Optimal Intermediated Investment in a Liquidity-Driven Business Cycle*

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Abstract

A general equilibrium model of a financial intermediary extends the model first introduced by D. Diamond and P. Dybvig (*JPE,* 1983) to an infinite-horizon environment. This extension enables the relationship between the real business cycle and the composition of assets held in the banking sector to be studied. As in the D-D model, the bank is an optimal financial intermediary coalition here. Moreover, the bank's optimal policy involves decisions about liquidity that vary systematically over the business cycle.

1 Introduction

The asset portfolio of the U.S. banking sector seems to be strongly correlated with the real business cycle. Motivated by this phenomenon, which is clearly reflected in figure 1, we construct here a general equilibrium model of the portfolio decisions of the banking sector. This model provides an explanation of why the liquidity of banks' portfolios varies systematically with the business cycle, and suggests that such variation is economically efficient.

Figure 1[1] shows 2 things about the portfolio of the banking sector. First, the portfolio becomes more liquid throughout the contraction period of the business cycle. Second,

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[1]Figure 1 shows the evolution of the illiquid asset portion of US banking sector from Jan.1973—Aug.2007. Federal Reserve Board H.8. (510) data (commercial banks in the United States, not seasonally adjusted) has been smoothed by an HP filter to produce the figure. We categorize treasury and agency

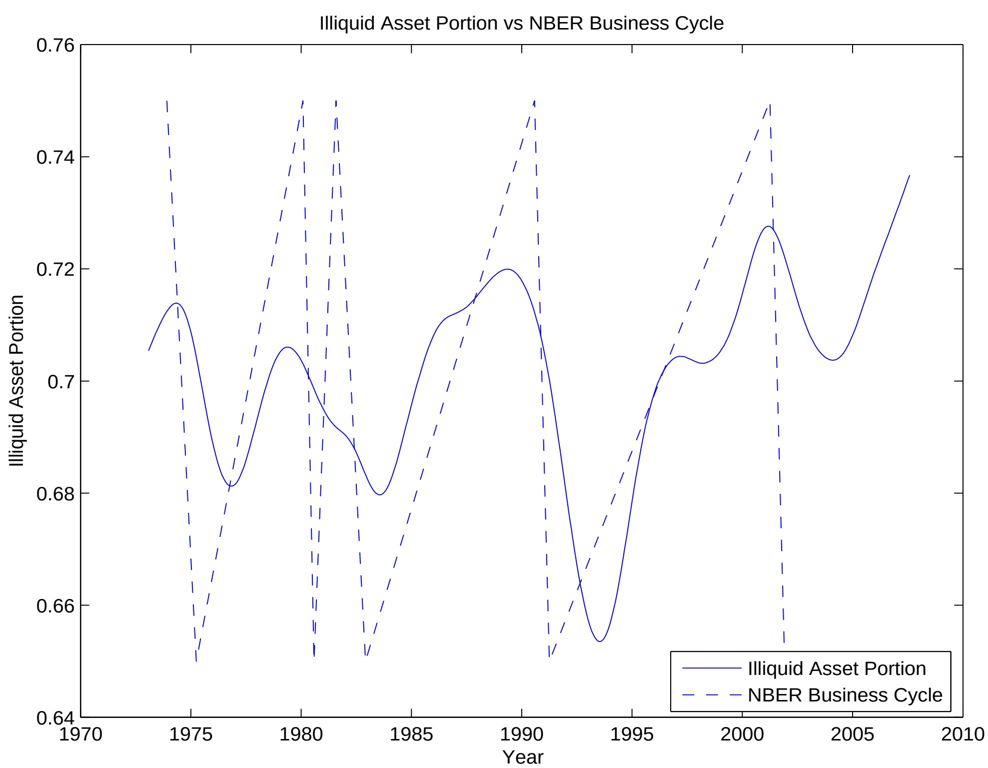

Figure 1: HP-filtered liquidity ratio of bank's portfolio from Jan.1973-Aug.2007.

to close approximation, the part of the sample period during which banks are increasing the liquidity of their portfolio strictly includes the NBER contraction episodes. In fact, between January 1973 and November 2001, banks increased the fraction of their portfolio held in liquid form during a total of 10 years and 1 month, while the NBER contractions span only 4 years and 6 months in total.

Our model provides a rigorous justification for this observation. It is an infinite-horizon, dynamic, stochastic model with two production technologies. One of these two technologies has a higher gross rate of return than the other does. However, adjusting investment in the higher return technology incurs a convex cost. In contrast, investment in the other technology is freely adjustable. We refer to these two technologies as the illiquid technology and the liquid technology respectively.

Section 2 presents a baseline model of an autarkic agent. The agent adjusts his/her portfolio in response to fluctuations in his/her rate of time preference. We show analytically (for a deterministic cycle) and by numerical simulation (for a random preference-shock process) that the agent may transfer some investment to the liquid technology in anticipation of choosing high consumption in the near future.

Section 3 presents the main model of the paper. In this model, a financial intermediary maximizes the aggregate expected discounted utility of a continuum of agents who are identical ex ante. These agents will face a random shock to time preference each period. These shocks are independent across the agents, so time preference risk can be fully diversified in principle. However, consumption preferences of each individual are private information. Patient agents have the incentive to misrepresent themselves to be impatient, even when they actually are patient, in order to collect an insurance indemnity. Since there is no way to detect this misrepresentation, no one would be willing to be a counterparty in such a direct market for insurance.

However, a financial intermediary that offers a deposit contract can provide a partial substitute for explicit insurance. As in Diamond and Dybvig's model, it is economically efficient for a financial intermediary to play this role. Instead of a 2-period model, our model has an infinite horizon. Thus we can use it to study the relationship between the real business cycle and the composition of assets held in the banking sector. We show banks' optimal portfolio evolves a in similar fashion to the autarkic agent's portfolio in the baseline model.

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securities, securities in loans and leases and cash assets as liquid assets. We omit the interbank loans, which are not an asset of the consolidated banking sector. The line segment connects peak and trough of business cycles between Nov 1973–Nov 2001 determined by NBER. Upward slope means expansion period, downward slope means contraction period.

2 Baseline Model

The model presented in this section is an infinite-horizon model with a single agent. There are two production technologies which differ in 2 respects. First, one technology has a higher gross rate of return than the other technology does. Second, the higher-gross-return technology is subject to an investment adjustment cost, from which the lower-gross-return technology is exempt. Thus we call the higher-gross-return technology the "illiquid" technology and the lower-gross-return technology the "liquid" technology. We would like to study the agent's optimal investment portfolio in three cases: the degenerate case in which there is no change in his /her consumption preference; the case in which the consumption preference is a deterministic 2-period cycle; and the case in which the preference shock sequence is a sequence of i.i.d random variables subject to some Bernoulli distribution. In the degenerate case we introduce, the agent will hold either purely liquid or purely illiquid investment at all times. Otherwise, if preferences fluctuate either deterministically or randomly, there will be some time at which the agent will hold both assets.

2.1 Shared Features of Different Versions of the Model

Environment There exists one composite good in the economy which is both the input and output of both technologies. In each period, the goods can be either reinvested in the current technology, or transfered to the other technology, or liquidated for consumption. Time is discrete $t = 1, 2, \cdots$.

Investment Choices There are two production technologies, illiquid and liquid[2], with one period gross return R_I and R_L, where $1 < R_L < R_I$. We use I and L to denote the level of investment in two technologies.

The crucial feature of the production technology is the following. In each period investment return from these two production technologies is divided into 2 parts: the agent's consumption of current period c_t and investment in 2 technologies in the next period I_{t+1} and L_{t+1}. In a typical model which does not have an adjustment cost, the feasibility constraint is given by $R_I I_t + R_L L_t - (c_t + I_{t+1} + L_{t+1}) \geq 0$. While in this model we assume the agent is not free to transfer the goods in or out of illiquid technology, i.e., the rate of transfer is not 1-1. Transfer is subject to a

[2]To understand this, imagine that the composite goods is corn. It can be grown or eaten by the agent. Two production technologies are like 2 farms in this case, one is more productive but far away from where the agent lives, the other one is less productive but very close to where the agent lives. Thus in order to consume the corn from the more productive farm the agent will have to pay transportation fee, while getting corn from the less productive but on site farm, no transportation fee is needed.

convex adjustment cost[3] deduction. The further I_{t+1} deviates from $R_I I_t$, the fewer composite goods there are to be allocated to $c_t + I_{t+1} + L_{t+1}$. In contrast, adjusting the investment level of the liquid technology will have no effect on $c_t + I_{t+1} + L_{t+1}$. Thus the feasible constraint for investment and consumption[4]is given by:

$$\begin{aligned} R_I I_t + R_L L_t - (c_t + I_{t+1} + L_{t+1}) &= \theta(1 - \frac{I_{t+1}}{R_I I_t})^2 I_t, \\ I_{t+1} &\geq 0, \\ L_{t+1} &\geq 0, \end{aligned}$$

where $\theta(1 - \frac{I_{t+1}}{R_I I_t})^2 I_t$ is the adjustment cost function and θ is a positive constant. The adjustment cost function is a convex function of $1 - \dfrac{I_{t+1}}{R_I I_t}$ and it is homogeneous of degree 1 of the investment leve I_t. To rule out trivial solutions of the investment problem, we assume $R_I - \theta < R_L$. Thus the return from liquid technology is not dominated all the time.

