Expectations, Bond Yields and Monetary Policy

Albert Lee Chun[*]

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Abstract

Through explicitly incorporating analysts' forecasts as observable factors in a dynamic arbitrage-free model of the yield curve, this paper proposes a framework for studying the impact of shifts in market sentiment on interest rates of all maturities. An empirical examination reveals that survey expectations about inflation, output growth and the anticipated path of monetary policy actions contain important information for explaining movements in bond yields. Although perceptions about inflation are largely responsible for movements in long-term interest rates, an explicit slope factor is necessary to adequately capture the dynamics of the yield curve. Macroeconomic forecasts play an important role in explaining time-variation in the market prices of risk, with forecasted GDP growth playing a dominant role. The estimated coefficients from a forward-looking monetary policy rule support the assertion that the central bank preemptively reacts to inflationary expectations while suggesting patience in accommodating real output growth expectations. Models of this type may provide traders and policymakers with a new set of tools for formally assessing the reaction of bond yields to shifts in market expectations due to the arrival of news or central bank statements and announcements.

1 Introduction

The expectations hypothesis asserts that yields on long bonds are functions of expected future short rates. Adjusting for the time varying nature of risk premia, this statement captures the prevailing view on the dynamic nature of the yield curve. Since future short rates are determined by future central bank policies, expectations about future yields are inseparably linked to expectations about future monetary policy. The federal funds rate is an instrument of monetary policy and through the use of open market operations the central bank can directly impact the short end of the yield curve. Indirectly, however, a policy action impacts yields of all maturities as a move at the short end of the curve quickly filters through the entire set of yields. One reason is the equilibrating nature of arbitrage - yields of all maturities respond to changes in the short rate as market forces intervene through the trades of investors and arbitrageurs. Yet interest rates often fluctuate outside of FOMC meetings, a response absent any relation to observable policy actions. What forces are driving these fluctuations? Although liquidity, currency and supply/demand pressures can certainly contort the shape of the yield curve, a more immediate answer may lie with the changing nature of market expectations. Bond prices are sensitive to fluctuations in expectations about the path of monetary policy, and as a consequence, are invariably linked to the perceived path of the macroeconomy.

The arrival of new information, such as a Federal Reserve announcement, or an unexpected policy move, may shift expectations to reflect innovations in the market's sentiment about future economic conditions.[1] This paper explicitly incorporates observable forecasts as factors into a dynamic model of the yield curve. These forecasts are assumed to proxy for market expectations and are compiled from a published survey of individual forecasts known as the Blue Chip Financial Forecasts. This monthly publication contains analysts' views on future interest rates and the macroeconomic variables that influence them. The analysts in the survey comprise professional economists at major banks, investment firms and economic consulting groups.

Understanding the nature of dynamic fluctuations in bond yields is of pivotal concern for both macroeconomists and financial economists alike. On one hand, empirical macroeconomists interested in the dynamics of interest rates have historically studied monetary policy rules or worked with vector-autoregressions (VARs). Taylor (1993) proposes a guideline for targeting the federal funds rate in which the central bank reacts to changes in inflation and output. Clarida, Gali, and Gertler (2000) and Orphanides (2003), among others, link expectations about future inflation and output to the central bank's target rate by proposing forward-looking variations on a Taylor-type rule.[2] By acknowledging that the monetary authority reacts preemptively to future economic scenarios, a forward-looking monetary policy rule places economic expectations at the forefront of the central bank's decision making. VAR studies, such as Evans and Marshall (1998), provide great insights in linking movements in interest rates with the underlying macroeconomy.[3] However, these studies present only a partial picture of the yield curve - they are not designed to explain movements in yields omitted from the model nor do they preserve a consistent arbitrage-free relationship across the enitre yield curve. On the other hand, traditional term structure research in theoretical and empirical finance focuses on jointly modeling the entire yield curve and proposes the existence of unobserved, latent risk-factors that govern yield dynamics under the absence of arbitrage. This literature is large and includes studies by Vasicek (1977), Cox, Ingersoll, and Ross (1985), Duffie and Kan (1996), Duffie and Singleton (1997), Dai and Singleton

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[1] Gurkaynak, Sack, and Swanson (2005a) find that a significant source of variation in long yields is the information content in Federal Reserve statements. They argue that a possible reason for this pattern is that statements "lead to a greater extent to positive revisions in investors' assessment of the future path of output and inflation."

[2] For a historical account of the development of policy rules and the forward-looking variants that followed see Orphanides (2003) and the references therein. See also Gavin and Mandal (2001), Orphanides and Williams (2003) and Batini and Haldane (1999).

