

Macro Stress and Worst Case Analysis of Loan Portfolios *

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Abstract

We introduce the technique of worst case search to macro stress testing. Among the macroeconomic scenarios satisfying some plausibility constraint we determine the worst case scenario which causes the most harmful loss in loan portfolios. This method has three advantages over traditional macro stress testing: First, it ensures that no harmful scenarios are missed and therefore prevents a false illusion of safety which may result when considering only standard stress scenarios. Second, it does not analyse scenarios which are too implausible and would therefore jeopardize the credibility of stress analysis. Third, it allows for a portfolio specific identification of key risk factors. Another lesson from this paper relates to the use of partial stress scenarios specifying the values of some but not all risk factors: The plausibility of partial scenarios is maximised if we set the remaining risk factors to their conditional expected values.

1 Introduction

Macro stress testing has become an important method of risk analysis for lending activities. This paper introduces the technique of worst case analysis to macro stress testing. Among the macroeconomic scenarios satisfying some plausibility constraint we determine the worst case scenario which causes the most harmful loss. In this way one can be sure not to miss out any harmful but plausible scenarios, which is a serious danger when considering only standard stress scenarios.

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This kind of systematic worst case analysis with plausibility constraints was developed for market risk stress testing, see ? and ??. The loss in the worst case scenario can also be regarded as risk measure. As such it was originally introduced under the name Maximum Loss by ??. Maximum Loss is a coherent risk measure in the sense of ?. Actually, it is the prototype of a coherent risk measure because by a duality argument every coherent risk measure can be represented as Maximum Loss over some set of generalised scenarios, see ? and ?. As a risk measure Maximum Loss has two advantages over Value at Risk. First, it is coherent and therefore can be the basis of economic capital allocation to subportfolios. Secondly, it provides information about which economic situations are really harmful and suggests possible counteraction to reduce risk in case it is considered unacceptable, see ?.

Stress testing started in market risk analysis but in recent years it has been applied to macro analysis as well. A brief introduction into macro stress testing as well as an overview of EU country-level macro stress testing practices is given in a special feature of the Financial Stability Report of the ?. According to the ECB, macro stress testing is a way of quantifying the link between macroeconomic variables and the health of either a single financial institution or the financial sector as a whole. A detailed introduction into the topic and an overview of related literature is given in ?. In many countries, central banks' endeavour with macro stress testing was boosted by the IMF running a Financial Sector Assessment Program (FSAP). For details see ? and ?. A stress analysis of sector concentration risk in credit portfolios is given in ?. Our paper adds to this literature by introducing the technique of worst case search to macro stress testing. However, we perform macro stress tests only of loan portfolios but not of a whole banking system.

The rest of the paper is structured as follows. First, in Section 2 we develop a methodology of macro stress testing and worst case analysis. In Section 3 we develop a model describing both macro and credit risk as well as their interaction in loan portfolios. On the basis of this model in Section 4 we apply the general methodology to loan portfolios and derive implications for their risk structure.

2 Macro Stress Testing Methods

We assume the following framework for our discussion of macro stress testing.

Assumption 1. *The value of the portfolio is a function v of n macro¹ risk*

¹The term 'macro' could sometimes be replaced by 'systematic' or by 'market', since the macro risk factors often play the role of systematic risk factors, and they include interest rates and exchange rates, which are market prices. We will use the term 'macro' risk factors without denying the appropriateness of other expressions.

factors $\mathbf{r} = (r_1, \dots, r_n)$ and of m idiosyncratic risk factors $\epsilon_1, \dots, \epsilon_m$, one for each counterparty. The macro risk factor changes are distributed elliptically with covariance matrix Σ and expectations $\boldsymbol{\mu}$. The idiosyncratic risk factors may be continuous or discrete.

The definition and some basic facts about elliptic distributions can be found in the Appendix.

2.1 Traditional Macro Stress Tests

Standard macro stress testing picks some macro scenarios, often historical scenarios or standard scenarios popular in the industry, or specific scenarios combining risk factor moves the bank considers dangerous to its sub-portfolios. Assigning values only to macro risk factors excludes a stress analysis of idiosyncratic risk factors. Analysing changes in the idiosyncratic risk factors referring to the most important counterparties is an important stress testing exercise, but it is not part of *macro* stress testing.

A macro stress scenario is *partial* if it fixes the values of some but not all macro risk factors. Let us call the risk factors whose value is specified by the partial macro scenario ‘the fixed risk factors’. For example, in the scenario ‘The € falls by 20% against the CHF’ the fixed risk factor is the CHF/€ rate. The standard stress testing procedure then is to analyse the implications of the scenario for the expected portfolio value, or for risk capital, or for capital ratios.

Good practice in stress testing is to identify scenarios which have harmful implications for the portfolio and at the same time are not completely implausible. The plausibility of macro scenarios will be measured by the Mahalanobis distance:

$$\text{Maha}(\mathbf{r}) := \sqrt{(\mathbf{r} - \boldsymbol{\mu})^T \cdot \Sigma^{-1} \cdot (\mathbf{r} - \boldsymbol{\mu})},$$

where \mathbf{r} , $\boldsymbol{\mu}$, and Σ only refer to the macro risk factors fixed by the scenario. Intuitively, $\text{Maha}(\mathbf{r})$ can be interpreted as the number of standard deviations of the multivariate move from $\boldsymbol{\mu}$ to \mathbf{r} . Maha takes into account the correlation structure between the risk factors. A high value of Maha implies a low plausibility of the scenario \mathbf{r} .

A macro scenario does not determine a unique portfolio value because it does not fix the values of the idiosyncratic risk factors. (Additionally, a scenario of Type D below does not fix the values of all macro risk factors.) We will analyse four stress distributions arising from different interpretations of some partial scenario:

- (A) The conditional profit distribution given the macro scenario \mathbf{r}_A , in which the fixed risk factors have the value specified by the partial scenario, and the other macro risk factors remain at their *last observed* value.

- (B) The conditional profit distribution given the macro scenario \mathbf{r}_B , in which the fixed risk factors have the value specified by the partial scenario, and the other macro risk factors take their *unconditional expectation* value.
- (C) The conditional profit distribution given the macro scenario \mathbf{r}_C , in which the fixed risk factors have the value specified by the partial scenario, and the other macro risk factors take their *conditional expected value* given the values of the fixed risk factors.
- (D) The conditional profit distribution given the partial macro scenario \mathbf{r}_D , in which the fixed risk factors have their value specified by the partial scenario, and the other macro factors are distributed according to the marginal distribution given the values of the fixed risk factors.

The term ‘stress distribution’ reflects the double nature of these conditional distributions. They are *distributions* which are derived from macro *stress* scenarios. The macro scenarios \mathbf{r}_A , \mathbf{r}_B , and \mathbf{r}_C have the full dimensionality of the macro model. The macro scenario \mathbf{r}_D has a lower number of dimensions because it consists just of the fixed risk factors.