Agent There is a single agent in the economy. In the initial period, the agent is endowed with a portfolio (I_1, L_1) which is exogenously invested in 2 technologies, where $I_1 + L_1 = 1$. We use $\log(\cdot)$ as the utility function of the agent. Since consumption is uniquely determined by the sequence of investment portfolio, the agent will choose the optimal portfolio sequence to maximize his /her discounted utility.

2.2 Constant Discount Factor

To provide an intuition before we expand our model, we assume the agent's discount factor is a constant, $\delta_t \equiv \delta \in [0, 1)$. This is a deterministic dynamic programming model. And the optimization problem of the agent is given by the following:

$$\max_{\{I_t, L_t\}_{t=1}^{\infty}} \sum_{t=2}^{\infty} \delta^{t-1} \log(c_t) \tag{1}$$

subject to

$$\begin{aligned} c_t &= R_I I_t + R_L L_t - \theta(1 - \frac{I_{t+1}}{R_I I_t})^2 I_t - I_{t+1} - L_{t+1}; \\ 0 &\leq I_{t+1}; \\ 0 &\leq L_{t+1}; \end{aligned}$$

[3]The adjustment cost here is a physical feature of the production technology, not merely a financial term.

[4]Since the goods liquidated from investment technologies are perishable, the agent will consume all liquidated goods from both technologies.

for all $t > 1$.

In order to study the composition of the agent's optimal investment portfolio, it is helpful to introduce another set of coordinates.

Definition 2.1 *Define B_t to be the portfolio size of the agent $B_t = I_t + L_t$ and q_t to be the liquidity ratio[5] of the portfolio $q_t = \frac{B_t - L_t}{B_t}$.*

Thus we can rewrite the constraints of the agent's problem:

$$\begin{aligned} c_t &= R_I B_t q_t - \theta(1 - \frac{B_{t+1}q_{t+1}}{R_I B_t q_t})^2 B_t q_t + R_L B_t(1 - q_t) - B_{t+1}; && (2) \\ 0 &\leq B_{t+1}; && (3) \\ 0 &\leq q_{t+1} \leq 1; && (4) \end{aligned}$$

for all $t \geq 1$.

Definition 2.2 *The agent's investment portfolio path is in a steady state (b, q) in period t, if for all $\tau \geq t$,*

- $\frac{B_{\tau+1}}{B_\tau} \equiv b > 0$ *and*
- $q_\tau \equiv q \in [0, 1]$.

The following proposition states that when the agent has constant consumption preference, he /she will hold purely liquid investment or purely illiquid investment in the optimal steady state.

Proposition 2.3 *q is 0 or 1 in the optimal steady state of the agent's investment portfolio.*

Proof The Proposition is proved by contradiction. If in the optimal steady state $q_t \equiv q \in (0, 1)$, then there exists a perturbation from the optimal steady state, which improves the agent's consumption.
Suppose the optimal steady state is given by $q \in (0, 1)$ and $b > 0$.

[5]The higher liquidity ratio is, the more illiquid the asset portfolio is.

Since the agent's portfolio sequence $(B_\tau, q_\tau)_{\tau=t}^{\infty}$ is in the optimal steady state, it must satisfy Euler's equation:

$$\begin{aligned}
\frac{1-\frac{2\theta}{R_I}(1-\frac{B_{\tau+1}q_{\tau+1}}{R_I B_\tau q_\tau})q_{\tau+1}}{c_\tau} - \delta\frac{(R_I-\theta+\theta(\frac{B_{\tau+2}q_{\tau+2}}{R_I B_{\tau+1}q_{\tau+1}})^2-R_L)q_{\tau+1}+R_L}{c_{\tau+1}} &= 0 \quad (5)\\
\frac{\frac{2\theta}{R_I}(1-\frac{B_{\tau+1}q_{\tau+1}}{RB_\tau q_\tau})}{c_\tau} + \delta\frac{R_I-\theta+\theta(\frac{B_{\tau+2}q_{\tau+2}}{RB_{\tau+1}q_{\tau+1}})^2-R_L}{c_{\tau+1}} &= 0 \quad (6)
\end{aligned}$$

Using the condition of a steady state, $\frac{B_{\tau+1}}{B_\tau} \equiv b$, and $q_\tau \equiv q$, we rewrite the above two equations:

$$\begin{aligned}
b(1-\frac{2\theta}{R_I}(1-\frac{b}{R_I})q)-\delta((R_I-\theta+\theta(\frac{b}{R_I})^2-R_L)q+R_L) &= 0 \quad (7)\\
b\frac{2\theta}{R_I}(1-\frac{b}{R_I})+\delta(R_I-\theta+\theta(\frac{b}{R_I})^2-R_L) &= 0 \quad (8)
\end{aligned}$$

Since b denotes the growth rate of B_τ, $b < R_I$. Thus we have $\frac{2\theta}{R_I}(1-\frac{b}{R_I}) > 0$. Since b is positive and $q \in (0,1)$, b and q are the solutions to (7) and (8). According to equation (8), $R_I - \theta + \theta(\frac{B}{R_I})^2 - R_L < 0$. Thus if $b > \delta$,

$$\begin{aligned}
R_I-\theta(1-\frac{b}{R_I})^2-R_L &= R_I-\theta+\theta(\frac{b}{R_I})^2-R_L+2\theta\frac{b}{R_I}(1-\frac{b}{R_I})\\
&= (1-\frac{\delta}{b})(R_I-\theta+\theta(\frac{b}{R_I})^2-R_L)\\
&< 0.
\end{aligned}$$

Consider a slight perturbation from steady state (b, q) with $\widetilde{q} = (1-\varepsilon)q$, where $\varepsilon > 0$. Now compare $\widetilde{c}_t$ when $q_t \equiv \widetilde{q}$ with c_t when $q_t \equiv q$. Since gross return from both production technologies are homogeneous of degree 1, without loss of generality we may assume $B_t = 1$. Then

$$\widetilde{c}_t - c_t = (-\varepsilon)(R_I - \theta(1-\frac{b}{R_I})^2 - R_L) > 0.$$

This perturbation can be applied again: let $\widetilde{\widetilde{q}} = (1-2\varepsilon)q$ and so on until $q \equiv 0$. Thus in the optimal steady state $q = 0$.
If $b < \delta$, we can consider a deviation with $\widetilde{q} = (1+\varepsilon)q$. Follow the same logic as above, we can conclude $q = 1$ in the optimal steady state.
If b is the solution to equation (8) and b satisfies $b = \delta$, the agent is indifferent between $q = 0$ or 1.
If the solution to equation (8) is less or equal to 0, then the constraint on q must be binding, $q = 0$ or 1.
So we can conclude that in the optimal steady state $q = 0$ or 1.

2.3 2-Period Cycle Discount Factor

In this section, we assume that the agent's discount factor follows a 2-period cycle, which reflects a cyclical consumption time preference [3]. Let $0 \leq \delta^l < \delta^h < 1$. Assume the discount factor δ_t discounts the utility after period t, in which case given the same portfolio, the agent is more willing to consume that period when $\delta_t = \delta^l$ than when $\delta_t = \delta^h$. Thus we call a period in which the agent has the discount factor δ^l an impatient period, and we call a period in which the agent has the higher discount factor δ^h a patient period.

Without loss of generality, assume period 1 is an impatient period, then

$$\delta_t = \begin{cases} \delta^l & : \text{ t is odd} \\ \delta^h & : \text{ t is even.} \end{cases}$$

where $0 < \delta^l < \delta^h < 1$.