[3] The macro VAR literature is large, see the references in Christiano, Eichenbaum, and Evans (1999) and Ang and Piazzesi (2004).

(2000), Dai and Singleton (2002) and Duffee (2002). Most of the models in this literature fall under a class known as affine term structure models.[4]

In a move toward establishing stronger links between macroeconomic fundamentals and asset pricing models of the yield curve, researchers began to augment latent variable models with observed macroeconomic information. Piazzesi (2001), and subsequently Piazzesi (2005), links bond yields with monetary policy actions by incorporating an observable federal funds target into a dynamic model of the yield curve. Ang and Piazzesi (2004) combine both latent and macro-variables in an affine setting, thus inspiring a series of papers. Some important papers include Ang, Piazzesi, and Wei (2005c), Hordahl, Tristani, and Vestin (2002), Rudebusch and Wu (2003), Rudebusch and Wu (2004), Duffee (2004), Ang, Dong, and Piazzesi (2005b), Dai and Philippon (2004), Bikbov and Chernov (2005), Dewachter and Lyrio (2004), Dewachter, Lyrio, and Maes (2005), Bekaert, Cho, and Moreno (2005), Law (2005), Garcia and Luger (2007) and Gallmeyer, Hollifield, and Zin (2005). These factor models are attractive in that they, through conditions of no-arbitrage, fully characterize the cross-sectional and time series properties of the entire yield curve while attributing significant yield dynamics to movements in the underlying macroeconomy. However, these models offer somewhat limited flexibility in capturing shifts in economic expectations. Macro-finance models of the term structure implicitly embed expectations of future short-rates based on estimated knowledge of the risk-neutral model parameters and the information in the state vector driving yields. Although the latent factors do pick up yield curve movements unrelated to changes in the economic fundamentals, these models do not explicitly capture changes in market sentiment due to news, information or the additional transparency afforded by Federal Reserve announcements.

Several macro-finance models of the yield curve accommodate the forward-looking behavior of the central bank by specifying the short-rate equation as a forward-looking monetary policy rule. Ang and Piazzesi (2004) refer to their model with lagged macroeconomic variables as a "forward-looking rule", where this interpretation hinges on the ability of lagged macroeconomic variables to forecast the future. Ang, Dong, and Piazzesi (2005b) estimate forward-looking Taylor rules using the short rate equation from a no-arbitrage model of the yield curve, in which forecasts are conditional expectations given current state variables. Rudebusch and Wu (2003) argue that expectations of forward-looking agents about the future dynamics of the economy are important determinants of both current and future yields. However, their work also captures these expectations indirectly through currently observed yield and macroeconomic information. Dewachter and Lyrio (2004) incorporate forecasts by using a filtering method to extract inflationary expectations which they find to be highly correlated with expectations taken from survey data. John Taylor in the introduction to the collection "Monetary Policy Rules"[5] points out a subtle issue with so-called forward-looking rules: "Rules that respond to the forecasts of inflation rather than actual inflation are frequently referred to as *forward-looking* rules, but since forecasts are based on current and lagged data, these rules are no more forward-looking than the so-called *backward-looking rules*". Given this observation, "forward-looking" models would not be expected to respond to news or announcements that are orthogonal to information in observed economic quantities, whereas observed forecasts would certainly embed this additional information. Ang, Bekaert, and Wei (2005a) find that survey forecasts outperform econometric methods at forecasting inflation. Our own preliminary research (to be reported in another paper) shows that mean survey forecasts of both inflation and the federal funds rate are more accurate than standard econometric approaches.

In this paper, we incorporate observable expectations directly into a dynamic term structure model.[6] We demonstrate that there is an explicit connection between bond yields and the expectations of

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[4]Duffie and Kan (1996) characterize the class of affine term structure models, and Dai and Singleton (2000) provide a system of classification. See Piazzesi (2003) for a review of affine term structure models. These models are further classified as "completely" affine or "essentially" affine, see Duffee (2002). For a survey of the empirical term structure literature see Dai and Singleton (2003).

[5]Taylor (1999).

