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Recovery Before Redemption: A Theory of Delays in Sovereign Debt Renegotiations

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ABSTRACT

Negotiations to restructure sovereign debts are *protracted*, taking on average almost 8 years to complete. In this paper we construct a new database (the most extensive of its kind covering ninety recent sovereign defaults) and use it to document that these negotiations are also *ineffective* in both repaying creditors and reducing the debt burden countries face. Specifically, we find that creditor losses average roughly 40 per-cent, and that the average debtor exits default more highly indebted than when they entered default. To explain this apparent large inefficiency in negotiations, we present a theory of sovereign debt renegotiation in which delay arises from the same commitment problems that lead to default in the first place. A debt restructuring generates surplus for the parties at both the time of settlement *and* in the future. However, a creditor's ability to share in the future surplus is limited by the risk that the debtor will default on the settlement agreement. Hence, the debtor and creditor find it privately optimal to delay restructuring until future default risk is low, even though delay means some gains from trade remain unexploited. We show that a quantitative version of our theory can account for a number of stylized facts about sovereign default, as well as the new facts about debt restructurings that we document in this paper. Finally, we argue that our findings shed light on the existence of delays in bargaining in a wider range of contexts.

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1 Introduction

In many economic environments, agents appear to have trouble reaching mutually advantageous agreements. In this paper, we document that this phenomenon is especially severe in the case of debt restructuring negotiations between a sovereign country in default and its international creditors. Using a new database of sovereign debt restructuring outcomes, the most extensive of its kind covering ninety recent sovereign defaults, we show that the average default takes more than 7 years to resolve, results in creditor losses (or "haircuts") of roughly 40 per-cent, and leaves the sovereign country more highly indebted than when they entered default. To explain this apparent inefficiency, we present a theory of sovereign borrowing, default, and debt restructuring in which delays in debt restructuring are the result of the same commitment problems that lead to default in the first place. As a debt restructuring agreement produces gains for the debtor country both in the period of the settlement, and in the future, the country would like to promise a share of these future gains as part of a settlement. However, there is a risk that the country will default on such a promise. As a result, both the country and its creditors find it privately optimal to delay restructuring until future default risk is low. We show that a quantitative version of the theory can account for a number of stylized facts about sovereign default, as well as the new facts on debt restructurings that we document in this paper. Finally, we use the theory to examine the efficacy of bailouts by multilateral institutions as a tool for both providing insurance to debtor countries, and for encouraging a prompt restructuring.

We begin by presenting our database of sovereign debt restructuring outcomes. Drawn from a variety of sources, the database covers 90 defaults by 73 countries that were settled during the period 1989 to 2006, and contains data on the occurrence of default and settlement, the outcomes of negotiations, as well as measures of economic performance and indebtedness. In addition to the three facts introduced above, we emphasize three facts about the relationship of these outcomes to economic activity, and to each other, that motivate the development of our theory below. Specifically, we find that longer defaults are correlated with larger haircuts, and that there is a modest (but only a modest) tendency for countries to enter default when output is relatively low, and to emerge from default once output has recovered to its trend. Finally, we also establish that longer defaults and larger haircuts are more likely when economic conditions in the defaulting country are weak at the time of default.

We then present our theory of sovereign borrowing, default and debt restructuring. In the theory, international debt markets are incomplete so that default offers the sovereign country partial (and costly) insurance against adverse economic outcomes. While in default, and until it has settled with its creditors, output in the country is reduced, and access to international financial markets is limited. As a result, surplus is wasted while the country is in default, and the country and its creditors bargain over shares of this surplus. Bargaining takes place under complete information, with the bargaining power of the parties fluctuating over time. A settlement consists of a transfer of current resources as well as a new debt issue which serves to share the future surplus generated by a settlement. The value of a settlement to creditors, therefore, depends on the market value of the new debt issue, which is in turn limited by the fact that the country may default on these debts. Delay arises as both the country and creditor find it optimal to wait until the value of any debt issued as part of a settlement has *recovered* before agreeing on a settlement and *redeeming* the old debts.

We next take the theory to the data and show that it is capable of matching the new debt restructuring facts above, as well as a number of facts about sovereign borrowing and default stressed in previous studies. Calibrating bargaining power in our model to the relationship between default and economic activity in the data, we generate some defaults when output is high as a result of a favorable bargaining position for the debtor. Other defaults occur following a sequence of low income levels. In such cases, the possibility of a settlement leads creditors to lend even when default risk is high, supporting higher levels of borrowing (at face value) at higher interest rates than in previous models. Defaults occur when the ability to raise debt levels in response to another negative income shock is limited. When debt levels are high, settlements consist largely of new debt issues, and occur only after significant improvements in economic circumstances or bargaining conditions that raise the value of new debt issues. This is the source of delay in our model. Likewise, when the face value of the defaulted debt is high we get large haircuts, generating a positive correlation between delay and haircuts. Since countries exit default when circumstances have improved, they are able to borrow more than they could just prior to default. Thus, debt levels often rise upon exit from default. The volatility of sovereign spreads is increased by both volatility in the size of the expected settlement, and the greater variability in debt levels.

Our paper contributes to a number of literatures. We believe we are the first to characterize the empirical relationship between delay, haircuts and debt levels for sovereign countries in default, while our characterization of creditor losses for ninety defaults triples the number of estimates previously available (e.g. Cline 1995 and Sturzenegger and Zettelmeyer 2007). Our theory contributes to the recent literature on debt and default in both an international (Arellano 2007, Kovrijnykh and Szentes 2007, Yue 2007, and Mendoza and Yue 2008) and domestic (Chatterjee et al 2007) context. Unlike all of these papers, our theory generates delays in bargaining, and does so without appealing to collective action problems among creditors (unlike Pitchford and Wright 2007, 2008), and while simultaneously explaining the evolution of debt during the default restructuring process (unlike Bi 2008 and d'Erasmo 2008). Finally, we view our work as a contribution to the broader literature on delays in bargaining. Our finding that delays are predictable leads us to focus on commitment problems with complete information, and abstract from the role of asymmetric information (unlike the work surveyed by Ausubel, Cramton, and Deneckere 2002). Our approach extends the abstract bargaining environment of Merlo and Wilson (1995) by allowing for outside options, flow payoffs, and an endogenous terminal payoff.

The rest of this paper is organized as follows. Section 2 describes our database of sovereign debt restructuring outcomes and presents our empirical findings. Section 3 presents our theory, first analyzing the debt restructuring process taking borrowing outcomes as given, before analyzing borrowing outcomes taking the debt restructuring process as given. We then combine the restructuring environment with the borrowing environment and provide a proof of existence of an equilibrium for the overall model. Section 4 shows that a calibrated version of the model can match the facts introduced in Section 2. Section 5 concludes by reinterpreting the phenomenon of worldwide sovereign debt crises in the light of our results, and considering the theories implications for negotiations in other contexts. An appendix collects proofs of theorems, while the working paper version of the paper provides further details on our database.

2 Sovereign Debt Restructuring Facts

In this section we describe our database of sovereign defaults and debt renegotiation outcomes, and present our empirical findings.

2.A Data Sources and Construction

In setting the limits of our database, we restrict attention to defaults on sovereign debts owed to private sector creditors, like banks and bondholders. The reason is that, in our model of debt restructuring below, creditors bargain with a view to maximizing the value of their settlement, while official creditors like the International Monetary Fund and creditor country governments are arguably motivated by broader concerns of equity. We define sovereign debts to include debts owed either directly by a country's national government, or owed indirectly by virtue of a government guarantee. The most comprehensive and widely used source of data on the dates of defaults on sovereign debts owed to private sector creditors, as well as the dates of settlements of these defaults, is published by the ratings agency Standard and Poors (Beers and Chambers 2006). Standard and Poors (S&P) defines a default on a debt contract to have occurred if a payment is not made within any grace period specified in the contract, or if debts are rescheduled on terms less favorable than those specified in the original debt contract. S&P defines the end of a default as occurring when a settlement occurs, typically in the form of an exchange of new debt for old debt, and when they conclude that "no further near-term resolution of creditors claims is likely" (page 22). Defining a default to have begun when debts are rescheduled on unfavorable terms, which is also related to the definition of a settlement, may result in an underestimate of actual delays in bargaining. Standard and Poors record only the year in which a default started and ended, and so we supplement these dates with data from Arteta and Hale (2007), Pitchford and Wright (2007) and Trebesch (2008), as well as a range of primary sources, to come up with the month, and in some cases the day, in which a default started and ended.

The most novel part of our dataset lies in its estimates of creditor losses, or haircuts, for a large number of defaults. Until now, there has existed only a small number of estimates produced by different researchers using different methods for largely non-overlapping samples of defaults[1]. In order to obtain the largest sample possible, and to ensure consistency of treatment, we base our measures on the World Bank's estimates of debt stock reduction, interest and principal forgiven, and debt buybacks, as published in *Global Development Finance*

[1] We have uncovered estimates of haircuts in 27 defaults, constructed by four different authors using five different methods. All of the estimates are tabulated for the purposes of comparison in Appendix C.

(GDF). We combine the World Bank's estimates of the reduction in the face value of the debt with estimates of the forgiveness of arrears on interest and principle. As the World Bank data do not make any distinction between forgiveness of debts by private creditors and forgiveness by official creditors, we scale the amount of forgiveness using estimates of the total amount of debt renegotiated, and on the proportion owed to private creditors, from both GDF and Institute for International Finance (2001). Losses in different years were added together and discounted back to the time of the default using a ten per-cent discount rate, following the practice of the OECD Development Assistance Committee and the World Bank (Dikhanov 2006). As shown in Appendix C, our estimates correlate closely with those of other studies.

The resulting series on private creditor haircuts covers ninety defaults and renegotiations by seventy-three separate countries that were completed after GDF data on debt forgiveness first became available in 1989 and that ended prior to 2006. Our data on default dates and haircuts were then combined with data on various indicators of economic activity taken from the World Bank's *World Development Indicators* publication, and with data on the stock of long term sovereign debt outstanding from GDF. Short term debt is not included because it is not available disaggregated by type of creditor.

2.B The Facts

Table 1: Delays and Haircuts

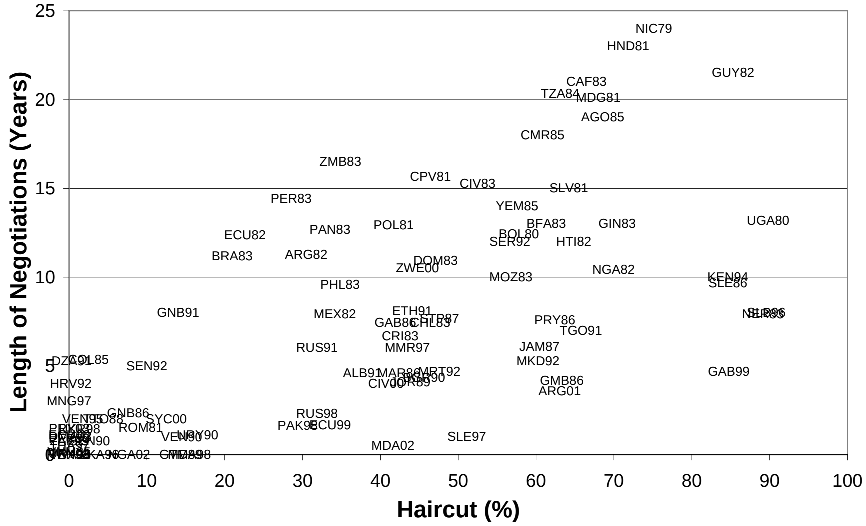
	Mean	Median	Correlation with $(e_T - e_{T-1})/e_{T-1}$	Correlation with $(e_{T+1} - e_{T-1})/e_{T-1}$	Correlation with $Debt/GDP$
Delay 1	7.4 years	6.0 years	-0.26	-0.21	0.03
Delay 2	7.6 years	6.7 years	-0.26	-0.21	0.03
Haircut 1	38%	42%	-0.25	-0.23	0.02
Haircut 2	38%	42%	-0.25	-0.23	0.02

Table 2: Output and Debt Levels Around Default

	mean % deviation from trend e	% of years e below trend	Debt/GDP (%) Mean	Debt/GDP (%) Median
years in default	-0.4	54	87	61
years out of default	0.4	43	51	33
year before default	1.0	39	52	40
year of default	-1.3	64	58	45
year of settlement	-0.2	51	73	50
year after settlement	0.1	48	72	47

Table 1 presents some summary statistics on the length of time taken to settle a default, which we refer to as delay, and on average haircuts weighted by the level of outstanding debt. There are three instances of defaults being contiguous in time, in the sense that S&P dates a

Figure 1: The Relationship Between Delays and Haircuts



default by a country as ending in the same year, or year before, another default begins[2]. We present results treating these defaults both as separate events ("delay 1"), and treating them as a single default episode ("delay 2"). Treating contiguous defaults as distinct defaults, there are ninety defaults in our sample lasting an average of 7.4 years. Delays rise to an average of 7.6 years if contiguous defaults are treated as a single default event. In our sample, delay is slightly higher than found in other studies, such as Pitchford and Wright (2008), who record an average delay of 6.5 years for a larger sample of defaults in the modern era. This leads to our first result:

Fact 1: *sovereign defaults are time consuming to resolve*, taking almost eight years on average in our sample.

Table 1 also presents evidence on the average size of haircuts, where the average is weighted by the value of outstanding debts for the case of contiguous defaults. As shown in the Table, the average creditor experienced a haircut of roughly 40 per-cent of the value of the debt. Further information on the sizes of haircuts and delays is presented in Figure 1 which contains a scatter plot of haircuts and delays for each of the ninety settlements contained in our sample. As shown in the Figure, haircuts in our sample have ranged from approximately

[2]The three episodes are: Ecuador, who S&P treat as being in default from 1999 to 2000, and again from 2000 to 2001; Russia, in default from 1991 to 1997, and from 1998 to 2000; and Venezuela, in default from 1995 to 1997, and in 1998.

zero all the way up to ninety per-cent of the value of creditors claims in the case of some African defaults. Likewise, there is a great deal of variation in delays with many defaults being settled almost immediately while others are settled in excess of two decades. There is also a noticeable positive relationship between the amount of delay in renegotiation and the size of the haircut, with the correlation coefficient between the two series equalling 0.66. This gives rise to our next two results:

Fact 2: *creditor losses (or* haircuts*) are substantial*, with the average creditor experiencing a reduction in the value of their claim of forty-four per-cent.

Fact 3: *longer defaults are associated with larger haircuts*, with a correlation between the length of the renegotiation process and the size of the creditor haircut of two-thirds.

One possible explanation for Fact 3 is that there is a common factor driving both longer defaults and larger haircuts. To examine this, Table 1 also presents evidence on the relationship between delays and haircuts and the level of economic activity in the year of the default. In particular, the third column shows that the larger is the decline in output in the year of default, the longer the delay and the larger the haircut, on average. The relationship is only modest, however, never rising above 0.3 in absolute value, with the correlation to haircuts barely different from zero. The fourth column Table 1 presents the relationship between delays and haircuts and the growth of output in the two years surrounding the default and finds a stronger negative relationship with haircuts. This leads to our fourth fact:

Fact 4: *larger output declines in the year of default are associated with modestly longer defaults and larger haircuts*, with correlation coefficients around -0.25

Table 2 provides further evidence on the relationship between defaults, settlements and output. As shown in the first column, there is a broad tendency for default to be associated with adverse economic conditions, with a mean level of output roughly one-half of one per-cent below trend[3], while output in non-default periods is above trend by an equal amount on average. Economic adversity is particularly likely in the first year of a default, when output

[3]Deviations from trend are calculated using a Hodrick-Prescott filter with smoothing parameter 6.25 for annual data (see the discussion in Ravn and Uhlig 2002). Tomz and Wright (2007) establish that these facts are robust to different filtering methods.

was on average 1.3 per-cent below trend, and tends to have dissipated by the time a country settles with its creditors when output is on average only 0.2% below trend. Nonetheless, there is a great deal of variation across country experiences so that the overall relationship between output and default is quite weak. In almost one-third of cases, a country defaults with output above trend. This confirms the earlier finding of Tomz and Wright (2007) for a larger sample of defaults, and leads to our fifth result:

Fact 5: *defaults are somewhat more likely to occur when output is below trend, and settlements tend to occur when output has returned to trend,* with 64% of defaults beginning when output is below trend, and 49% ending when output is above trend. The average deviation of output from trend is -1.3% in the first year of a default, and -0.2% in the year of the settlement.

Table 2 also explores the relationship between defaults and debt levels for the defaulting country. As shown in the table, being in default is associated with levels of debt to GDP that are more than seventy per-cent higher than for when a country is not in default, bearing in mind that our sample of countries is conditioned upon having defaulted once during this period. Strikingly, the table reveals that the average country exits default with levels of debt that are 25 per-cent higher than they possessed when they entered default. This figure is accentuated by some outlier countries, but even the median country exits default with 5 per-cent more debt. From this we conclude that renegotiations are ineffective at reducing the indebtedness of a debtor country. This leads to our sixth result:

Fact 6: *default resolution is* ***not*** *associated with decreased country indebtedness*, with the median and average country exiting default with a debt to GDP ratio 5 and 25 per-cent higher than before they entered default, respectively.

Finally, Table 1 also shows that delays and haircuts are essentially unrelated to the initial level of indebtedness of a country. In our theory, which we begin to outline in the next section, we therefore do not focus upon differences in debt levels as a major factor in negotiations.

3 A Theory of Sovereign Debt, Default, and Debt Restructuring

In this section, we present our theory of sovereign borrowing and default. We begin by first describing the decisions facing a sovereign country that is in good standing with its creditors, before moving on to a description of international credit markets, and then to the debt restructuring environment, devoting the most detail to the latter.

3.A The Borrowing and Default Environment

The Sovereign Borrower

Consider a world in which time is discrete and lasts forever. In each period $t = 0, 1, ..,$ a sovereign country receives an endowment of the single non-storable consumption good $e(s)$ that is a function of the exogenous state s which takes on values in the finite set S. Thus, the endowment also takes on only a finite number, N_e, of values. The state s summarizes all sources of uncertainty in the model and evolves according to a first order Markov process with transition probabilities given by a transition matrix with representative element $\pi(s'|s)$. Below, the evolution of the state s will also govern the evolution of the country's bargaining position with creditors.

The sovereign country is represented by an agent that maximizes the discounted expected value of its utility from consuming state contingent sequences of the single consumption good $\{c_t(s^t)\}$ according to

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t|s_0} \pi\left(s^t|s_0\right) u\left(c_t\left(s^t\right)\right).$$

Here, the felicity function u is twice continuously differentiable, strictly increasing and strictly concave so that the country is averse to fluctuations in its consumption. The notation $s^t|s_0$ is used to denote a history of the state that begins with state s_0, while $\pi(s^t|s_0)$ is the product of the associated one-period ahead conditional probabilities. The discount factor β lies between zero and one and is assumed to imply a discount rate in excess of the world interest rate. As a result, international borrowing may be motivated by both a desire to smooth consumption, as well as a desire to tilt a country's consumption profile forward in time.

A sovereign country that is not in default enters a period with a new value of the state s, and a level of international debt b. It is assumed that b must lie in the set of debt levels, B, which is finite with cardinality N_b, and contains both negative and positive elements, as well as the zero element, where negative elements are interpreted as savings by the country. We let $V(b, s)$ denote the value function of a country of that enters the period with debt b and state s, *before* the country has decided whether or not to default, which is an N_e by N_b

vector of real numbers.

The sovereign's first decision is whether or not to default on its debts. If the sovereign defaults, they receive a payoff given by $\tilde{V}^D(b,s)$, which is a N_e by N_b vector of real numbers, and which will be determined below when we describe the process by which a country in default bargains with its creditors. If we let $V^R(b,s)$ denote the value function of a country that enters the period with debt b and state s, *after* it has decided to repay it's debts, which is an N_e by N_b vector of real numbers, then the value function $V(b,s)$ satisfies

$$V(b,s) = \max\left\{V^R(b,s), \tilde{V}^D(b,s)\right\}. \tag{1}$$

If the sovereign country repays its debts, it must decide how much to consume c and how much debt $b' \in B$ to take into the next period. The value function associated with the repayment of debt, V^R, is defined by

$$V^R(b,s) = \max_{c,b' \in B} u(c) + \beta \sum_{s' \in S} \pi(s'|s) V(b', s'), \tag{2}$$

subject to $c - q(b', s) b' \leq e(s) + b$. Here, $q(b', s)$ is a $N_e \times N_b$ vector of prices today of a bond that pays one unit tomorrow as long as the country does not default, and that depends on the current state s and total borrowing b'. It is determined by competition in international credit markets, which we describe next.

International Credit Markets

We assume that international credit markets are populated by a large number of risk neutral creditors that behave competitively. The opportunity cost of funds for a creditor is given by the world interest rate r^w, which we assume is constant. Competition in the international credit market ensures that creditors expect to earn the world interest rate from their investments in the sovereign borrower's bonds.

To understand the determinants of the price of a country's bonds, suppose the country issues a total of b claims, each of which pays one unit tomorrow as long as the country does not default. If a creditor were to buy one unit of the country's bonds at price $q(b,s)$, then competition ensures that they must expect to receive $(1 + r^w) q(b,s)$ on average tomorrow.