Proposition 1. *Assume the distribution of macro risk factors is elliptical with density strictly decreasing as a function of Maha. Then:*

(1) *The plausibility of the full macro scenario \mathbf{r}_C (with non-fixed macro factors assigned their conditional expectation) is equal to the plausibility of the partial macro scenario \mathbf{r}_D (which does not assign any value to the non-fixed macro factors):*

$$\text{Maha}(\mathbf{r}_C) = \text{Maha}(\mathbf{r}_D).$$

(2) *This is the maximal plausibility (i.e. the minimal Maha) which can be achieved among all macro scenarios which agree on the fixed risk factors.*

(3) *The same plausibility is achieved by macro scenarios which assign to some of the non-fixed risk factors their conditional expected values given the fixed risk factors, and to other non-fixed risk factors no value. Assigning to non-fixed risk factors other value than their conditional expectation given the fixed risk factors yields scenarios of lower plausibility.*

A proof of this proposition is given in the Appendix. This proposition implies that two choices of macro stress distributions are preferable, namely (C) or (D). Assigning to the non-fixed risk factors other values than the conditional expected values given the fixed risk factors leads to less plausible macro stress scenarios.

This proposition is of high practical relevance. It is the basis of partial scenario analysis. Typically portfolios are modelled with hundreds or thousands of risk factors. But for the purpose of macro stress testing one

is interested only in a handful of risk factors. How should the other risk factors be treated? Proposition 1 tells us which values to assign to the other risk factors in order to maximise the plausibility of scenarios.

A second aspect of partial scenarios analysis is the severeness of the scenarios. The implications of an interesting stress scenario should be harmful. The harm caused by a scenario is related to the conditional profit distribution given the scenario. It may be measured in terms of the expected value of the conditional profit distribution, the capital requirement implied by the conditional profit distribution via some risk measure, or the capital ratio. For the purpose of this paper we will measure harm by low conditional expected profits (CEP) of the stress distributions.

Proposition 2. *If the portfolio value function v is concave in the non-fixed macro risk factors, then*

$$CEP(\mathbf{r}_D) \leq CEP(\mathbf{r}_C).$$

If v is convex in the non-fixed risk factors the opposite inequality holds. If v is neither concave nor convex $CEP(\mathbf{r}_D)$ may be higher or lower than $CEP(\mathbf{r}_C)$.

This proposition is the second element of partial scenario analysis. From Proposition 1 we know that $\mathbf{r}_C, \mathbf{r}_D$ both have the maximal plausibility among macro scenarios with the specified values of the fixed risk factors. Proposition 2 tells us which of the two is more harmful.

2.2 Worst Case Analysis

An important disadvantage of standard stress testing is the danger to miss out harmful but plausible scenarios. This may result in a false illusion of safety. A way to overcome this disadvantage is to search systematically for those macro scenarios in some plausible admissibility domain which are most harmful to the portfolio. By searching systematically over admissibility domains of plausible macro scenarios one can be sure not to miss out any harmful but plausible scenarios. The goal is to try to find the macro scenarios which are most relevant in that they are most harmful and at the same time are above the minimal plausibility threshold.

Finding the scenario which does maximal harm is an optimisation problem under noise because the goal function is itself a random variable even if macro risk factors are specified. Luckily in some models the goal function can be calculated explicitly without determining the full conditional profit distribution, as in Lemma 1 below. This reduces the problem to a deterministic optimisation problem.

As admissibility domain for the macro scenarios it is natural to take an ellipsoid whose shape is determined by the covariance matrix of macro risk

factor changes:

$$\text{Ell}_k := \{\mathbf{r} : \text{Maha}(\mathbf{r}) \leq k\}, \quad (1)$$

Finding a macro scenario in the ellipsoid Ell_k which has minimal conditional expectation of the profit distribution is a deterministic non-convex optimisation problem. Using an algorithm of ? this can be solved numerically.

In the early days of systematic stress testing, see ?? or ?, plausibility of a scenario was measured by the probability of the set of scenarios with smaller Maha. But Maximum Loss over such admissibility domains with a specified probability mass shows a peculiar kind of dimensional dependence: for a fixed portfolio and fixed probability of the admissibility domain, the inclusion of additional irrelevant risk factors increases Maximum Loss. Since we characterise the admissibility domain by its Mahalanobis radius instead of its probability mass, the inclusion of irrelevant risk factors, or of risk factors which are highly correlated to other risk factors does not affect Maximum Loss.

What is the advantage of worst case search over standard stress testing? First, the worst case scenarios are superior to the standard stress scenarios in the sense that they are more harmful and equal or higher plausibility. Secondly, worst case scenarios reflect portfolio specific dangers. What is a worst case scenario for one portfolio might be a harmless scenario to another portfolio. This is not taken into account by standard stress testing. Thirdly, worst case scenarios allow for an identification of the key risk factors which contribute most to the loss in the worst case scenario.

3 A Market and Credit Risk Model of Loan Portfolios

Before we illustrate the use of these techniques on loan portfolios we need to specify a model determining the profit or loss of a loan portfolio as a function of macro and idiosyncratic risk factors.

We consider a portfolio of foreign currency loans for obligors $i = 1, \dots, m$ at a time horizon of one year. At time 0, in order to receive the home currency amount l the customer takes a loan of $l/e(0)$ units in a foreign currency, where $e(0)$ is the home currency value of the foreign currency at time 0. The bank borrows $l/e(0)$ units of the foreign currency at the interbank market. After one period, at time 1, which we take to be one year, the loan expires and the bank repays the foreign currency at the interbank market with an interest rate r_f , e.g. LIBOR, and it receives from the customer a home currency amount which is exchanged at the rate $e(1)$ to the foreign currency amount covering repayment of the principal plus interest rolled over from four quarters, plus a spread s . So the customer's payment

obligation to the bank at time 1 in home currency is

$$o_f = l \prod_{i=1}^4 (1 + r_f(i/4)/4) E + s_f l E,$$

where $r_f(i/4)$ are the LIBOR rates in the foreign currency in quarter i and $E := e(1)/e(0)$. In order to reduce the number of dimensions one can introduce an average 3 months LIBOR rate r_f over the year defined by $1 + r_f = \prod_{i=1}^4 (1 + r_f(i/4)/4)$. This yields for the payment obligation in home currency

$$o_f = l(1 + r_f) E + s_f l E \quad (2)$$

The first term on the right hand side is the part of the payment which the bank passes on to the interbank market. The second term is profits remain with the bank. For a customer taking a home currency loan the payment obligation is

$$o_h = l(1 + r_h) + l s_h,$$

where r_h is the home interest rate. The spreads s_f, s_h demanded from the customer depend on the rating class and the loan type. From the model the spreads will be set in such a way that the bank achieves 20% return on the standard regulatory capital charge of 8%, which amounts to an expected profit of €160 on a loan of €10 000. For all loans in the portfolio we assume they expire at time 1. The model can be extended to a multi period setting allowing for loans maturing not at the same time and requiring payments at intermediate times.

In order to evaluate credit and macro risk of a portfolio of such loans we use a one-period structural model specifying default frequencies and losses given default endogeneously. We present the simplest possible specification for the model rather than the most general.

Assumption 2. *Customers default in case their payment ability a at the expiry of the loan is smaller than their payment obligation o . In case of default the customer pays a .*

This assumption implies that the profit or loss the bank makes with a customer is

$$v_i := \min(a_i, o) - l(1 + r_f) E. \quad (3)$$

In this profit function the first term is what the customer pays to the bank and the second term is what the bank has to repay on the interbank market. Even if the customer defaults the bank might make a profit because o includes the spread over the LIBOR. Both, PD and LGD depend on the macro risk factors via the payment obligation o and the payment ability a .