[6]Survey expectations from the Blue Chip Financial Forecasts are also used by Kim and Orphanides (2005) to help

market participants. Using different subsets of the forecasts, we estimate models with the purpose of dissecting the relationship between data on expectations and bond yields. In the spirit of the forward-looking Taylor rule of Ang, Dong, and Piazzesi (2005b), we model the reaction function of the monetary authority as a function of expectations, the difference being that our expectations are observed from forecast data. By estimating the short-rate equation within the formal framework of a dynamic term-structure model, information in yields of various maturities can be used to estimate the policy rule parameters. To gain insights into the relationship between the yield curve and observable expectations, we estimate two classes of models which are denoted as "McCallum-based" models and "Taylor-based" models.[7]

McCallum (1994) suggests a policy rule where the Federal Reserve reacts to the lagged interest rate, that plays the role of a smoothing parameter, and the yield spread, which contains information about the future macroeconomy. We propose a forward-looking variant of this reaction function as the baseline rule characterizing the first class of "McCallum-based" models. Using the language of traditional term structure studies, the forecasted funds rate acts as a "level" factor by shifting yields of all maturities nearly equally, while the forecasted yield spread plays the role of a "slope" factor by shifting one end of the curve more than the other. Since these two factors already incorporate much information about the future macroeconomy, additional information embedded in forecasts of real output growth or inflation only contributes in a marginal way to the ability of a model to fit historical movements in bond yields. This is consistent with the finance literature where a 2-factor model, comprising a "level" and a "slope" factor, captures most of the variation in yields. Although these models do well at fitting the historical yield curve, we find that macroeconomic intuition is obtained by examining a 3 or 4 factor model that allows the macroeconomic forecasts the first chance at explaining the time dynamics of yields.

We denote the second class of models as "Taylor-based" models. In this class, forecasted inflation acts as a "level" factor while GDP growth plays the role of a weak "slope" factor. The component of the federal funds rate forecast that has been purged of the linear effect of the macroeconomic forecasts takes on the interpretation of an "anticipated monetary policy" factor. A change to this factor embeds expectations about movements in the future federal funds rate that is free from the "endogenous" response of forecasted inflation and output. This adds another dimension to the longstanding problem of separation between anticipated and unanticipated components of monetary policy changes. As a consequence, we are able to quantify the impact of changes in the anticipated component of monetary policy on the dynamics of the term structure of interest rates. A shock to this factor results in a response to the yield curve that is strong when compared with other studies that examine the relationship between monetary policy changes and interest rates. Although this third factor does introduce an additional element of "slope" into the model, the above factors alone are unable to fully capture movements at the long end of the curve. We thus find that a stronger "slope" factor is essential for fitting the dynamics of long maturity yields. This final factor, which is given by the expected 10-year yield minus the fed funds rate spread (that has also been purged of the linear influence of the first 2 factors), captures expectations about long-horizon macroeconomic conditions not present in the other forecasts. The use of orthogonal factors preserves the interpretation and role of the macroeconomic forecasts in the reaction function, while allowing the yield-based forecasts to pick up the remaining variation.

Policy rules of this type have an advantage in that they are "operational", as forecasts can be obtained at a higher frequency than difficult to measure macroeconomic variables such as the output gap. By studying the estimated coefficients from a forward-looking policy rule we find that the Federal Reserve accommodates GDP growth expectations while preemptively counteracting inflationary expec-

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estimate a formal 3-factor latent variable model. They find that survey yield forecasts add useful information for effectively estimating model parameters when faced with a small sample problem. Pennacchi (1991) and Brennan, Wang, and Xia (2004) both augment their asset pricing models with survey forecasts of inflation, as does an earlier version of Dai and Philippon (2004).

[7]See Gallmeyer, Hollifield, and Zin (2005) for a study on the equivalence of McCallum and Taylor rules within an affine term structure setting.

tations. The negative relationship between target moves and growth expectations is partially due to the Federal Reserve's policy of continuing to lower interest rates well after the market has started to anticipate a recovery. However, impulse response analysis, which considers the model-implied correlation structure of the innovations in the forecasts, shows virtually no initial response of yields to a shock in real output expectations followed by a delayed, positive impulse response path that suggests patience on the part of the monetary authority.

Changes in expected future short rates are not the only important component of interest rate movements, bond prices are also sensitive to the dynamic nature of market risk premia. A risk-averse market will demand a premium for bearing exposure to the risk associated with future fluctuations in the short rate. Dai and Singleton (2002) find that violations of the expectations hypothesis can be resolved when one accounts for time-varying risk premia. However, what if the size and direction of risk premia are determined by how the market views future economic conditions? It then seems sensible that a model of the yield curve specify the *market price of risk* process as a function of market expectations. Campbell and Shiller (1991) suggest that deviations from the expectations hypothesis may be due to the correlation of time-varying risk premia with *expected* short rate increases. We thus study the forces driving the time-varying nature of risk premia by examining how market expectations shape the tolerance for risk over time. We find that macroeconomic forecasts play a significant role in explaining time-variation in risk-premia, with expected real output growth playing a dominant role that alters the dynamic configuration of risk prices whenever it is incorporated into a model.