The actual return they receive has two components. First, with some probability $1-p(b,s)$ the country is expected to repay-in-full which yields a total of one unit. Second, with probability $p(b,s)$ the country defaults. In this case, the country will commence bargaining with its creditors and the creditor will receive a one-in-b share of any returns from this bargaining process. If we let $\tilde{W}(b,s')$ be a $N_e \times N_b$ vector of the total expected discounted values of any settlement on a default on b bonds in state s' tomorrow, viewed from the perspective of tomorrow, then the equilibrium bond price must satisfy

$$q(b,s) = \frac{1 - p(b,s) + p(b,s) \sum_{s' \in S} \pi(s'|s) \tilde{W}(b,s')/b}{1 + r^w}.$$

The total expected discounted value of any settlement, viewed from tomorrow, $\tilde{W}(b,s')$ will be determined along with the $N_e \times N_b$ vector of values to the country from default $\tilde{V}^D(b,s)$, as a result of the bargaining process which we describe in the next section. For now, we assume that $\tilde{W}(b,s')$ is bounded below by zero and above by b, which in turn ensures that the bond price function takes values in the interval $[0, 1/(1+r^w)]$; we prove that $\tilde{W}$ has these properties below. We let $\mathcal{Q}(B \times S)$ be the set of all functions on $B \times S$ taking values in $[0, 1/(1+r^w)]$.

It remains to describe the probability of default $p(b,s)$, which is determined by the sovereign's decision to default described in (1) above. For most values of (b,s), the sovereign country will strictly prefer defaulting over repaying, or repaying over defaulting. However, it is possible that for some values of (b,s) that the country is indifferent. To deal with this possibility, we define an indicator correspondence for default with debt b in state s, $\Phi(b,s)$, as

$$\Phi(b,s) = \begin{cases} 1 & \text{if } \tilde{V}^D(b,s) > V^R(b,s) \\ 0 & \text{if } \tilde{V}^D(b,s) < V^R(b,s) \\ [0,1] & \text{if } \tilde{V}^D(b,s) = V^R(b,s) \end{cases}.$$

From this we can define the default probability correspondence for debt b and state s, $P(b,s)$, as the set of all $p(b,s)$ constructed as $p(b,s) = \sum_{s' \in S} \phi(b,s') \pi(s'|s)$, for some $\phi(b,s) \in \Phi(b,s)$.

Debt Restructuring Negotiations

In this subsection, we specify the process by which a sovereign country in default bargains with its creditors over a settlement. We abstract from the coordination problems in debt restructuring negotiations studied by Pitchford and Wright (2007, 2008), and assume that creditors are able to perfectly coordinate in bargaining with the country. Hence, our restructuring negotiations are modeled as a game between two players: the sovereign borrower in default, and a single creditor.

Environment We assume that the country is in autarky in the period in which the default actually occurs. Hence, the relationship between the total value to creditors from a settlement $\tilde{W}(b, s')$ and the value to the country from default $\tilde{V}^D(b, s)$, that we introduced above, and the $N_e \times N_b$ vectors of outcomes of bargaining that we derive below, $W(b, s')$ and $V^D(b, s)$, is given by $\tilde{W}(b, s) = \delta E[W(b, s')|s]]$, and $\tilde{V}^D(b, s) = u\left(e^{def}(s)\right) + \beta E\left[V^D(b, s')|s\right]$. Here, $\delta = 1/(1 + r^w)$ while $e^{def}(s)$ is used to denote the possibility that the endowment process may be lower in the event of a default (reflecting any direct costs of default). The output loss, combined with one period of autarky, ensure that there is always *some* cost to default, and deter the country from continually renegotiating its debts.

Negotiations begin with a sovereign country that has previously entered default with a level of debt b. At stake is the ability of the country to re-access credit markets. The value to the country of settling today in state s with its creditors and re-accessing capital markets with a new level of debt b' is given by $\sum_{s' \in S} \pi(s'|s) V(b', s')$, where V was described above and is treated as exogenous for the purposes of bargaining.

Neither player is able to commit to a split of surplus beyond the current period. Instead, the players can only agree to a current transfer of resources that may be partially (or wholly) financed by the issue of new debt securities. The ability to share future surplus is therefore limited by the fact that the country may default on these new debt securities in the future. Delay can occur as both the creditor and the debtor wait for an improvement in the terms under which new debt securities can be issued. Importantly, the same commitment problem that leads to default also drives the outcome of the renegotiation.

If no agreement is reached this period, the bargaining game continues with a new state s' and the same level of debt b. The assumption that the amount of debt in default, b, is unchanged throughout negotiations captures the fact that for most of the period under study, interest on missed payments was not a part of default settlements[4].

Negotiations between the creditors and the debtor are efficient, in the sense that agreements are optimal for the two parties subject to the constraints on negotiations implied by future default risk. To capture this fact, we say that negotiations are privately optimal ex post. Nonetheless, delay may be said to be socially wasteful ex post, as the country is unable to access capital markets while in default, and thus forfeits potential gains from trade in tilting and smoothing its consumption.

Timing and Actions Bargaining occurs according to a randomly alternating offer bargaining game with an outside option available to the debtor. At any point, the debtor country has the option of paying off the defaulted debt in full, using any desired mix of current transfers and new debt securities issued at the market price. We refer to this action as the *outside option* of the debtor, although we stress that this is strictly only an outside option for the game conditional on default, and not for the entire borrowing environment. In addition to being a feature of the actual environment governing sovereign debt renegotiations, this assumption guarantees that the total value of the settlement never exceeds b which serves to bound our bond price function.

In every period and in each state of the world s, either the sovereign borrower or the creditor is selected to be the *proposer* who is then allowed to make a settlement offer. A *proposal* consists of a transfer of resources τ to the creditor in the current period, and an issue of new debt securities b'. The proposer's action is therefore given by an offer of two values $(\tau, b') \in \mathbb{R} \times B$. We do not place any additional bounds on the issue of new debt, although debt issues will continue to be limited by the price that new creditors will be prepared to offer for these new bonds. Importantly, we allow for the possibility that the settlement may contain an amount of "new money" in which case the country receives a positive flow of the consumption good in the period in which they settle (this corresponds to a negative τ).

Once a proposal is made, the non-proposing agent chooses to either *accept* or *reject* the current proposal. If the proposal is accepted, or if the debtor country's outside option is

[4]In cases that went to court, the courts did not award interest on missed payments until 1997 as part of the legal proceedings involving Elliott Associated and Peru.

taken, the bargaining concludes and the country emerges from default with the new negotiated debt level. If the proposal is rejected and the outside option is not taken, the game continues to the next period, and we say that there has been *delay in bargaining*. In the next period, the realization of the state determines the identity of the proposer, and the timing repeats with the next proposer suggesting an offer.

A history of the bargaining game is a list of all previous actions and states that have occurred after a country's most recent default. That is, we are assuming that each debt restructuring is not affected by previous borrowing, default or debt restructurings, except insofar as these decisions have determined the debt level b. If no offer has been accepted, and if t indexes stages, a history up to the beginning of stage t is defined by the sequence of realizations for the state variable and the sequence of rejected offers:

$$h^t = \left\{ s^t = (s_0, s_1, ..., s_{t-1}), (\tau, b')^t = \left((\tau_0, b'_0), (\tau_1, b'_1), ..., \left(\tau_{t-1}, b'_{t-1}\right)\right) \right\}.$$

We let H^t denotes the set of all histories to stage t.

Strategies Strategies map the level of the defaulted debt b and the history into a choice of actions. The current state determines the identity of the current proposer, and the set of feasible actions depends on which player is the proposer. A strategy for the creditor when they are the proposer is a function $\sigma^{C,P} : B \times H^t \times S \to \mathbb{R} \times B$. The situation is more complicated when the debtor is the proposer due to the fact that the debtor may elect to take the outside option. In particular, a strategy for the debtor when they are the proposer is a function $\sigma^{D,P} : B \times H^t \times S \to \mathbb{R} \times B \times \{0, 1\}$, where the third element takes on the value one if the debtor takes the outside option; whether or not the debtor takes the outside option, there is an associated transfer and new debt level (τ, b'). A strategy for the creditor when they are not the proposer depends on whether or not the debtor has taken the outside option. If the debtor has *not* taken the outside option, a strategy for a non-proposing creditor is a function $\sigma^{C,NP} : B \times H^{t+1} \to \{0, 1\}$ where 0 denotes rejection of the proposal, and 1 acceptance of the proposal. If the debtor has taken the outside option, the creditor has no choice but to accept the proposed settlement and so a strategy for a non-proposing creditor is a function $\sigma^{C,NP} : B \times H^{t+1} \to \{1\}$. A strategy for the debtor when they are not the proposer is a function $\sigma^{D,NP} : B \times H^{t+1} \rightarrow \{0\} \cup \{1\} \cup \{2\} \times \{(\tau, b') \in \mathbb{R} \times B : \tau + q(b', s_{t+1}) b' \geq b\}$ where the first element 0 indicates a rejection, 1 indicates acceptance, and the third element indicates that the outside option was chosen with associated transfer and new debt levels (τ, b'). A *strategy profile* is a pair of strategies, one for each player.

Payoffs and Equilibrium Next we discuss outcomes and payoffs and define an equilibrium. An outcome is a termination of negotiations plus the final accepted offer. That is, an *outcome of the bargaining game* is a stopping time t^* and the associated proposal (τ, b'). At any history, a strategy profile induces an outcome and hence a payoff for each player. The payoff to the debtor given outcome $\varphi = \{t^*, (\tau, b')\}$ after history s^{t^*} is

$$V^D\left(t^*, s^{t^*}, (\tau, b')\right) = \sum_{r=0}^{t^*-1} \beta^r u\left(e^{def}(s_r)\right) + \beta^{t^*}\left\{u\left(e^{def}(s_{t^*}) - \tau\right) + \beta E\left[V(b', s_{t^*+1}) | s_{t^*}\right]\right\},$$

while to the creditor it is given by $W\left(t^*, s^{t^*}, (\tau, b')\right) = \delta^{t^*}\{\tau + q(b', s_{t^*}) b'\}$.

Let $G(b, h^t)$ denote the game from date t onwards starting from history h^t. Let $|h^t$ denote the restriction to the histories consistent with h^t. Then $\sigma|h^t$ is a strategy profile on $G(b, h^t)$. We let $\varphi(\sigma|h^t)$ be the outcome generated by the strategy profile $\sigma|h^t$ in game $G(b, h^t)$. A strategy profile is subgame perfect (SP) if, for every history h^t, $\sigma|h^t$ is a Nash equilibrium of $G(b, h^t)$, or $W(\varphi(\sigma|h^t)) \geq W\left(\varphi\left(\sigma^D|h^t, \sigma^{C\prime}|h^t\right)\right)$, and $V^D(\varphi(\sigma|h^t)) \geq V^D\left(\varphi\left(\sigma^{D\prime}|h^t, \sigma^C|h^t\right)\right)$, for all σ, t, and h^t.

As is customary in the literature, we impose the restriction of stationarity. A strategy profile is stationary if the actions prescribed at any history depend only on the current state and proposal. That is a *stationary strategy profile* satisfies $\sigma^D(b, h^t, s_t) = \sigma^D(b, s_t)$, and $\sigma^C(b, (h^t, (s_t, (\tau_t, b'_t)))) = \sigma^C(b, s_t, (\tau_t, b'_t))$, for all h^t and all t when s_t is such that the debtor proposes, and $\sigma^C(b, h^t, s_t) = \sigma^C(b, s_t)$, and $\sigma^D(b, (h^t, (s_t, (\tau_t, b'_t)))) = \sigma^D(b, s_t, (\tau_t, b'_t))$, for all h^t and all t when s_t is such that the creditor proposes. A *stationary subgame perfect equilibrium* (SSP) outcome and payoff are the outcome and payoff generated by an SSP strategy profile. We define a stationary outcome as $((B \times S)^\mu, \mu)$,where $\mu = (\tau, b')$ and where $(B \times S)^\mu$ is the set of debt levels b and states s on which an agreement occurs or the outside option is taken, and where $(B \times S) \setminus (B \times S)^\mu$ is the disagreement set.

3.B Solution to the Bargaining Model

The solution to the overall model involves solving a fixed point problem. First, taking as given the solution to the bargaining problem, we solve for the solution to the debtor countries default problem and update the market price of debt. Second, we take the market price of debt and the debtor's value function from repayment and then use these to solve the bargaining problem. An equilibrium is a fixed point of the composition of the two operators. In this section, we focus on the bargaining model, taking as given the form of the solution to the borrowing problem.

Recursive Problem Statement

For this section, we take the solution to the borrowing problem as given. That is, the debtor country's value of accessing capital markets $V(b,s)$ is assumed to be a fixed element of the set all real valued N_e by N_b vectors, and the equilibrium bond price function $q(b,s)$ is assumed to be a fixed element of $\mathcal{Q}(B \times S)$. Given these assumptions, we then show that the SSP values of the bargaining game are fixed points of a particular functional equation. As is usual, the key to the approach is that we focus directly on SSP payoffs, rather than on the SSP itself.

Our approach is recursive, and relies upon the following operator $\hat{T}$. Given any pair of functions (f_1, f_2) with $f_i : B \times S \to \mathbb{R}$ for $i = 1, 2$, we define the mapping $\hat{T}$ such that: If s is such that the debtor is the proposer

$$\hat{T}f_1(b,s) = \max\left\{ \begin{array}{c} \max_{\tau,b'} u(e^{def}(s) - \tau) + \beta E[V(b',s')|s] \\ s.t \quad \tau + b'q(b',s) \geq \min\{b, \delta E[f_2(b,s')|s]\} \end{array}, u(e^{def}(s)) + \beta E\left[f_1(b,s')|s\right] \right\},$$

and

$$\hat{T}f_2(b,s) = \min\{b, \delta E[f_2(b,s')|s]\},$$

while if s is such that the creditor is the proposer

$$\hat{T}f_2(b,s) = \max\left\{ \min\left\{ b, \begin{array}{c} \max_{\tau,b'} \tau + b'q(b',s) \\ s.t \quad u(e^{def}(s) - \tau) + \beta E[V(b',s')|s] \\ \geq u(e^{def}(s)) + \beta E\left[f_1(b,s')|s\right] \end{array} \right\}, \delta E\left[f_2(b,s')|s\right] \right\},$$

and

$$\hat{T}f_1(b,s) = \max\left\{ u(e^{def}(s)) + \beta E\left[f_1(b,s')|s\right], \begin{array}{c} \max_{\tau,b'} u(e^{def}(s) - \tau) + \beta E[V(b',s')|s] \\ s.t \quad \tau + b'q(b',s) \geq b \end{array} \right\}.$$

Intuitively, the $\hat{T}$ mapping yields the values from bargaining at a given stage with defaulted debt b and current state s, given that the continuation values associated with not reaching agreement this period are determined by f_1, for the debtor, and f_2 for the creditor. To understand this mapping, note that if the debtor is the proposer, they have three options. First, they could make an offer which will not be accepted. In this case, the debtor consumes the autarky endowment level this period and moves on the next stage with defaulted debt still at b, new state s' and payoffs encoded in f_1, while the creditor receives nothing today and a future payoff encoded by f_2. This payoff is the right hand component of the debtor-proposer half of the operator, for both the debtor and the creditor.

Second, the debtor could take the outside option, in which case the creditor receives the value of the defaulted debt b, and the debtor receives the maximum value achievable while still delivering a payoff of b to the creditor. This corresponds to the left hand side of the creditors part of the debtor-proposer half of the operator, and to the left hand side of the debtor's part of the operator given the constraint on creditor utility defined by b.

Third, the debtor could make an offer that is accepted. In this case, since the debtor makes the offer, the creditor receives none of the surplus from the agreement, and hence receives the same payoff as if the offer was not accepted (the right hand side of the creditor part of the debtor-proposer half of the operator). The debtor, on the other hand, receives the maximum value that can be achieved while delivering this value to the creditor (the left hand side of the debtor's part of the operator with the constraint defined by the reservation payoff of the creditor). Since the debtor would never take the outside option when it can do better by making an offer that is accepted, the minimum over the value of the debt and the creditors reservation value is the relevant determinant of the constraint.

Similar logic underlies the half of the operator that applies to states in which the creditor is the proposer, noting that the creditor will extract all of the surplus from an accepted proposal up to a maximum value of b at which level the debtor will take the outside option.

The following theorem, which can be thought of as a version of the principle of optimality for our problem, establishes an equivalence between SSP payoffs and fixed points of the $\hat{T}$ operator. As the proof is standard, but notationally cumbersome, the details are relegated to Appendix B.

Theorem 1. *The functions $f = (f_1, f_2)$ are SSP payoffs if and only if $\hat{T}f = f$.*

Proof. See Appendix B. □

This operator forms the basis for our theoretical and numerical analysis of the bargaining problem below. In the next subsection we establish existence of an equilibrium bargain, and provide a sufficient condition under which this bargain is unique, by studying the properties of the $\hat{T}$ operator.

Existence and Uniqueness of Symmetric Subgame Perfect Equilibria

Next we show that an SSP equilibrium exists, by demonstrating that our $\hat{T}$ mapping operates on a bounded space of functions, and is monotone. As the details are standard, they are relegated to Appendix B.

Theorem 2. *An SSP equilibrium exists.*

Proof. See Appendix B. □

The uniqueness of the values of the equilibrium bargain could be easily established if $\hat{T}$ is a contraction mapping. However, as in many multi-agent problems, this is not straightforward. The difficulty results from two issues. First, changes in one agent's continuation value function will affect the result of the operator on the other agents continuation value function, because continuation values act as constraints on the proposals that will be accepted. Second, and more importantly, the rates at which changes in one agent's continuation value affect the operator on the other agents continuation value can vary when payoffs are non-linear functions of outcomes.

To understand this difficulty, it is instructive to consider how these issues appear in an attempt to establish Blackwell's sufficient conditions for a contraction mapping, and in particular by affecting the proof of the discounting condition. Suppose we change the value of the creditor's and debtor's continuation values by a small constant amount. The discounting property requires that the operator produce functions that are bounded by the modulus of the contraction mapping, which is strictly less than one. Since the country's felicity function is non-linear, it is possible that a small increase in the creditor's continuation value, which would lead to a small change in the settlement value, could lead to a large change in the country's payoff if the marginal utility of consumption was high near the solution of the debtor's problem in the debtor's half of the $\hat{T}$ operator. Moreover, it is also possible that a small change in the debtor's continuation value could result in a large change in the value of the settlement (and hence also a large change in creditor payoffs) if the marginal utility of consumption is low near the solution of the creditor's problem in the creditor's half of the $\hat{T}$ operator.

The following theorem states a condition that is sufficient to prove uniqueness, by imposing bounds on the rate at which resources can be transformed into utility, and the rate at which utility can be transformed into resources. As a consequence of the fact that we have imposed few restrictions on the shape of the V and q functions, the condition is stated in terms of bounds on the slope of the utility function of the debtor. In our numerical work below, as in much of the quantitative literature on sovereign debt and default, we focus on discount factors for the country that are substantially less than one, reflecting political economy problems in developing countries that lead to impatient policy making. For sufficiently low β, we can typically show that the sufficient condition is satisfied.

Theorem 3. *Let $u : \mathbf{R} \to \mathbf{R}$ be differentiable. If there exists $K_L > \beta$ and $K_U < 1/\delta$ such that $K_L \leq u'(c) \leq K_U$, for all c, then the SSP equilibrium values are unique.*

Proof. See Appendix A. □

3.C Solution to Borrowing Model

In the previous section, we establish the existence and uniqueness of a solution to the debt restructuring bargaining problem, taking as given the value to the country from re-accessing capital markets with new debt b', $E[V(b', s')|s]$, and the value of new debt to creditors $q(b', s)$. In this section, we take as given the solution to the bargaining model, and hence the value to the country and the creditor from being in default, and then establish existence of a solution to the borrowing problem. That is, we take as given the $N_e \times N_b$ vectors of payoffs to the country, $\tilde{V}^D(b, s)$, and the creditor, $\tilde{W}(b, s)$, in default, that are elements of $\mathcal{B}(B \times S)$.

The solution of the borrowing problem is established as the composition of two operators. The first takes a value to the country from default and an equilibrium bond price function, and then solves the country's problem to obtain a value to the country for access to capital markets, and a default policy function, which is a selection from a default policy correspondence. The second takes the default policy function and combines it with the value to the creditors from default to obtain a new bond price function. The proof of existence of a solution is standard, and follows from the monotonicity of the composition of these operators.

Theorem 4. *Given* $\left(\tilde{V}^D(b,s), \tilde{W}(b,s)\right) \in \mathcal{B}(B \times S)$ *and* $q(b,s) \in \mathcal{Q}(B \times S)$, *there exists a value function for the country,* $V(b,s)$, *and an equilibrium bond price function* $q(b,s) \in \mathcal{Q}(B \times S)$, *that solve the borrowing problem.*

Proof. See Appendix B. □

Given the result of this Theorem, it is tempting to try to prove existence of an equilibrium for our entire model by iterating successively on the T^V, T^q and $\hat{T}$ operators. However, this approach need not converge. Specifically, although iterating on the T^V and T^q operators produces a monotone operator, when combined with the bargaining operator, the compounded operator need not be monotone. Intuitively, it can be the case that a high value to the creditor in default, and a low value to debtor, leads to a high bond price, which in turn leads to a high value to the country from repayment. This high value to repayment can lead to a high value from default, which then leads to a low bond price in the next iteration. That is, we cannot rule out cycles in the successive application of these operators.

In the next section, we describe an alternative method for proving existence.

3.D Existence of Equilibrium

In this section, we establish the existence of a recursive equilibrium for our economy. First, we define an equilibrium for our economy.