The maximization problem of the agent is:

$$\max_{(I_t, L_t)_{t=2}^{\infty}} \sum_{t=1}^{\infty} (\delta^l)^{\lfloor \frac{t}{2} \rfloor} (\delta^h)^{\lfloor \frac{t-1}{2} \rfloor} \log(c_t) \tag{9}$$

subject to (2) (3) (4), where $\lfloor \cdot \rfloor$ denotes the floor function, i.e. $\lfloor n + \frac{1}{2} \rfloor = n, \forall n \in N$

Since the agent's consumption preference is cyclical, there may exist a cycle in a steady state of agent's investment portfolio.

Definition 2.4 *The agent's investment portfolio path is in a steady state*$((b^l, q^l), (b^h, q^h))$ *in period* t*, if* $\forall \tau \geq \frac{t}{2}$

- $\frac{B_{2\tau+1}}{B_{2\tau}} \equiv b^l > 0, \frac{B_{2\tau+2}}{B_{2\tau+1}} \equiv b^h > 0;$ *and*
- $q_{2\tau+1} \equiv q^l \in [0, 1], q_{2\tau+2} \equiv q^h \in [0, 1].$

Assumption 2.5 *Assume* $R_I - R_L < \theta < \frac{R_I^2}{R_I - 1}$.

For example if $R_I = 1.2$ we only require $0.02 < \theta < 7.2$.

Proposition 2.6 *If assumption 2.5 is satisfied, then the liquidity ratio is identically* 0 *or* $0 < q^l \leq q^h = 1$ *in the optimal steady state.*

Lemma 2.7 *For any $t > 0$, if $q_t = 0$, then $q_\tau = 0$ for all $\tau \geq t$.*

Proof According to the adjustment function, if $q_t = 0$ and $q_{t+1} > 0$, then the adjustment cost in period $t+1$ is $+\infty$, which is a contradiction. Therefore, q_{t+1}.

Lemma 2.8 *In the optimal steady state, if the liquidity ratio q_t is not identically 0, then either $q^l = 1$ or $q^h = 1$.*

Proof We prove this lemma by contradiction.
Suppose $0 < q^l < 1$ and $0 < q^h < 1$.

Here we only consider when the portfolio path is in the optimal steady state. Thus $q_t = q^l$ if t is odd or $q_t = q^h$ if t is even.

For convenience, we introduce the following notation: define c^l to be the consumption in an impatient period with portfolio size $B = 1$.

$$c^l = R_I q^l + R_L(1 - q^l) - \theta(1 - \frac{b^h q^h}{R_I q^l})^2 q^l - b^h.$$

Correspondingly define c^h to be the consumption in a patient period with portfolio size $B = 1$.

$$c^h = R_I q^h + R_L(1 - q^h) - \theta(1 - \frac{b^l}{R_I q^h})^2 q^h - b^l.$$

Since the portfolio path is in the optimal steady state with $q^l < 1$ and $q^h < 1$, consider Euler's equation of this problem:

$$\frac{1 - \frac{2\theta}{R_I}(1 - \frac{b^l q^l}{R_I q^h})q^l}{c^h} = \delta^h \frac{(R_I - \theta + \theta(\frac{b^h q^h}{R_I q^l})^2 - R_L)q^l + R_L}{b^l c^l}; \tag{10}$$

$$\frac{1 - \frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I q^l})q^h}{c^l} = \delta^l \frac{(R_I - \theta + \theta(\frac{b^l q^l}{R_I q^h})^2 - R_L)q^h + R_L}{b^h c^h}; \tag{11}$$

and

$$\frac{\frac{2\theta}{R_I}(1 - \frac{b^l q^l}{R_I q^h})}{c^h} + \delta^h \frac{(R_I - \theta + \theta(\frac{b^h q^h}{R_I q^l})^2 - R_L)}{b^l c^l} = 0; \tag{12}$$

$$\frac{\frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I q^l})}{c^l} + \delta^l \frac{(R_I - \theta + \theta(\frac{b^l q^l}{R_I q^h})^2 - R_L)}{b^h c^h} = 0. \tag{13}$$

Multiply equation (12) by q^l and equation (13) by q^h and add the resulting equations to equations (10) and (11) respectively:

$$\frac{1}{c^h} = \delta^h \frac{R_L}{b^l c^l}; \tag{14}$$
$$\frac{1}{c^l} = \delta^l \frac{R_L}{b^h c^h}. \tag{15}$$

Substituting equation (14) and (15) into equation (12) and (13) correspondingly:

$$\frac{2\theta}{R_I}(1 - \frac{b^l q^l}{R_I q^h}) + \frac{1}{R_L}(R_I - \theta + \theta(\frac{b^h q^h}{R_I q^l})^2) - 1 = 0; \tag{16}$$
$$\frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I q^l}) + \frac{1}{R_L}(R_I - \theta + \theta(\frac{b^l q^l}{R_I q^h})^2) - 1 = 0. \tag{17}$$

By subtracting equation (16) from (17), we conclude

$$\frac{b^h q^h}{q^l} = \frac{b^l q^l}{q^h}.$$

Then according to the expression of c^l and c^h:

$$c^l = (R_I - R_L - \theta(1 - \frac{b^h q^h}{R_I q^l})^2)q^l + R_L - b^h \tag{18}$$
$$c^h = (R_I - R_L - \theta(1 - \frac{b^l q^l}{R_I q^h})^2)q^h + R_L - b^l. \tag{19}$$

Depending on whether $R_I - R_L - \theta(1 - \frac{b^l q^l}{R_I q^h})^2$ is positive or negative, simultaneously increasing and decreasing q^l or q^h will increase the consumption of the agent in both odd and even periods. Since $q_t \in [0, 1]$,if decreasing q improves the agent's consumption, then $q^l = q^h = 0$, while if increasing q improves the consumption, then either q^l or q^h is 1.

The next lemma states that if the agent holds purely illiquid investment in an impatient period, then he/she holds purely illiquid investment in a patient period. Intuitively, with a convex adjustment cost, the agent tries to liquidate assets in illiquid technology gradually over 2 periods rather than all at once. Specifically, the agent starts to decrease the illiquid investment level in patient period and consumes up all liquid assets in patient periods. Thus $q^h = 1$ and $q^l < 1$ would imply the agent liquidate excess the illiquid investment in an impatient period and consume all the liquid assets in a patient period which contradicts the intuition.

Lemma 2.9 *If assumption 2.5 is satisfied, then in the optimal steady state, $q^l = 1$ implies $q^h = 1$.*

Proof We will prove this lemma by contradiction. Suppose the optimal steady state of the liquidity ratio is given by $q^l = 1$ and $q^h < 1$. Define c^l and c^h to be the same as in the previous proof.

$$\frac{1 - \frac{2\theta}{R_I}(1 - \frac{b^l}{R_I q^h})}{c^h} = \delta^h \frac{R_I - \theta + \theta(\frac{b^h q^h}{R_I})^2}{b^l c^l}; \tag{20}$$

$$\frac{1 - \frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I})q^h}{c^l} = \delta^l \frac{(R_I - \theta + \theta(\frac{b^l}{R_I q^h})^2 - R_L)q^h + R_L}{b^h c^h}; \tag{21}$$

$$\frac{\frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I})}{c^l} + \delta^l \frac{(R_I - \theta + \theta(\frac{b^l}{R_I q^h})^2 - R_L)}{b^h c^h} = 0. \tag{22}$$

As before we have

$$\frac{1}{c^l} = \delta^l \frac{R_L}{b^h c^h} \tag{23}$$

Since the constraint on q^l is binding

$$\frac{1}{c^h} > \delta^h \frac{R_L}{b^l c^l} \tag{24}$$

By (23) and (24) we can conclude:

$$(\frac{c^h}{c^l})^2 < \frac{\delta^l b^l}{\delta^h b^h} \tag{25}$$

Consider the following 2 cases:

1) Assume $b^h q^h < \dfrac{b^l}{q^h}$.

According to equation (22) and (23), we have:

$$\frac{2\theta}{R_I}(1 - \frac{b^h q^h}{R_I}) + \frac{R_I - \theta - R_L}{R_L} + \frac{\theta}{R_L}(\frac{b^l}{R_I q^h})^2 = 0 \tag{26}$$

By assumption,

$$\frac{2\theta}{R_I}(1 - \frac{b^l}{R_I q^h}) + \frac{\theta}{R_L}(\frac{b^l}{R_I q^h})^2 + \frac{R_I - \theta - R_L}{R_L} < 0 \tag{27}$$

Let $x = \dfrac{b^l}{q^h}$, and rewrite inequality (27) as follows:

$$\frac{\theta}{R_L}(\frac{x}{R_I})^2 + \frac{2\theta}{R_I}(1 - \frac{x}{R_I}) + \frac{R_I - \theta - R_L}{R_L} < 0 \tag{28}$$

According to assumption 2.5, $\theta < \dfrac{R_I^2}{R_I - 1} < \dfrac{R_I^2}{R_I - R_L}$. Thus the discriminant of inequality (28) is

$$\frac{4\theta}{R_I^4 R_L^2}(R_I - R_L)(\theta(R_I - R_L) - R_I^2) < 0,$$

and so there are no solutions to inequality (28), which is a contradiction.