A key advantage of arbitrage-free models is their ability to explicitly model the impact of risk premia on asset prices. The sensitivity of long term interest rates is linked with this adjustment for risk as yields are intimately tied to the expected path of the short rate under the risk-neutral probability measure Q. For example, a shock to inflation expectations are much more persistent under the risk-neutral measure Q than under the actual, data-generating measure P, explaining expected inflation's role as a "level" factor. Macroeconomic models do not account for this risk explicitly, whereas no-arbitrage finance models do! So we are able to offer a new explanation to a puzzle from the macro literature - that of the strong sensitivity of the long end of the yield curve to certain shocks that are simply not persistent enough under P.[8] We demystify the response of long yields by examining the impulse response path of the short-rate under the risk-adjusted measure Q.

Unlike most previous studies, we choose to not include any latent variables in the model. Duffee (2004) and Ang, Piazzesi, and Wei (2005c) also specify models without the help of latent variables, depending only on observable factors to drive the yield curve. Naturally, the absence of latent factors means that our ability to capture the dynamics of yield movements is somewhat diminished. Thus a contribution of this paper is to explore the performance limits of a model using only observable forecasts as state variables, without resorting to the help of latent variables. In certain cases expectations-based models improve the fit of the short and middle parts of the yield curve when compared with equivalent benchmark models using corresponding realized factors. The inclusion of the fed funds rate forecast improves the fit across all maturities when compared with an equivalent model using the lagged funds rate.

Our model provides a tool for assessing the arbitrage-free reaction of yields to shifts in market expectations. Central bankers should take great interest in these relationships. Armed with knowledge of how various yield segments respond to changes in expectations, policymakers could have a more precise influence on investors and consumers with differing planning horizons. Research in this direction will lead to a better understanding of the role that expectations play in facilitating the transmission of monetary policy through the economy. This framework may one day enable central banks to more powerfully leverage their ability to influence asset prices.

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[8]See Evans and Marshall (2002), Roley and Sellon (1995), Mehra (1996) and Gurkaynak, Sack, and Swanson (2005b). The fact that a persistent factor under Q drives movements in long yields has probably always been known in a finance setting.
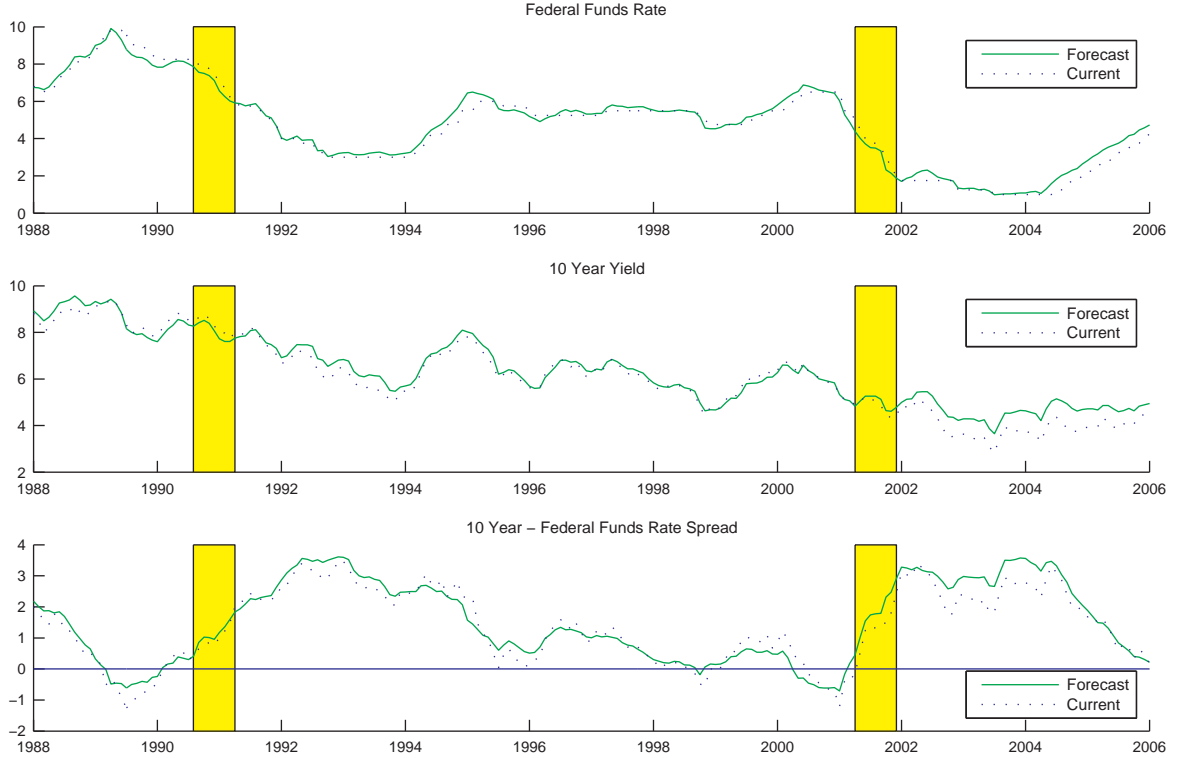
Figure 1: The mean Blue Chip Financial forecasts are shown by the solid lines. The historical, realized values at the time of the forecasts are shown by the dotted lines. The yield forecasts used in this paper are the federal funds rate and the 10-year CMT yield. The bottom panel plots the yield spread. All forecasts are for 1 quarter averages that have been adjusted to have a forecast horizon of 3-months. The two shaded regions mark the last 2 NBER recessions from July 1990 to March 1991 and March 2001 to November 2001.