Definition 1. *An* ***equilibrium*** *for our economy is a value function for the country from borrowing* $V(b,s)$, *a value function for the country in default* $V^D(b,s)$, *a value function for the creditor in default* $W(b,s)$ *and a bond price function* $q(b,s)$ *such that:*

1. *Given the bond price function $q(b,s)$ and the value to the country from re-accessing capital markets $V(b,s)$, the country and the creditor optimally bargain over re-access to financial markets. That is, V^D and W are fixed points of the* ***inside default*** *operator $\hat{T}$;*
2. *Given the value to the country and from default $V^D(b,s)$, and the bond price $q(b,s)$, the country makes optimal borrowing and default decisions. That is, $V(b,s)$ is a fixed point of T^V with associated default policy correspondence $\Phi(b,s)$*
3. *Given the payoff to the creditor in default W and the optimal default policy correspondence, the bond price function $q(b,s)$ satisfies the no arbitrage condition for creditors. That is, $q(b,s)$ is a fixed point of the operator T^q.*

The latter two conditions may equivalently be written as: Given $V^D(b,s)$ and $W(b,s)$, $V(b,s)$ and $q(b,s)$ are a fixed point of the **outside default** operator, which is the composition of the T^V and T^q operators.

We prove existence by using the operators defined above to construct a new mapping from the space of value functions for the country and creditor in default, and the space of bond price functions, into itself, and establishing that it possesses a fixed point. Specifically, define the mapping H from $\mathcal{B}(B\times S)\times\mathcal{Q}(B\times S)$ into itself as follows. First, given V^D, W and q, iterate on the outside default operator to convergence to obtain a new bond price function $q'(b,s)$. Second, given V^D and q, iterate on the T^V operator to convergence to produce a value function V. Then, given the old q and this V, iterate on the $\hat{T}$ operator to convergence to find new $V^{D\prime}$ and W'. We establish that the combination of these operators defines an upper hemi-continuous correspondence with non-empty and convex values. Then, noting that $\mathcal{B}(B\times S)\times\mathcal{Q}(B\times S)$ is a compact and convex space of functions, the result then follows by application of the Kakutani-Fan-Glicksberg fixed point theorem.

Theorem 5. *If the SSP equilibrium values of the bargaining model are unique, then there exists an equilibrium of our borrowing economy.*

Proof. See Appendix A. □

4 Calibration and Numerical Results

In this section, we present results from several numerical solutions of the model that vary only in the calibration of the bargaining power process for the debtor and creditor. These examples are used to illustrate some comparative static properties of the model, and to build intuition for the elements of the model that produce delay. We then present our benchmark case in which the parameters of the model governing bargaining power are calibrated to some aspects of the relationship between default and output observed in the data. The model is then assessed according to it's ability to match the other facts discussed in the introduction.

4.A Calibration

The first step in calibrating the model is the choice of a period length. On the one hand, as debt contracts in our theory are one period in duration, calibration to a long period length is necessary in order to match the maturity of observed debt issues. A long period length is also desirable given that the information on which bargaining positions are formed is revealed at best quarterly, and in some cases only annually. On the other hand, the bargaining process plausibly operates at a high frequency, suggesting that we should calibrate to a shorter period length. To balance these concerns, and given that most other studies in the literature calibrate to quarterly data, we adopt a quarterly calibration.

In some cases, our data is only available at an annual frequency. To construct annual outcomes, we simulate on a quarterly calendar for 11000 periods beginning with the March quarter, and drop the first thousand periods to eliminate the effect of initial conditions. All model variables are treated identically to the data, with flow variables such as output being summed, and stock variables such as debt calculated as of the end of the year. Trend output is computed from the annualized data using a Hodrick-Prescott filter with smoothing parameter equal to 6.25 as in Ravn and Uhlig (2002). If a country exits and re-enters default in successive quarters, we count this as one default[5]. Since our data measures the start and end of a default

[5] This is conventional. Standard and Poors classify periods in which a country has engaged in a series of successive renegotiations as one default episode. For example, S&P defines Mexico to have defaulted once in the past three decades, starting in 1982 and ending in 1990, despite the fact that Arteta and Hale (2008) record 3 separate negotiations and 23 separate rescheduling agreements for commercial bank debt during this period. Likewise, Beim and Calomiris (2001, p.35) treat defaults occurring within five years of each other as one default episode. By only merging defaults that occur within one quarter of each other, our estimates of delay are conservative.

at a high frequency, we calculate the duration of default in the model from the quarterly data. In comparing our annual data on output and debt to the timing of defaults and settlements, we follow the practice of S&P and label a country as being in default in a given year if it was in default at any point in that year.

Table 3: Parameter Values for Calibration

Name	Meaning	Value
β	Discount factor	0.945
$1 + r^w$	World Interest rate	1.01
γ	CRRA	2
ρ_e	Persistence	0.945
σ_ϵ	Std Dev	0.02

Most parameters in the model are held constant in every experiment and, as shown in Table 3, are set to values that are standard in the literature. Following Arellano (2007), Aguiar and Gopinath (2006), Yue (2007), and Tomz and Wright (2007), the world interest rate is set to 1% per quarter, and the coefficient of relative risk aversion is set to two. The income process is assumed to follow a log normal AR(1) process, which is chosen to match Argentinean output data. One non-standard parameter value is the discount factor, which in the rest of the literature often takes on values as low as 0.8 for quarterly data implying annual discount factors around 0.4. Although a low value can be plausibly motivated by political economy considerations that lead developing country governments to act myopically, we view a choice of 0.8 as too extreme and use a more modest 0.945 at our quarterly frequency.

The remaining parameters describe the evolution of the proposer identity during bargaining, and the loss of output experienced by the country during default. For our benchmark case, these parameters are calibrated to the features of the data summarized in Table 4. First, we calibrate the output loss during default to match the observed average ratio of debt to GDP; higher output costs support higher borrowing levels. In matching model data on debt to the World Bank data on debt, we must confront a measurement issue, raised earlier and described in more detail in Appendix C. The issue is that the World Bank reports debt at face values, which is defined as the undiscounted sum of all future amortization payments. However, two equivalent debts (that is, two debts with exactly the same streams of debt service) will have different face values if debt service is divided into principal and interest in different ways. In our presentation of the theory, all debt is issued at a discount without a coupon, so the face value of the debt is given by the state variable b. However, these zero-coupon bonds with face value b are equivalent to par bonds with face value equal to their market value, $q(b,s)\,b$, and with total coupons worth $b - q(b,s)\,b$. We calibrate the output cost of default so that a 2 : 1 weighted average of face value and market value debt matches the average debt-to-GDP level of 65%, reported by the World Bank. This produces an output cost of 1% of GDP, which is about half the level assumed by other studies.

Finally, we need to choose values for the parameters governing the evolution of the identity of the proposer in bargaining. Obviously, the ability to make today's offer can be thought of as giving the proposer more bargaining power. Less obviously, the agent's expectations about who will propose offers in the future has the strongest effect on bargaining power because it determines the reservation value of the non-proposing agent. Hence, we refer to these probabilities as "bargaining power" parameters. Since the economics literature provides little guidance as to how to set these parameters, we experiment with a number of alternatives.

Table 4: Calibration Targets

Target	Data	Model Outcome
mean($(e_t - e_t^{trend})/e_t^{trend} \| t$ is period before default)	0.01	0.01
mean($(e_{t+1} - e_t)/e_t \| t+1$ is period default ends)	0.004	0.005
prob($e_t < e_t^{trend} \| t$ is period in default)	54%	54%
mean($(e_t - e_{t-1})/e_{t-1} \| t$ is period in default)	-0.0016	-0.0014
mean(debt to gdp)	0.65	0.65

In our benchmark case, we calibrate bargaining power to some aspects of the relationship between default and income in the data. Specifically, we divide output realizations equally into 'high', 'medium' and 'low' levels, and describe the evolution of proposer identity in terms of the probability that the debtor proposes conditional on the output level and identity of the proposer in the previous period. This leaves us six probabilities to calibrate. We assume a limited form of symmetry to set two of these parameters; we assume that the probability that the creditor proposes tomorrow given that they proposed today and output was high (low) is equal to the probability that the debtor will propose tomorrow given that they proposed today and output was low (high). The remaining probabilities are set to match the first four moments in Table 4, with the resulting probabilities tabulated in Table 5.

In order to build intuition for the effect of bargaining power on the outcomes of the model, we also present numerical results for five stylized example processes for bargaining power. The economics literature that considers bargaining typically assumes that bargaining power is constant (for example, when the cooperative Nash bargaining solution is imposed, as in Yue (2007) in her study of sovereign debt restructuring). To capture this case, we present results for two examples in which the identity of the proposer is i.i.d., and which differ only as to whether it is the creditor (probability 0.99) or debtor (probability 0.55) who is always likely to propose, and hence who has most of the bargaining power. We follow these i.i.d. cases with a "persistent" regime where the proposer today is very likely (probability 0.99) to remain proposer tomorrow, so that there will be random cycles in bargaining power.

Last, we present two examples in which the bargaining power of the agents depends on economic conditions in the country. In the first case, which we follow the international relations literature in referring to as "strength through weakness", the likelihood that the country is able to make offers in the future is higher when output is low, so that the debtors bargaining power is greatest when the economy is weak. This case attempts to capture the idea that in countries with weak economies, the politicians negotiating the debt restructuring are too weak politically at home to propose significant concessions to its lenders, which acts as a form of bargaining power. In the second – "strength through strength" – the debtor country has more bargaining power when output is high, capturing the idea that strong economic performance insulates a political leader from domestic political pressures. In both of these cases, the agent that has higher bargaining power when output in the debtor country is high proposes with a probability greater than one half (probability 0.96). Inspection of the calibrated probabilities in Table 5 reveals that the benchmark case possesses elements of the "persistent" and "strength through weakness" examples.

Table 5: Parameters Calibrated to Match Targets

Name	Value
$\pi\left(D \mid D, e(s)\ low\right)$	0.86
$\pi\left(D \mid D, e(s)\ medium\right)$	0.93
$\pi\left(D \mid D, e(s)\ high\right)$	0.61
$\pi\left(D \mid C, e(s)\ medium\right)$	0.05
Output Cost	0.01

4.B Intuition For Delays in Bargaining

Before turning to our results for the benchmark model, it is instructive to examine bargaining outcomes for two of the example bargaining power processes introduced above. We begin with an i.i.d. case in which the creditor is very likely to propose each period and hence has the greater bargaining power. As the likelihood of making future offers is constant,bargaining power is constant, and defaults are driven primarily by fluctuations in

Figure 2: Simulation of i.i.d. Creditor Example

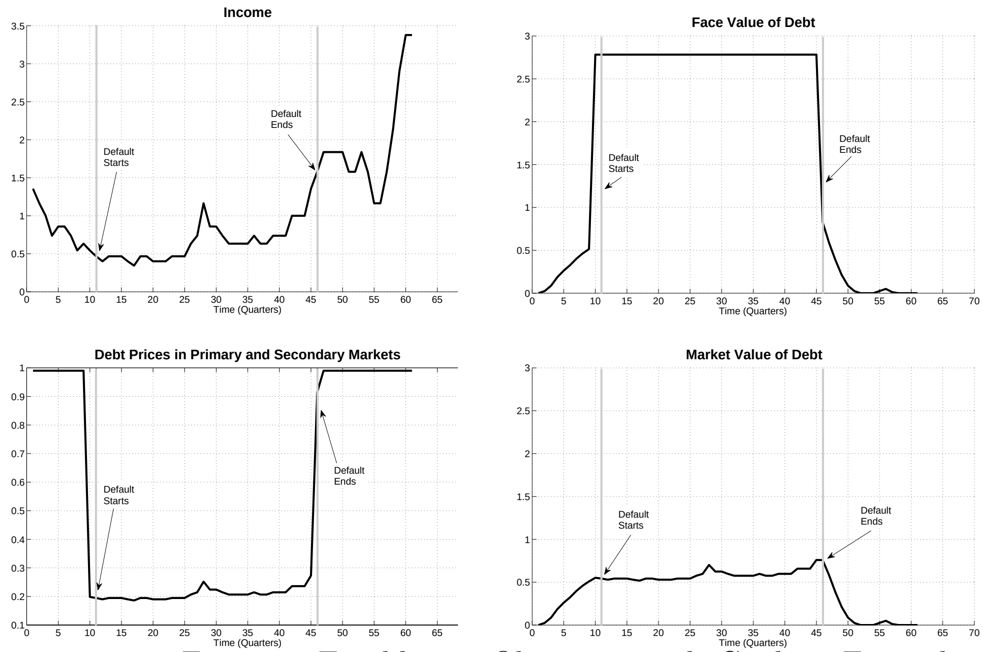
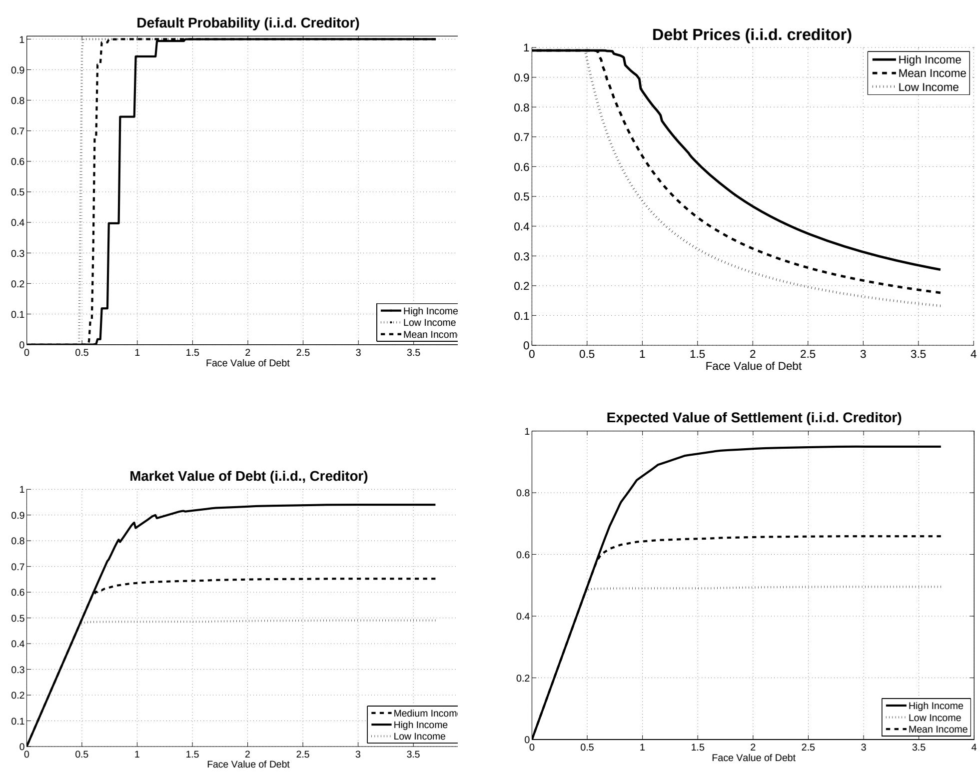
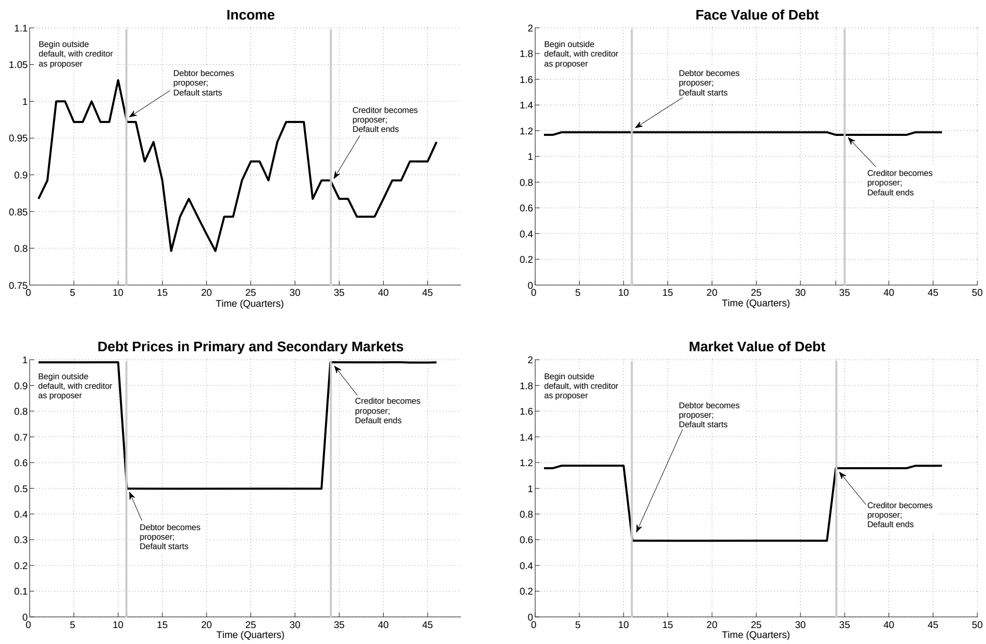


Figure 3: Equilibrium Objects in i.i.d. Creditor Example



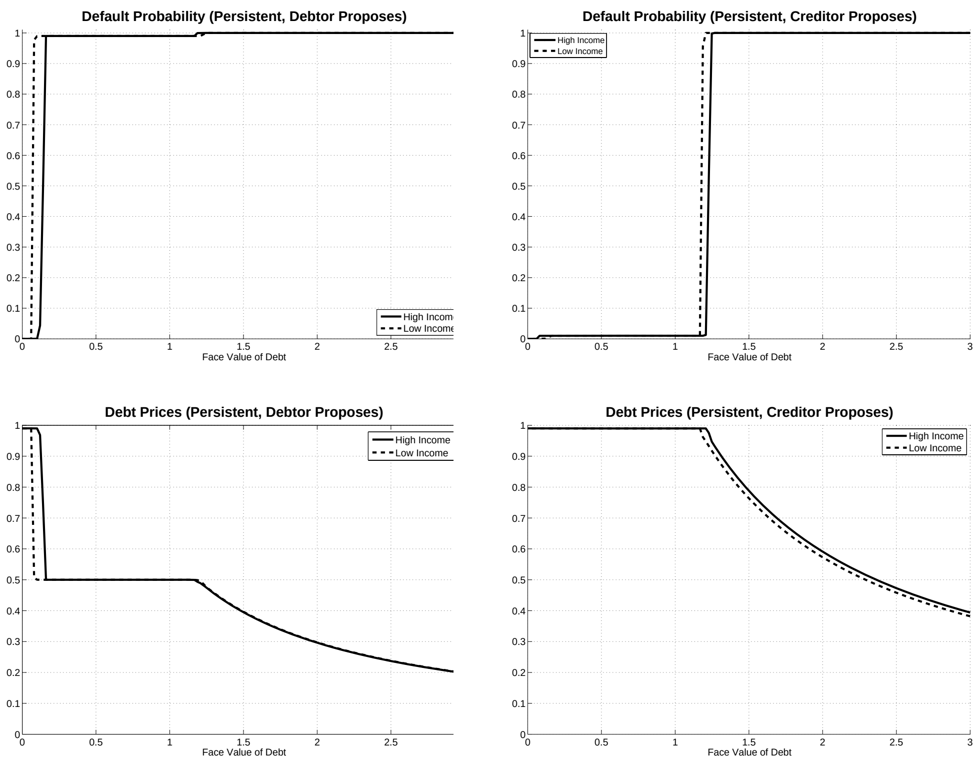
output. To see this, Figure 2 plots a sample time series of income levels for this case, along with the corresponding debt choices of the country at both face and market values. For the first two and one-half years (that is, the first nine periods), income levels are low and falling, on average, leading the country to increase its debt level to smooth consumption. Market and face values of debt move together reflecting a low and stable level of default risk. In the tenth period, income falls again and the risk of default rises, leading to a large rise in spreads (that is, fall in the bond price as shown in Figure 2) so that the country must now dramatically increase the face value of its debt in order to generate enough resources from foreign borrowing (the market value of debt) to smooth its consumption. When output falls again in period 11, the country enters default.