? perform macro stress tests of foreign currency loan portfolios, as do we. They assume that exchange rate changes affect loan loss provisions via

disposable income, which in turn is proxied by GDP. In contrast our model translates exchange rate changes via its effect on the payment obligation into default probability changes.

Assumption 3. *The payment ability at final time 1 for each single obligor i is distributed according to*

$$a_i(1) = a_i(0) \cdot \frac{GDP(1)}{GDP(0)} \cdot \epsilon, \quad (4)$$

$$\log(\epsilon) \sim N(\mu, \sigma) \quad (5)$$

where $a_i(0)$ is a constant, and $\mu = -\sigma^2/2$, ensuring $\mathbb{E}(\epsilon) = 1$. The realisations ϵ_i are independent of each other and of the macro risk factors.

GDP(0) is the known GDP at time $t = 0$, GDP(1) is a random variable. The distribution of ϵ_i reflects obligor specific random events, like losing or changing job. The support of ϵ_i is $(0, \infty)$ reflecting the fact that the amount a_i available for repayment of the loan cannot be less than zero if the obligor has no lines of credit open with the bank. Since the expected value of ϵ_i is one and ϵ is independent of GDP, the expectation of $a_i(1)$ is $a_i(0)$ times the expectation of $GDP(1)/GDP(0)$. ? use a model of this type for the returns of firm value.

Assuming that for different customers the realizations of ϵ_i are independent is the doubly stochastic hypothesis.² Conditional on the path of macro risk factors which determine the default intensities, customer defaults are independent.

The initial payment ability $a_i(0)$ is a customer specific parameter determined in the loan approval procedure. For example, to be on the safe side the bank can extend loans only to customers with $a_i(0)$ equal to 1.2 times the loan amount. This extra margin is taken into account in the rating. From a rating system the bank determines the default probability p_i of the customer. In the loan approval procedure both the present payment ability $a_i(0)$ and the rating (implying the default probability) are determined. They are input to our valuation model.

The payment ability distribution must satisfy the following condition:

$$p_i = P[a_i(1) < o_i]. \quad (6)$$

$a_i(1)$ is a function of σ and o_i is a function of the spreads. Spreads are set to achieve some target expected profit for each loan:

$$\mathbb{E}v_i(\sigma, s) = \text{EP}_{\text{target}}, \quad (7)$$

²See ?. Note also that there is some empirical evidence that the doubly stochastic hypothesis might be violated, see ?.

where v_i is the profit with obligor i and EP_{target} is some target expected profit. The two free parameters σ and s (s_f resp. s_h) are determined from these two conditions.

A GDP increase shifts the payment ability distribution to the right, as shown in Fig. 1. It increases distance to default and reduces default probabilities, provided the payment obligation is unchanged.

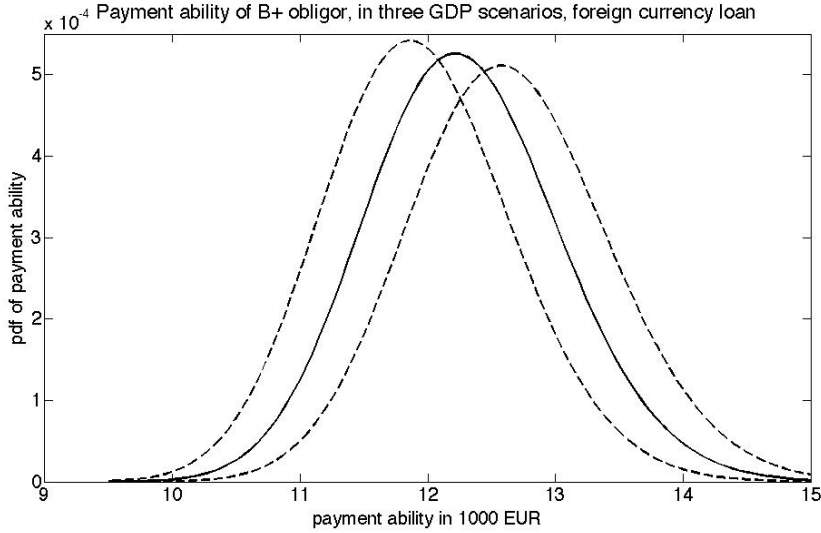


Figure 1: Plots of density function of the payment ability distribution, with GDP equal to its expected value (solid line), and GDP equal to ± 3 standard deviations. We observe that the payment ability distribution for higher GDP values stochastically dominates the distribution for lower GDP values.

The macroeconomic risk factors entering the portfolio valuation are GDP, r_f , r_h , and E . To model the dynamics of these we use a GVAR model, as ?, ?, ?, and ?.

Assumption 4. *The dynamics of the four risk factors GDP, home and foreign interest rate, and exchange rate is determined by the GVAR model specified below.*

Since we are considering a loan portfolio in Austria, in the GVAR we model the economies of Austria and its most important trading partners Switzerland, France, Germany, Italy, and the US. As domestic variables for each economy we used the logarithm Y of deseasonalized, real gross domestic product, the logarithm E of the exchange rate USD/(domestic currency), and the logarithm RS of (3 month maturity interbank interest rate per annum), divided by 100. The exchange rate was not included for the US. The strengths of import and export trade relationships between the countries was used to build the foreign counterpart of each domestic variable.

Table 1: Estimated mean and covariances of logarithms of macro risk factors in the GVAR models.

	GDP	r_{EUR}	r_{CHF}	$e(1/4)$
GVAR				
mean	5.446	1.246	0.556	0.423
std. dev.	0.0097	0.1870	0.6301	0.0387
correlations	1.000	0.291	0.217	-0.040
		1.000	0.519	0.140
			1.000	0.007
				1.000

The individual country models were estimated allowing for unit roots and cointegration assuming that the foreign variables are weakly exogenous. More precisely, for each country a weakly exogenous VECM with no deterministic terms and auto-regressive lag 2 (i.e. lag 1 in the VECM equations) was estimated using maximum likelihood reduced rank regression, as in ? and ?. The six VECMs were then combined to a global VAR model including all and only domestic variables. This global model can be iterated recursively to obtain future scenarios of all variables. The GVAR model allows for cross-country as well as inter-country cointegration.

The distribution of the macro risk factors was estimated from quarterly data 1989–2005. Nominal GDP data for Austria were from the IFS of the International Monetary Fund. For the logs of risk factors, mean and covariance matrix of the estimated distribution are given in Table 1.

To sum up, in our model the macro risk factors \mathbf{r} are the logs of $GDP(1), r_f$, and $e(1)$ for the foreign currency loans and $GDP(1), r_h$ for the home currency loans. They are elliptically distributed with mean and covariances as specified above. The idiosyncratic risk factors are the ϵ_i in the payment ability distribution for each customer.