2)Assume $b^h q^h \geq \dfrac{b^l}{q^h}$.

According to equation (20) and inequality (24),

$$1 - \frac{2\theta}{R_I}(1 - \frac{b^l}{R_I q^h}) > R_I - \theta + \theta(\frac{b^h q^h}{R_I})^2.$$

Let $x = b^h q^h$. The condition $b^h q^h \geq \dfrac{b^l}{q^h}$ implies:

$$\theta(\frac{x}{R_I})^2 - \frac{2\theta}{(R_I)^2}x + (R_I - \theta - 1 + \frac{2\theta}{R_I}) < 0. \qquad (29)$$

By assumption 2.5, $\theta < \dfrac{R_I^2}{R_I - 1}$. Thus the discriminant of inequality (29) is

$$\frac{4\theta}{R_I^2}[\theta(\frac{1}{R_I} - 1)^2 - (R_I - 1)] < 0.$$

Again there are no solutions to this inequality (29), which is a contradiction. Thus we can conclude that $q^l = 1$ implies $q^h = 1$. This completes the proof.

To find out when the liquidity ratio has a nontrivial cycle in the optimal steady state, we need an auxiliary problem of the bank: in this problem, assume the illiquid technology is the only investment choice, thus B_t is composed of purely illiquid investment.

The auxiliary problem can be described as follows:

$$\max_{(B_t)_{t=2}^{\infty}} \sum_{t=1}^{\infty} (\delta^l)^{\lfloor \frac{t}{2} \rfloor} (\delta^h)^{\lfloor \frac{t-1}{2} \rfloor} \log(c_t) \qquad (30)$$

subject to

$$\begin{aligned} c_t &= R_I B_t - \theta(1 - \frac{B_{t+1}}{R_I B_t})^2 B_t - B_{t+1} \\ 0 &\leq B_{t+1} \end{aligned}$$

for all $t \geq 1$. [6]

Since the gross returns from both technologies are homogeneous of degree 1, it is sufficient to consider the case with initial investment $B_1 = 1$.

Definition 2.10 *The agent's investment portfolio path of the auxiliary problem is in a steady state* (x^l, x^h) *in period* t*, if* $\forall \tau \geq \frac{t}{2}$

- $\frac{B_{2\tau+1}}{B_{2\tau}} \equiv x^l > 0,\ \frac{B_{2\tau+2}}{B_{2\tau+1}} \equiv x^h > 0$

The following theorem provides a sufficient condition for the existence of a nontrivial case with a cyclical liquidity ratio in the optimal steady state. It is proved that if a perturbation from the investment with purely illiquid asset is profitable, then the liquidity ratio in the optimal steady state is cyclical with $q^h = 1$ and $q^l < 1$.

Theorem 2.11 *Let* (x^l, x^h) *be the optimal auxiliary steady state we defined in 2.10. If*

$$x^l \frac{-\frac{2\theta}{R_I}(1 - \frac{x^l}{R_I})}{R_I - \theta(1 - \frac{x^l}{R_I})^2 - x^l} - \delta^h \frac{(R_I - \theta + \theta(\frac{x^h}{R_I})^2 - R_L)}{R_I - \theta(1 - \frac{x^h}{R_I})^2 - x^h} > 0, \tag{31}$$

and assumption 2.5 are satisfied, then the liquidity ratio is cyclical with $q^l < 1$ *and* $q^h = 1$ *for all* $t \geq 1$ *in the optimal steady state .*

Proof Suppose $q^l = q^h = 1$, so the optimal steady state of the agent's original problem coincides with the optimal steady state of the auxiliary problem. According to definition 2.10, if the investment portfolio path is in the optimal auxiliary steady state in period t, then $x^l = \frac{B_{2\tau+1}}{B_{2\tau}}$ and $x^h = \frac{B_{2\tau+2}}{B_{2\tau+1}}$ for $2\tau > t$.
Since the gross returns from both production technologies are homogeneous of degree 1, without loss of generality, we may assume $B_{2\tau} = 1$. Consider a two-period perturbation from the optimal auxiliary steady state with $B_{2\tau+1} = x^l, \widetilde{q}_{2\tau+1} = 1 - \varepsilon$, $B_{2\tau+2} = x^l x^h$, and $\widetilde{q}_{2\tau+2} = 1$, for a given $\varepsilon > 0$.
Then

$$c_{2\tau} = R_I - \theta(1 - \frac{x^l}{R_I})^2 - x^l$$

and the consumption in period 2τ after the perturbation is given by

$$\widetilde{c}_{2\tau} = R_I - \theta(1 - \frac{x^l(1-\varepsilon)}{R_I})^2 - x^l.$$

[6] $\lfloor \cdot \rfloor$ is the floor function defined previously.

The difference in consumption due to the perturbation is

$$\widetilde{c}_{2\tau} - c_{2\tau} = -x^l \varepsilon \frac{2\theta}{R_I}(1 - \frac{x^l}{R_I}) + O(\varepsilon^2); \tag{32}$$

Similarly, we have $c_{2\tau+1}$ and $\widetilde{c}_{2\tau+1}$ as follows:

$$c_{2\tau+1} = x^l[R_I - \theta(1 - \frac{x^h}{R_I})^2 - x^h],$$

$$\widetilde{c}_{2\tau+1} = x^l[R_I(1-\varepsilon) + R_L\varepsilon - \theta(1 - \frac{x^h}{R_I(1-\varepsilon)})^2(1-\varepsilon)] - x^l x^h.$$

So

$$\widetilde{c}_{2\tau+1} - c_{2\tau+1} = x^l \varepsilon[-R_I - \theta + \theta(\frac{x^h}{R_I})^2 + R_L] + O(\varepsilon^2) \tag{33}$$

According to inequality (31),

$$\frac{\widetilde{c}_{2\tau} - c_{2\tau}}{c_{2\tau}} + \delta^h \frac{(\widetilde{c}_{2\tau+1} - c_{2\tau+1})}{c_{2\tau+1}} > 0,$$

In order to compare $[\log(\widetilde{c}_{2\tau}) - \log(c_{2\tau})] + \delta^h[\log(\widetilde{c}_{2\tau+1}) - \log(c_{2\tau+1})]$, we use the first order approximation:

$$[\log(\widetilde{c}_{2\tau}) - \log(c_{2\tau})] + \delta^h[\log(\widetilde{c}_{2\tau+1}) - \log(c_{2\tau+1})] = [\frac{\widetilde{c}_{2\tau} - c_{2\tau}}{c_{2\tau}} + \delta^h \frac{(\widetilde{c}_{2\tau+1} - c_{2\tau+1})}{c_{2\tau+1}}]\varepsilon + O(\varepsilon^2) > 0.$$

Since the amount of $B_{2\tau+2}$ is not affected, this perturbation can be applied to any 2-period cycle. Thus it generates an investment portfolio path with which the agent will have a higher consumption than in the investment portfolio path generated by the optimal steady state of the auxiliary problem. This is a contradiction. Thus we can conclude that the liquidity ratio of the optimal steady state is cyclical with $q^l < 1$ and $q^h = 1$.

2.4 Random Discount Factor

In this section we assume that the discount factor $\{\Delta_t\}_{t=1}^{\infty}$ is a sequence of independent identically distributed random variables. For each period t, Δ_t is subject to a Bernoulli distribution. In particular, for all $t > 1$, the support of $\Delta_t = \{\delta^l, \delta^h\}$ and $P(\Delta_t = \delta^l) = p$.[7]

The optimization problem of the agent is:

[7] Since we do not discount the consumption in period 1, without loss of generality, let $\Delta_0 \equiv 1$.