Figure 4: Simulation of Persistent Example



To help understand the motivation of the country, Figure 3 plots the probability that the country will default as a function of income and debt levels, as well as the corresponding bond price function. As shown, for low income levels, the probability of default rises quickly from zero to one in the neighborhood of the debt choices plotted in Figure 2. In much of the previous literature, where there is no possibility of a settlement, this would cause the price of new bonds issued by the country to drop to zero, curtailing borrowing at low levels with a low probability of default and low spreads over the risk free rate. However, in our model, the possibility of a debt settlement gives creditors the incentive to keep lending even when default risk is high, albeit at higher interest rates. It is this fact that enables our framework to produce greater borrowing levels at higher spreads than in the previous literature. To understand the dramatic rise in debt levels, the Figure also plots the market value of debt associated with a given face value of debt in this case, which shows that beyond a certain point

Figure 5: Equilibrium Objects in Persistent Example: Part I

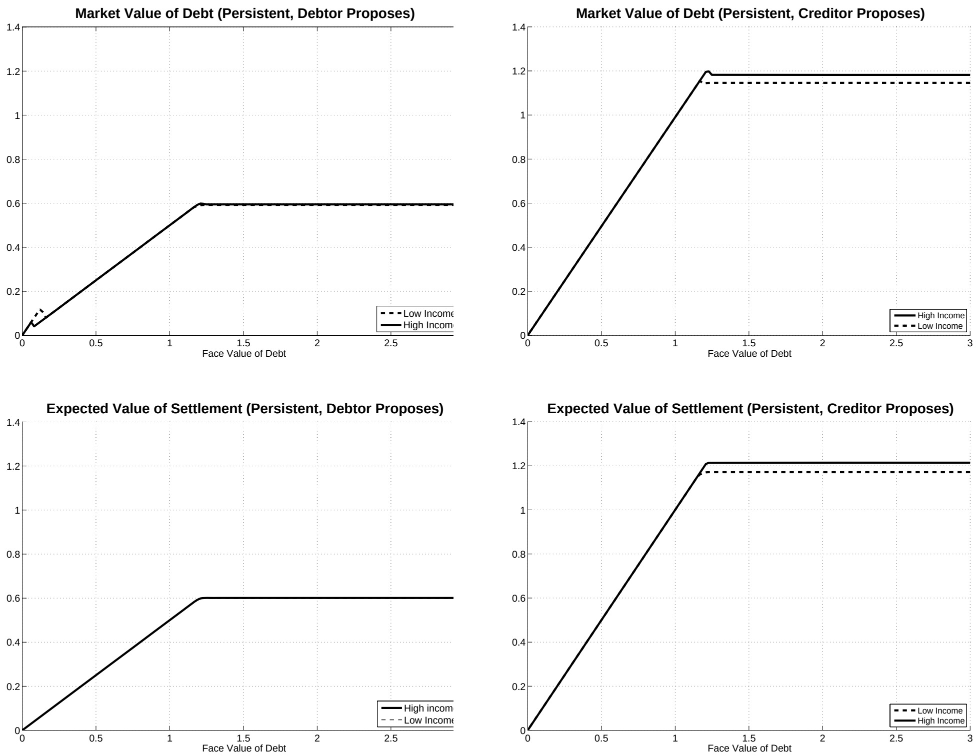


bond prices fall as borrowing levels rise to keep the market value of debt roughly constant. As shown in the Figure, at high debt levels, the country correctly anticipates that the outcome of bargaining will not vary greatly with the face value of the debt, and so dramatically increases the face value issued, whereas the actual amount of resources borrowed increases only slightly. This is profitable for the creditor because it increases the settlement obtained in states where the outside option in bargaining is taken.

After nine and one-half years (that is, by period 45), both output and bond prices have reached levels significantly higher than when the country entered default (see Figure 2). This allows the country and creditor to agree on a settlement in which the creditor receives a substantial share of the surplus: as current income is high, a larger current transfer of resources is possible for a smaller utility cost to the country, while high bond prices make debt issues more valuable to the creditor. Note that, in the first full year *after* exiting default, debt levels at face value are higher than in periods 5 through 9 *before* entering default; likewise, at market values, the country exits default with more debt than when they entered default (although scaled by GDP, indebtedness falls).

In the previous example, delays in bargaining were caused solely by persistence in

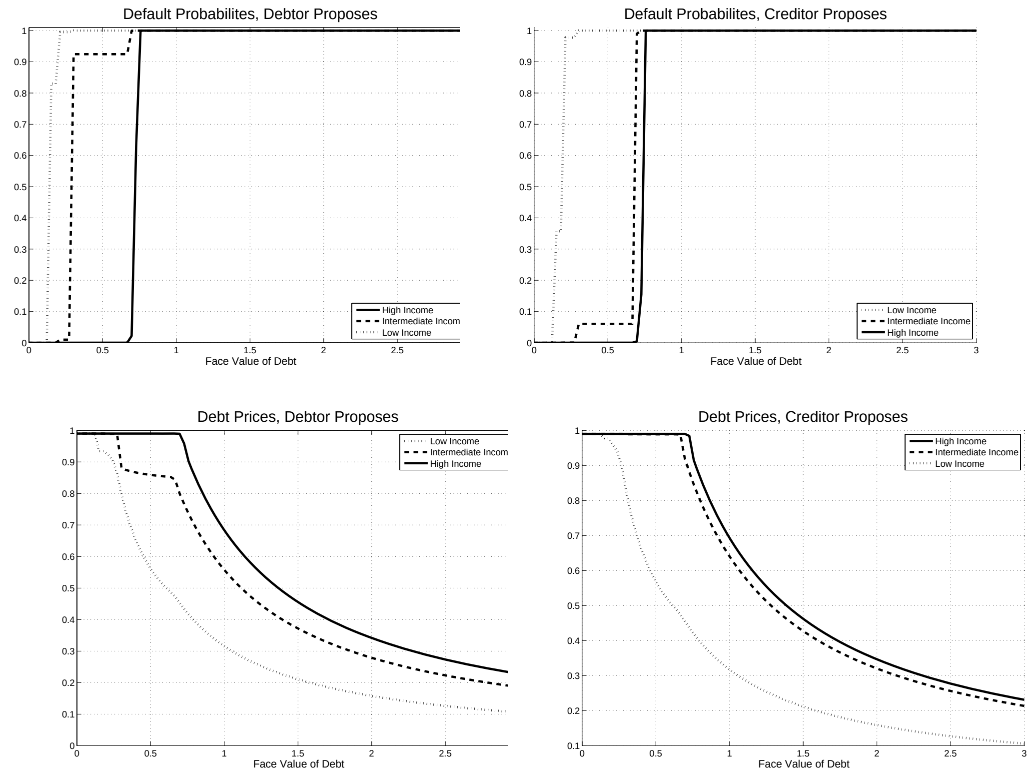
Figure 6: Equilibrium Objects in Persistent Example: Part II



income levels. Next we consider a case in which there are persistent fluctuations in bargaining power. In this case, default may be driven by a change in bargaining power, and Figure 4 illustrates a simulation in which this outcome occurs. Although output is relatively high in the first two and one-half years of this simulation, and debt remains roughly constant, a shift in bargaining power in favor of the debtor causes a collapse in bond prices so that debts with the same face value would now raise a smaller amount of resources (they would have a lower market value), which in turn leads the country to default in period 11.

To see why this occurs, Figures 5and 6 plot the equilibrium objects for the persistence example. As these figures show, for this regime, changes in bargaining power have a dramatic effect on default likelihoods, whereas income has little effect. These differences also show up in the prices at which the country can issue debt, which are much higher when the creditor has the bargaining power. Even when default is certain, debt prices are positive as creditors expect a settlement. However, when the debtor has the bargaining power, they are likely to extract most of the surplus from such a settlement, and debt prices are lowest. Consequently, market values of debt tend to be much higher when creditors have the bargaining power (compare the 'kink' in the plots of the market values of debts when the creditor, as opposed

Figure 7: Equilibrium Objects of the Benchmark Model I



to the debtor, makes the proposal). This means that the value of credit market access to the debtor, when they have the bargaining power, is low, and so default is the more attractive option. The Figures also show that expected settlements are large when the creditor has the bargaining power, explaining why bond prices are high. Default ends after roughly nine years when bargaining power switches back to the creditor, and the country's debt price recovers. This is in part because the debtor can issue more debt as part of a settlement, but mostly because capital market access is now more valuable to the debtor, making a settlement more attractive.

Finally, Figures 7 and 8 present the equilibrium objects for our benchmark case. Relative to the persistent example, bond prices in the benchmark vary more with income, while relative to the i.i.d. case they vary with bargaining power quite substantially at intermediate income levels. This is driven primarily by differences in default probabilities, although as seen in Figure 8 changes in the expected settlement also play a role. The importance of intermediate income levels is driven by the fact that bargaining power is more persistent at these income levels. The benchmark regime also allows the face value of debt to vary more than in the persistent example, and allows the market value of debt to vary more than in the i.i.d.

Figure 8: Equilibrium Objects of the Benchmark Model II

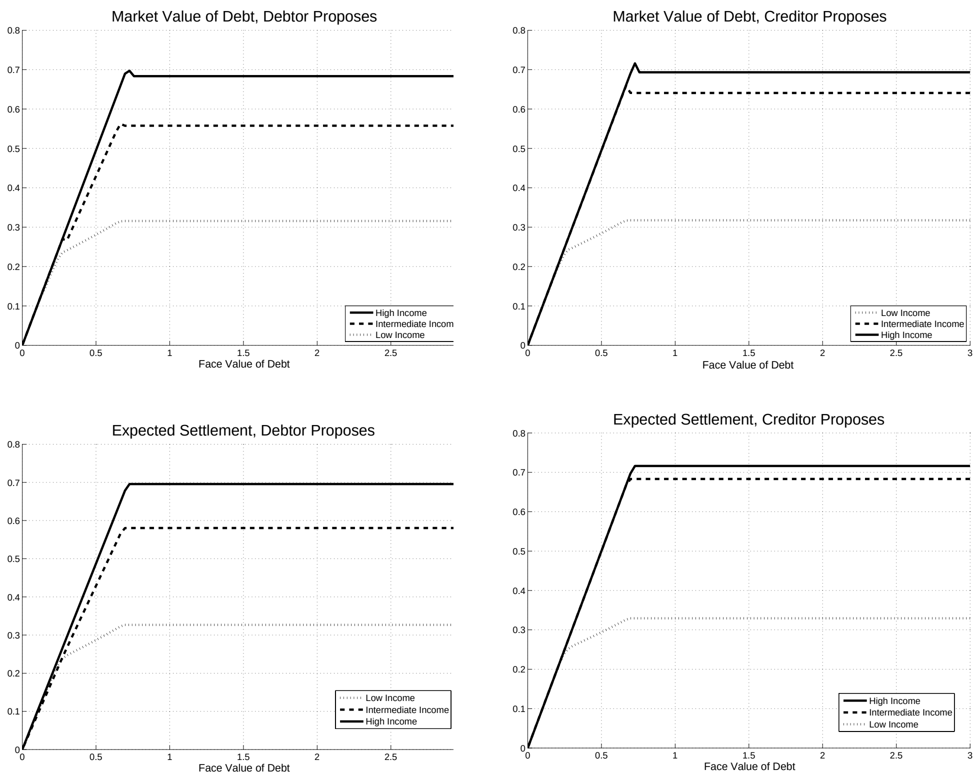


Table 6: Numerical Results for Delays and Haircuts

	Probability of Default (%)	Mean Delay (Years)	Mean Haircut (%)	Correlation		
				Delays & Haircuts	$(e_T - e_{T-1})/e_{T-1}$ & Delays	$(e_T - e_{T-1})/e_{T-1}$ & Haircuts
i.i.d creditor	3.9	6.7	84	-0.21	-0.03	0.01
i.i.d debtor	4.4	1.0	68	-0.18	-0.06	0.06
strength through strength	3.6	5.0	63	0.44	0.02	-0.05
strength through weakness	3.9	6.7	78	0.89	-0.05	0.10
persistent	4.0	10.7	35	0.54	0.22	0.27
benchmark	4.4	7.2	28	0.70	-0.15	-0.08
data	4.4	7.6	38	0.66	-0.25	-0.26

example. This can be seen by comparing the market value of debt panels in Figure 8, with the corresponding panels in Figures 3 and 6. The borrower optimally chooses a face value of debt at or near the kink in the market value figure; changes in bargaining power in the benchmark produce twice the change in market values possible in the i.i.d. regime (the height of the line in the figure), as well as much large changes in face values than the persistent regime (as seen by the horizontal location of the kink).

4.C Results

Tables 6 through 8 compare the performance of our benchmark model to the facts documented in Section 2. For comparison, we also collect the predictions of our various

Table 7: Numerical Results on Default, Debt and Economic Activity

	% Countries Below Trend in Year of:		Mean Trend Deviation of Output in Year of:		Change in Indebtedness: $\frac{Debt/GDP\ Year\ After\ Settlement}{Debt/GDP\ Year\ of\ Default}$	
	Default	Settlement	Default	Settlement	Face Value	Market Value
i.i.d debtor	47	53	2.9	-0.6	0.79	0.95
i.i.d creditor	48	60	2.8	-4.9	0.56	0.85
strength through strength	49	50	0.1	0.6	0.72	0.80
strength through weakness	49	62	0.2	-1.1	0.70	0.81
persistent	39	66	0.2	-0.3	0.99	1.08
benchmark	53	76	-0.3	-0.8	0.97	1.03
data (mean)	64	52	-1.3	-0.2	1.24	
data (median)					1.04	

Table 8: Numerical Results on the Proportion of Countries Below Trend

Regime	Period Before Default	Period Of Default	Period Of Settlement	Period After Settlement
	Quarterly			
i.i.d debtor	97	99	87	84
i.i.d creditor	99	97	52	56
strength through strength	1	5	42	40
strength through weakness	98	99	57	53
persistent	60	89	89	58
benchmark	46	90	73	19
	Annual			
i.i.d debtor	4	47	53	86
i.i.d creditor	3	48	60	55
strength through strength	96	49	50	40
strength through weakness	3	49	62	40
persistent	15	39	66	62
benchmark	25	53	76	29
data	40	64	52	49

Table 9: Numerical Results on Spreads, Income and Capital Flows: Quarterly Data

Model		Correlation Between Spreads & NX/Y	Spreads & NX	Spreads & Income	Income & NX/Y	Income & NX	Std Dev (C)/ Std Dev (Y)	Face Value of Debt/ Income
Arellano (2007)			0.43	-0.29		-0.25	1.10	6
Aguiar &	Trend Stat	-0.21		0.51	-0.33		0.99	27
Gopinath (2007)	Diff. Stat.	0.11		-0.03	-0.19		0.98	19
Yue (2007)			0.54	-0.18		-0.14	1.03	10
Bi (2008)				-0.51	0.12		0.99	9
i.i.d debtor		0.23		-0.24	0.19		0.99	49
i.i.d creditor		0.12		-0.45	0.27		0.98	246
strength through strength		0.17		0.31	-0.20		1.27	51
strength through weakness		0.19		-0.51	0.24		1.09	66
persistent		0.25		-0.09	0.01		1.03	83
benchmark		0.32		-0.44	-0.24		1.09	69
data range	min (abs. val.)	0.05	0.49	-0.12	-0.17	-0.64	1.03	65
	max (abs. val.)	0.86	0.70	-0.88	-0.89	-0.88	1.19	

Sources: Arellano (2007) Tables 1 and 4. Aguiar and Gopinath (2007) Tables 1 and 3. Yue (2008) Table 4. Bi (2008) Table 4.

example bargaining power processes for debt restructuring outcomes. As shown in Table 6, our benchmark model produces exactly the same default probability as observed in our sample. At 4.4%, this default probability is roughly twice the level (2%) assumed in other studies on the basis of observed default over the entire 20th Century. In our view, this default probability is artificially low: the middle of the Century contains more than three decades in which capital flows were very small due to controls under Bretton-Woods and the adoption of inward-looking development policies in many developing countries. With little debt to default upon, little default was observed. Our 4.4% number describes the modern period, and also seems consistent with other periods of well functioning international capital markets, such as the Gold Standard era. For the benchmark output process, both i.i.d. regimes produced little delay; for the i.i.d. regimes only, we increased the variance of output innovations seven-fold in order to study the effect of longer delays in the creditor regime.

In examining our sample of debt restructuring outcomes in practice, we found that *sovereign defaults were time consuming to resolve*, taking almost eight years on average (Fact 1). Table 6 shows that our benchmark regime produces an average delay in excess of seven years, slightly less than the delay observed in our sample, and slightly more than the average for the modern era documented in Pitchford and Wright (2008) using a larger sample of defaults. This result follows from both the persistence in output fluctuations as well as the persistence in bargaining power.

We next documented that, in practice, *debt restructurings were costly to creditors* with the average restructuring generating creditor losses, or "haircuts", of roughly 40% (Fact 2). Table 6 shows that our benchmark model is able to produce haircuts of 28%; although this is less than the unweighted average haircut, is almost equal to the debt weighted average haircut of 30%. Haircuts in many of our other example regimes are even larger, and in some cases total more than 75% of the value of creditors claims. Counterintuitively, average haircuts are largest in the i.i.d. regime where *creditors* have more bargaining power, because this regime supports higher debt levels; for a fixed debt level, haircuts are larger in the i.i.d. debtor regime, but in both regimes haircuts increase with debt levels.

In the data, we also found that *longer defaults are associated with larger haircuts* (Fact 3), with a correlation coefficient of 0.66. This is almost exactly matched by our benchmark model, which produces a correlation coefficient of 0.7. Both the i.i.d. regimes generate a negative correlation between delay and haircuts, because defaults in these regimes only occur at the lowest output levels: when output is very low, reversion to the mean output level is fast and settlements occur quickly; however, when output is low, haircuts are large. This also explains why there is a negative correlation between output declines and delay for the i.i.d. models, with the largest output declines being associated with the fastest reversion to the mean, and hence the shortest delays. The other example bargaining power regimes produce little or no relationship between output declines in the year of default and debt restructuring outcomes, in contrast to the data where *larger output declines in the year of default are associated with longer defaults and larger haircuts* (Fact 4). Our benchmark model produces negative correlations between the change in output and both haircuts and delays. The reason is that, in the benchmark model, defaults occur for two reasons: when bargaining power changes at intermediate income levels, defaults are resolved with little delay and small haircuts because creditors regain bargaining power relatively quickly; when income falls to low levels, defaults are resolved more slowly and with larger haircuts, because there is more persistence in the output process. In our other example regimes, only one effect is present so that time aggregation tends to reduce or eliminate the correlation between outcomes and output.

In calibrating the bargaining power process, we chose parameter values to match some aspects of the relationship between output and default in the data, although not the aspects that we had emphasized in Section 2, where we had confirmed the finding of Tomz and Wright (2007) that *defaults are somewhat more likely to occur when output is below trend, and settlements tend to occur when output has returned to trend* (Fact 5). Table 7 shows that our benchmark model also produces a weak relationship between output and default, as observed in the data, although settlements tend to occur before output returns to trend. The weak relationship between default and output is also a feature of each of our example bargaining regimes. However, whereas for many of our example regimes this is solely a product of time aggregation, for our benchmark regime it is also due to the role of switches in bargaining power in generating defaults. As shown in Table 8, at a quarterly frequency, all of our examples except for the strength-through-strength regime produce a near perfect negative relationship between default and output. Moving from quarterly to annual data not only weakens this pattern, but occasionally reverses it. By contrast, our benchmark regime displays a strong negative relationship in only the quarter of default, and almost no relationship with output in the preceding quarter.

Finally, we documented that in practice *debt restructuring negotiations are ineffective at reducing country indebtedness* (Fact 6) with the median country exiting default with 5%

more debt (scaled by GDP) than when it entered default, and the mean country exiting with 25% more debt. Table 7 presents results on the evolution of indebtedness throughout default for each of the bargaining regimes. Data on debt is presented at both at face values and market values in order to bracket the World Bank data which, as discussed above and in Appendix C, is an average of the two. Our benchmark regime predicts that a country should exit default with roughly the same level of indebtedness as when they entered default, regardless of which measure of the value of debt is used. This is because settlements tend to be associated with both improved economic conditions and more bargaining power for the creditor, which increase the country's ability to borrow.

To summarize, we conclude that our benchmark model is able to explain almost all of the delay observed in the data and more than two-thirds of observed haircuts, while also producing the relationships between bargaining outcomes and economic activity documented above in Section 2. In addition to matching these new facts, our model is also able to match the features of the data emphasized by the previous quantitative theoretical literature on sovereign default. Specifically, this literature has emphasized four facts, all established using quarterly (and chiefly Argentine) data; we collect these facts in Table 9. First, capital flows tend to be "procyclical", with net exports relatively high (and hence capital inflows small) when output is relatively low; the correlation coefficient in the quarterly Argentine data has been reported between -0.2 and -0.9. Second, the difference between interest rates charged on developing country and developed country debts, typically referred to the 'spread' on developing country debt, is counter-cyclical, rising in recessions and falling in booms, with estimates of the correlation between spreads and output for Argentina ranging from -0.1 to -0.9. Third, high spreads are associated with capital outflows, with correlation coefficients between spreads and net exports reported between 0.05 and 0.85. Fourth and finally, consumption in many developing countries is more volatile than output, with the standard deviation of consumption roughly 10 percentage points higher than the standard deviation of output.

Table 9 documents the performance of both our benchmark model and our expository examples along these dimensions and compares them to results reported in a sample of other studies. All moments were calculated from the quarterly model data. Our benchmark model, like some of the leading previous studies, produces consumption that is almost exactly 10%

more volatile than output. This arises due to the large movements in consumption that are predicted in both the year of default and the year of settlement. All regimes, except for strength-through-strength, produce countercyclical spreads, with the benchmark model producing a correlation between spreads and income that is almost exactly in the center of the range of published targets. Likewise, the benchmark model also produces procyclical capital flows with the correlation between income and capital flows the same as that produced by Arellano (2007). All of the regimes produce a positive relationship between spreads and capital flows, with the benchmark model producing a stronger relationship, closer to that observed in the data, than any of the alternative studies.

Finally, a number of studies have emphasized the inability of quantitative models of sovereign debt to produce spreads with the same level, and volatility, as observed in the data (see, for example, Arellano and Ramnarayan 2008, Chatterjee and Eyigungor 2008, and Hatchondo and Martinez 2008). In assessing the ability of this class of models to match data on spreads, it is important to note that the available data on spreads are limited. Prior to the early 1990s, most developing country borrowing took the form of bank loans that were not traded on liquid markets. Moreover, many developing countries spent the decade of the 1980s in default, so that the data on spreads in the 1990s should be viewed as being conditional on a recent settlement of a default. In the case of Argentina, spreads in the 1990s should also be viewed as conditional on being in the lead-up to another default[6]. Bearing these caveats in mind, a coarse summary of the Argentine data over this period shows that Argentine spreads averaged 5 percentage points during the 1990s, peaking near 15 percentage points in the middle of 1995 and again just prior to the default at the end of 2001. The benchmark calibration of our theory produces spreads on the order of those observed in the Argentine data. In the quarter that a default occurs, spreads average 20%, while a country emerges from a default with spreads at roughly 8%.