4 Application to Loan Portfolios

With this model at hand we will now perform standard macro stress tests and analyse worst case scenarios. The portfolios we consider consist of 100 loans of $l = \text{€} 10\,000$ taken out in CHF from an Austrian bank by Austrian customers in the rating class B+, corresponding to a default probability of $p_i = 2\%$, or in rating class BBB+, corresponding to a default probability of $p_i = 0.1\%$. Obligors are assumed to have an initial payment ability of $a_i(0) = 1.2 l$. The spreads s_f and s_h for each rating class were set in such

a way that the expected profit of each loan amounts to a 20% return on regulatory capital. Under the minimal capital requirement of 8% the bank aims at an expected profit EP_{target} of €160 for a loan of €10 000. The resulting spreads are:

rating	loan type	spread [bp]
BBB+	home	160.14
B+	home	165.64
BBB+	foreign	158.06
B+	foreign	163.88

The profit distribution was calculated in a Monte Carlo simulation by generating 100 000 scenario paths of four steps each. The resulting distribution of risk factors after the last quarter, which is not normal, was used to estimate the covariance matrix of 1yr macro risk factor changes. In each macro scenario defaults of the customers were determined by 100 draws from the ϵ in the payment ability distribution. From these we evaluated the profit distribution at the one year time horizon.

4.1 Traditional Macro Stress Tests

For the standard stress scenario “The € falls by 20% against the CHF” Table 2 compares the expected values and plausibility of the stress distributions (A) to (D). What do we learn about our loan portfolios using this standard stress test?

- Expected loss is much higher for the foreign currency loan portfolio than for the home currency loans. This is true for all stress distribution types. Home currency loan portfolios in the FX stress scenario have an expected value between €15 797 and €16 150, which is not far from the unconditional expected profit of €16 000. The foreign currency loan portfolios have a stress expected values of around €-60 000 for B+ obligors and €-30 000 for BBB+ obligors, which amounts to a loss of 6% (resp. 3%) of the total exposure of the bank. From this stress test the bank learns that foreign currency loans are hit far more forcefully by an exchange rate shock than home currency loans. The economic rationale is clear. Rising exchange rates increase payment obligations and thereby default probabilities. This is a paradigmatic case of a dangerous interaction between credit and market risk.
- The second last column of Table 2 shows that the macro scenarios of types (C) and (D) have exactly the same plausibility, and that the

scenarios of types (A) and (B) have higher Maha, i.e. lower plausibility. This is not a coincidence but a consequence of Proposition 1.

- Conditional expected profits CEP from Type D stress distributions are lower than for Type C stress distributions. This hold for all four portfolio types considered. This is a consequence of Proposition 2 and the fact that v_i is concave in the non-fixed risk factors (GDP, r_f) .

Table 2: Analysis of standard exchange rate scenario with various assumptions about unspecified risk factors. Various stress distributions are compared by their expectation value and the plausibility of their stress scenario. All conditional distributions are characterised by an exchange rate move of +20%, but they differ in assumptions about values of the remaining macro risk factors. Standard deviations of CEP numbers in brackets. CEP numbers for stress distribution were calculated analytically with the formula in Lemma 1 for distribution types A–C, and approximated with Monte Carlo simulation for type D. We observe that for the stress distributions of types C and D Maha values are equal and lowest among the considered macro scenarios, as implied by Proposition 1. CEP values are lower for distribution type D than for distribution type C, as implied by Proposition 2.

Portfolio		Stress distribution							CEP
rating	curr.	type	CHF/€	GDP	IR foreign	IR home	Maha		
B+	for.	A	-20%	last obs	last obs	last obs	5.587	-64 204 (0)	
B+	for.	B	-20%	uncd exp	uncd exp	uncd exp	4.979	-56 293 (0)	
B+	for.	C	-20%	cond exp	cond exp	cond exp	4.905	-53 337 (0)	
B+	for.	D	-20%	not spec	not spec	not spec	4.905	-54 209 (56.1)	
BBB+	for.	A	-20%	last obs	last obs	last obs	5.587	-58 134 (0)	
BBB+	for.	B	-20%	uncd exp	uncd exp	uncd exp	4.979	-48 225 (0)	
BBB+	for.	C	-20%	cond exp	cond exp	cond exp	4.905	-44 587 (0)	
BBB+	for.	D	-20%	not spec	not spec	not spec	4.905	-45 136 (62.3)	
B+	home	A	-20%	last obs	last obs	last obs	5.587	15 797 (0)	
B+	home	B	-20%	uncd exp	uncd exp	uncd exp	4.979	16 037 (0)	
B+	home	C	-20%	cond exp	cond exp	cond exp	4.905	16 150 (0)	
B+	home	D	-20%	not spec	not spec	not spec	4.905	16 130 (1.5)	
BBB+	home	A	-20%	last obs	last obs	last obs	5.587	15 991 (0)	
BBB+	home	B	-20%	uncd exp	uncd exp	uncd exp	4.979	16 004 (0)	
BBB+	home	C	-20%	cond exp	cond exp	cond exp	4.905	16 008 (0)	
BBB+	home	D	-20%	not spec	not spec	not spec	4.905	16 006 (0.2)	

Next let us compare the conditional profit distributions in two stress scenarios to the unconditional profit distribution. In addition to the exchange rate scenario we consider an economic recession scenario in which GDP shrinks by 3%. Table 3 compares the unconditional and the two stress distributions by their expected values and their Expected Shortfall based risk capital at the 10%, 5%, 1%, and 0.1% quantiles. For a profit loss distribution X risk capital is

$$\text{RC}_\alpha(X) := \mathbb{E}(X) - \text{ES}_\alpha(X), \quad (8)$$

where ES_α is Expected Shortfall at some confidence level α , as defined e.g. in [?, Def. 2.6]. Standard deviations of approximation errors of ES are calculated using the method of ?.

The comparison of stress distributions gives additional information about the loan portfolios:

1. The FX shock has serious consequences on the foreign currency loan portfolio. In mean it wipes out around €60 000 for the B+ portfolio, which amounts to more than 6% of the exposure. The FX shock also has serious consequences on the capital required. Risk capital requirements at the 1% confidence level for the BBB+ portfolio increase from around €3 765 in the unconditional case to €89 280, which is more than 8% of the exposure. This can be put into relation with the 8% capital ratio.
2. The FX shock has a weak but positive influence on the home currency loan portfolio. This is due to the positive correlation between exchange rate and home interest rate changes. A EUR depreciation tends to be accompanied by a reduction in EUR interest rates, which reduces PDs and LGDs.
3. The effects of a GDP shock depend on the rating of obligors rather than on the loan type. Expected profits are reduced from around €16 000 to under €15 000 for the B+ portfolios. Expected profits of the BBB+ portfolio are reduced only very slightly to around €15 800.