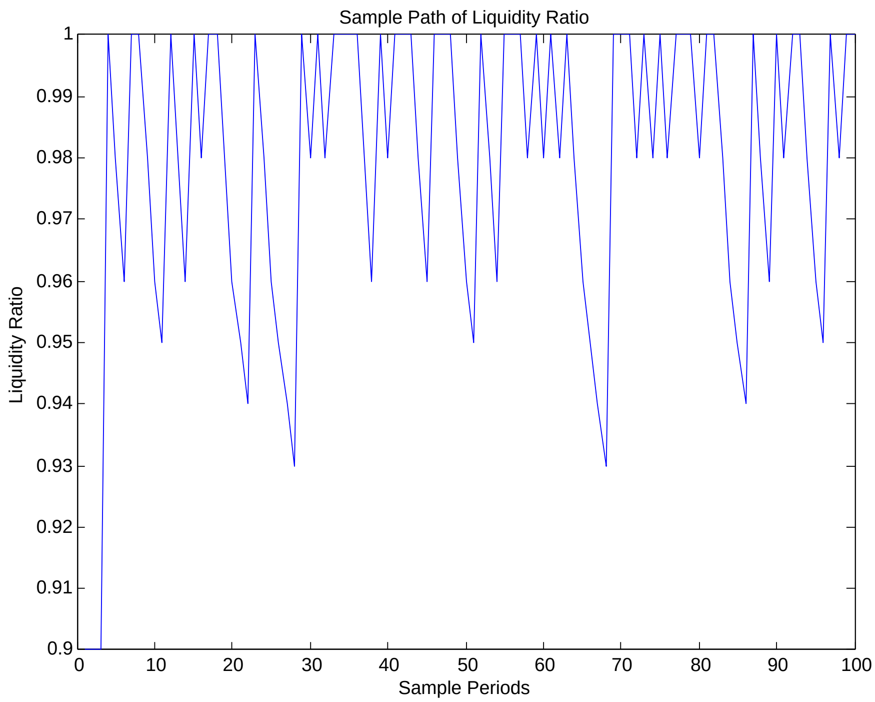

Figure 2: This picture shows A sample path of the liquidity ratio of the agent's investment portfolio for $R_I = 1.2$, $R_L = 1.18$, $\theta = 2$, with $\delta^l = 0.05$, $\delta^h = 0.95$ and $P(\delta_t = \delta^l) = 0.5$.

$$E[\max_{(I_t,L_t)_{t=2}^{\infty}} \sum_{t=1}^{\infty} (\prod_{\tau=0}^{t-1} \Delta_\tau) \log(c_t)] \tag{34}$$

subject to the constraints (2) (3) (4) again.
As the previous section, during a patient period, the agent will cautiously build up liquid investment stock to smooth out the liquidation process. To illustrate the statement, we will use a conceptual numerical example .(See figure 2)

3 Main Model

In this section, we will present the main model of the paper which has the same production technologies as the previous baseline model. With this model, we study how the liquidity of asset portfolio in banking sector varies with the consumption preference of its depositors.

Agents and Bank There are a continuum of individual agents and a financial intermediary which we call a bank. Formally let the measure space $(A, \mathcal{A}, \mu)$ be the space of agents. Assume $\mu(\cdot)$ is a non-atomic measure with $\mu(A) = 1$.

Each agent in this model is specified in the same way as the agent in the single agent economy. In every period, each agent will face a random consumption-preference shock, which is independent of the other agents' consumption-preference shocks. For given $t \geq 1$, Δ_t^α has a Bernoulli distribution with support $\{\delta^l, \delta^h\}$ and $P(\Delta_t^\alpha = \delta^l) = p_t$. [8] Assume[9] for every period t, the measure of agents with discount factor δ^l

$$\mu\{\alpha | \Delta_t^\alpha = \delta^l\} = p_t. \tag{35}$$

Each agent privately learns his/her own preference shock at the beginning of each period t. Agents who have $\Delta_t = \delta^l$ are called impatient agents in period t, since they value future consumption less than those agents with discount factor δ^h in period t. Correspondingly, agents with discount factor δ^h are called patient agents.

Agents are risk averse and would potentially gain from sharing risk with one another. From an ex post perspective, insurance against such shocks is a subsidy provided

[8] Later in this section, we will consider 2 different cases of p_t: first, we let p_t be a constant p for all periods; second, assume p_t is subject to a deterministic cycle with $p_t = p_1$ when t is an odd number, and $p_t = p_2$ when t is an even number.

[9] This in the same spirit of the law of the large number and its logical consistency with probability theory is shown by Robert M Anderson.[1]

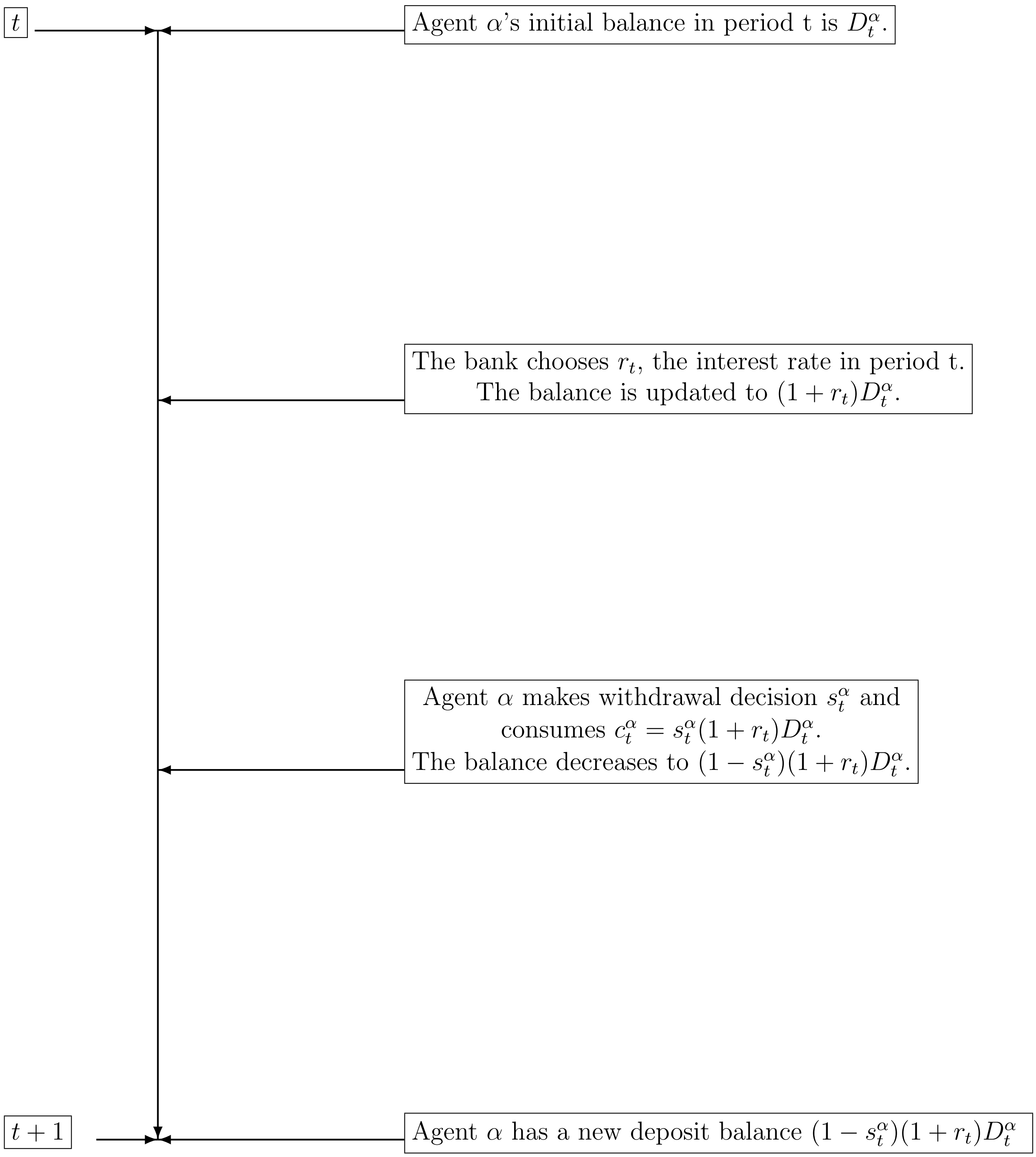

Figure 3: Sequence of events in period t.

by patient agents to impatient agents. The opportunities for mutually beneficial trade are restricted, however by an incentive problem due to private information. Specifically, an individual preference shock is private information of that agent, thus agents can collect the subsidy by claiming to be impatient, and no one can detect whether this is a truthful report. Therefore, such an insurance market will not work since truthful reporting violates agents' incentive constraints. In contrast, the bank can provide at least partial insurance in an incentive compatible way. So agents will only make contracts with the bank.