[6]In recognition of this problem, most authors follow Arellano (2007, p.22) and restrict attention to simulations that are less than eighty quarters in length and terminate just prior to a default, so as to match the available quarterly data for Argentina. As shown in her Figure 5, the spreads generated by her model for this period are typically zero, and exceed the spreads observed in the data at only four data points.

5 Conclusion and Future Work

In this paper, we documented that negotiations to restructure sovereign debts are both time consuming and costly, leading to creditor losses in excess of forty per-cent and leaving the defaulting country more highly indebted than when they entered default. We also documented the relationships between these outcomes, as well as their relationship to economic activity. We then proposed a theory of these delays in which the very same risk of default that gave rise to these negotiations is also the factor that leads to negotiations being prolonged. Intuitively, the conclusion of a debt restructuring negotiation generates surplus to be shared at both the time of the settlement and in the future. However, the debtor country cannot be trusted to honor promises to share future surplus. Hence, both the creditor and the country find it optimal to wait until a future time period in which the risk of default is low; low default risk facilitates the sharing of future surplus, and is also directly associated with a greater amount of surplus to be shared as access to capital markets by the country is more valuable. We show that our model is capable of explaining the bulk of the observed delay in reaching a settlement, as well as about three-quarters of the observed creditor losses.

Our theory also suggests a reinterpretation of the modern history of worldwide sovereign default crises in which multiple countries default at the same time. The phenomenon of concurrent defaults has often been explained by appealing to common negative economic shocks. However, this is hard to reconcile with the modest declines in output observed at the time of default. Our alternative emphasizes changes in the institutional structure governing negotiations over sovereign debt restructuring with all countries. The rise in sovereign borrowing in the late 1970s coincides with the weakening of the "absolute view" of sovereign immunity and movement to a more "restrictive view" which allowed suit against a sovereign in default, and weakened the bargaining position of debtors. In the mid 1980's, the IMF's policy of "lending into arrears" combined with the weak financial position of international banks, strengthened the position of debtors in default, and it was not until these banks improved their financial position, and hence their bargaining power, that the crisis was resolved[7]. Similarly, the rise of

[7] In 1987, John Reed, chairman of Citicorp wrote that "Through building up their reserves and capital, U.S. banks' exposure to troubled debtor nations now accounts for a much smaller portion of capital and earnings than it did in 1982 ... [and that this] increases the strength of the banks and is putting a great strain on the bank restructuring process. Banks are now more able to lend new money, but they are also more able to 'walk away' from the process entirely" (Reed 1987 p.427).

litigious “vulture creditors” in the 1990s has been associated with fewer and shorter defaults. Such an interpretation suggests that current efforts to curtail legal action by creditors, such as the introduction of collective action mechanisms into bond contracts, may lead to more default, and lower borrowing levels, in the future.

We intend to pursue three extensions of this project in future work. First, our model makes predictions about the behavior of secondary market prices for sovereign bonds while a country is in default. We have begun the collection of data on secondary market prices to evaluate these predictions. Second, as discussed in length in the paper, we calibrated the model on a quarterly frequency. Although this is standard when examining the timing of investment decisions in macroeconomics, it is arguably too long a time horizon when thinking about the frequency with which parties may make proposals in bargaining. However, shortening the time horizon also limits the set of assets available to the country; calibrated to a monthly frequency, the country can only issue thirty-day treasury bills. Adding more assets, however, expands the dimension of the state vector for the model, and hence requires greater computational power. In future work we intend to explore the approach of Hatchondo and Martinez (2008), Arellano and Ramnarayan (2008) and Chatterjee and Eyigungor (2008) to computing models with multiple debt maturities. Third, the model can be used as a laboratory to examine the effect of multinational bailout policies. In the working paper version of this paper, we show that the negative effect of bailouts on default incentives outweigh any benefits from both greater risk sharing and a greater incentive to reach a settlement.

Finally, we argue that our findings may be useful in understanding the presence of delays in other bargaining contexts. As one, but by far from the only, possibility, consider bargaining between a firm and its workers in which changes in current work practices are sought in return for future wage and pension benefits. If firm profits are currently low (or negative) the firms’ workers may delay agreeing to changes in work practices that are potentially mutually beneficial, if they anticipate that the firm will declare bankruptcy in the future in order to avoid honoring these future benefits.

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6 Appendix A: Proofs of Key Theorems

To prove Theorem 3, we will require the following definitions. Let $\mathcal{F}(B \times S)$ be the space of all functions mapping $B \times S$ into $\mathbb{R}^2$, and let $\mathcal{B}(B \times S)$ be the subset of $\mathcal{F}(B \times S)$ that satisfies the following bounds

$$\begin{aligned} \min_{s\in S} \frac{u\left(e^{def}(s)\right)}{1-\delta} &\equiv V_{\min} \leq \min_{(b,s)\in B\times S} f_1(b,s) \leq \max_{(b,s)\in B\times S} f_1(b,s) \\ &\leq V_{\max} \equiv \max_{(b,s,s')\in B\times S\times S} u(e(s)-b) + \beta V(b,s'), \\ b_{\min} &\equiv \min B \leq \min_{(b,s)\in B\times S} f_2(b,s) \leq \max_{(b,s)\in B\times S} f_2(b,s) \leq b_{\max} \equiv \max B. \end{aligned}$$

We endow $\mathcal{B}(B \times S)$ with the supremum (in this case, maximum) norm.

Theorem 3. *Let $u : \mathbf{R} \to \mathbf{R}$ be differentiable. If there exists $K_L > \beta$ and $K_U < 1/\delta$ such that $K_L \leq u'(c) \leq K_U$, for all c, then the SSP equilibrium values are unique.*

Proof. Let $f^1 = (f_1^1, f_2^1)$ and $f^2 = (f_1^2, f_2^2)$ be elements of $\mathcal{B}(B \times S)$. To establish the result, we need to show that there exists a $\gamma \in (0,1)$ such that

$$\begin{aligned} & ||\hat{T}f^1 - \hat{T}f^2||_\infty \\ = & \max_{(b,s)\in B\times S} \left\{ \max\left\{ \left| \left(\hat{T}f_1^1\right)(b,s) - \left(\hat{T}f_1^2\right)(b,s) \right|, \left| \left(\hat{T}f_2^1\right)(b,s) - \left(\hat{T}f_2^2\right)(b,s) \right| \right\} \right\} \\ \leq & \gamma \max_{(b,s)\in B\times S} \left\{ \max\left\{ \left| f_1^1(b,s) - f_1^2(b,s) \right|, \left| f_2^1(b,s) - f_2^2(b,s) \right| \right\} \right\} \\ \leq & \gamma ||f^1 - f^2||_\infty. \end{aligned}$$

The argument varies according to whether the outside offer is taken, no proposal is accepted, or a proposal is accepted.

First, fix (b,s) and consider the case in which s is such that the debtor proposes. If the outside option is taken for both f^1 and f^2, then we have

$$\left| \left(\hat{T}f^1\right)(b,s) - \left(\hat{T}f^2\right)(b,s) \right| = 0,$$

since the creditor's payoff is b, and the debtor's payoff solves

$$\begin{aligned} \max\nolimits_{\tau,b'} u(e^{def}(s) - \tau) + \beta E[V(b',s')|s] \\ s.t \quad \tau + b'q(b',s) \geq b, \end{aligned}$$

neither or which depends on the continuation values f^1 and f^2.

If no proposal is accepted for both f^1 and f^2, then we have

$$\left| \left(\hat{T}f_2^1\right)(b,s) - \left(\hat{T}f_2^2\right)(b,s) \right| = \left| \delta E\left[f_2^1(b,s')|s \right] - \delta E\left[f_2^2(b,s')|s \right] \right| \leq \delta ||f_2^1 - f_2^2||_\infty,$$

for the creditor's continuation value function, and

$$\begin{aligned} & \left| \left(\hat{T}f_1^1\right)(b,s) - \left(\hat{T}f_1^2\right)(b,s) \right| \\ = & \left| u(e^{def}(s)) + \beta E\left[f_1(b,s')|s\right] - u(e^{def}(s)) - \beta E\left[f_1(b,s')|s\right] \right| \\ = & \beta \left| E\left[f_1(b,s')|s\right] - E\left[f_1(b,s')|s\right] \right| \\ \leq & \beta \left\| f_1^1 - f_1^2 \right\|_\infty. \end{aligned}$$

for the debtor's continuation value function.

If a proposal is accepted for both f^1 and f^2, consider first the case in which s is such that the debtor proposes. In this case, the creditor's continuation values satisfy

$$\left| \left(\hat{T}f_2^1\right)(b,s) - \left(\hat{T}f_2^2\right)(b,s) \right| = \left| \delta E\left[f_2^1(b,s')|s \right] - \delta E\left[f_2^2(b,s')|s \right] \right| \leq \delta ||f_2^1 - f_2^2||_\infty.$$

Using this fact, the debtor's continuation values satisfy

$$
\begin{aligned}
& \left|\left(\hat{T} f_1^1\right)(b, s)-\left(\hat{T} f_1^2\right)(b, s)\right| \\
= & \left| \begin{array}{c} \max_{\tau, b'} u\left(e^{def}(s)-\tau\right)+\beta E\left[V\left(b', s'\right) | s\right] \\ s.t \quad \tau+b' q\left(b', s\right) \geq \delta E\left[f_2^1\left(b, s'\right) | s\right] \end{array} \right. \\
& - \left. \begin{array}{c} \max_{\tau, b'} u\left(e^{def}(s)-\tau\right)+\beta E\left[V\left(b', s'\right) | s\right] \\ s.t \quad \tau+b' q\left(b', s\right) \geq \delta E\left[f_2^2\left(b, s'\right) | s\right] \end{array} \right| \\
\leq & \left| \begin{array}{c} \max_{\tau, b'} u\left(e^{def}(s)-\tau\right)+\beta E\left[V\left(b', s'\right) | s\right] \\ s.t \quad \tau+b' q\left(b', s\right) \geq \delta E\left[f_2^2\left(b, s'\right) | s\right]+\delta||f_2^1-f_2^2||_{\infty} \end{array} \right. \\
& - \left. \begin{array}{c} \max_{\tau, b'} u\left(e^{def}(s)-\tau\right)+\beta E\left[V\left(b', s'\right) | s\right] \\ s.t \quad \tau+b' q\left(b', s\right) \geq \delta E\left[f_2^2\left(b, s'\right) | s\right] \end{array} \right| .
\end{aligned}
$$

Now suppose that (τ^2, b'^2) attain the maximum for f_2^2. Then exploiting the fact that U is defined over negative consumptions and that its slope is bounded we can find a feasible $\hat{\tau}$ such that

$$
\hat{\tau}=\tau^2+\delta\left\|f_2^1-f_2^2\right\|_{\infty},
$$

yielding

$$
\begin{aligned}
& \left|\left(\hat{T} f_1^1\right)(b, s)-\left(\hat{T} f_1^2\right)(b, s)\right| \\
\leq & \left|u\left(e^{def}(s)-\tau^2+\delta\left\|f_2^1-f_2^2\right\|_{\infty}\right)+\beta E\left[V\left(b^2, s'\right) | s\right]-u\left(e^{def}(s)-\tau^2\right)\right. \\
& \left.-\beta E\left[V\left(b^2, s'\right) | s\right]\right| \\
\leq & \left|u\left(e^{def}(s)-\tau^2\right)+u'\left(e^{def}(s)-\tau^2\right) \delta\left\|f_2^1-f_2^2\right\|_{\infty}-u\left(e^{def}(s)-\tau^2\right)\right| \\
\leq & \delta K_U\left\|f_2^1-f_2^2\right\|_{\infty} .
\end{aligned}
$$

Next consider the case in which s is such that the creditor proposes. In this case, the debtor's continuation values satisfy

$$
\begin{aligned}
& \left|\left(\hat{T} f_1^1\right)(b, s)-\left(\hat{T} f_1^2\right)(b, s)\right| \\
= & \left|\beta E\left[f_1^1\left(b, s'\right) | s\right]-\beta E\left[f_1^2\left(b, s'\right) | s\right]\right| \leq \beta||f_1^1-f_1^2||_{\infty}.
\end{aligned}
$$

Using this fact, the creditor's continuation values satisfy

$$
\begin{aligned}
& \left|\left(\hat{T} f_2^1\right)(b, s)-\left(\hat{T} f_2^2\right)(b, s)\right| \\
= & \left| \begin{array}{c} \max_{\tau, b'} \tau+b' q(b', s) \\ s.t \quad u(e^{def}(s)-\tau)+\beta E[V(b', s')|s] \\ \geq u(e^{def}(s))+\beta E\left[f_1^1(b, s')|s\right] \end{array} - \begin{array}{c} \max_{\tau, b'} \tau+b' q(b', s) \\ s.t \quad u(e^{def}(s)-\tau)+\beta E[V(b', s')|s] \\ \geq u(e^{def}(s))+\beta E\left[f_1^2(b, s')|s\right] \end{array} \right| \\
\leq & \left| \begin{array}{c} \max_{\tau, b'} \tau+b' q(b', s) \\ s.t \quad u(e^{def}(s)-\tau)+\beta E[V(b', s')|s] \\ \geq u(e^{def}(s))+\beta E\left[f_1^1(b, s')|s\right] \end{array} \right. \\
& - \left. \begin{array}{c} \max_{\tau, b'} \tau+b' q(b', s) \\ s.t \quad u(e^{def}(s)-\tau)+\beta E[V(b', s')|s] \\ \geq u(e^{def}(s))+\beta E\left[f_1^1(b, s')|s\right]+\beta\left\|f_1^1-f_1^2\right\|_{\infty} \end{array} \right| .
\end{aligned}
$$

Now suppose that (τ^1, b'^1) attain the maximum for f_1^1. Then there exists a $\hat{\tau}$ such that

$$\left|\hat{\tau} - \tau^1\right| \leq \beta K_L \left\|f_1^1 - f_1^2\right\|_\infty ,$$

and that $(\hat{\tau}, b'^2)$ is feasible for f_1^1 and so

$$\left|\left(\hat{T} f_2^1\right)(b,s) - \left(\hat{T} f_2^2\right)(b,s)\right| \leq \beta \frac{1}{K_L} \left\|f_1^1 - f_1^2\right\|_\infty .$$

It remains to consider cases that involve combinations of the outside option, no proposal being accepted, and a proposal being accepted. Suppose the outside option is taken for one of the f^i and no proposal is accepted for f^{-i}. The argument is analogous regardless of whether the debtor proposes or the creditor proposes at s. Without loss of generality we can order the creditor's continuation value functions such that

$$\begin{aligned}
\left|\left(\hat{T} f_2^1\right)(b,s) - \left(\hat{T} f_2^2\right)(b,s)\right| &= \left|b - \delta E\left[f_2^2(b,s')\,|s\right]\right| \\
&\leq \left|\delta E\left[f_2^1(b,s')\,|s\right] - \delta E\left[f_2^2(b,s')\,|s\right]\right| \\
&\leq \delta ||f_2^1 - f_2^2||_\infty,
\end{aligned}$$

while for the debtor, if we define

$$V^{oo}(b,s) = \begin{array}{c} \max_{\tau,b'} u(e^{def}(s) - \tau) + \beta E[V(b',s')|s] \\ s.t \;\; \tau + b'q(b',s) \geq b, \end{array}$$

we have

$$\begin{aligned}
&\left|\left(\hat{T} f_1^1\right)(b,s) - \left(\hat{T} f_1^2\right)(b,s)\right| \\
=\;& \left|V^{oo}(b,s) - u(e^{def}(s)) + \beta E\left[f_1^2(b,s')|s\right]\right| \\
\leq\;& \left|u\left(e^{def}(s)\right) + \beta E\left[f_1^2(b,s')\,|s\right] - u(e^{def}(s)) - \beta E\left[f_1^1(b,s')|s\right]\right| \\
\leq\;& \beta \left\|f_1^1 - f_1^2\right\|_\infty .
\end{aligned}$$

where the first inequality follows from the fact that the debtor did not take the outside option for f^2 and the fact that the value of the outside option is independent of the continuation values.

Now suppose the outside option is taken for one of the f^i and a proposal is accepted for f^{-i}. If s is such that the debtor proposes, then the argument for the creditor is the same as in the previous case since they earn their autarky value from an accepted proposal. For the debtor, we have

$$\begin{aligned}
&\left|\left(\hat{T} f_1^1\right)(b,s) - \left(\hat{T} f_1^2\right)(b,s)\right| \\
=\;& \left|\begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s) - \tau\right) + \beta E\left[V(b',s')\,|s\right] \\ s.t \;\; \tau + b'q(b',s) \geq \delta E\left[f_2^1(b,s')\,|s\right] \end{array} - V^{oo}(b,s)\right| \\
\leq\;& \left|\begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s) - \tau\right) + \beta E\left[V(b',s')\,|s\right] \\ s.t \;\; \tau + b'q(b',s) \geq \delta E\left[f_2^1(b,s')\,|s\right] \end{array} - \begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s) - \tau\right) + \beta E\left[V(b',s')\,|s\right] \\ s.t \;\; \tau + b'q(b',s) \geq \delta E\left[f_2^2(b,s')\,|s\right] \end{array}\right| \\
\leq\;& \delta K_U \left\|f_2^1 - f_2^2\right\|_\infty ,
\end{aligned}$$

where the first inequality follows from the fact that the debtor did not take the outside option for f^2. If s is such that the creditor proposes, the argument for the debtor's continuation value function is the same as in the previous case because the debtor receives their autarky value from an accepted proposal. For the creditor, the result follows from an argument similar to the debtor proposer case.

Finally, consider the case where no agreement occurs for f^1 and an agreement occurs for f^2. Non-proposers receive their autarky values in both cases, implying no difference in continuation value functions under the $\hat{T}$ operator. For the proposer, the fact that no agreement is chosen over agreement for f^2 means we can apply the same argument as in the previous case.

Since the result holds for arbitrary (b,s), the operator T is a contraction with modulus

$$\gamma = \max\{\delta, \beta, \delta K_U, \beta/K_L\}. \qquad \square$$

Theorem 5. *If the SSP equilibrium values of the bargaining model are unique, then there exists an equilibrium of our borrowing economy.*

Proof. Let $q \in \mathcal{Q}(B \times S)$ and $\left(V^D, W\right) \in \mathcal{B}(B \times S)$. We construct the first part of our mapping, $\mathcal{H}_1\left(V^D, q\right)$ as follows. Fix (b,s) and think of the $q(b',s)$ and $V^D(b,s)$ as a set of $N_b + 1$ parameters for the country's borrowing problem. Let $C(X)$ be the set of continuous and bounded functions defined on $X = [0, 1/(1+r^w)]^{N_b+1} \times [V_{\min}, V_{\max}]$. Let $f \in \mathcal{C}(X)$ and define the operator $\hat{T}^V$ by

$$\left(\hat{T}^V\right) f = \max\left\{\max_{b' \in B} u\left(e(s) - b + b'q(b',s)\right) + \beta E\left[V(b',s')\,|s\right], \tilde{V}^D(b,s)\right\}.$$

Next define $H_1\left(T^V, q\right)$ as the fixed point of the bargaining operator, given a default value of T^V and a bond price of q.

The finiteness of B ensures that a solution to the country's borrowing problem exists, and that it is bounded, while the Theorem of the Maximum implies that $\left(\hat{T}^V\right) f$ is continuous in x. For any $f^1, f^2 \in \mathcal{C}(X)$ analogues of the arguments provided above ensure that the fixed points of the bargaining operator defined on $\mathcal{C}(X)$ are also continuous in X. Select the largest such fixed point. Then the mapping $\mathcal{H}_1\left(V^D, q\right)(b,s)$ is a continuous (and hence upper hemi-continuous) single valued, and hence compact and convex valued, correspondence. From this, we can construct the product correspondence

$$\mathcal{H}_1\left(V^D, q\right) = \prod_{(b,s) \in B \times S} \mathcal{H}_1\left(V^D, q\right)(b,s).$$

By Theorem 17.28 of Aliprantis and Border (2006), this product correspondence is continuous and compact valued.

Now consider the second part of our mapping $\mathcal{H}_2\left(V^D, W, q\right)$ defined as follows. First, think of the $q(b',s)$, $V^D(b,s)$ and $W(b,s)$ as a finite set of parameters for the country's borrowing problem, with each $q(b',s)$ belonging to the compact interval $[0, 1/(1+r^w)]$, each $V^D(b,s)$ belonging to $[V_{\min}, V_{\max}]$, and each $W(b,s)$ belonging to $[b_{\min}, b_{\max}]$. Let $\mathcal{C}(X)$ be the space of all continuous functions defined on

$$X = [0, 1/(1+r^w)]^{N_b \times N_e} \times [V_{\min}, V_{\max}]^{N_b \times N_e} \times [b_{\min}, b_{\max}]^{N_b \times N_e}.$$

Let $f \in \mathcal{C}(X)$ and define the operator $\hat{T}^V$ be defined by

$$\left(\hat{T}^V\right) f = \max\left\{\max_{b' \in B} u\left(e\left(s\right) - b + b'q\left(b', s\right)\right) + \beta E\left[V\left(b', s'\right)|s\right], \tilde{V}^D\left(b, s\right)\right\}.$$

As above, the fixed point V is continuous on X; the calculations also define the function $V^R(b, s)$.