Table 3: Standard macro stress tests of the home and foreign currency loan portfolios. The unconditional profit distribution of the two portfolios is compared to the Type D profit distribution conditional on a -3% change of GDP, and conditional on a +20% change of the exchange rate. Profit distributions are compared by their means CEP and ES-based risk capital at various quantiles. Standard deviations in brackets.

scenario	CEP	RC _α				
		10%	5%	1%	0.1%	
B+ foreign						
unconditional	16 001 (6.5)	2 797 (65.0)	4 672 (119.4)	12 423 (490.5)	39 183 (3818)	
GDP -3%	14 812 (8.4)	6 046 (70.5)	9 111 (110.1)	17 961 (291.7)	33 616 (957)	
FX -20%	-54 209 (56.1)	36 909 (301.7)	48 796 (449.6)	82 523 (1 206.4)	148 638 (5 416)	
BBB+ foreign						
unconditional	15 999 (4.8)	1 264 (47.4)	1 664 (89.6)	3 765 (426.3)	23 404 (4 176)	
GDP -3%	15 821 (2.7)	1 275 (22.0)	1 716 (40.2)	4 050 (182.7)	16 866 (1 100)	
FX -20%	-45 136 (62.3)	40 982 (324.6)	53 899 (476.8)	89 280 (1 236.1)	155 276 (5 472)	
B+ home						
unconditional	16 001 (1.8)	1 259 (8.2)	1 587 (10.7)	2 315 (20.7)	3 312 (69)	
GDP -3%	14 139 (3.8)	2 419 (14.3)	2 961 (18.0)	4 114 (33.0)	5 580 (87)	
FX -20%	16 130 (1.5)	1 100 (7.3)	1 396 (9.6)	2 055 (19.4)	2 989 (58)	
BBB+ home						
unconditional	16 001 (0.2)	121 (2.1)	224 (3.0)	450 (5.9)	743 (16)	
GDP -3%	15 789 (0.9)	626 (4.2)	797 (5.4)	1 167 (10.8)	1 679 (32)	
FX -20%	16 006 (0.2)	74 (1.6)	155 (2.8)	373 (5.5)	639 (14)	

4.2 Worst Case Analysis

As a second way to perform stress tests we search systematically for those macro scenarios in the admissibility domain Ell_k which lead to the worst conditional expected profit CEP. (Other objective function for the worst case search, like RC_α , could also be considered.) By searching systematically over admissibility domains of plausible macro scenarios one can be sure not to miss any harmful but plausible scenarios. The goal is to try to find the macro scenarios which are most relevant in that they are most harmful but remain over a minimal plausibility threshold.

This is an optimisation problem under noise if the objective function CEP is determined in a Monte Carlo simulation. But luckily the problem can be reduced to a deterministic problem. Minimising CEP amounts to minimising the profit with the payment ability replaced by its deterministic part.

Lemma 1. *The conditional expectation of the profit distribution (3), given the values $E := e(1)/e(0)$, $G := \text{GDP}(1)/\text{GDP}(0)$, and r_f , is*

$$\text{CEP}(E, G, r_f) := l E s_f + a(0) G \mathbb{E}(\epsilon 1_{\epsilon < E_0}) - l E (1 + r_f + s_f) P[\epsilon \leq E_0],$$

where $E_0 := l E (1 + r_f + s) / (a(0)G)$. For the CEP of the home currency loan there is a similar formula.

What is the advantage of worst case search over standard stress testing? First, the worst case scenarios are superior to the standard stress scenarios. This is reflected by Table 4. This table compares the expected portfolio values of the standard scenarios with those of the worst case scenarios of same plausibility. We see that the worst case scenarios are substantially more harmful than the standard scenarios. For example, the expected value of the home currency B+ portfolio in the scenario ‘GDP -3%’ is €14 249 as compared to €13 291 in the worst case scenario of the same plausibility. Although we know that GDP is an important risk factor for the home currency loan portfolio, there are plausible macro scenarios which are more harmful to the portfolio than a 3% reduction in the GDP.

What is a worst case scenario for one portfolio might be a harmless scenario to another portfolio. This is not taken into account by standard stress testing. For example, the home currency B+ loan portfolio is more or less insensitive to moves in the FX rate. (The 20% depreciation of the EUR increases CEP to €16 150.) The FX move not just harmless but even positive for the home currency loan portfolio.

Stress testing is relevant only if the choice of scenarios takes into account the portfolio. In a systematic way this is done by worst case search. The combination of a -3.25σ move in GDP and a $+2.56\sigma$ move in the home interest rate reduces expected profits to €13 766. This move has the same

plausibility (Maha equal to 5.42) as the FX-20% move, but it causes a considerably worse reduction in expected profits.

Table 4: Comparison of standard stress scenarios with worst case scenarios of the same plausibility. Conditional expected profits for the standard scenarios ‘GDP -3% and other risk factors at their conditional expected value’ and ‘CHF/€ +20% and other risk factors at their conditional expected value’, and of worst case scenarios of the same plausibility. Values of risk factors in worst case scenarios are given in absolute terms and changes in standard deviations. We observe that for all portfolios the expected profits are considerably lower in the worst case scenarios than in the standard scenarios.

name	Maha	Scenario				CHF/€				CEP
		GDP		home IR		foreign IR		CHF/€		
		abs.	stdv	abs.	stdv	abs.	stdv	abs.	stdv	
foreign B+										
FX -20%	4.91	232.3	0.20			0.0204	-0.07	1.238	-4.91	-53 337
worst case	4.91	230.9	-0.08			0.0422	1.37	1.254	-4.66	-68 023
GDP -3%	5.42	219.6	-5.42			0.0048	-1.04	1.541	0.22	15 950
worst case	5.42	230.9	-0.05			0.0442	1.50	1.225	-5.16	-98 101
foreign BBB+										
FX -20%	4.91	232.3	0.20			0.0204	-0.07	1.238	-4.91	-44 587
worst case	4.91	230.9	-0.09			0.0421	1.37	1.254	-4.66	-62 139
GDP -3%	5.42	219.6	-5.42			0.0048	-1.04	1.541	0.22	15 870
worst case	5.42	230.8	-0.07			0.0437	1.47	1.224	-5.17	-95 591
home B+										
FX -20%	4.91	232.3	0.20	0.0308	-0.68			1.238	-4.91	16 150
worst case	4.91	223.1	-3.25	0.0471	2.56			1.566	0.44	13 766
GDP -3%	5.42	219.6	-5.42	0.0249	-1.56			1.541	0.22	14 249
worst case	5.42	222.1	-3.60	0.0482	2.82			1.570	0.49	13 291
home BBB+										
FX -20%	4.91	232.3	0.20	0.0308	-0.68			1.238	-4.91	16 008
worst case	4.91	223.0	-3.28	0.0468	2.53			1.565	0.42	15 726
GDP -3%	5.42	219.6	-5.42	0.0249	-1.56			1.541	0.22	15 811
worst case	5.42	222.0	-3.64	0.0479	2.78			1.569	0.47	15 626

A second advantage of systematic worst case analysis is that the worst case scenario allows for an identification of the key risk factors, to which the portfolio reacts most sensitively. We define key risk factors as the risk factors with the highest Maximum Loss Contribution (MLC). The MLC of risk factor i is

$$MLC(i) := \frac{CEP(\mathbb{E}r_1, \mathbb{E}r_2, \dots, r_i^{WC}, \mathbb{E}r_{i+1}, \dots, \mathbb{E}r_n) - CEP(\mathbb{E}\mathbf{r})}{CEP(\mathbf{r}^{WC}) - CEP(\mathbb{E}\mathbf{r})}.$$

$MLC(i)$ is the loss if risk factor i takes its worst case value and the other risk factors take their expected values, as a percentage of MaxLoss. Table 5 gives for different sizes of the admissibility domain the worst macro scenarios together with the expected profit in the worst macro scenario. For each scenario the risk factor with the highest MLC are printed in bold face. Table 5 displays the results of the worst case search.