At the the beginning of the initial period, each agent is endowed with a portfolio (I_1, L_1) of assets which are exogenously invested in 2 production technologies. At the same time, the agent will make an irreversible choice whether to retain the endowment portfolio and manage it directly themselves or to transfer it to the bank in return for a deposit balance D_1.[10] This deposit will accumulate interest and agents can subsequently withdraw from it to satisfy their consumption needs.

Figure 3 shows the sequence of events that happen in period t. Let D_t^α be the deposit balance of agent α at the beginning of period t, and let s_t^α be the withdrawal decision of agent α, which is the fraction of the deposit balance that agent α wants to withdraw in period t, so $0 \leq s_t^\alpha \leq 1$. After each agent α learns his/her consumption preference of that period, he/she comes to the bank and withdraws a fraction s_t^α of his/her deposit balance to consume. The consumption constraint[11] is

$$c_t^\alpha = (1 + r_t) s_t^\alpha D_t^\alpha.$$

The law of motion of the deposit balance is

$$D_{t+1}^\alpha = (1 - s_t^\alpha)(1 + r_t) D_t^\alpha.$$

Since the bank is mutually owned by its depositors, it makes investment decisions to maximize the aggregate expected discounted of their utilities. Let $(\bar{I}_t, \bar{L}_t)$ be the investment portfolio of the bank in period t. To meet the withdrawal demand, the bank will adjust its investment portfolio to $(\bar{I}_{t+1}, \bar{L}_{t+1})$.

Since all agents are identical ex ante, they will make the same decision whether or not they would transfer their endowment to the bank. If an agent chooses to

[10]Due to the parametric specification of the model, there is no loss of generality that an agent choose to transfer all endowment or none of it to the bank and assuming deposits are only accepted at the beginning of the initial period (before agents learn their own consumption type of period 1).

[11]As in the previous model, assume that goods withdrawn from the bank are perishable and since if agents choose to transfer their endowments to the bank, then they will not invest directly any more. Thus agents will consume all withdrawn goods.

retain his/her endowment portfolio rather than exchange it for a bank deposit, then his/her investment decision is the same as the single agent economy. If an agent chooses to transfer his/her endowment to the bank, then the relevant optimization problem is

$$\max_{\{s_t\}_{t=1}^{\infty}} E[\sum_{t=1}^{\infty}(\prod_{\tau=0}^{t-1}\Delta_\tau)\log\left(s_t D_t(1+r_t)\right)], \tag{36}$$

subject to

$$0 \leq s_t \leq 1$$

for all $t \geq 1$.

Since the utility function is given by a logarithmic function, the withdrawal decisions of the agent are proportional to his/her wealth level. Thus s_t only depends on the discount factor in period $t : s_t \in \{s^l, s^h\}$, where s^l (s^h) is the optimal withdrawal decision when being impatient (patient.)

Given the withdrawal decisions of depositors, the bank will maximize the aggregate discounted utility values by choosing an optimal sequence of interest rates.[12] The interest rate[13] given by the bank is uniquely determined by the bank's investment portfolio,

$$(1+r_t)\int_A s_t^\alpha D_t^\alpha d\alpha = R_I\overline{I}_t - \theta(1-\frac{\overline{I}_{t+1}}{R_I\overline{I}_t})^2\overline{I}_t + R_L\overline{L}_t - \bar{I}_{t+1} - \overline{L}_{t+1}.$$

Thus the maximization problem of the bank is given by:

$$\max_{\{\bar{I}_t,\bar{L}_t\}_{t=2}^{\infty}} \int_A \sum_{t=1}^{\infty}\left(\prod_{\tau=0}^{t-1}\Delta_\tau^\alpha\right)\log(c_t^\alpha)\, d\alpha, \tag{37}$$

subject to

$$\begin{aligned}
c_t^\alpha &= s_t^\alpha D_t^\alpha(1+r_t); \\
(1+r_t)\int_A s_t^\alpha D_t^\alpha d\alpha &= R_I\overline{I}_t - \theta(1-\frac{\overline{I}_{t+1}}{R_I\overline{I}_t})^2\overline{I}_t + R_L\overline{L}_t - \bar{I}_{t+1} - \overline{L}_{t+1}; \\
0 &\leq \bar{I}_{t+1}; \\
0 &\leq \bar{L}_{t+1}
\end{aligned}$$

for all $t \geq 1$.

[12]With this payment mechanism, just as [2], the run equilibrium still exists. In this paper we will focus on the welfare of the no run equilibrium, we do not discuss the bank run equilibrium here.

[13]As in the baseline model, goods adjusted from investment are perishable and this constraint is binding in the optimal case.

In the remainder of this section we first show that transferring the endowment to the bank is beneficial to an individual agent and then study the evolution of the bank's optimal investment portfolio of two variations of p_t.

3.1 Direct investing vs Depositing in the bank

Proposition 3.1 *The expected discounted utility level that agents can achieve by depositing in the bank is at least as high as that which they would achieve by investing directly.*

Proof It is sufficient to show that a deviation from directly investing to depositing in the bank will not lower the agents' consumption .
Since all agents are identical ex ante, they will make the same decision at the initial period. Assume agents will only deposit in the bank for the first period and then stop depositing by taking back the same investment portfolio as they would have when investing directly. This deviation is feasible due to the following:

Since all agents transfer their endowments to the bank at the beginning of period 1, $\bar{I}_1 = \int_A I_1 d\alpha = I_1$, and $\bar{L}_1 = \int_A L_1 d\alpha = L_1$, which means that the bank will start with the same initial portfolio as any individual agent.
If agents choose to directly invest, then all those agents with the same discount factor in period 1 will make the same optimizing investment choice. Let $(I_2^l, L_2^l), (I_2^h, L_2^h)$ be the optimal investment choice of an individual agent with discount factor δ^l and δ^h respectively in the direct investing case.
Consider the following bank's suboptimal investment choice:

$$\bar{I}_2 = pI_2^l + (1-p)I_2^h,$$

and

$$\bar{L}_2 = pL_2^l + (1-p)L_2^h.$$

This is a feasible investment plan since

$$\begin{aligned}
\bar{I}_2 + \bar{L}_2 &= (pI_2^l + (1-p)I_2^h) + (pL_2^l + (1-p)L_2^h) \\
&= R_I I_1 + R_L L_1 - (p\theta(1 - \frac{I_2^l}{R_I I_1})^2 + (1-p)\theta(1 - \frac{I_2^h}{R_I I_1})^2) I_1 - pc_1^l - (1-p)c_1^h \\
&\leq R_I I_1 + R_L L_1 - \theta(1 - \frac{pI_1^l + (1-p)I_2^h}{R_I I_1})^2 I_1 - pc_1^l - (1-p)c_1^h
\end{aligned}$$

with equality only when $I_2^l = I_2^h$.
Also

$$\bar{c}_1 = R_I I_1 + R_L L_1 - \theta(1 - \frac{pI_1^l + (1-p)I_2^h}{R_I I_1})^2 I_1 - (\bar{I}_2 + \bar{L}_2),$$

and so we can conclude $\bar{c}_1 \geq pc_2^l + (1-p)c_2^h$.
The above deviation allows agents to take back the same investment portfolio as they directly invest, and to obtain at least the same level of consumption in period 1 as they would have in the first period by investing directly.
Depositing in the bank for a longer time is the same as applying a similar deviation in period 2, period 3 etc. Thus we proved that depositing in the bank will allow agents to have at least the same consumption as they would achieve by investing directly.

Example 3.2 *Let $R_I = 1.2$, $R_L = 1.18$, $\theta = 2$, $\delta^l = 0.05$, $\delta^h = 1.0$ and $P(\delta_t = \delta^l) = 0.05$. All agents are endowed with investment portfolio $(I_1, L_1) = (1, 0)$. By numerical simulation, the expected discounted utility value of the autarkic case is -6.8804, while the expected discounted utility value of depositing in the bank is -0.2551, which is higher than the autarkic case.*[14]

3.2 Bank's Optimal Portfolio

3.2.1 Constant p_t

In this section, we consider the simplest case of p_t, $P(\delta_t = \delta^l) = p_t \equiv p$. This implies $\mu\{\alpha \in A | \delta_t^\alpha = \delta^l\} \equiv p$ for all t.
For the convenience of studying the bank's optimal portfolio, as in the single agent economy, we will change the coordinates.