Define the default indicator correspondence

$$\Phi\left(b, s\right) = \begin{cases} 1 & \text{if } \tilde{V}^D\left(b, s\right) > V^R\left(b, s\right) \\ 0 & \text{if } \tilde{V}^D\left(b, s\right) < V^R\left(b, s\right) \\ [0, 1] & \text{if } \tilde{V}^D\left(b, s\right) = V^R\left(b, s\right) \end{cases}.$$

From this we can define a default probability correspondence, $P(b', s)$, as the set of all $p(b', s)$ constructed as

$$p\left(b', s\right) = \sum_{s' \in S} \phi\left(b', s'\right) \pi\left(s'|s\right),$$

for some $\phi(b', s'; x) \in \Phi(b', s'; x)$., Hence, for any fixed (b', s) we can define the bond price correspondence from points in X to $[0, 1/(1 + r^w)]$ as

$$\mathcal{H}_2\left(V^D, W, q\right)\left(b', s\right) = \left\{y : y = \frac{1 - p + p\sum_{s' \in S} \pi\left(s'|s\right) \tilde{W}(b', s')/b}{1 + r^w} \text{ for some } p \in P\left(b', s\right)\right\},$$

where $\tilde{W}(b', s')$ was defined above.

It is straightforward to show that for (b', s) and $\left(V^D, W, q\right)$ fixed, this is a closed interval contained in $[0, 1]$. Hence, it is compact valued. A straightforward adaptation of App Lemma 8 from Chatterjee, Corbae, Nakajima and Rios-Rull (2002) shows that it is also upper-hemi continuous. Therefore, viewed as a correspondence from points in X to $[0, 1/(1 + r^w)]$ this is upper-hemi continuous. Then for any $\left(V^D, W, q\right)$, we can define the product correspondence

$$\mathcal{H}_2\left(V^D, W, q\right) = \prod_{(b,s) \in B \times S} \mathcal{H}_2\left(V^D, W, q\right)\left(b, s\right).$$

By Theorem 17.28 of Aliprantis and Border (2006), this product correspondence is continuous and compact valued.

Finally, form

$$\mathcal{H}\left(V^D, W, q\right) = \left[\mathcal{H}_1\left(V^D, q\right), \mathcal{H}_2\left(V^D, W, q\right)\right].$$

By Theorem 17.23 of Aliprantis and Border (2006), $\mathcal{H}$ is upper hemi-continuous. Using the fact that $\mathcal{H}_1$ is single valued, it is also straightforward to show that it is convex valued. Hence, by Kakutani's fixed point theorem there exists a fixed point of $\mathcal{H}$.

Using the fixed points for q^* and V^{D*}, we can then iterate to convergence to find V^*. The collection V^*, V^{D*}, W^* and q^* satisfies the definition for an equilibrium of our borrowing economy, and hence there exists an equilibrium for our borrowing economy. □

Extra Appendices For

Recovery Before Redemption: A Theory of Delays in Sovereign Debt Renegotiations

by

David Benjamin and Mark L. J. Wright

NOT FOR PUBLICATION

7 Appendix B: Extra Details on Proofs of Theorems

7.A Solution to the Bargaining Model

Recursive Problem Statement

In this section, we prove the equivalence between SSP payoffs and fixed points of the $\hat{T}$ operator defined in the text. This requires some notation, and an intermediate Lemma.

Take the SSP outcome, which consists of a set of states in which acceptance occurs and the proposal that is accepted in that state, $((B \times S)^{\mu}, \mu)$, as given. We can define the value of this outcome as follows. First, fix the value of the defaulted debt to b. Then, given a sequence of realizations of the state, define the stopping time for an agreement by t^* where $(b, s_{t^*}) \in (B \times S)^{\mu}$ and $(b, s_t) \in (B \times S) \setminus (B \times S)^{\mu}$ for all $t = 0, ..., t^* - 1$. Then we can define the value of this outcome in state s as

$$v^{\mu}(b,s) \equiv \begin{pmatrix} v_1^{\mu}(b,s) \\ v_2^{\mu}(b,s) \end{pmatrix}$$

$$= \begin{pmatrix} E\left[\sum_{t=0}^{t^*-1} \beta^t U\left(e^{def}(s_t)\right) + \beta^{t^*}\left\{U\left(e^{def}(s_{t^*}) - \tau(s_{t^*})\right) + \beta V\left(b(s_{t^*}), s_{t^*}\right)\right\} |s\right] \\ E\left[\delta^{t^*}\left\{\tau(s_{t^*}) + b(s_{t^*})\, q\left(b(s_{t^*}), s_{t^*}\right)\right\} |s\right] \end{pmatrix}.$$

First, we establish that the value function $v^{\mu}(b,s)$ is the unique function defined on $B \times S$ taking values in $\mathbb{R}^2$ satisfying a particular functional equation. The proof relies on the following mapping which is defined for an arbitrary stationary outcome. Specifically, consider the mapping T on the set of functions $f : B \times S \to \mathbb{R}^2$ into itself defined by:

$$Tf_1(b,s) = \begin{cases} u(e^{def}(s) - \tau(b,s)) + \beta E[V(b'(b,s), s'|s)] & \text{if } (b,s) \in (B \times S)^{\mu} \\ u(e^{def}(s)) + \beta E[f_1(b,s')|s] & \text{if } (b,s) \in (B \times S) \setminus (B \times S)^{\mu} \end{cases},$$

and

$$Tf_2(b,s) = \begin{cases} \tau(b,s) + b'(b,s)\, q(b'(b,s), s) & \text{if } (b,s) \in (B \times S)^{\mu} \\ \delta E[f_2(b,s')|s] & \text{if } (b,s) \in (B \times S) \setminus (B \times S)^{\mu} \end{cases}.$$

The first operator applies to the payoff of the debtor country, and simply states that if (b,s) is in the set $(B \times S)^{\mu}$, which is the set of debt levels and states in which either the outside option is taken or a proposal is accepted, then the payoff to the country is found by evaluating the value of that proposal. Conversely, if (b,s) is not in the acceptance set, the debtor country consumes its endowment in default today and the discounted value of the expected payoff from continuing the bargaining game tomorrow. The second operator is similar and applies to the payoff of the creditors.

Lemma B. 1. *Given an outcome $((B \times S)^{\mu}, \mu)$ where $\mu = (\tau, b)$, v^{μ} is the unique function defined on $B \times S$ taking values in $\mathbb{R}^2$ for which*

$$\begin{pmatrix} v_1^{\mu}(b,s) \\ v_2^{\mu}(b,s) \end{pmatrix} = \begin{cases} \begin{pmatrix} u(e^{def}(s) - \tau(b,s)) + \beta E[V(b'(b,s), s')|s] \\ \tau(b,s) + b'(b,s)\, q(b'(b,s), s) \end{pmatrix} & \textit{if } s \in S^{\mu} \\ \begin{pmatrix} u(e^{def}(s)) + \beta E[v_1^{\mu}(b,s')|s] \\ \delta E[v_2^{\mu}(b,s')|s] \end{pmatrix} & \textit{if } s \in S \setminus S^{\mu} \end{cases}.$$

Proof. The proof requires us to show that v^{μ} is a fixed point of the operator T, and that the operator T has a unique fixed point. First, to see that v^{μ} is a fixed point, note that if

$(b, s_0) \in (B \times S)^{\mu}$ then

$$Tv^{\mu}(b, s_0) = \begin{pmatrix} u(e^{def}(s_0) - \tau(b, s_0)) + \beta E[V(b'(b, s_0), s_1 | s_0)] \\ \tau(b, s_0) + b'(b, s_0) q(b'(b, s_0), s_0) \end{pmatrix},$$

which is precisely the definition of v^{μ} on states for realizations in which the stopping time is zero. Alternatively, suppose that $(b, s_0) \in (B \times S) \setminus (B \times S)^{\mu}$. Then by definition of T we have

$$T \begin{pmatrix} v_1^{\mu}(b, s_0) \\ v_2^{\mu}(b, s_0) \end{pmatrix} = \begin{pmatrix} u(e^{def}(s_0)) + \beta E[v_1^{\mu}(b, s_1) | s_0] \\ \delta E[v_2^{\mu}(b, s_1) | s_0] \end{pmatrix}.$$

Define a stopping time t^*,such that if (b, s_0) is the initial state, t^* is the period in which agreement is reached. That is, $(b, s_{t^*}) \in (B \times S)^{\mu}$ and $(b, s_t) \in (B \times S) \setminus (B \times S)^{\mu}$ for all $t < t^*$. Then iterating on the operator T we have

$$\begin{aligned} & T \begin{pmatrix} v_1^{\mu}(b, s_0) \\ v_2^{\mu}(b, s_0) \end{pmatrix} \\ = & \begin{pmatrix} u(e^{def}(s_0)) + \beta E[v_1^{\mu}(b, s_1) | s_0] \\ \delta E[v_2^{\mu}(b, s_1) | s_0] \end{pmatrix} \\ = & \begin{pmatrix} u(e^{def}(s_0)) + \beta E[\sum_{t=1}^{t^*-1} \beta^t u\left(e^{def}(s_t)\right) + \beta^{t^*} E[v_1^{\mu}(b, s_{t^*}) | s_1] | s_0] \\ \delta\left[E\left[E[\delta^{t^*} v_2^{\mu}(b, s_{t^*}) | s_1]\right] | s_0\right] \end{pmatrix} \\ = & \begin{pmatrix} E\left[\sum_{t=0}^{t^*-1} \beta^t U\left(e^{def}(s_t)\right) + \beta^{t^*}\left\{U\left(e^{def}(s_{t^*}) - \tau(b, s_{t^*})\right) V(b'(b, s_{t^*}), s_{t^*})\right\} | s\right] \\ E\left[\delta^{t^*}\left\{\tau(b, s_{t^*}) + b'(b, s_{t^*}) q(b'(b, s_{t^*}), s_{t^*})\right\} | s\right] \end{pmatrix} = v^{\mu}. \end{aligned}$$

Second, to show that T has a unique fixed point it is sufficient to show that T is a contraction on the metric space of functions defined on $B \times S$ taking values in $\mathbb{R}^2$ endowed with the sup (or in this case, the max) norm. That is, we require that if f^1 and f^2 are each functions mapping $B \times S$ into $\mathbb{R}^2$, then

$$||T(f^1) - T(f^2)||_{\infty} \leq \delta ||f^1 - f^2||_{\infty}.$$

To see this, note that if $(b, s) \in (B \times S)^{\mu}$, then Tf is independent of the function f and hence

$$\left|Tf^1(b, s) - Tf^2(b, s)\right| = \max\left\{\left|Tf_1^1(b, s) - Tf_1^2(b, s)\right|, \left|Tf_2^1(b, s) - Tf_2^2(b, s)\right|\right\} = 0.$$

Otherwise,

$$\begin{aligned} & \left|Tf^1(b, s) - Tf^2(b, s)\right| \\ = & \max\left\{\left|\beta E[f_1^1(b, s') | s] - \beta E[f_1^2(b, s') | s]\right|, \left|\delta E[f_2^1(b, s') | s] - \delta E[f_2^2(b, s') | s]\right|\right\} \\ = & \beta \max\left\{\left|E[f_1^1(s') - f_1^2(s') | s]\right|, \left|E[f_2^1(s') - f_2^2(s') | s]\right|\right\} \\ \leq & \beta ||f^1 - f^2||_{\infty}, \end{aligned}$$

where we have exploited our assumption that $\beta < \delta < 1$. But then

$$\left\|Tf^1 - Tf^2\right\|_{\infty} = \max_{b,s}\left|Tf^1(b, s) - Tf^2(b, s)\right| \leq \beta ||f^1 - f^2||_{\infty}.$$

□

Using the result of the previous Lemma, the following theorem establishes an equivalence between SSP payoffs and fixed points of the $\hat{T}$ operator.

Theorem 1. *The functions $f = (f_1, f_2)$ are SSP payoffs if and only if $\hat{T}f = f$.*

Proof. First, suppose that f are SSP payoffs. Fix $(b, s) \in B \times S$. Suppose that no proposal is accepted at (b, s), and the outside option is not taken. Then the SSP payoffs f satisfy the relationships

$$\begin{aligned} f_1(b, s) &= u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right], \\ f_2(b, s) &= \delta E\left[f_2(b, s')|s\right]. \end{aligned}$$

If a proposal is accepted at (b, s), it must be that it gives the agent who receives the proposal at least their reservation utility. If the debtor is proposing, then it must be that the proposal (τ, b') satisfies

$$\tau + b'q(b', s) \geq \min\{b, \delta E[f_2(b, s')|s]\} = \delta E[f_2(b, s')|s],$$

while if the creditor is proposing, it must satisfy

$$\begin{aligned} & u(e^{def} - \tau) + \beta E[V(b', s')|s] \\ \geq \; & \max\left\{u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right], \begin{array}{c} \max_{\tau, b'} u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ s.t \;\; \tau + b'q(b', s) \geq b \end{array} \right\}. \end{aligned}$$

Moreover, as the proposal is part of a SSP, it must give the proposer the largest payoff over all such feasible proposals. Hence, if the debtor proposes in a state where a proposal is accepted

$$\begin{aligned} f_1(b, s) = \; & \max_{\tau, b'} u(e^{def}(s) - \tau) + \beta E[V(b, s')|s], \\ & s.t. \; \tau + b'q(b', s) \geq \min\{b, \delta E[f_2(b, s')|s]\}, \end{aligned}$$

while if a creditor proposes, it must be that

$$f_2(b, s) = \min\left\{b, \begin{array}{l} \max_{\tau, b'} \tau + b'q(b', s), \\ s.t. \; u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \geq u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right] \end{array} \right\}.$$

Finally, as the proposer can always guarantee themselves their reservation payoff (or the outside option in the case of the debtor) by proposing something that will not be accepted, it must be that

$$f_1(b, s) = \max\left\{ \begin{array}{c} \max_{\tau, b'} u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ s.t \;\; \tau + b'q(b', s) \geq \min\{b, \delta E[f_2(b, s')|s]\} \end{array}, u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right] \right\}$$

when the debtor proposes, and

$$\begin{aligned} f_2(b, s) = \; & \max\left\{ \min\left\{ b, \begin{array}{c} \max_{\tau, b'} \tau + b'q(b', s) \\ s.t \;\; u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \geq u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right] \end{array} \right\} \right. \\ & \left. , \delta E\left[f_2(b, s')|s\right] \right\} \end{aligned}$$

when the creditor proposes. But then $\hat{T}f = f$.

Second, suppose that $\hat{T}f = f$. We will construct a SSP outcome $((B \times S)^{\mu}, \mu)$ for which $f = v^{\mu}$. We construct $(B \times S)^{\mu}$ by noting that, if for a given (b, s) there exists (τ, b')

such that

$$\begin{aligned} f_1(b,s) &= u(e^{def}(s)-\tau)+\beta E[V(b',s')|s] \\ f_2(b,s) &= \tau+b'q(b',s), \end{aligned}$$

then (b,s) is an agreement state and hence $(b,s)\in(B\times S)^{\mu}$. Then for that state we let

$$\mu(b,s)=(\tau,b').$$

Otherwise, we say $(b,s)\in(B\times S)\setminus(B\times S)^{\mu}$.

We need to show that the value of the outcome $((B\times S)^{\mu},\mu)$, v^{μ}, is equal to f and that it is a SSP outcome. To show that the value of the outcome is v^{μ}, consider any state (b,s). Since $\hat{T}f=f$, for the non-proposing player we have

$$\begin{aligned} f_1(b,s) &= u(e^{def}(s))+\beta E\left[f_1(b,s')|s\right], \\ f_2(b,s) &= \min\left\{b,\delta E\left[f_2(b,s')|s\right]\right\}, \end{aligned}$$

while for the proposing country we have

$$\hat{T}f_1(b,s)=\max\left\{\begin{array}{c}\max_{\tau,b'}u(e^{def}(s)-\tau)+\beta E[V(b',s')|s] \\ s.t\ \ \tau+b'q(b',s)\geq\delta E[f_2(b,s')|s]\end{array},u(e^{def}(s))+\beta E\left[f_1(b,s')|s\right]\right\},$$

with an analogous result for the creditor. If $\tau+b'q(b',s)=f_2(b,s)$, then $(b,s)\in(B\times S)^{\mu}$ by construction and

$$f_1(b,s)=u(e^{def}(s)-\tau)+\beta E[V(b',s')|s].$$

If $\tau+b'q(b',s)<f_2(b,s)$, then $(b,s)\notin(B\times S)^{\mu}$ and

$$f_1(b,s)=u(e^{def}(s))+\beta E\left[f_1(b,s')|s\right].$$

but in Lemma 1 we showed that v^{μ} was the unique function satisfying these conditions. Hence $f=v^{\mu}$.

Finally, to show that $((B\times S)^{\mu},\mu)$ is a SSP outcome, consider a strategy designed as follows: (i) if $(b,s)\in(B\times S)^{\mu}$, then propose $\mu(b,s)$, otherwise propose an outcome that delivers the other player strictly less than $v^{\mu}(b,s)$; (ii) accept any proposal as long as it delivers at least $v^{\mu}(b,s)$. To see that this is a subgame perfect equilibrium, consider a node at which a player has yet to propose. $\mu(b,s)$ delivers at least $v^{\mu}(b,s)$ by the previous result and so will be accepted. Moreover, as $\hat{T}v^{\mu}=v^{\mu}$, this proposal maximizes the payoff of the proposer subject to delivering this utility level. Hence a proposer cannot gain by deviating to any other proposal. Next, consider a node at which a proposal has been made. If the proposal gives strictly less than $v^{\mu}(b,s)$, the player can only lose by accepting it. If the proposal gives exactly $v^{\mu}(s)$, then by construction it also delivers exactly the reservation payoff of the agent, which is the value they expect from rejecting the offer. Hence, a one stage rejection of a proposal gives the same expected payoff. Familiar arguments show that by iterating on this argument we can rule out finite stage deviations, while boundedness and discounting rule out infinite deviations. □

Existence and Uniqueness of SSP Equilibria of the Bargaining Model

Next we show that an SSP equilibrium exists, and provide a condition under which the SSP equilibrium is unique. Existence is proven by demonstrating that our $\hat{T}$ mapping operates on a bounded set of functions, and is monotone. Let $\mathcal{F}(B\times S)$ be the space of all functions mapping $B\times S$ into $\mathbb{R}^2$, and let $\mathcal{B}(B\times S)$ be the subset of $\mathcal{F}(B\times S)$ that satisfies the following bounds

$$
\begin{aligned}
\min_{s\in S}\frac{u\left(e^{def}\left(s\right)\right)}{1-\delta} &\equiv V_{\min}\leq\min_{(b,s)\in B\times S}f_{1}\left(b,s\right)\leq\max_{(b,s)\in B\times S}f_{1}\left(b,s\right)\\
&\leq V_{\max}\equiv\max_{(b,s,s')\in B\times S\times S}u\left(e\left(s\right)-b\right)+\beta V\left(b,s'\right),\\
b_{\min} &\equiv \min B\leq\min_{(b,s)\in B\times S}f_{2}\left(b,s\right)\leq\max_{(b,s)\in B\times S}f_{2}\left(b,s\right)\leq b_{\max}\equiv\max B.
\end{aligned}
$$

We endow $\mathcal{B}\left(B\times S\right)$ with the supremum (in this case, maximum) norm.

Lemma B. 2. *The operator $\hat{T}$ maps $\mathcal{B}\left(B\times S\right)$ into itself.*

Proof. To see that if $f\in\mathcal{B}\left(B\times S\right)$ then $\hat{T}f\in\mathcal{B}\left(B\times S\right)$, first consider the creditors continuation value function. Fix b. Then if s is such that the debtor proposes

$$
\hat{T}_{2}(f_{1},f_{2})\left(b,s\right)=\min\left\{b,\delta E\left[f_{2}\left(b,s'\right)|s\right]\right\}\in\left[b_{\min},b_{\max}\right].
$$

If s is such that the creditor proposes

$$
\begin{aligned}
\hat{T}_{2}\left(f_{1},f_{2}\right)\left(b,s\right) &= \max\left\{\min\left\{b,\begin{array}{c}\max_{\tau,b'}\tau+b'q(b',s)\\ s.t\ \ u(e^{def}\left(s\right)-\tau)+\beta E[V(b',s')|s]\\ \geq u(e^{def}\left(s\right))+\beta E\left[f_{1}(b,s')|s\right]\end{array}\right\},\delta E\left[f_{2}(b,s')|s\right]\right\}\\
&\leq \max\left\{b,\delta E\left[f_{2}(b,s')|s\right]\right\}\leq b_{\max},
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{T}_{2}\left(f_{1},f_{2}\right)\left(b,s\right) &= \max\left\{\min\left\{b,\begin{array}{c}\max_{\tau,b'}\tau+b'q(b',s)\\ s.t\ \ u(e^{def}\left(s\right)-\tau)+\beta E[V(b',s')|s]\\ \geq u(e^{def}\left(s\right))+\beta E\left[f_{1}(b,s')|s\right]\end{array}\right\},\delta E\left[f_{2}(b,s')|s\right]\right\}\\
&\geq \delta E\left[f_{2}(b,s')|s\right]\geq b_{\min},
\end{aligned}
$$

since $b_{\min}\leq 0$.