1. For the foreign currency loan portfolio the exchange rate is clearly the key risk factor. This becomes apparent from Table 5. In the worst case scenario the FX rate alone contributes between 59% and 100% of the losses in the worst case scenarios. This indicates that the FX rate is the key risk factor of the foreign currency loan portfolio. The left hand plots in Fig. 2 confirm this.
2. For the home portfolio GDP is the key risk factor, but the home interest rate is also relevant. The moves in GDP alone contribute between 46% and 73% of the losses in the worst case scenarios. The diagnosis is confirmed by the right hand plot in Fig. 2, which shows the expected profits in dependence of single macro risk factor moves, keeping the other macro risk factors fixed at their expected values. Note the different scales of the two plots. Expected losses of the FX loan are vastly larger than for the home currency loan.

The dependence of expected profits of both loan types on the relevant risk factors is clearly non-linear. The profiles of expected profits in Fig. 2 resemble those of short options. A home currency loan behaves like a short put on GDP together with a short call on the home interest rate. A foreign currency loan behaves largely like a short call on the FX rate.

3. There is another interesting effect. The dependence of expected profits of foreign currency loans on the CHF/€ rate is not only non-linear, but also not monotone. For the BBB+ FX loan portfolio (bottom left plots in Figure 2), focusing on changes smaller than 4σ it becomes evident that a small increase in the exchange rate has a positive influence on the portfolio value, but large increases have a very strong negative influence. Correspondingly, in Table 5, if we restrict ourselves

to small moves (Maha smaller than 4σ) the worst case scenario is in the direction of increasing exchange rates, but if we allow larger moves the worst case scenario is in the direction of decreasing exchange rates. This effect also shows up in the worst macro scenarios of Table 5. The reason for this non-monotonicity is that a small decrease in the FX rate increases the EUR value of spread payments received. For larger moves of the FX rate this positive effect is outweighed by the increases in defaults due to the increased payment obligations of customers. For the bad quality B+ portfolios the positive effect of a small FX rate decrease persists only up to a maximal Maha radius of $k = 2$.

4. The Maximum Loss Contributions of the macro risk factors in general do not add up to 100%. Sometimes the sum is larger, sometimes it is greater. The reason for this is the non-linear dependence of CEP on the macro risk factors, or more precisely the fact that the cross derivatives of the CEP function do not vanish. Because of the curvature of the CEP surface the effect of a combined move in several risk factors may be larger or smaller than the sum of effects of individual risk factor moves.

One could ask why the effort to search for worst case scenarios is necessary to identify key risk factors. Wouldn't it be easier to read the key risk factors from the plots in Fig. 2? This would be true if losses from moves in different risk factors *added up*. But for certain kinds of portfolios the worst case is a *simultaneous* move of several risk factors—and the loss in this worst case might be considerably worse than adding up the losses resulting from moves in single risk factors. The effects of simultaneous moves are not reflected in Fig. 2.

As an example consider a B+ home currency loan, and assume we are restricting ourselves to moves with Maha smaller than $k = 6$. From Table 5 we see that the worst macro scenario involves a simultaneous move of -4.01σ in GDP and $+3.10\sigma$ in the home interest rate. The right hand plot in Fig. 2 shows that a -4.01σ move in GDP would reduce CEP from €160 to roughly €145, and a $+3.10\sigma$ move in the home interest rate would reduce CEP to roughly €155. Summing up these two losses would imply a CEP of €140 per loan, or a portfolio profit of €14 000 for the portfolio. This dangerously overestimates the true CEP resulting from the simultaneous move, which is €12 676 according to Table 5.

5 Conclusion

We introduce the technique of worst case search to macro stress testing. Among the macroeconomic scenarios satisfying some plausibility constraint we determined the worst case scenario which causes the most harmful loss

in loan portfolios. This method has three advantages over traditional macro stress testing: First, it ensures that no harmful scenarios are missed and therefore prevents a false illusion of safety which may result when considering only standard stress scenarios. Second, it does not analyse scenarios which are too implausible and would therefore jeopardize the credibility of stress analysis. Third, it allows for a portfolio specific identification of key risk factors. Another lesson from this paper relates to the use of partial stress scenarios specifying the values of some but not all risk factors: The plausibility of partial scenarios is maximised if we set the remaining risk factors to their conditional expected values.

In order to carve out the basic insights we presented the approach in the most basic framework. For practical purposes the framework has to be generalised to a multi-period setup, requiring scenario paths instead of one step scenarios. Admissibility domains also have to be defined for scenario paths instead of one step scenarios. In a multi period setup one can analyse portfolios of loans maturing at different times and requiring payments at intermediate times. The computational burden in the multi-period framework is by far heavier.

Table 5: Systematic macro stress tests of the home and foreign currency loan portfolios. We search for the macro scenarios with the worst expectation value of the profit distribution, under the condition that macro scenarios lie in elliptic admissibility domain of maximal Mahalanobis radius k . Macro scenarios are specified by the macro risk factors GDP, exchange rate, and interest rates. We give the absolute values of these risk factors in the worst case scenario, as well as their change in standard deviations, and their Maximum Loss Contributions MLC. For the key risk factors MLC is printed in bold face.

		Worst Macro Scenario												
max. Maha		GDP			home IR			foreign IR			CHF/€			
		abs.	stdv	MLC	abs.	stdv	MLC	abs.	stdv	MLC	abs.	stdv	MLC	CEP
foreign	B+													
1	231.73	-0.07	0.6%		0.022	0.04	0.8%	1.587	1	100.0%	1.587	1	100.0%	15 959
2	231.64	-0.14	0.5%		0.022	0.04	0.4%	1.646	2	100.0%	1.646	2	100.0%	15 400
3	231.08	-0.11	0.1%		0.035	0.93	0.8%	1.363	-2.81	59.7%	1.363	-2.81	59.7%	1 631
4	230.98	-0.1	0.1%		0.039	1.17	0.4%	1.306	-3.78	65.3%	1.306	-3.78	65.3%	-26 084
5	230.85	-0.09	0.0%		0.042	1.39	0.2%	1.249	-4.76	71.2%	1.249	-4.76	71.2%	-73 257
6	230.86	-0.03	0.0%		0.046	1.59	0.2%	1.191	-5.74	77.0%	1.191	-5.74	77.0%	-136 000
foreign	BBB+													
1	231.75	-0.07	0.0%		0.022	0.01	0.0%	1.587	1	100.0%	1.587	1	100.0%	15 409
2	231.65	-0.14	0.0%		0.022	0.03	0.0%	1.646	2	100.0%	1.646	2	100.0%	14 855
3	231.54	-0.21	0.0%		0.022	0.04	0.0%	1.707	3	100.0%	1.707	3	100.0%	14 339
4	231.47	-0.27	0.0%		0.022	0.07	0.0%	1.765	4	100.0%	1.765	4	100.0%	13 859
5	230.88	-0.07	0.0%		0.043	1.39	0.0%	1.249	-4.76	63.3%	1.249	-4.76	63.3%	-68 164
6	230.83	-0.04	0.0%		0.045	1.58	0.0%	1.191	-5.74	75.8%	1.191	-5.74	75.8%	-135 203
home	B+													
1	230.07	-0.65	72.8%	0.038	0.54	21.2%	0.038	0.54	0.53	17.7%	0.038	0.53	17.7%	15 804
2	228.31	-1.3	70.0%	0.04	1.07	18.5%	0.04	1.07	1.06	12.9%	0.04	1.06	12.9%	15 482
3	226.51	-1.97	67.8%	0.043	1.59	15.8%	0.043	1.59	1.57	9.1%	0.043	1.57	9.1%	15 044
4	224.7	-2.64	65.7%	0.045	2.1	13.5%	0.045	2.1	2.07	6.4%	0.045	2.07	6.4%	14 458
5	222.88	-3.32	63.6%	0.047	2.61	11.6%	0.047	2.61	2.57	4.5%	0.047	2.57	4.5%	13 684
6	221.04	-4.01	62.0%	0.049	3.1	9.9%	0.049	3.1	3.07	3.3%	0.049	3.07	3.3%	12 676
home	BBB+													
1	230.06	-0.65	70.1%	0.038	0.53	17.7%	0.038	0.53	0.53	17.7%	0.038	0.53	17.7%	15 992
2	228.28	-1.31	63.9%	0.04	1.06	12.9%	0.04	1.06	1.06	12.9%	0.04	1.06	12.9%	15 969
3	226.47	-1.99	58.8%	0.043	1.57	9.1%	0.043	1.57	1.57	9.1%	0.043	1.57	9.1%	15 926
4	224.65	-2.67	54.2%	0.045	2.07	6.4%	0.045	2.07	2.07	6.4%	0.045	2.07	6.4%	15 848
5	222.82	-3.35	50.2%	0.047	2.57	4.5%	0.047	2.57	2.57	4.5%	0.047	2.57	4.5%	15 710
6	220.99	-4.03	46.8%	0.049	3.07	3.3%	0.049	3.07	3.07	3.3%	0.049	3.07	3.3%	15 476