Definition 3.3 *Define $\bar{B}_t$ to be the portfolio size of the bank in period t,*

$$\bar{B}_t = \bar{I}_t + \bar{L}_t$$

and define $\bar{q}_t$ to be the liquidity ratio of the bank's portfolio,

$$\bar{q}_t = \frac{\bar{B}_t - \bar{L}_t}{\bar{B}_t}.$$

Thus $\bar{I}_t = \bar{B}_t\bar{q}_t$ and $\bar{L}_t = \bar{B}_t(1-\bar{q}_t)$. With the new set of coordinates $(\bar{B}_t, \bar{q}_t)$, we rewrite the aggregate consumption constraint as follows:

[14] All formulae for computing the expected discounted value of agents are provided in the appendix. The interest rate is given by $\frac{\bar{B}_{t+1}}{\bar{B}_t} = \frac{(1+r_t)\int_A (1-s_t^\alpha) D_t^\alpha d\alpha}{\int_A D_t^\alpha d\alpha}$. The initial deposit balance is given by $\bar{c}_1 = (1+r_1)\int_A s_1^\alpha D_1 d\alpha$. For this example, the withdrawal choice is $s^l = 0.4872$ and $s^h = 0.0453$, the steady state interest rate is $r_t = 0.2301$ and the initial deposit of each agent is $D_1 = 0.6018$.

$$\bar{c}_t = (1+r_t)\int_\alpha s_t^\alpha D_t^\alpha d\alpha = R_I \overline{B}_t \bar{q}_t - \theta(1 - \frac{\overline{B}_{t+1}\bar{q}_{t+1}}{R_I \overline{B}_t \bar{q}_t})^2 \overline{B}_t \bar{q}_t + R_L \overline{B}_t (1-\bar{q}_t) - \overline{B}_{t+1}$$

Since we use a logarithmic utility function in this model, it can be shown that the optimal investment decision of the bank is independent of the distribution of agents' wealth level and withdrawal choices.

Lemma 3.4 *The solution to the bank's problem* (37) *is equivalent to the Simplified Bank's Problem [SBP] stated below:*

$$\max_{\{\overline{B}_t, \bar{q}_t\}_{t=2}^\infty} \sum_{t=1}^\infty \left(\int_A \prod_{\tau=0}^{t-1} \Delta_\tau^\alpha d\alpha \right) \log(\bar{c}_t) \tag{38}$$

subject to

$$\bar{c}_t = R_I \overline{B}_t \bar{q}_t - \theta(1 - \frac{\overline{B}_{t+1}\bar{q}_{t+1}}{R_I \overline{B}_t \bar{q}_t})^2 \overline{B}_t \bar{q}_t + R_L \overline{B}_t (1-\bar{q}_t) - \overline{B}_{t+1} \tag{39}$$

$$0 \leq \bar{I}_{t+1} \tag{40}$$

$$0 \leq \bar{L}_{t+1} \tag{41}$$

for all $t \geq 1$.

Proof Consider an agent α, and let σ_t^α be the ratio of D_t^α to the mean balance of all agents $\overline{D}_t$.[15] By exchanging the order of integration and summation[16], we rewrite the bank's maximizing problem (37) as follows:

$$\max_{\{\overline{B}_t, \bar{q}_t\}_{t=2}^\infty} \sum_{t=1}^\infty \int_A (\prod_{\tau=0}^{t-1} \Delta_\tau^\alpha) \log(\sigma_t^\alpha \bar{c}_t) d\alpha, \tag{42}$$

Since $\log(\sigma_t^\alpha \bar{c}_t) = \log(\sigma_t^\alpha) + \log(\bar{c}_t)$, problem (42) can be written as follows:

$$\max_{\{B_t, q_t\}_{t=2}^\infty} \{ \sum_{t=1}^\infty \left(\int_A (\prod_{\tau=0}^{t-1} \Delta_\tau^\alpha) \log(\sigma_t^\alpha) d\alpha \right) + \sum_{t=1}^\infty \left(\int_A \prod_{\tau=0}^{t-1} \Delta_\tau^\alpha d\alpha \right) \log(\bar{c}_t) \}, \tag{43}$$

We notice the first term in (43) is not affected by the bank's investment choice. This completes the proof of lemma 3.4.

According to the lemma 3.4, the bank's investment choice only depends on the aggregate consumption preference.

[15]Since we normalize $\mu(A) = 1$, the mean balance coincides with the aggregate balance of the agents.

[16]This exchange is justified by the Dominated Convergence Theorem.

Corollary 3.5 *The equivalent functional problem to SBP is*

$$V(\overline{B}_t, \overline{q}_t) = \max_{\{\overline{B}_{t+1}, \overline{q}_{t+1}\}} \{\log(\overline{c}_t) + (p\delta^l + (1-p)\delta^h) V(\overline{B}_{t+1}, \overline{q}_{t+1})\} \tag{44}$$

subject to constraints (39) (40) (41).

Proof Since $p_t \equiv p$ is a constant,

$$\int_A (\prod_{\tau=0}^{t} \Delta_\tau^\alpha) d\alpha = \sum_{k=0}^{t} ({}_t\mathbf{C}_k) p^k (1-p)^{t-k} (\delta^l)^k (\delta^h)^{t-k} = (p\delta^l + (1-p)\delta^h)^t.$$

The SBP can be written:

$$\max_{\{\overline{B}_t, \overline{q}_t\}_{t=2}^{\infty}} \sum_{t=1}^{\infty} (p\delta^l + (1-p)\delta^h)^{t-1} \log(\overline{c}_t)$$

By the principle of optimality, we can write the equivalent value functional problem of the SBP

$$V(\overline{B}_t, \overline{q}_t) = \max_{\{\overline{B}_{t+1}, \overline{q}_{t+1}\}} \{\log(\overline{c}_t) + (p\delta^l + (1-p)\delta^h) V(\overline{B}_{t+1}, \overline{q}_{t+1})\}$$

Definition 3.6 *The bank's investment portfolio path is in steady state $(\bar{b}, \bar{q})$ in period t, if for all $\tau \geq t$,*

- $\dfrac{\bar{B}_{\tau+1}}{\bar{B}_\tau} \equiv \bar{b} > 0$ *and*
- $\bar{q}_\tau \equiv \bar{q} \in [0, 1]$.

Proposition 3.7 *$\overline{q}$ is 0 or 1 in the optimal steady state of the bank's investment portfolio.*

Proof According to corollary 3.5, the proof is similar to the proof of proposition 2.3.

3.2.2 Cyclical p_t

In this section, we assume that the probability of an agent being impatient is cyclical over the period: $P(\delta_t = \delta^l) = p_i$, where $i = 1$ if t is odd and $i = 2$ if t is even. Without loss of generality, we assume $p_1 > p_2$, which implies that more agents are impatient in an odd period than in an even period.

Corollary 3.8 *The equivalent functional equation to SBP is*

$$V(\overline{B}_t, \overline{q}_t) = \max_{\{\overline{B}_{t+1}, \overline{q}_{t+1}\}} \{\log(\bar{c}_t) + \bar{\delta}_t V(\overline{B}_{t+1}, \overline{q}_{t+1})\}, \tag{45}$$

subject to constraints (39) (40) (41), and where $\bar{\delta}_t$ *is given by*

$$\bar{\delta}_t = \begin{cases} p_1\delta^l + (1-p_1)\delta^h & : \ t \text{ is odd} \\ p_2\delta^l + (1-p_2)\delta^h & : \ t \text{ is even.} \end{cases}$$

Proof For any $t > 0$,

$$\begin{aligned}
\int_A \prod_{\tau=0}^{t} \Delta_\tau \, d\alpha &= \sum_{k_1=0}^{t-[\frac{t}{2}]} \sum_{k_2=0}^{[\frac{t}{2}]} ({}_{t-[\frac{t}{2}]}\mathbf{C}_{k_1})({}_{[\frac{t}{2}]}\mathbf{C}_{k_2}) p_1^{k_1}(1-p_1)^{t-[\frac{t}{2}]-k_1} p_2^{k_2}(1-p_2)^{[\frac{t}{2}]-k_2}(\delta^l)^{k_1+k_2}(\delta^h)^{t-k_1-k_2} \\
&= \left(\sum_{k_1=0}^{t-[\frac{t}{2}]} ({}_{t-[\frac{t}{2}]}\mathbf{C}_{k_1})(\delta^l)^{k_1}(\delta^h)^{t-[\frac{t}{2}]-k_1} p_1^{k_1}(1-p_1)^{t-[\frac{t}{2}]-k_1} \right) \cdot \\
&\quad \left(\sum_{k_2=0}^{[\frac{t}{2}]} ({}_{[\frac{t}{2}]}\mathbf{C}_{k_2}) p_2^{k_2}(1-p_2)^{[\frac{t}{2}]-k_2}(\delta^l)^{k_2}(\delta^h)^{[\frac{t}{2}]-k_2} \right) \\
&= \left(p_1\delta^l + (1-p_1)\delta^h\right)^{t-[\frac{t}{2}]} \left(p_2\delta^l + (1-p_2)\delta^h\right)^{[\frac{t}{2}]}.
\end{aligned}$$