Next consider the debtor's continuation value function. Fix b. Then if s is such that the creditor proposes

$$
\begin{aligned}
&\hat{T}_{1}\left(f_{1},f_{2}\right)\left(b,s\right)\\
=\ &\max\left\{u\left(e^{def}\left(s\right)\right)+\beta E\left[f_{1}\left(b,s'\right)|s\right],\begin{array}{c}\max_{\tau,b'}u\left(e^{def}\left(s\right)-\tau\right)+\beta E\left[V\left(b',s'\right)|s\right]\\ s.t\ \ \tau+b'q\left(b',s\right)\geq b\end{array}\right\}\\
\leq\ &\max\left\{u\left(e^{def}\left(s\right)-b_{\min}\right)+\beta V_{\max},V_{\max}\right\}\leq V_{\max},
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{T}_{1}\left(f_{1},f_{2}\right)\left(b,s\right) &= \max\left\{u\left(e^{def}\left(s\right)\right)+\beta E\left[f_{1}\left(b,s'\right)|s\right],\begin{array}{c}\max_{\tau,b'}u\left(e^{def}\left(s\right)-\tau\right)+\beta E\left[V\left(b',s'\right)|s\right]\\ s.t\ \ \tau+b'q\left(b',s\right)\geq b\end{array}\right\}\\
&\geq u\left(e^{def}\left(s\right)\right)+\beta E\left[f_{1}\left(b,s'\right)|s\right]\geq V_{\min}.
\end{aligned}
$$

If s is such that the debtor proposes

$$\begin{aligned}
& \hat{T}_1\left(f_1, f_2\right)(b, s) \\
= \quad & \max \left\{ \begin{array}{c} \max_{\tau, b'} u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ s.t \quad \tau + b'q(b', s) \geq \min\left\{b, \delta E\left[f_2\left(b, s'\right)|s\right]\right\}, \end{array}, u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right] \right\} \\
\geq \quad & u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right] \geq V_{\min},
\end{aligned}$$

and

$$\begin{aligned}
& \hat{T}_1\left(f_1, f_2\right)(b, s) \\
= \quad & \max \left\{ \begin{array}{c} \max_{\tau, b'} u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ s.t \quad \tau + b'q(b', s) \geq \min\left\{b, \delta E\left[f_2\left(b, s'\right)|s\right]\right\}, \end{array}, u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right] \right\} \\
\leq \quad & \max\left\{V_{\max}, u\left(e^{def}(s) - b_{\min}\right) + \beta V_{\max}\right\} \leq V_{\max}. \qquad \square
\end{aligned}$$

Lemma B. 3. *The operator $\hat{T}$ is monotone. That is, if there exists functions $f_1, f_1', f_2, f_2' \in \mathcal{F}(B \times S)$ such that $f_1 > f_1'$ and $f_2' > f_2$ then*

$$\hat{T}_1(f_1, f_2) \geq \hat{T}_1(f_1', f_2') \quad \text{and} \quad \hat{T}_2(f_1, f_2) \leq \hat{T}_2(f_1', f_2').$$

Proof. Take the functions f_1, f_1', f_2, f_2' as given. Fix b and consider a state s in which the debtor proposes. Then it follows immediately that the creditor's value satisfies

$$\hat{T}_2(f_1, f_2)(b, s) = \min\left\{b, \delta E\left[f_2\left(b, s'\right)|s\right]\right\} \leq \min\left\{b, \delta E\left[f_2'\left(b, s'\right)|s\right]\right\} = \hat{T}_2(f_1', f_2')(b, s).$$

For the debtor's value, we have

$$\begin{aligned}
& \hat{T}_1\left(f_1, f_2\right)(b, s) \\
= \quad & \max \left\{ \begin{array}{c} \max_{\tau, b'} u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ s.t \quad \tau + b'q(b', s) \geq \min\left\{b, \delta E\left[f_2\left(b, s'\right)|s\right]\right\}, \end{array}, u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right] \right\} \\
\geq \quad & \max \left\{ \begin{array}{c} \max_{\tau, b'} u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ s.t \quad \tau + b'q(b', s) \geq \min\left\{b, \delta E\left[f_2'\left(b, s'\right)|s\right]\right\}, \end{array}, u(e^{def}(s)) + \beta E\left[f_1'(b, s')|s\right] \right\} \\
= \quad & \hat{T}_1(f_1', f_2')(b, s).
\end{aligned}$$

As this is true for all (b, s), monotonicity holds for this region of the state space.

Now consider s such that the creditor proposes. The debtor's value satisfies

$$\begin{aligned}
& \hat{T}_1\left(f_1, f_2\right)(b, s) \\
= \quad & \max \left\{ u\left(e^{def}(s)\right) + \beta E\left[f_1\left(b, s'\right)|s\right], \begin{array}{c} \max_{\tau, b'} u\left(e^{def}(s) - \tau\right) + \beta E\left[V\left(b', s'\right)|s\right] \\ s.t \quad \tau + b'q\left(b', s\right) \geq b \end{array} \right\} \\
\geq \quad & \max \left\{ u\left(e^{def}(s)\right) + \beta E\left[f_1'\left(b, s'\right)|s\right], \begin{array}{c} \max_{\tau, b'} u\left(e^{def}(s) - \tau\right) + \beta E\left[V\left(b', s'\right)|s\right] \\ s.t \quad \tau + b'q\left(b', s\right) \geq b \end{array} \right\} \\
= \quad & \hat{T}_2\left(f_1', f_2'\right)(b, s).
\end{aligned}$$

Similarly, the creditor's value satisfies

$$\begin{aligned}
& \hat{T}_2\left(f_1, f_2\right)(b, s) \\
= & \max \left\{ \min \left\{ b, \begin{array}{c} \max_{\tau, b'} \tau + b'q(b', s) \\ s.t \quad u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ \geq u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right] \end{array} \right\}, \delta E\left[f_2(b, s')|s\right] \right\} \\
\leq & \max \left\{ \min \left\{ b, \begin{array}{c} \max_{\tau, b'} \tau + b'q(b', s) \\ s.t \quad u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ \geq u(e^{def}(s)) + \beta E\left[f_1(b, s')|s\right] \end{array} \right\}, \delta E\left[f_2'(b, s')|s\right] \right\} \\
\leq & \max \left\{ \min \left\{ b, \begin{array}{c} \max_{\tau, b'} \tau + b'q(b', s) \\ s.t \quad u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ \geq u(e^{def}(s)) + \beta E\left[f_1'(b, s')|s\right] \end{array} \right\}, \delta E\left[f_2'(b, s')|s\right] \right\} \\
= & \hat{T}_2\left(f_1', f_2'\right)(b, s),
\end{aligned}$$

where the last inequality comes from the fact that $f_1' \leq f_1$ which loosens the constraint on the creditor's maximization problem and thus weakly increases the value of the program. □

The proof of existence then follows by applying the $\hat{T}$ operator to a suitable initial f^0 within the space $\mathcal{B}(B \times S)$.

Theorem 2. *An SSP equilbrium exists.*

Proof. Choose $f^0 = (f_1^0, f_2^0)$ such that for all (b, s), $f_1^0(b, s) = V_{\max}$ and $f_2^0(b, s) = b_{\min}$and successively apply the operator $\hat{T}$ to obtain the sequence of functions $\{f^n\}_{n=0}^{\infty}$ where $f^{n+1} = \hat{T} f^n$. By Lemma 4 $\hat{T}$ is monotone, and by Lemma 3 $\hat{T}$ maps $\mathcal{B}(B \times S)$ into itself, so that this is a monotone sequence of functions in $\mathcal{B}(B \times S)$. Hence, the sequence converges to a SSP equilibrium values and by Theorem 2 and Lemma 1 there exists a SSP equilibrium. □

The following theorem provides bounds on the rate at which resources can be transformed into utility, and the rate at which utility can be transformed into resources which, if satisfied, are sufficient to establish uniqueness of this fixed point.

Theorem 3. *Let $u : \mathbf{R} \to \mathbf{R}$ be differentiable. If there exists $K_L > \beta$ and $K_U < 1/\delta$ such that $K_L \leq u'(c) \leq K_U$, for all c, then the SSP equilibrium values are unique.*

Proof. Let $f^1 = (f_1^1, f_2^1)$ and $f^2 = (f_1^2, f_2^2)$ be elements of $\mathcal{B}(B \times S)$. To establish the result, we need to show that there exists a $\gamma \in (0, 1)$ such that

$$\begin{aligned}
& ||\hat{T} f^1 - \hat{T} f^2||_\infty \\
= & \max_{(b,s) \in B \times S} \left\{ \max \left\{ \left| \left(\hat{T} f_1^1\right)(b, s) - \left(\hat{T} f_1^2\right)(b, s) \right|, \left| \left(\hat{T} f_2^1\right)(b, s) - \left(\hat{T} f_2^2\right)(b, s) \right| \right\} \right\} \\
\leq & \gamma \max_{(b,s) \in B \times S} \left\{ \max \left\{ \left| f_1^1(b, s) - f_1^2(b, s) \right|, \left| f_2^1(b, s) - f_2^2(b, s) \right| \right\} \right\} \\
\leq & \gamma ||f^1 - f^2||_\infty.
\end{aligned}$$

The argument varies according to whether the outside offer is taken, no proposal is accepted, or a proposal is accepted.

First, fix (b, s) and consider the case in which s is such that the debtor proposes. If the outside option is taken for both f^1 and f^2, then we have

$$\left|\left(\hat{T}f^1\right)(b,s) - \left(\hat{T}f^2\right)(b,s)\right| = 0,$$

since the creditor's payoff is b, and the debtor's payoff solves

$$\begin{array}{c} \max_{\tau,b'} u(e^{def}(s) - \tau) + \beta E[V(b', s')|s] \\ s.t \ \ \tau + b'q(b', s) \geq b, \end{array}$$

neither or which depends on the continuation values f^1 and f^2.

If no proposal is accepted for both f^1 and f^2, then we have

$$\left|\left(\hat{T}f_2^1\right)(b,s) - \left(\hat{T}f_2^2\right)(b,s)\right| = \left|\delta E\left[f_2^1(b,s')\,|s\right] - \delta E\left[f_2^2(b,s')\,|s\right]\right| \leq \delta||f_2^1 - f_2^2||_\infty,$$

for the creditor's continuation value function, and

$$\begin{aligned} & \left|\left(\hat{T}f_1^1\right)(b,s) - \left(\hat{T}f_1^2\right)(b,s)\right| \\ = & \left|u(e^{def}(s)) + \beta E\left[f_1(b,s')|s\right] - u(e^{def}(s)) - \beta E\left[f_1(b,s')|s\right]\right| \\ = & \ \beta\left|E\left[f_1(b,s')|s\right] - E\left[f_1(b,s')|s\right]\right| \\ \leq & \ \beta\left\|f_1^1 - f_1^2\right\|_\infty . \end{aligned}$$

for the debtor's continuation value function.

If a proposal is accepted for both f^1 and f^2, consider first the case in which s is such that the debtor proposes. In this case, the creditor's continuation values satisfy

$$\left|\left(\hat{T}f_2^1\right)(b,s) - \left(\hat{T}f_2^2\right)(b,s)\right| = \left|\delta E\left[f_2^1(b,s')\,|s\right] - \delta E\left[f_2^2(b,s')\,|s\right]\right| \leq \delta||f_2^1 - f_2^2||_\infty.$$

Using this fact, the debtor's continuation values satisfy

$$\begin{aligned} & \left|\left(\hat{T}f_1^1\right)(b,s) - \left(\hat{T}f_1^2\right)(b,s)\right| \\ = & \left| \begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s) - \tau\right) + \beta E\left[V(b',s')\,|s\right] \\ s.t \ \ \tau + b'q(b',s) \geq \delta E\left[f_2^1(b,s')\,|s\right] \end{array} \right. \\ & \left. - \begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s) - \tau\right) + \beta E\left[V(b',s')\,|s\right] \\ s.t \ \ \tau + b'q(b',s) \geq \delta E\left[f_2^2(b,s')\,|s\right] \end{array} \right| \\ \leq & \left| \begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s) - \tau\right) + \beta E\left[V(b',s')\,|s\right] \\ s.t \ \ \tau + b'q(b',s) \geq \delta E\left[f_2^2(b,s')\,|s\right] + \delta||f_2^1 - f_2^2||_\infty \end{array} \right. \\ & \left. - \begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s) - \tau\right) + \beta E\left[V(b',s')\,|s\right] \\ s.t \ \ \tau + b'q(b',s) \geq \delta E\left[f_2^2(b,s')\,|s\right] \end{array} \right| . \end{aligned}$$

Now suppose that (τ^2, b'^2) attain the maximum for f_2^2. Then exploiting the fact that U is defined over negative consumptions and that its slope is bounded we can find a feasible $\hat{\tau}$ such that

$$\hat{\tau} = \tau^2 + \delta\left\|f_2^1 - f_2^2\right\|_\infty ,$$

yielding

$$\begin{aligned}
&\left|\left(\hat{T}f_1^1\right)(b,s)-\left(\hat{T}f_1^2\right)(b,s)\right| \\
\leq\ & \left|u\left(e^{def}(s)-\tau^2+\delta\left\|f_2^1-f_2^2\right\|_\infty\right)+\beta E\left[V\left(b^2,s'\right)|s\right]-u\left(e^{def}(s)-\tau^2\right)\right. \\
& \left.-\beta E\left[V\left(b^2,s'\right)|s\right]\right| \\
\leq\ & \left|u\left(e^{def}(s)-\tau^2\right)+u'\left(e^{def}(s)-\tau^2\right)\delta\left\|f_2^1-f_2^2\right\|_\infty-u\left(e^{def}(s)-\tau^2\right)\right| \\
\leq\ & \delta K_U\left\|f_2^1-f_2^2\right\|_\infty .
\end{aligned}$$

Next consider the case in which s is such that the creditor proposes. In this case, the debtor's continuation values satisfy

$$\begin{aligned}
&\left|\left(\hat{T}f_1^1\right)(b,s)-\left(\hat{T}f_1^2\right)(b,s)\right| \\
=\ & \left|\beta E\left[f_1^1(b,s')|s\right]-\beta E\left[f_1^2(b,s')|s\right]\right|\leq\beta||f_1^1-f_1^2||_\infty .
\end{aligned}$$

Using this fact, the creditor's continuation values satisfy

$$\begin{aligned}
&\left|\left(\hat{T}f_2^1\right)(b,s)-\left(\hat{T}f_2^2\right)(b,s)\right| \\
=\ & \left|\begin{array}{c}\max_{\tau,b'}\tau+b'q(b',s)\\ s.t\ \ u(e^{def}(s)-\tau)+\beta E[V(b',s')|s]\\ \geq u(e^{def}(s))+\beta E\left[f_1^1(b,s')|s\right]\end{array}-\begin{array}{c}\max_{\tau,b'}\tau+b'q(b',s)\\ s.t\ \ u(e^{def}(s)-\tau)+\beta E[V(b',s')|s]\\ \geq u(e^{def}(s))+\beta E\left[f_1^2(b,s')|s\right]\end{array}\right| \\
\leq\ & \left|\begin{array}{c}\max_{\tau,b'}\tau+b'q(b',s)\\ s.t\ \ u(e^{def}(s)-\tau)+\beta E[V(b',s')|s]\\ \geq u(e^{def}(s))+\beta E\left[f_1^1(b,s')|s\right]\end{array}\right. \\
& \left.-\begin{array}{c}\max_{\tau,b'}\tau+b'q(b',s)\\ s.t\ \ u(e^{def}(s)-\tau)+\beta E[V(b',s')|s]\\ \geq u(e^{def}(s))+\beta E\left[f_1^1(b,s')|s\right]+\beta\left\|f_1^1-f_1^2\right\|_\infty\end{array}\right| .
\end{aligned}$$

Now suppose that (τ^1,b'^1) attain the maximum for f_1^1. Then there exists a $\hat{\tau}$ such that

$$\left|\hat{\tau}-\tau^1\right|\leq\beta K_L\left\|f_1^1-f_1^2\right\|_\infty ,$$

and that $(\hat{\tau},b'^2)$ is feasible for f_1^1 and so

$$\left|\left(\hat{T}f_2^1\right)(b,s)-\left(\hat{T}f_2^2\right)(b,s)\right|\leq\beta\frac{1}{K_L}\left\|f_1^1-f_1^2\right\|_\infty .$$

It remains to consider cases that involve combinations of the outside option, no proposal being accepted, and a proposal being accepted. Suppose the outside option is taken for one of the f^i and no proposal is accepted for f^{-i}. The argument is analogous regardless of whether the debtor proposes or the creditor proposes at s. Without loss of generality we can order the creditor's continuation value functions such that

$$\begin{aligned}
\left|\left(\hat{T}f_2^1\right)(b,s)-\left(\hat{T}f_2^2\right)(b,s)\right| &= \left|b-\delta E\left[f_2^2(b,s')|s\right]\right| \\
&\leq \left|\delta E\left[f_2^1(b,s')|s\right]-\delta E\left[f_2^2(b,s')|s\right]\right| \\
&\leq \delta||f_2^1-f_2^2||_\infty ,
\end{aligned}$$

while for the debtor, if we define

$$V^{oo}(b,s) = \begin{array}{c} \max_{\tau,b'} u(e^{def}(s)-\tau) + \beta E[V(b',s')|s] \\ s.t \quad \tau + b'q(b',s) \geq b, \end{array}$$

we have

$$\begin{aligned} & \left| \left(\hat{T} f_1^1\right)(b,s) - \left(\hat{T} f_1^2\right)(b,s) \right| \\ = \ & \left| V^{oo}(b,s) - u(e^{def}(s)) + \beta E\left[f_1^2(b,s')|s\right] \right| \\ \leq \ & \left| u\left(e^{def}(s)\right) + \beta E\left[f_1^2(b,s')|s\right] - u(e^{def}(s)) - \beta E\left[f_1^1(b,s')|s\right] \right| \\ \leq \ & \beta \left\| f_1^1 - f_1^2 \right\|_\infty . \end{aligned}$$

where the first inequality follows from the fact that the debtor did not take the outside option for f^2 and the fact that the value of the outside option is independent of the continuation values.

Now suppose the outside option is taken for one of the f^i and a proposal is accepted for f^{-i}. If s is such that the debtor proposes, then the argument for the creditor is the same as in the previous case since they earn their autarky value from an accepted proposal. For the debtor, we have

$$\begin{aligned} & \left| \left(\hat{T} f_1^1\right)(b,s) - \left(\hat{T} f_1^2\right)(b,s) \right| \\ = \ & \left| \begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s)-\tau\right) + \beta E\left[V(b',s')|s\right] \\ s.t \quad \tau + b'q(b',s) \geq \delta E\left[f_2^1(b,s')|s\right] \end{array} - V^{oo}(b,s) \right| \\ \leq \ & \left| \begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s)-\tau\right) + \beta E\left[V(b',s')|s\right] \\ s.t \quad \tau + b'q(b',s) \geq \delta E\left[f_2^1(b,s')|s\right] \end{array} - \begin{array}{c} \max_{\tau,b'} u\left(e^{def}(s)-\tau\right) + \beta E\left[V(b',s')|s\right] \\ s.t \quad \tau + b'q(b',s) \geq \delta E\left[f_2^2(b,s')|s\right] \end{array} \right| \\ \leq \ & \delta K_U \left\| f_2^1 - f_2^2 \right\|_\infty , \end{aligned}$$

where the first inequality follows from the fact that the debtor did not take the outside option for f^2. If s is such that the creditor proposes, the argument for the debtor's continuation value function is the same as in the previous case because the debtor receives their autarky value from an accepted proposal. For the creditor, the result follows from an argument similar to the debtor proposer case.

Finally, consider the case where no agreement occurs for f^1 and an agreement occurs for f^2. Non-proposers receive their autarky values in both cases, implying no difference in continuation value functions under the $\hat{T}$ operator. For the proposer, the fact that no agreement is chosen over agreement for f^2 means we can apply the same argument as in the previous case.

Since the result holds for arbitrary (b,s), the operator T is a contraction with modulus

$$\gamma = \max\{\delta, \beta, \delta K_U, \beta/K_L\}. \qquad \square$$

7.B Solution to the Borrowing Problem

In this subsection, we prove the following theorem:

Theorem 4. *Given* $\left(\tilde{V}^D(b,s), \tilde{W}(b,s)\right) \in \mathcal{B}(B \times S)$ *and* $q(b,s) \in \mathcal{Q}(B \times S)$, *there exists a value function for the country,* $V(b,s)$, *and an equilibrium bond price function* $q(b,s) \in$

$\mathcal{Q}(B \times S)$, *that solve the borrowing problem.*

The proof proceeds by establishing the following two Lemmata. The first takes the bond price function as given and establishes the existence of a unique solution to the country's problem.

Lemma B. 4. *Given* $\left(\tilde{V}^D(b,s), \tilde{W}(b,s)\right) \in \mathcal{B}(B \times S)$ *and* $q(b,s) \in \mathcal{Q}(B \times S)$, *there exists a unique solution to the country's borrowing problem,* $V(b,s)$.

Proof. Let $\mathcal{G}(B \times S)$ be the space of all real functions on $B \times S$, bounded below by $\tilde{V}^D(b,s)$, and above by $U(\max_{s \in S} e(s) - b_{\max} + b_{\min}/(1+r^w))/(1-\beta)$. It is straightforward to show that T^V maps $\mathcal{G}(B \times S)$ into itself.

