alpha_\ell \left(1 + \frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right) \frac{\Gamma_{\ell m} m}{\Gamma_\ell} \\
&= -\frac{\Gamma_{mm} m}{\Gamma_m} + \alpha_\ell m \left(\frac{\Gamma_{\ell m}}{\Gamma_\ell} - \frac{\Gamma_{mm}}{\Gamma_m}\right) - ... \\
& \frac{\alpha_\ell \ell m}{\Gamma_\ell \Gamma_m} \left[\Gamma_{\ell\ell}\Gamma_{mm} - (\Gamma_{\ell m})^2\right] \\
&> 0,
\end{aligned}
$$

where the last inequality is obtained using (A.9) and (A.12).

Appendix D: Robustness Analysis

D.1 Matrices $\mathbf{A}^r$ and $\mathbf{B}^r$

$$
\begin{aligned}
\mathbf{A}^r \equiv & \begin{bmatrix} \frac{1+\beta}{\beta} - \frac{\kappa_m}{\beta} & \frac{-1}{\beta} & \frac{\kappa_y}{\beta} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{\beta\sigma} - \delta_m - \frac{\kappa_m}{\beta\sigma} & \frac{-1}{\beta\sigma} & 1 + \delta_y + \frac{\kappa_y}{\beta\sigma} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + (\nu_P - 1) \begin{bmatrix} -\frac{\kappa_m}{\beta} & 0 & \frac{\kappa_y}{\beta} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + ... \\
& \frac{\nu_P - 1}{\nu_P} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \nu_y \begin{bmatrix} \delta_m + \frac{\kappa_m}{\beta\sigma} & 0 & \frac{1}{\beta} - \delta_y - \frac{\kappa_y}{\beta\sigma} & \frac{-1}{\beta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + ... \\
& \frac{\nu_y}{\nu_P} \begin{bmatrix} \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta\sigma} & \frac{-1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{\nu_y^2}{\nu_P} \begin{bmatrix} 0 & 0 & \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\
\mathbf{B}^r \equiv & \begin{bmatrix} 0 & 0 & \frac{-\kappa_g}{\beta} \\ 0 & 0 & 0 \\ \frac{1}{\sigma} & 1 & -\left(1 + \delta_g + \frac{\kappa_g}{\beta\sigma}\right) \\ 0 & 0 & 0 \end{bmatrix} + (\nu_P - 1) \begin{bmatrix} 0 & 0 & \frac{-\kappa_g}{\beta} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \nu_y \begin{bmatrix} \frac{-1}{\sigma} & -1 & 1 + \delta_g + \frac{\kappa_g}{\beta\sigma} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

D.2 Proof of Lemma 5

After some algebra, we get

$$\det (\mathbf{A}^r - X\mathbf{I}_4) = X \left[\mathcal{P}(X) + \mathcal{Z}(X)\right],$$

where $\mathbf{I}_4$ denotes the 4×4 identity matrix and

$$\begin{aligned} \mathcal{Z}(X) \equiv & -\left[\left(\delta_m + \frac{\kappa_m}{\beta\sigma}\right)\nu_y - \frac{\kappa_m}{\beta}(\nu_P - 1)\right]X^2 + \left\{\left[\frac{(1+\beta)\delta_m}{\beta} + \frac{\kappa_m}{\beta\sigma}\right]\nu_y + ... \right. \\ & \left. \left(\frac{\delta_m\kappa_y - \delta_y\kappa_m}{\beta} - \frac{\kappa_m}{\beta}\right)(\nu_P - 1)\right\}X - \frac{\delta_m}{\beta}\nu_y. \end{aligned}$$

Therefore, the eigenvalues of $\mathbf{A}^c$ are 0 and the roots of

$$\mathcal{D}(X) \equiv \mathcal{P}(X) + \mathcal{Z}(X).$$

We have

$$\mathcal{Z}(0) = \frac{-\delta_m\nu_y}{\beta}, \tag{D.1}$$

$$\mathcal{Z}(1) = \frac{(\delta_m\kappa_y - \delta_y\kappa_m)(\nu_P - 1)}{\beta} = \frac{K_3(\nu_P - 1)}{\beta}, \tag{D.2}$$

so that, using (C.12) and (C.13), we get

$$\begin{aligned} \mathcal{D}(0) &= \frac{-(1+K_1)}{\beta} - \frac{\delta_m\nu_y}{\beta} < -1, \\ \mathcal{D}(1) &= \frac{K_3\nu_P}{\beta} > 0. \end{aligned}$$

Therefore, $\mathcal{D}(X)$ has either one root or three roots in the real-number interval $[0,1]$. Now, the product of the three roots of $\mathcal{D}(X)$ is equal to $-\mathcal{D}(0) > 1$, so that $\mathcal{D}(X)$ has at least one root outside the unit circle. As a consequence, $\mathcal{D}(X)$ has exactly one root inside the real-number interval $[0,1]$.