It takes only a little imagination to see the potential impact of our model on the financial industry and the world's trading desks. Bond traders are constantly refining trading strategies where model inputs are based on anticipation of uncertain economic events such as monetary policy shocks or the release of new inflation, output and unemployment numbers. Perhaps in the near future, a trader can input a set of forecasts into a forward-looking model of the term structure and be presented with a system of prices that are, in theory, consistent with the absence of arbitrage.

2 Data

The Blue Chip Financial Forecasts are a survey of professional economists at leading investment banks, financial firms and economic advisory firms. The survey data, provided by Randell Moore of Blue Chip, contains analysts' views on future interest rates and the macroeconomic factors that influence them. The forecasted interest rate variables are given as the average yield over a particular quarter on Constant Maturity Treasuries (CMT) as defined by the Federal Reserve Statistical Release (FRSR) H.15 report and also includes the federal funds rate. In this study, we work with a subset of the forecasts focusing on the federal funds rate and the 10-year Treasury forecasts. We also utilize forecasts of macroeconomic variables provided in the survey: real GNP/GDP, GNP/GDP Price Index and CPI. Our data set on
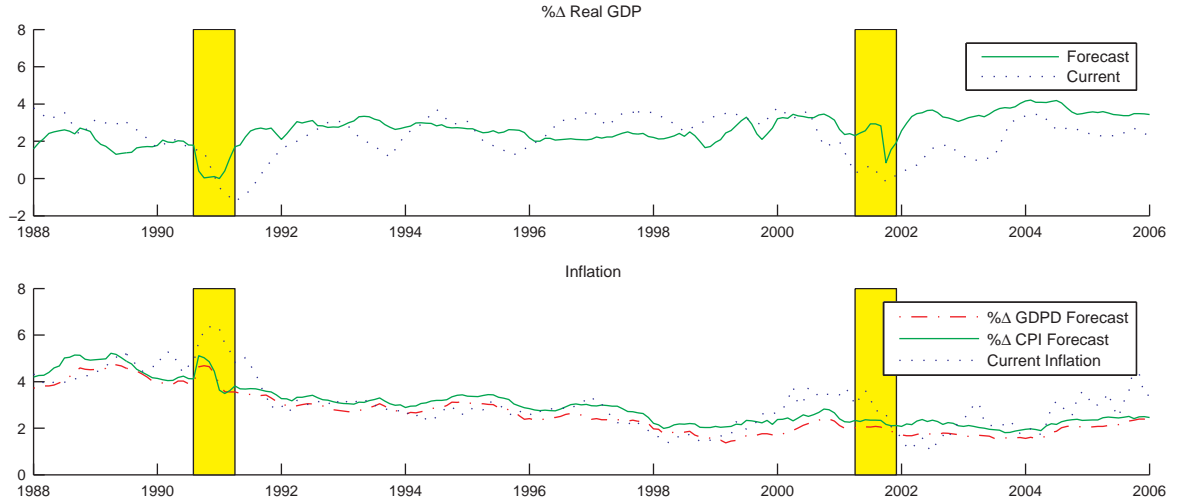
Figure 2: The macroeconomic forecasts used in the paper are the mean forecasts of the percentage changes in real GDP, GDP deflator and CPI. The historical, realized values at the time of the forecasts are shown by the dotted lines and are defined as year-over-year percentage changes in real GDP (top plot) and CPI (bottom plot). All forecasts are for 1 quarter averages that have been adjusted to have a forecast horizon of 3-months. The two NBER recessions are marked by the shaded regions.

interest rate forecasts begins in January of 1983, although the 10-year yield forecast first begins with the 1988 survey. Prior to 1992, the real output series forecast is real GNP, after 1992 the real output series forecast is real GDP. Although we will henceforth refer to the real output series as GDP in this paper, the reader should be mindful of the definition of the series prior to 1992. Forecasts of CPI, real GDP and the GDP Price Index are expressed as seasonally adjusted annualized percentage changes of quarterly averages. This says that for values of X_1 and X_2 representing seasonally adjusted averages over subsequent quarters, the forecasters are asked to forecast, $[X_2/X_1]^4 - 1$.