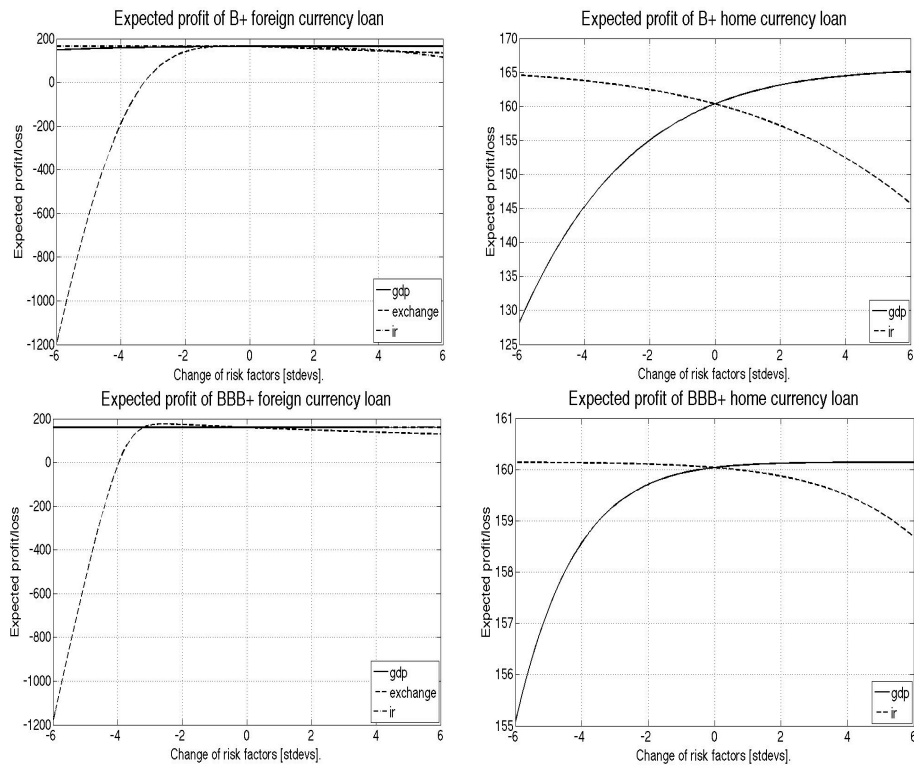


Figure 2: Key risk factors of foreign and home currency loans. Expected profit or loss of a single foreign (left) and home currency (right) loans as a function of changes of the macro risk factors with other macro risk factors fixed at their expected values. Top: B+ loans. Bottom: BBB+ loans. The left hand plot shows that for the foreign portfolio the exchange rate is the key risk factor if we restrict ourselves to small moves, but that the foreign interest rate becomes the dominant risk factor if we allow for larger moves. We also observe the negative effect of small foreign currency depreciations, which is particularly pronounced for the BBB+ portfolio. The right hand plot shows that for the home portfolio GDP is the key risk factor. Note the different scales of the two plots.

References

Appendix

Proof of Proposition 1

Let us assume that we have n macro risk factors, whose change is governed by a multivariate elliptically symmetric distribution with covariance matrix Σ and mean $\boldsymbol{\mu}$. Let us assume that the risk factors are indexed in such a way that the fixed risk factors have numbers $1, 2, \dots, k$. Let us denote by r_{k+1}^*, \dots, r_n^* the conditional expected values of risk factors r_{k+1}, \dots, r_n given that r_1, \dots, r_k have their fixed values. We will show that $\text{Maha}(r_1, \dots, r_k, r_{k+1}^*, \dots, r_n^*) = \text{Maha}(r_1, \dots, r_k, r_{k+1}^*, \dots, r_{n-1}^*)$ and that $r_n = r_n^*$ minimises Maha among all scenarios with the values of the first k risk factors equal to $r_1, \dots, r_k, r_{k+1}^*, \dots, r_{n-1}^*$. Repeating this argument for the risk factors r_{n-1} down to r_{k+1} yields the Proposition.

Denote the n -dimensional density function of the macro risk factors by f^n . Since this distribution is assumed to be elliptical with strictly decreasing density we can write it as $f^n(\mathbf{r}) = g(\text{Maha}(\mathbf{r}))$ for some strictly decreasing non-negative function g . The level surfaces of the density function agree with the ellipsoids determined by the covariance matrix.

When we fix the value of some remaining risk factor, say risk factor n , the distribution of the remaining $n-1$ risk factors is described by the marginal distribution $f^{n-1}(\mathbf{r}')$, resulting from integration over r_n . Here \mathbf{r}' is the $(n-1)$ -dimensional vector resulting from deleting the last component from \mathbf{r} . The conditional distribution of r_n given some fixed \mathbf{r}' is

$$h(r_n|\mathbf{r}') = \frac{f^n(\mathbf{r}', r_n)}{f^{n-1}(\mathbf{r}')}.$$

The expected value of the conditional distribution $h(r_n|\mathbf{r}')$ is r_n^* .

Lemma 2. *Assume Σ is a positive definite $n \times n$ -matrix. Then we have*

$$\text{Maha}(\mathbf{r})^2 - \text{Maha}(\mathbf{r}')^2 = \left(\sum_{i=1}^n C(\Sigma)^{-1}_{in} r_i \right)^2, \quad (9)$$

where $C(\Sigma)^{-1}_{in}$ is the element in row i and column n of the inverse matrix of the Cholesky decomposition of Σ .

Proof. For an arbitrary symmetric positive definite matrix M denote by $C(M)$ its Cholesky decomposition. $C(M)$ is the upper triangular matrix satisfying

$$M = C(M)^T C(M). \quad (10)$$

Here are some properties of the Cholesky decomposition.

$$C(M)^{-1} C(M)^{-1T} = M^{-1} \quad (11)$$

In other words, the transpose of the inverse of the Cholesky decomposition of M is the Cholesky decomposition of M^{-1} .