The other roots of $\mathcal{D}(X)$ are either two real numbers outside $[0,1]$, or two conjugate complex numbers. In the latter case, both are outside the unit circle, since $\mathcal{D}(X)$ has at least one root outside it. Therefore, $\mathcal{D}(X)$ has exactly two roots outside the unit circle if and only if it has no root inside the real-number interval $[-1,0)$. Since $\mathcal{D}(0) < 0$, the latter condition is equivalent to $\mathcal{D}(X) < 0$ for all $X \in [-1,0]$.

Now rewrite $\mathcal{P}(X)$ and $\mathcal{Z}(X)$ as

$$\begin{aligned} \mathcal{P}(X) &= \frac{\kappa_y}{\beta}[\mathcal{P}_1(X) + \mathcal{P}_2(X)], \\ \mathcal{Z}(X) &= \frac{\kappa_y}{\beta}[\mathcal{Z}_1(X) + \mathcal{Z}_2(X)], \end{aligned}$$

where

$$\begin{aligned} \mathcal{P}_1(X) &\equiv \frac{-1}{\kappa_y}(1-X)(1-\beta X)(1+\delta_y - X), \\ \mathcal{P}_2(X) &\equiv \left(\frac{1}{\sigma} - \psi\right)X(1-X) + (\delta_m - \delta_y\psi)X, \\ \mathcal{Z}_1(X) &\equiv \frac{-\delta_m\nu_y}{\kappa_y}(1-X)(1-\beta X), \\ \mathcal{Z}_2(X) &\equiv \left[\frac{\psi\nu_y}{\sigma} - \psi(\nu_P - 1)\right]X(1-X) + [(\delta_m - \delta_y\psi)(\nu_P - 1)]X. \end{aligned}$$

Whatever $X \in [-1,0]$, $\mathcal{P}_1(X)$ and $\mathcal{Z}_1(X)$ are decreasing functions of θ, while $\mathcal{P}_2(X)$ and $\mathcal{Z}_2(X)$ do not depend on θ. Therefore, $\mathcal{D}(X) < 0$ for all $X \in [-1,0]$ and all $\theta \in (0,1)$ if and only if $\mathcal{D}(X) < 0$ for all $X \in [-1,0]$ as $\theta \to 0$. Since $\mathcal{P}_1(X)$ and $\mathcal{Z}_1(X)$ converge towards zero as θ goes to zero whatever $X \in [-1,0]$, the latter condition is equivalent to $\mathcal{D}_2(X) < 0$ for all $X \in [-1,0)$, where

$$\mathcal{D}_2(X) \equiv \mathcal{P}_2(X) + \mathcal{Z}_2(X).$$

Now, we have

$$\mathcal{D}_2(X) = X\left\{Z^r X + \left[(\delta_m - \delta_y \psi)\nu_P - Z^r\right]\right\},$$

where

$$Z^r \equiv \psi\nu_P - \frac{1+\psi\nu_y}{\sigma},$$

so that $\mathcal{D}_2(X) < 0$ for all $X \in [-1,0)$ if and only if

$$Z^r < \frac{(\delta_m - \delta_y\psi)\nu_P}{2}. \tag{D.3}$$

Since

$$\begin{aligned} \psi &\equiv -\alpha_\ell \frac{\Gamma_{\ell m} m}{\Gamma_\ell}\left[\alpha_\ell \frac{\Gamma_{\ell\ell}\ell}{\Gamma_\ell}\left(-\frac{u'' y}{u'} + \frac{v'' h}{v'}\frac{f}{f' h} + \frac{f}{f' h}\right) - (1+\alpha_\ell)\frac{u'' y}{u'} + \frac{v'' h}{v'}\frac{f}{f' h} - \frac{f f''}{(f')^2}\right]^{-1} \\ &< -\alpha_\ell \frac{\Gamma_{\ell m} m}{\Gamma_\ell}\left[\frac{-u'' y}{u'}\right]^{-1} = -\frac{\alpha_\ell}{\sigma}\frac{\Gamma_{\ell m} m}{\Gamma_\ell}, \end{aligned} \tag{D.4}$$