For any $f \in \mathcal{G}(B \times S)$ define the operator T^V by

$$\left(T^V f\right)(b,s) = \max\left\{ \begin{array}{c} \max_{c,b' \in B} U(c) + \beta \sum_{s' \in S} \pi(s'|s) f(b',s'), \\ s.t.\ c - q(b',s) b' \leq e(s) + b. \end{array}, \tilde{V}^D(b,s) \right\}.$$

First, we show that the operator T^V is monotone. Let $f^1, f^2 \in \mathcal{G}(B \times S)$ such that $f^1 \geq f^2$. Then for all (b,s)

$$\begin{aligned} \left(T^V f^1\right)(b,s) &= \max\left\{ \begin{array}{c} \max_{c,b' \in B} U(c) + \beta \sum_{s' \in S} \pi(s'|s) f^1(b',s'), \\ s.t.\ c - q(b',s) b' \leq e(s) + b. \end{array}, \tilde{V}^D(b,s) \right\} \\ &\geq \max\left\{ \begin{array}{c} \max_{c,b' \in B} U(c) + \beta \sum_{s' \in S} \pi(s'|s) f^2(b',s'), \\ s.t.\ c - q(b',s) b' \leq e(s) + b. \end{array}, \tilde{V}^D(b,s) \right\} \\ &\geq \left(T^V f^2\right)(b,s). \end{aligned}$$

Next, we show that the operator T^V satisfies the discounting property. Let $a \in R$. Then for all $f \in \mathcal{G}(B \times S)$ and all (b,s) we have

$$\begin{aligned} & \left|T^V(f+a)(b,s) - T^V(f)(b,s)\right| \\ = & \left|\max\left\{ \begin{array}{c} \max_{c,b' \in B} U(c) + \beta \sum_{s' \in S} \pi(s'|s) f(b',s') + \beta a \\ s.t.\ c - q(b',s) b' \leq e(s) + b \end{array}, \tilde{V}^D(b,s) \right\} - T^V(f)(b,s)\right| \\ \leq & \left|\max\left\{ \begin{array}{c} \max_{c,b' \in B} U(c) + \beta \sum_{s' \in S} \pi(s'|s) f(b',s') \\ s.t.\ c - q(b',s) b' \leq e(s) + b \end{array}, \tilde{V}^D(b,s) \right\} + \beta a - T^V(f)(b,s)\right| \\ = & \beta a. \end{aligned}$$

Hence, T^V is a contraction with modulus β, and there exists a unique fixed point in $\mathcal{G}(B \times S)$. □

The second Lemma constructs a new operator and shows that, in combination with the result of the first Lemma, that the composition of these operators is monotone, and hence that an equilibrium exists.

Lemma B. 5. *Given* $\left(\tilde{V}^D(b,s), \tilde{W}(b,s)\right) \in \mathcal{B}(B \times S)$, *there exists an equilibrium bond price function* $q(b,s) \in \mathcal{Q}(B \times S)$.

Proof. For any $g^n \in \mathcal{Q}(B \times S)$, define the operator T^q as follows. First, given g^n, apply the operator T^V (which is defined for a given g) until convergence to V^n with associated $\left(V^R\right)^n$.

Then define

$$\phi^n(b,s) = \begin{cases} 1 & \text{if } \tilde{V}^D(b,s) > \left(V^R\right)^n(b,s) \\ 0 & \text{if } \tilde{V}^D(b,s) \le \left(V^R\right)^n(b,s) \end{cases},$$

which embodies the behavioral assumption that when indifferent between default and repayment the country always repays, from which can be constructed the default probability

$$p^n(b,s) = \sum_{b\in B, s'\in S} \phi^n(b,s')\,\pi(s'|s),$$

and a new bond price function

$$g^n(b,s) = \frac{1 - p(b,s) + p(b,s)\sum_{s'\in S}\pi(s'|s)\,\tilde{W}(b,s')/b}{1+r^w},$$

which is an element of $\mathcal{Q}(B\times S)$ given the bounds on $\tilde{W}(b,s)$.

Then define the sequence $\{g^n\}_{n=0}^{\infty}$ by applying T^q successively from the initial $g^0 = 1/(1+r^w)$. To see that this is a monotone sequence in $\mathcal{Q}(B\times S)$, note that $g^1 \le g^0$ and moreover that $\phi^n(b,s) = 0$ whenever $b \le 0$. Hence, the interest rate on borrowings is increasing at each stage, while the interest rate on savings is unchanged, and consequently the fixed points of the associated T^V operators are ordered. But this produces an ordered sequence of default probabilities p^n and, given our restriction on $\tilde{W}(b,s)$, a monotonically decreasing sequence of g^n. As this sequence is bounded below by zero, it converges to a fixed point in $\mathcal{Q}(B\times S)$. □

7.C Existence of Equilibrium

Theorem 5. *If the SSP equilibrium values of the bargaining model are unique, then there exists an equilibrium of our borrowing economy.*

Proof. Let $q \in \mathcal{Q}(B\times S)$ and $\left(V^D, W\right) \in \mathcal{B}(B\times S)$. We construct the first part of our mapping, $\mathcal{H}_1\left(V^D, q\right)$ as follows. Fix (b,s) and think of the $q(b',s)$ and $V^D(b,s)$ as a set of N_b+1 parameters for the country's borrowing problem. Let $C(X)$ be the set of continuous and bounded functions defined on $X = [0, 1/(1+r^w)]^{N_b+1} \times [V_{\min}, V_{\max}]$. Let $f \in \mathcal{C}(X)$ and define the operator $\hat{T}^V$ by

$$\left(\hat{T}^V\right) f = \max\left\{\max_{b'\in B} u\left(e(s) - b + b'q(b',s)\right) + \beta E\left[V(b',s')\,|s\right], \tilde{V}^D(b,s)\right\}.$$

Next define $H_1\left(T^V, q\right)$ as the fixed point of the bargaining operator, given a default value of T^V and a bond price of q.

The finiteness of B ensures that a solution to the country's borrowing problem exists, and that it is bounded, while the Theorem of the Maximum implies that $\left(\hat{T}^V\right) f$ is continuous in x. For any $f^1, f^2 \in \mathcal{C}(X)$ analogues of the arguments provided above ensure that the fixed points of the bargaining operator defined on $\mathcal{C}(X)$ are also continuous in X. Select the largest such fixed point. Then the mapping $\mathcal{H}_1\left(V^D, q\right)(b,s)$ is a continuous (and hence upper hemicontinuous) single valued, and hence compact and convex valued, correspondence. From this, we can construct the product correspondence

$$\mathcal{H}_1\left(V^D, q\right) = \prod_{(b,s)\in B\times S} \mathcal{H}_1\left(V^D, q\right)(b,s).$$

By Theorem 17.28 of Aliprantis and Border (2006), this product correspondence is continuous and compact valued.

Now consider the second part of our mapping $\mathcal{H}_2\left(V^D, W, q\right)$ defined as follows. First, think of the $q\left(b', s\right)$, $V^D\left(b, s\right)$ and $W\left(b, s\right)$ as a finite set of parameters for the country's borrowing problem, with each $q\left(b', s\right)$ belonging to the compact interval $[0, 1/\left(1+r^w\right)]$, each $V^D\left(b, s\right)$ belonging to $\left[V_{\min}, V_{\max}\right]$, and each $W\left(b, s\right)$ belonging to $\left[b_{\min}, b_{\max}\right]$. Let $\mathcal{C}\left(X\right)$ be the space of all continuous functions defined on

$$X = [0, 1/\left(1+r^w\right)]^{N_b \times N_e} \times \left[V_{\min}, V_{\max}\right]^{N_b \times N_e} \times \left[b_{\min}, b_{\max}\right]^{N_b \times N_e}.$$

Let $f \in \mathcal{C}\left(X\right)$ and define the operator $\hat{T}^V$ be defined by

$$\left(\hat{T}^V\right) f = \max\left\{\max_{b' \in B} u\left(e\left(s\right) - b + b'q\left(b', s\right)\right) + \beta E\left[V\left(b', s'\right)|s\right], \tilde{V}^D\left(b, s\right)\right\}.$$

As above, the fixed point V is continuous on X; the calculations also define the function $V^R\left(b, s\right)$.

Define the default indicator correspondence

$$\Phi\left(b, s\right) = \begin{cases} 1 & \text{if } \tilde{V}^D\left(b, s\right) > V^R\left(b, s\right) \\ 0 & \text{if } \tilde{V}^D\left(b, s\right) < V^R\left(b, s\right) \\ [0, 1] & \text{if } \tilde{V}^D\left(b, s\right) = V^R\left(b, s\right) \end{cases}.$$

From this we can define a default probability correspondence, $P\left(b', s\right)$, as the set of all $p\left(b', s\right)$ constructed as

$$p\left(b', s\right) = \sum_{s' \in S} \phi\left(b', s'\right) \pi\left(s'|s\right),$$

for some $\phi\left(b', s'; x\right) \in \Phi\left(b', s'; x\right)$., Hence, for any fixed $\left(b', s\right)$ we can define the bond price correspondence from points in X to $[0, 1/\left(1+r^w\right)]$ as

$$\mathcal{H}_2\left(V^D, W, q\right)\left(b', s\right) = \left\{y : y = \frac{1 - p + p\sum_{s' \in S} \pi\left(s'|s\right) \tilde{W}(b', s')/b}{1+r^w} \text{ for some } p \in P\left(b', s\right)\right\},$$

where $\tilde{W}(b', s')$ was defined above.

It is straightforward to show that for $\left(b', s\right)$ and $\left(V^D, W, q\right)$ fixed, this is a closed interval contained in $[0, 1]$. Hence, it is compact valued. A straightforward adaptation of App Lemma 8 from Chatterjee, Corbae, Nakajima and Rios-Rull (2002) shows that it is also upper-hemi continuous. Therefore, viewed as a correspondence from points in X to $[0, 1/\left(1+r^w\right)]$ this is upper-hemi continuous. Then for any $\left(V^D, W, q\right)$, we can define the product correspondence

$$\mathcal{H}_2\left(V^D, W, q\right) = \prod_{(b,s) \in B \times S} \mathcal{H}_2\left(V^D, W, q\right)\left(b, s\right).$$

By Theorem 17.28 of Aliprantis and Border (2006), this product correspondence is continuous and compact valued.

Finally, form

$$\mathcal{H}\left(V^D, W, q\right) = \left[\mathcal{H}_1\left(V^D, q\right), \mathcal{H}_2\left(V^D, W, q\right)\right].$$

By Theorem 17.23 of Aliprantis and Border (2006), $\mathcal{H}$ is upper hemi-continuous. Using the fact that $\mathcal{H}_1$ is single valued, it is also straightforward to show that it is convex valued. Hence, by Kakutani's fixed point theorem there exists a fixed point of $\mathcal{H}$.

Using the fixed points for q^* and V^{D*}, we can then iterate to convergence to find V^*. The collection V^*, V^{D*}, W^* and q^* satisfies the definition for an equilibrium of our borrowing economy, and hence there exists an equilibrium for our borrowing economy. □

8 Appendix C: Data

In this appendix, we tabulate our data on delays and haircuts, and study the relationship between our estimates of haircuts and those computed by other authors. We also discuss the some issues that arise with the use of World Bank debt stock data.

8.A Data on Haircuts

The data on haircuts are presented in Figure 9, for all ninety defaults and settlements. Table 10 then presents the correlations between our measures of haircuts, and those computed by other authors for smaller samples of countries. As shown in Table 10, the correlation with the World Bank and Cline estimates is around 0.9, which presumably follows from the similar sources of data. The correlation with the Sturzenegger and Zettelmeyer preferred estimate (calculated as a debt value weighted average over the estimates for all instruments in a restructuring) is also 0.86. Interestingly, the correlations with the market estimates of Sturzenegger and Zettelmeyer, and with the estimates produced by the Global Committee of Argentine Bondholders, are the smallest.

The differences in estimates result for a number of reasons. One reason is the rate of discounting. Another is that not all estimates subtract "new money" (new loans made as part of a restructuring), although as pointed out by Cline (1995 p. 236), new money typically amounted to less than two per-cent of the debt stock, and should have little impact on the results. Another reason is that some estimates are intended as estimates of the reduction in total debt, rather than just the debts owed to private sector creditors. For example, the World Bank (1993) estimates of "debt reduction equivalents" for nine countries subtract the value of new loans by the official sector. The estimates of the private sector Global Committee of Argentine Bondholders, 2004, were intended as evidence in support of their claim that the restructuring of Argentine debts after the 2001 default was particularly severe. Since the methodology for their computation was not reported, it is not possible to verify whether or not they focused on measures that would tend to understate the estimates. Finally, the World Bank estimates also focus on the reduction in the face value of the debt, which neglects the effect of any extension of the maturity of the loans being rescheduled. The most rigorous measurement is by Sturzenegger and Zettelmeyer (2005, 2007), who provide careful instrument-by-instrument estimates of creditor losses for 246 debts, but for only six defaults, and who are careful to adjust for the effect of maturity extensions. The high correlation between their estimates and ours suggests that this adjustment is often not significant.

Table 10 also presents results for the relationship between delays and the different measures of haircuts. As shown in the table, the range of estimates brackets the one produced for the large sample (0.66). The most reliable estimates, produced by Sturzenegger and Zettelmeyer, have the highest correlation with delays at 0.88.

8.B Data on Debt

In its *Global Development Finance* (GDF) publication, the World Bank publishes estimates of the *face value* of sovereign debt of a country, which is defined to be the sum of all future principal repayments on the debt. This creates a problem when matching the model to the data because different debt contracts with precisely the same payment stream will have different face values depending on the way the payments streams are divided into ‘principal’ and ‘interest’.

To see this in the context of our model, note that we have assumed that all debts take the form of a zero-coupon discount bond. The face value of such a bond is therefore equal to the amount b of payments promised in the next period, since all payments for such a bond are regarded as principal. An alternative contract that produces the same payment stream as these zero-coupon discount bonds would be a bond issued at par (a ‘par-bond’) in the amount $bq(b,s)$ and that bore a coupon of $(1-q(b,s))$ per bond, generating total interest payments of $b(1-q(b,s))$. For such a contract, face value of the debt outstanding would be reported as $bq(b,s)$, which is the market value of the debt. Of course, there are also a continuum of other equivalent contracts that divide debt service into principal and interest in different proportions, and that have face values that lie between $bq(b,s)$ and b.

In the data, over the period we study, there has been a shift away from bank loans, which are typically issued at par, towards bonds issued at a discount. There are also difference in the financing mix across countries. As a consequence, we examine the models implications for *both* the face value *and* the market value of debt, before comparing both to the GDF data.

Table 10: Comparison of Alternate Haircut Estimates

	World Bank (1993)	Cline (1995)	Sturzenegger and Zettelmeyer (2005) Preferred	Sturzenegger and Zettelmeyer (2005) Market	GCAB (2004)
no. obs.	13	17	6	6	17
Correlation with Authors' Estimates	0.87	0.90	0.86	0.77	0.50
Correlation with Delay	0.40	0.55	0.88	0.72	0.42

Figure 9: Data on Delays and Haircuts

			Length (Years)			Haircuts (%)					
	Country	Default	Default	Default	Default	Authors	World Bank	Cline	Sturzenegger & Zettelmeyer (2005)		GCAB
Country	Code	Code	Start	End	Length	Estimate	(1993)	(1995)	Preferred	Market	(2004)
Albania	ALB	ALB91	1991	1995	4.6	38					
Algeria	DZA	DZA91	1991	1996	5.2	0					
Angola	AGO	AGO85	1985	2004	19.0	69					
Argentina	ARG	ARG82	1982	1993	11.2	30	32	29			35
Argentina	ARG	ARG01	2001	2005	3.6	63			55	63	63
Bolivia	BOL	BOL80	1980	1993	12.4	58	78				
Brazil	BRA	BRA83	1983	1994	11.2	21	18	28			35
Bulgaria	BGR	BGR90	1990	1994	4.3	46	44	50			50
Burkina Faso	BFA	BFA83	1983	1996	13.0	61					
Cameroon	CMR	CMR85	1985	2003	18.0	61					
Cape Verde	CPV	CPV81	1981	1996	15.7	46					
Central African Republic	CAF	CAF83	1983	2004	21.0	66					
Chile	CHL	CHL83	1983	1990	7.4	46					
Colombia	COL	COL85	1985	1991	5.3	2					
Costa Rica	CRI	CRI83	1983	1990	6.7	43	62	61			
Croatia	HRV	HRV92	1992	1996	4.0	0					
Dominica	DMA	DMA03	2003	2004	1.0	0					
Dominican Republic	DOM	DOM83	1983	1994	10.9	47	63	50			
Ecuador	ECU	ECU82	1982	1995	12.3	23		45			45
Ecuador	ECU	ECU99	1999	2000	1.7	34			27	60	
Ecuador	ECU	ECU00	2000	2001	1.1	0					40
El Salvador	SLV	SLV81	1981	1996	15.0	64					
Ethiopia	ETH	ETH91	1991	1999	8.1	44					
Gabon	GAB	GAB86	1986	1994	7.4	42					
Gabon	GAB	GAB99	1999	2004	4.7	85					
Gambia	GMB	GMB86	1986	1990	4.2	63					
Guatemala	GTM	GTM89	1989	1989	0.0	14					
Guinea	GNB	GNB86	1986	1988	2.3	8					
Guinea	GNB	GNB91	1991	1998	8.0	14					
Guinea-Bissau	GIN	GIN83	1983	1996	13.0	70					
Guyana	GUY	GUY82	1982	2004	21.5	85		86			
Haiti	HTI	HTI82	1982	1994	12.0	65					
Honduras	HND	HND81	1981	2004	23.0	72					
Ivory Coast	CIV	CIV83	1983	1998	15.2	52					
Ivory Coast	CIV	CIV00	2000	2004	4.0	41					
Jamaica	JAM	JAM87	1987	1993	6.1	60					
Jordan	JOR	JOR89	1989	1993	4.1	44	42	33			35
Kenya	KEN	KEN94	1994	2004	10.0	85					
Macedonia	MKD	MKD92	1992	1997	5.2	60					
Madagascar	MDG	MDG81	1981	2002	20.1	68					
Mauritania	MRT	MRT92	1992	1996	4.7	48					
Mexico	MEX	MEX82	1982	1990	7.9	34	35	30			35
Moldova	MDA	MDA98	1998	1998	0.0	15					
Moldova	MDA	MDA02	2002	2002	0.5	42					
Mongolia	MNG	MNG97	1997	2000	3.0	0					

Figure 9 (Continued): Data on Delays and Haircuts

			Length (Years)			Haircuts (%)					
	Country	Default	Default	Default	Default	Authors	World Bank	Cline	Sturzenegger & Zettelmeyer (2005)		GCAB
Country	Code	Code	Start	End	Length	Estimate	(1993)	(1995)	Preferred	Market	(2004)
Morocco	MAR	MAR86	1986	1990	4.6	42					
Mozambique	MOZ	MOZ83	1983	1992	10.0	57		58			
Myanmar	MMR	MMR97	1997	2003	6.0	43					
Nicaragua	NIC	NIC79	1979	2003	24.0	75					
Niger	NER	NER83	1983	1991	7.9	89		82			
Nigeria	NGA	NGA82	1982	1992	10.4	70	80	49			
Nigeria	NGA	NGA02	2002	2002	0.0	8					
Pakistan	PAK	PAK98	1998	1999	1.6	29			31	30	
Panama	PAN	PAN83	1983	1996	12.7	34					45
Paraguay	PRY	PRY86	1986	1993	7.6	62					
Paraguay	PRY	PRY03	2003	2004	1.4	0					
Peru	PER	PER80	1980	1980	0.9	0					
Peru	PER	PER83	1983	1997	14.4	29					45
Philippines	PHL	PHL83	1983	1992	9.6	35	44	36			
Poland	POL	POL81	1981	1994	12.9	42	58	45			45
Romania	ROM	ROM81	1981	1983	1.5	9					
Russia	RUS	RUS91	1991	1997	6.0	32					
Russia	RUS	RUS98	1998	2000	2.3	32			53	65	38
Rwanda	RWA	RWA95	1995	1995	0.0	0					
Sao Tome and Principe	STP	STP87	1987	1994	7.7	48					
Senegal	SEN	SEN90	1990	1990	0.7	3					
Senegal	SEN	SEN92	1992	1996	5.0	10					
Serbia and Montenegro	SER	SER92	1992	2004	12.0	57					
Seychelles	SYC	SYC00	2000	2002	2.0	12					
Sierra Leone	SLE	SLE86	1986	1995	9.7	85					
Sierra Leone	SLE	SLE97	1997	1998	1.0	51					
Solomon Islands	SLB	SLB96	1996	2004	8.0	90					
South Africa	ZAF	ZAF93	1993	1993	0.7	0					
Sri Lanka	LKA	LKA96	1996	1996	0.0	4					
Tanzania	TZA	TZA84	1984	2004	20.3	63					
Thailand	THA	THA97	1997	1998	0.5	0					
Togo	TGO	TGO91	1991	1997	7.0	66					
Trinidad and Tobago	TTO	TTO88	1988	1989	2.0	4					
Uganda	UGA	UGA80	1980	1993	13.2	90		76			
Ukraine	UKR	UKR98	1998	2000	1.4	1			18	28	
Uruguay	URY	URY90	1990	1991	1.1	16	41	31			
Uruguay	URY	URY03	2003	2003	0.0	0			16	29	
Venezuela	VEN	VEN90	1990	1990	1.0	14	23	20			30
Venezuela	VEN	VEN95	1995	1997	2.0	2					
Venezuela	VEN	VEN98	1998	1998	0.0	0					
Venezuela	VEN	VEN05	2005	2005	0.1	0					
Vietnam	VNM	VNM85	1985	1998	14.0	58					
Yemen	YEM	YEM85	1985	2001	16.5	35					
Zambia	ZMB	ZMB83	1983	1994	10.5	45					
Zimbabwe	ZWE	ZWE00	2000	2004	4.0	19					