Furthermore we have $C(M)'^T C(M)' = M'$. So we may write

$$C(M') = C(M)'. \quad (12)$$

Deleting the n -th row and column of M and then making the Cholesky decomposition amounts to the same as making the Cholesky decomposition of M and then deleting the n -th row and column.

For an arbitrary triangular matrix C we have

$$(C^{-1})' = (C')^{-1}. \quad (13)$$

Now let us calculate the squares of the Mahalanobis distances.

$$\begin{aligned} \mathbf{r}'^T \cdot \Sigma'^{-1} \cdot \mathbf{r}' &\stackrel{(12)}{=} \mathbf{r}'^T \cdot (C(\Sigma)'^T C(\Sigma)')^{-1} \cdot \mathbf{r}' \\ &\stackrel{(11)}{=} \mathbf{r}'^T \cdot C(\Sigma)'^{-1} C(\Sigma)'^{-1T} \cdot \mathbf{r}' \\ &\stackrel{(13)}{=} \mathbf{r}'^T \cdot C(\Sigma)^{-1'} C(\Sigma)^{-1'T} \cdot \mathbf{r}'. \end{aligned} \quad (14)$$

Similarly we have

$$\begin{aligned} \mathbf{r}^T \cdot \Sigma^{-1} \cdot \mathbf{r} &\stackrel{(11)}{=} \mathbf{r}^T \cdot C(\Sigma)^{-1} C(\Sigma)^{-1T} \cdot \mathbf{r} \\ &= (\mathbf{r}^T \cdot C(\Sigma)^{-1}) (\mathbf{r}^T \cdot C(\Sigma)^{-1})^T \end{aligned} \quad (15)$$

The inverse of the triangular matrix $C(\Sigma)$ is again triangular, so we can write

$$C(\Sigma)^{-1} = \begin{pmatrix} & & & C(\Sigma)^{-1}_{1n} \\ & C(\Sigma)^{-1}' & & \vdots \\ & & & \vdots \\ 0 & \dots & 0 & C(\Sigma)^{-1}_{nn} \end{pmatrix}.$$

Writing $\mathbf{r}^T = (\mathbf{r}'^T, r_n)$ equation (15) reads

$$\begin{aligned} \mathbf{r}^T \cdot \Sigma^{-1} \cdot \mathbf{r} &= \left(\mathbf{r}'^T C(\Sigma)^{-1}', \sum_{i=1}^n C(\Sigma)^{-1}_{in} r_i \right) \left(\mathbf{r}'^T C(\Sigma)^{-1}', \sum_{i=1}^n C(\Sigma)^{-1}_{in} r_i \right)^T \\ &= \mathbf{r}'^T \cdot C(\Sigma)^{-1}' C(\Sigma)^{-1'T} \cdot \mathbf{r}' + \left(\sum_{i=1}^n C(\Sigma)^{-1}_{in} r_i \right)^2. \end{aligned} \quad (16)$$

Subtracting (14) from (16) yields the Lemma. \square

Lemma 3. *The expected value of the conditional distribution $h(r_n|\mathbf{r}')$ is given by*

$$r_n^* = - \frac{\sum_{i=1}^{n-1} C(\Sigma)^{-1}_{in} r_i}{C(\Sigma)^{-1}_{nn}}.$$

Furthermore,

$$g(\text{Maha}(\mathbf{r})^2) = g\left(\text{Maha}(\mathbf{r}')^2 + (C(\Sigma)^{-1}_{nn})^2 (r_n - r_n^*)^2\right),$$

which implies that $\text{Maha}(\mathbf{r})$ as a function of r_n is minimal, namely equal to $\text{Maha}(\mathbf{r}')$, at $r_n = r_n^*$.

Proof: As a function of r_n , the conditional density $h(r_n|\mathbf{r}') = f^n(\mathbf{r}', r_n)/f^{n-1}(\mathbf{r}')$ is a constant times the n -dimensional density $f^n(\mathbf{r}', r_n)$. By eq. (9) $\text{Maha}(\mathbf{r})$ as a function of r_n is minimal, namely equal to $\text{Maha}(\mathbf{r}')$, at

$$r_n^* = -\frac{(\sum_{i=1}^{n-1} C(\Sigma)^{-1}_{in} r_i)}{C(\Sigma)^{-1}_{nn}}.$$

So the conditional density $h(r_n|\mathbf{r}')$ is maximal at r_n^* , where $\text{Maha}(\mathbf{r})$ is minimal. We also have

$$\begin{aligned} f^n(\mathbf{r}', r_n) &= g(\text{Maha}(\mathbf{r})^2) = g\left(\text{Maha}(\mathbf{r}')^2 + \left(\sum_{i=1}^n C(\Sigma)^{-1}_{in} r_i\right)^2\right) \\ &= g\left(\text{Maha}(\mathbf{r}')^2 + \left(\sum_{i=1}^{n-1} C(\Sigma)^{-1}_{in} r_i + C(\Sigma)^{-1}_{nn} r_n\right)^2\right) \\ &= g\left(\text{Maha}(\mathbf{r}')^2 + (-C(\Sigma)^{-1}_{nn} r_n^* + C(\Sigma)^{-1}_{nn} r_n)^2\right) \\ &= g\left(\text{Maha}(\mathbf{r}')^2 + (C(\Sigma)^{-1}_{nn})^2 (r_n - r_n^*)^2\right) \end{aligned}$$

This implies that $f^n(\mathbf{r}', r_n)$, and consequently $h(r_n|\mathbf{r}')$ is symmetric around its maximum r_n^* . Thus the expected value of $h(r_n|\mathbf{r}')$ if it exists is r_n^* . \square

This finishes the proof of Proposition 1.

Proof of Proposition 2

Denote by \mathbf{r}^f the vector of risk factors which are fixed by a partial scenario and by \mathbf{r}^{nf} the macro risk factor which are not fixed by the scenario. The set of all risk factors $(\mathbf{r}^f, \mathbf{r}^{nf}, \epsilon)$ defines a N -variate random variable on the measure space $(\mathbb{R}^N, \mathcal{F}, P)$. The generalised Jensen inequality implies that for a concave portfolio value function v we have

$$v(\mathbb{E}\{\mathbf{r}^{nf}|\mathcal{B}\}) \geq \mathbb{E}\{v(\mathbf{r}^{nf}|\mathcal{B})\}$$

for all σ -subalgebras \mathcal{B} of \mathcal{F} . Choosing for \mathcal{B} the subalgebra generated by \mathbf{r}^f and ϵ we have

$$v(\mathbb{E}\{\mathbf{r}^{nf}|(\mathbf{r}^f, \epsilon)\}) \geq \mathbb{E}\{v(\mathbf{r}^{nf}|(\mathbf{r}^f, \epsilon))\}$$

Taking expectations with respect to ϵ this implies

$$\mathbb{E}_\epsilon v(\mathbb{E}\{\mathbf{r}^{nf} | (\mathbf{r}^f, \epsilon)\}) \geq \mathbb{E}_\epsilon \mathbb{E}\{v(\mathbf{r}^{nf} | (\mathbf{r}^f, \epsilon))\}.$$

The right hand side is $CEP(\mathbf{r}_D)$ and the left hand side is $CEP(\mathbf{r}_C)$. \square