Condition (59) implies $\psi\nu_P \leq 1/\sigma$ and hence $Z^r < 0$, which in turn implies (D.3), which finally implies that $\mathcal{D}(X)$ has exactly two roots outside the unit circle.

D.3 Phillips Curve in the Model With Cash

The derivation of the Phillips curve (62) follows the same steps as in Appendix C.1. The only change is that the objective of firm i is now to maximize

$$\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k \frac{\beta\lambda_{t+k+1}}{\lambda_t \Pi_{t,t+k+1}}\left[\widetilde{P}_t(i)\, y_{t+k}(i) - I^\ell_{t+k} W_{t+k} f^{-1}\left[y_{t+k}(i)\right]\right]\right\},$$

since the firm has to wait until the next period to exchange its cash. The log-linearization of the first-order condition around the unique zero-inflation steady state thus gives

$$\widetilde{p}_t - p_t = (1-\beta\theta)\,\mathbb{E}_t\left\{\sum_{k=0}^{+\infty}(\beta\theta)^k\left(i^\ell_{t+k} + \widehat{w}_{t+k} + p_{t+k} - p_t - \widehat{mp}_{t+k|t}\right)\right\},$$

which corresponds to (C.1) without the $-i^b_{t+k}$ term. Rewriting this equation then leads to the Phillips curve (62) with

$$\kappa_b \equiv \frac{(1-\theta)(1-\beta\theta)}{\theta\left[1 - \frac{\varepsilon f f''}{(f')^2}\right]} > 0.$$

D.4 Matrices $\mathbf{A}^c$ and $\mathbf{B}^c$

Defining $\eta \equiv 1-\alpha_g$ and $\widetilde{\kappa}_j \equiv \kappa_j + \sigma\delta_j\kappa_b > 0$ for $j \in \{y,m,g\}$, we can write $\mathbf{A}^c$ and $\mathbf{B}^c$ as

$$\mathbf{A}^c \equiv \begin{bmatrix} \frac{1+\beta}{\beta} - \frac{\widetilde{\kappa}_m}{\beta} & \frac{-1}{\beta} & \frac{\widetilde{\kappa}_y}{\beta} & 0 \\ 1 & 0 & 0 & 0 \\ \frac{1}{\beta\sigma} - \delta_m - \frac{\widetilde{\kappa}_m}{\beta\sigma} & \frac{-1}{\beta\sigma} & 1+\delta_y + \frac{\widetilde{\kappa}_y}{\beta\sigma} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \alpha_c \begin{bmatrix} \frac{-1}{\beta\eta\sigma} & \frac{1}{\beta\eta\sigma} & \frac{1}{\beta\eta^2\sigma} & \frac{-1}{\beta\eta^2\sigma} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\beta\sigma} & \frac{1}{\beta\sigma} & \frac{1}{\beta\eta\sigma} & \frac{-1}{\beta\eta\sigma} \\ 0 & 0 & 0 & 0 \end{bmatrix} + \ldots$$

$$\frac{\alpha_c}{1-\alpha_c} \begin{bmatrix} \frac{\delta_m}{\eta} + \frac{(1-\eta\sigma)\widetilde{\kappa}_m}{\beta\eta\sigma} & 0 & \frac{-(1-\eta\sigma)}{\beta\eta^2\sigma} - \frac{\delta_y}{\eta} - \frac{(1-\eta\sigma)\widetilde{\kappa}_y}{\beta\eta\sigma} & \frac{1-\eta\sigma}{\beta\eta^2\sigma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B}^c \equiv \begin{bmatrix} \frac{\kappa_b}{\beta} & 1 & \frac{-1}{\beta} & 0 & \frac{-\widetilde{\kappa}_g}{\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sigma} + \frac{\kappa_b}{\beta\sigma} & 0 & \frac{-1}{\beta\sigma} & 1 & -\left(1+\delta_g + \frac{\widetilde{\kappa}_g}{\beta\sigma}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \alpha_c \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{-1}{\beta\eta^2\sigma} & \frac{1}{\beta\eta^2\sigma} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{\beta\eta\sigma} & \frac{1}{\beta\eta\sigma} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \ldots$$