Each month approximately 50 forecasters are asked to make several sets of forecasts. Each analyst provides their forecast of the average realization over a particular quarter beginning with the current quarter and extending 4 to 5 quarters into the future. For example, in January the one quarter ahead forecast is for the average expected realization over April, May and June of that year. In March, the one quarter ahead forecast also pertains to the average over the same period - April, May and June. One undesirable result of this survey format is the introduction of a time-varying forecast horizon into the data. We thus make the following compromise for the purpose of extracting a time series.[9] We will fix the forecast horizon at 3 months and extract a time series as follows. In the first month of the quarter, the 3-month ahead forecast is simply the 1 quarter ahead forecast given by the survey. In the second month of the quarter, we approximate the 3 month ahead forecast by taking the average of the 1 and 2 quarter ahead forecasts with weights equal to 2/3 and 1/3, respectively. The 3 month ahead forecast made at the beginning of the final month of the quarter is taken to be the average forecast over the next 2 quarters with corresponding weights equal to 1/3 and 2/3.

The Blue Chip Financial Forecasts report the consensus forecast as the simple mean rounded to one decimal point. As opposed to employing this consensus forecast in our study, we construct a mean

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[9]This solution method is clearly not the most rigorous, yet is used because of its simplicity and practical value. A more advanced filtering method has been derived for dealing with the time-varying forecast horizon issue, yet the contribution in this setting may only be marginal and introduce other technical issues. Thus, the simpler method is chosen here and the more formal method will be reserved for a subsequent paper.

series for each variable after dropping the high and low forecaster at every point in time. This method serves to eliminate outliers and increases the numerical precision of our data. This mean forecast is taken to proxy for what we refer to as the "market expectation". Figure 1 plots the federal funds rate, the 10-year CMT yield and the spread forecasts along with the actual values of these variables at the time the forecasts were made. Note the negative yield spread immediately preceding the two NBER recessions as marked by the shaded regions. Figure 2 plots the forecasts of the macroeconomic variables used in the paper. The top plot shows the real GDP forecast along with the realized year-over-year real GDP growth at the time of the forecast, where the series has been interpolated to facilitate plotting at a monthly frequency. The bottom plot shows the 2 forecasts of inflation along with the realized year-over-year % change in CPI at the time of the forecast.[10]

The source of the historical data on the Consumer Price Index is the Department of Labor's Bureau of Labor Statistics (BLS). Historical data on real GDP and the GDP price index are taken from the Bureau of Economic Analysis (BEA). Zero-coupon yields used in the estimation are constructed from data on CMT yields, details are given in the Appendix. We constrain our study to the sample period between January 1988 and January 2006. The main reason is that the 10 year yield forecasts begin in 1988. This date conveniently also limits the impact of parameter instability across different monetary policy regimes as it marks the approximate beginning of the Greenspan era.