Thus we can write the SBP:

$$\max_{\{\overline{B}_t, \overline{q}_t\}_{t=2}^{\infty}} \sum_{t=1}^{\infty} (p_1\delta^l + (1-p_1)\delta^h)^{t-1-[\frac{t-1}{2}]}(p_2\delta^l + (1-p_2)\delta^h)^{[\frac{t-1}{2}]} \log(\bar{c}_t).$$

By the principle of optimality,

$$V(\overline{B}_t, \overline{q}_t) = \max_{\{\overline{B}_{t+1}, \overline{q}_{t+1}\}} \{\log(\bar{c}_t) + \bar{\delta}_t V(\overline{B}_{t+1}, \overline{q}_{t+1})\},$$

where $\bar{\delta}_t$ is the average discount factor of each period,

$$\bar{\delta}_t = \begin{cases} p_1\delta^l + (1-p_1)\delta^h & : \ \text{if t is odd} \\ p_2\delta^l + (1-p_2)\delta^h & : \ \text{if t is even.} \end{cases}$$

This completes the proof.

Definition 3.9 *The investment portfolio path of the bank is in steady state* $((\bar{b}_1, \bar{q}_1), (\bar{b}_2, \bar{q}_2))$ *in period* t*, if* $\forall \tau \geq \dfrac{t}{2}$,

- $\dfrac{\overline{B}_{2\tau+1}}{\overline{B}_{2\tau}} \equiv \bar{b}_1 > 0, \dfrac{\overline{B}_{2\tau+2}}{\overline{B}_{2\tau+1}} \equiv \bar{b}_2 > 0$
- $\overline{q}_{2\tau+1} \equiv \overline{q}_1 \in [0,1], \overline{q}_{2\tau+2} \equiv \overline{q}_2 \in [0,1]$.

As in the cyclical case of the baseline model, we study the auxiliary problem of the bank's investment to find out when the liquidity ratio of the bank's investment portfolio is cyclical in the optimal steady state.
The corresponding auxiliary problem for the bank is: assume the illiquid technology is the only one investment choice,

$$V(\overline{B}_t) = \max_{\overline{B}_{t+1}} \{\log(\overline{c}_t) + \overline{\delta}_t V(\overline{B}_{t+1})\} \tag{46}$$

subject to

$$\begin{aligned} \bar{c}_t &= R_I \overline{B}_t - \theta(1 - \frac{\overline{B}_{t+1}}{R_I \overline{B}_t})^2 \overline{B}_t - \overline{B}_{t+1}. \\ 0 &\leq B_{t+1} \end{aligned}$$

for all $t \geq 1$.

Since the return from illiquid technology is homogeneous of degree 1, without loss of generality we assume $\bar{B}_1 = \bar{I}_1 = 1$.

Definition 3.10 *The investment portfolio path is in the steady state $(\bar{x}_1, \bar{x}_2)$ of the auxiliary problem in period τ, if $\forall t \geq \frac{\tau}{2}$:*

- $\dfrac{\overline{B}_{2t+1}}{\overline{B}_{2t}} \equiv \bar{x}_1 > 0, \dfrac{\overline{B}_{2t+2}}{\overline{B}_{2t+1}} \equiv \bar{x}_2 > 0$ *for all* $t \geq \dfrac{\tau}{2}$.

Let

$$\begin{aligned} \bar{\delta}^l &= p_1 \delta^l + (1 - p_1) \delta^h \\ \bar{\delta}^h &= p_2 \delta^l + (1 - p_2) \delta^h \end{aligned}$$

As in the single agent economy we have the following theorem:

Theorem 3.11 *Let $(\bar{x}_1, \bar{x}_2)$ be the optimal steady state of auxiliary problem. If*

$$\bar{x}_1 \frac{-\frac{2\theta}{R_I}(1 - \frac{\bar{x}_1}{R_I})}{R_I - \theta(1 - \frac{\bar{x}_1}{R_I})^2 - \bar{x}_1} - \bar{\delta}^h \frac{(R_I - \theta + \theta(\frac{\bar{x}_2}{R_I})^2 - R_L)}{R_I - \theta(1 - \frac{\bar{x}_2}{R_I})^2 - \bar{x}_2} > 0 \tag{47}$$

and assumption 2.5 are satisfied, then in the optimal steady state of the bank's investment portfolio, the liquidity ratio is cyclical with $0 < \overline{q}_1 < 1$ and $\overline{q}_2 = 1$.

Proof According to corollary 3.8, this proof is similar to the proof of theorem 2.11.

Example 3.12 *Let* $R_I = 1.2, R_L = 1.18, \theta = 0.5, p_1 = 0.9, p_2 = 0, \delta^l = 0$ *and* $\delta^h = 0.99$. *By computation, we have* $\bar{q}_1 = 0.936$ *and* $\bar{q}_2 = 1.000$, *and the average growth rate of portfolio size is given by* $\sqrt{\bar{b}_1 \bar{b}_2} = \sqrt{0.7143 * 0.3327} = 0.4876$. *The portfolio size is shrinking because* 90% *of the agents are impatient during odd periods.*

4 Conclusion

The model developed here makes a beginning towards explaining features of the asset portfolio of the U.S banking sector, which we described in the introduction. Since the aggregate preference shock is a deterministic 2-period cycle in the model, the priority for further research is to generalize it to a Markovian process and the ultimate goal is to calibrate or estimate such a generalized model.

5 Appendix

To compare the expected discount utility level achieved by an agent investing directly with that of depositing in the bank.

Since all agents are identical ex ante, we consider a generic agent. Let $v(D_t)$ be the expected discounted value function of the generic agent with current deposit D_t. According to the principle of optimality

$$v(D_t) = \max_{s_t} E[u(s_t D_t(1 + r_t)) + \Delta_t v(D_{t+1})] \tag{48}$$

subject to $0 \leq s_t \leq 1$.

Since we use a logarithmic utility function, the withdrawal choice is independent of the agent's deposit balance. In other words, the withdrawal choice s_t depends only on the utility discount factor Δ_t, that is $s_t \in \{s^l, s^h\}$.
The law of motion is

$$D_{t+1} = (1 - s_t)(1 + r_t)D^t.$$

In order to find the optimizing value of s^l and s^h, we solve equation (47) by making a conjecture and verifying it. Assume the value function has the form $v(D_t) = a \log(D_t) + b$.

Equation (47) induces a contraction mapping for a, b with the following solution:

$$\begin{aligned} a &= \frac{1}{1-p\delta^l-(1-p)\delta^h} \\ b &= \frac{\log(1-p\delta^l-(1-p)\delta^h)}{1-p\delta^l-(1-p)\delta^h} + \frac{\log(1+r_t)}{(1-p\delta^l-(1-p)\delta^h)^2} \\ &+ \frac{p\delta^l\log(\delta^l)+(1-p)\delta^h\log(\delta^h)}{(1-p\delta^l-(1-p)\delta^h)^2} \\ &- \frac{p(1-(1-p)(\delta^h-\delta^l))\log(1-(1-p)(\delta^h-\delta^l))}{(1-p\delta^l-(1-p)\delta^h)^2} \\ &- \frac{(1-p)(1+p(\delta^h-\delta^l))\log(1+p(\delta^h-\delta^l))}{(1-p\delta^l-(1-p)\delta^h)^2} \end{aligned}$$

This solution implies the optimal withdrawal choice s^l, s^h:

$$s^l = \frac{1-p\delta^l-(1-p)\delta^h}{1-(1-p)(\delta^h-\delta^l)},$$

$$s^h = \frac{1-p\delta^l-(1-p)\delta^h}{1+p(\delta^h-\delta^l)}.$$

References

[1] Robert M. Anderson. Non-standard analysis with applications to economics. In W. Hildenbrand and H. Sonnenschein, editors, *Handbook of Mathematical Economics*, volume 4 of *Handbook of Mathematical Economics*, chapter 39, pages 2145–2208. Elsevier, October 1991.

[2] Douglas W Diamond and Philip H Dybvig. Bank runs, deposit insurance, and liquidity. *Journal of Political Economy*, 91(3):401–419, June 1983.

[3] Neil Wallace. Narrow banking meets the Diamond-Dybvig model. *Quarterly Review*, (Win):3–13, 1996.