$$\frac{\alpha_c}{1-\alpha_c} \begin{bmatrix} \frac{-1}{\eta\sigma} + \frac{(1-\eta\sigma)\kappa_b}{\beta\eta\sigma} & 1 & \frac{1-\eta\sigma}{\beta\eta\sigma} & 0 & \frac{1-\eta\sigma}{\beta\eta^2\sigma} + \frac{\delta_g}{\eta} + \frac{(1-\eta\sigma)\widetilde{\kappa}_g}{\beta\eta\sigma} & \frac{1-\eta\sigma}{\beta\eta^2\sigma} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

D.5 Proof of Lemma 6

Since $\mathbf{A}^c$ can be obtained by replacing ν_p, ν_y, κ_y, κ_m, and κ_g by $1/(1-\alpha_c)$, $\alpha_c/[(1-\alpha_c)\eta]$, $\widetilde{\kappa}_y$, $\widetilde{\kappa}_m$, and $\widetilde{\kappa}_g$ respectively in $\mathbf{A}^r$, and since $\widetilde{K}_3 \equiv \delta_m\widetilde{\kappa}_y - \delta_y\widetilde{\kappa}_m = \delta_m\kappa_y - \delta_y\kappa_m = K_3 > 0$, Appendix D.2 implies that $\mathbf{A}^c$ has two eigenvalues inside the unit circle and two eigenvalues outside whatever $\theta \in (0,1)$ if and only if

$$Z^c < \frac{\delta_m - \delta_y\widetilde{\psi}}{2(1-\alpha_c)}, \tag{D.5}$$

where

$$Z^c \equiv \frac{-1}{\sigma} + \left[1 + \left(\frac{\alpha_c}{1-\alpha_c}\right)\left(\frac{\eta\sigma - 1}{\eta\sigma}\right)\right]\widetilde{\psi} \quad \text{and} \quad \widetilde{\psi} \equiv \frac{\widetilde{\kappa}_m}{\widetilde{\kappa}_y}.$$

Now, we have

$$\begin{aligned} Z^c &< \frac{-1}{\sigma} + \frac{\widetilde{\psi}}{1-\alpha_c} \\ &< \frac{1}{\sigma}\left[-1 + \frac{1}{1-\alpha_c}\left(-\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell} + \alpha_m\frac{\Gamma_{mm}m}{\Gamma_m}\right)\right] \\ &\leq \overline{Z}^c \equiv \frac{1}{\sigma}\left[-1 + \frac{1}{1-\alpha_c}\left(-\alpha_\ell\frac{\Gamma_{\ell m}m}{\Gamma_\ell} - \alpha_m\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right)\right], \end{aligned}$$

where the last inequality follows from (A.11). Using first (11) and (12), and then (15) with equality and (23), we get sequentially

$$\begin{aligned}\overline{Z}^c &= \frac{1}{\sigma}\left[-1+\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left(\frac{I^b}{I^\ell}\frac{\Gamma_{\ell m}m}{\Gamma_m}+\frac{1}{\beta I^m}\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right)\right] \\ &= \frac{1}{\sigma}\left\{-1+\left(\frac{1-\beta I^m}{1-\alpha_c}\right)\left[\left(\frac{\varepsilon}{\varepsilon-1}\right)\left(\frac{f}{f'h}\right)\left(\frac{m}{y}\right)+\frac{1}{\beta I^m}\right]\frac{\Gamma_{\ell m}\ell}{\Gamma_m}\right\}.\end{aligned}$$

Therefore, Condition (66) is equivalent to $\overline{Z}^c < 0$, which implies $Z^c < 0$, which in turn implies (D.5), which finally implies that $\mathbf{A}^c$ has two eigenvalues inside the unit circle and two eigenvalues outside.