The Blue Chip survey provides forecasts of two variables that measure inflation - percentage change in CPI and percentage change in the GDP deflator. We will use the (scaled) first principal component of these two forecasted series as our measure of forecasted inflation. The top panel of Figure 3 shows both inflation forecasts and the extracted inflation factor. As will be further explained in the next section, for the "Taylor-based" models, we orthogonalize the two yield forecasts to give the macro forecasts the first opportunity to explain movements in bond prices. The resulting factors are shown in the bottom panel of Figure 3.
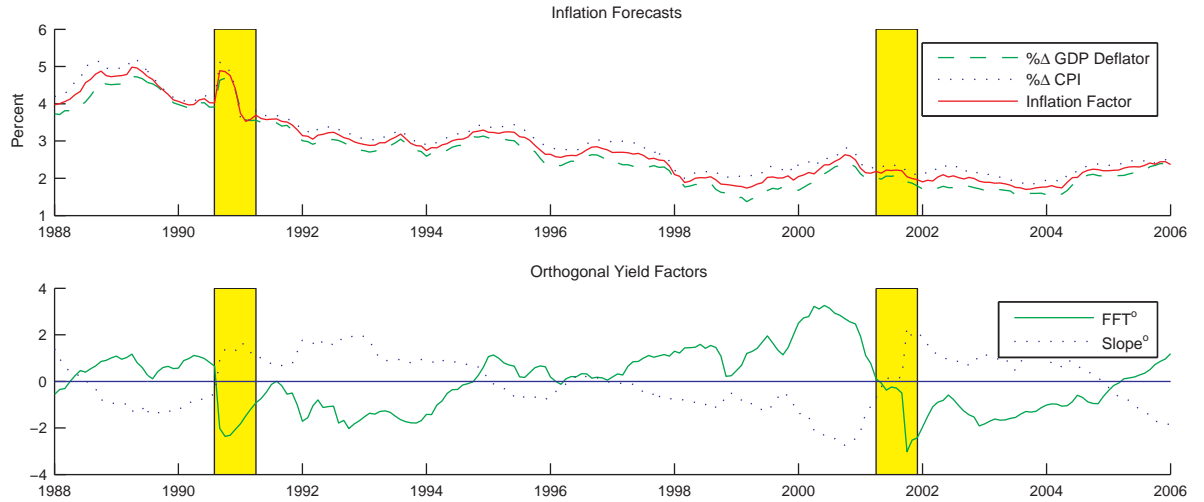
Figure 3: The top panel shows the inflation factor that is taken to be the scaled principal component of the mean forecasts of quarter-over-quarter percentage changes in the GDP deflator and CPI. The bottom panel shows the two orthogonal factors used in Models T2 and T3. The last 2 NBER recessions are shown as shaded regions.

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[10]To illustrate the level of inflation, we plot the realized year-over-year % change in CPI as opposed to the quarter-over-quarter % change in CPI as the former series is more stable whereas the latter series quite volatile. Note that for monetary policy concerns the more appropriate series for inflation would be one that measures core inflation which excludes volatile energy prices, however the Blue Chip Financial Forecasts do not survey core inflation forecasts.

Table 1: Summary of Monetary Policy Rules

Model	Instantaneous Short Rate
M1	$r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h})$
M2	$r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h}) + \rho_{13} E_t^m(\bar{g}_{t+h})$
M3	$r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h}) + \rho_{13} E_t^m(\bar{\pi}_{t+h})$
T1	$r_t = \rho_0 + \rho_{11} E_t^m(\bar{g}_{t+h}) + \rho_{12} E_t^m(\bar{\pi}_{t+h})$
T2	$r_t = \rho_0 + \rho_{11} E_t^m(\bar{g}_{t+h}) + \rho_{12} E_t^m(\bar{\pi}_{t+h}) + \rho_{13} E_t^m(\bar{f}_{t+h})^o$
T3	$r_t = \rho_0 + \rho_{11} E_t^m(\bar{g}_{t+h}) + \rho_{12} E_t^m(\bar{\pi}_{t+h}) + \rho_{13} E_t^m(\bar{f}_{t+h})^o + \rho_{14} E_t^m(\bar{s}_{t+h})^o$
Benchmark Models	
M1b	$r_t = \rho_0 + \rho_{11} s_t + \rho_{12} f_{t-}$
M3c	$r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h}) + \rho_{13} E_t^m(\bar{cpi}_{t+h})$
M3b	$r_t = \rho_0 + \rho_{11} s_t + \rho_{12} f_{t-} + \rho_{13}\, cpi_t$
T2b	$r_t = \rho_0 + \rho_{11} E_t^m(\bar{g}_{t+h}) + \rho_{12} E_t^m(\bar{\pi}_{t+h}) + \rho_{13}\, f_{t-}$

Each model in the paper is characterized by the choice of X_t in the short rate equation $r_t = \rho_0 + \rho_1 X_t$ that characterizes a continuous-time representation of a monetary policy rule. The table lists the 6 main models used in this paper plus the models used for the benchmark comparisons. E_t^m is used to denote the mean survey expectation operator proxying for the market forecast, the bar over the variables indicate that the forecasts are for a quarterly average. The factors are h-period ahead (h=3/12 or 3 months) forecasts for quarterly averages of g_t - real GDP growth, π_t - inflation, f_t - fed funds rate, s_t - 10 year minus fed funds rate. The o superscript denotes that the factor is orthogonal to the preceding factors. f_{t-} denotes the lagged funds rate.