Appendix E: Model With a Satiation Level

E.1 Steady-State Analysis

As in the benchmark model, we assume that I_t^m can vary exogenously around a given value $I^m \geq 1$, μ_t around the value $\mu = 1$, and g_t around a given value $g \geq 0$. In any steady state, both nominal reserves (because $\mu = 1$) and real reserves (by definition of a steady state) are constant over time. Therefore, prices are also constant over time, and the set of steady states is the same under sticky prices ($\theta > 0$) as under flexible prices ($\theta = 0$).

Now, the introduction of a finite satiation level of reserves brings three changes to Subsection 3.1's flexible-price analysis. First, Equation (30) is replaced by

$$\lim_{h_t \to \overline{h}_t} \mathcal{M}(h_t) = \underline{m}\left[\mathcal{L}\left(\overline{h}_t\right)\right],$$

so that we have

$$h_t < \overline{h}_t \quad \text{and} \quad m_t = \mathcal{M}(h_t) < \underline{m}\left[\mathcal{L}\left(\overline{h}_t\right)\right]$$

when the economy is outside the satiation range at date t. Second, the economy can now also be inside the satiation range at date t, in which case we have

$$h_t = \overline{h}_t \quad \text{and} \quad m_t \geq \underline{m}\left[\mathcal{L}\left(\overline{h}_t\right)\right].$$

Third, the dynamic equation (31) is replaced by

$$1+\frac{\Gamma_m\left[\mathcal{L}(h_t), m_t\right]}{u'\left[f(h_t)-g_t\right]} = \beta I_t^m \mathbb{E}_t\left\{\frac{u'\left[f(h_{t+1})-g_t\right]m_{t+1}}{\mu_{t+1}u'\left[f(h_t)-g_t\right]m_t}\right\}. \tag{E.1}$$

For convenience, we extend the domain of definition of $\mathcal{M}$ from $\left(\underline{h}_t, \overline{h}_t\right)$ to $\left(\underline{h}_t, \overline{h}_t\right]$, and define $\mathcal{M}(\overline{h}_t) \equiv \underline{m}[\mathcal{L}(\overline{h}_t)]$.

When $(h_{t+1}, m_{t+1}) = (h_t, m_t)$ and $(I_t^m, \mu_t, g_t) = (I^m, 1, g)$, the dynamic equation (E.1) boils down to the same static equation (32) as previously, where the function $\mathcal{F}$ still has all the properties stated in Lemma 2. The only novelty is that $\mathcal{F}$ is now also defined at point $\overline{h}$, with

$\mathcal{F}(\overline{h}) = 0$. Therefore, the set of steady states is characterized as follows. When $I^m > \beta^{-1}$, there is no steady state. When $I^m = \beta^{-1}$, there is an infinity of steady states; in each of these steady states, the employment level is equal to $\overline{h}$; these steady states differ from each other only in terms of the constant values of real reserves and the price level. When $1 \leq I^m < \beta^{-1}$, there is a unique steady state; in this steady state, the employment level is lower than $\overline{h}$.

E.2 Log-Linearization

We now assume that $I^m \in [1, \beta^{-1}]$ for a steady state to exist. If $I^m < \beta^{-1}$, then the model has a unique steady state; this steady state lies outside the satiation range and has zero inflation. Naturally, log-linearizing the model around this steady state gives exactly the same reduced form as the reduced form of our benchmark model (without a satiation level).

Alternatively, if $I^m = \beta^{-1}$, then the model has an infinity of steady states; all these steady states lie inside the satiation range and have zero inflation. Log-linearizing the model around any of these steady states leads to the reduced form made of the Phillips curve (34) with $\kappa_m = 0$ (because $\Gamma_{\ell m} = 0$), the IS equation (35), the spread equation (36) with $\delta_y = \delta_m = \delta_g = 0$ (because $\Gamma_m = 0$ and $\alpha_m = 0$), and the reserve-market-clearing condition (37). This reduced form is isomorphic to the reduced form of the basic NK model. If $\Gamma_\ell > 0$, then the parameters κ_y and κ_g of our model's Phillips curve (defined in Appendix C.1) are larger than their respective counterparts $\overline{\kappa}_y$ and $\overline{\kappa}_g$ in the basic NK model's Phillips curve (defined in Appendix C.4). If $\Gamma_\ell = 0$, then these parameters are equal to each other ($\kappa_y = \overline{\kappa}_y$ and $\kappa_g = \overline{\kappa}_g$), and the reduced form of our model becomes exactly identical to the reduced form of the basic NK model.

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