3   Monetary Policy Rules

Each term structure model is characterized by a particular choice of state variables that drive the short-rate process, $r_t = \rho_0 + \rho_1 X_t$. If X_t is a vector of forecasts, then the short rate equation takes on the interpretation of a forward-looking policy reaction function. In this section, we develop 2 classes of models that differ in the choice of forecasts taken as policy variables. E_t^m is used to denote the mean survey expectation operator that represents the market forecast. This may not necessarily represent a mathematical expectation in the rigorous sense[11] and the forecast data are simply taken to be primitives of the model. The forecast horizon h is set equal to 3 months and the forecasts are denoted by $E_t^m[\bar{X}_{t+h}]$, where the bar denotes the quarterly average of the variables. We let g_t be real output growth, π_t inflation, f_t the fed funds rate and s_t the spread between the 10 year yield and the fed funds rate.

We denote the first class of models as "McCallum-based" models, since the first 2 factors are motivated by a forward-looking interpretation of the policy reaction function proposed by McCallum (1994). These policy variables also take on a yields-only finance-based designation as a "level" and "slope" factor. The second-class of models are denoted as "Taylor-based" models, as the baseline-model is motivated by a variant of the forward-looking Taylor-rule of Clarida, Gali, and Gertler (2000). We construct the models in this class so as to allow the macroeconomic forecasts the first opportunity to explain variations in yields.

[11]Under the assumption of a quadratic loss function the survey expectation operator is the conditional expectation of the mean forecaster. However there is some evidence that forecasters may have asymmetric loss functions, see for instance Elliot, Komunger, and Timmermann (2004).

3.1 "McCallum-based" Models

Traditional term structure studies find that a model with 2 latent factors, identified as "level" and "slope", explains much of cross-sectional and time-series movements in interest rates. Litterman and Scheinkman (1991) find that a model with 3 factors, which they call "level", "steepness" and "curvature", can capture nearly all the variation in yields. Chen and Scott (1993) conclude that although more than one factor is needed to explain changes in bond yields, in most instances a 2-factor model does almost as well as a 3 factor model. These findings are intimately linked to the policy reaction function proposed by McCallum (1994), in which the Federal Reserve smoothes interest rate changes based on the lagged interest rate, a "level" factor, while reacting to information about the future macroeconomy in the current term spread, a "slope" factor. This policy reaction function may be written as

$$r_t = \rho_0 + \rho_{11} s_t + \rho_{12} f_{t-\delta}. \tag{M1b}$$

where $f_{t-\delta}$ denotes the lagged federal funds rate, and s_t is the 10-year yield minus the fed funds rate spread. In this paper, we propose a forward-looking interpretation of this model, which takes the form

$$r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h}) \tag{M1}$$

where $E_t^m(\bar{s}_{t+h})$ is a "slope" factor and $E_t^m(\bar{f}_{t+h})$ a "level" factor. By using the federal funds rate forecast we explicitly incorporate expectations about monetary policy 3 to 6 months off into the future, while the slope factor naturally picks up longer horizon forecasts about the path of the fed funds rate and the future state of the economy.

To see if macroeconomic forecasts contain useful information once we have accounted for the first two factors, we propose a set of 3-factor models:

$$r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h}) + \rho_{13} E_t^m(\bar{g}_{t+h}) \tag{M2}$$

$$r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h}) + \rho_{13} E_t^m(\bar{\pi}_{t+h}). \tag{M3}$$

Model M2 is a forward-looking interpretation of the model proposed by Ang, Piazzesi, and Wei (2005c). In this model, we postulate that the short rate is determined by a function where the first two factors are defined as in Model M1 and the third factor, $E_t^m(\bar{g}_{t+h})$, is the forecast of real output growth. Similarly, Model M3 augments Model M1 with forecasted inflation.

3.2 "Taylor-based Models"

The second class of models ties the policy reaction function to the macroeconomy through a forward-looking Taylor (1993) rule. Clarida, Gali, and Gertler (2000) propose a forward-looking monetary policy rule where the central bank reacts to the deviation of expected inflation and expected output from their target levels. A forecast-based version of the rule can be written as follows

$$\begin{aligned} r_t &= \rho^* + \rho_{11}[E_t^m(\bar{g}_{t+h}) - g^*] + \rho_{12}[E_t^m(\bar{\pi}_{t+h}) - \pi^*] \\ &= \rho_0 + \rho_{11} E_t^m(\bar{g}_{t+h}) + \rho_{12} E_t^m(\bar{\pi}_{t+h}) \end{aligned} \tag{T1}$$

where g^* is the target growth rate of output, π^* the target inflation rate and $\rho_0 = \rho^* - \rho_{11} g^* - \rho_{12} \pi^*$. This variant of the Taylor rule suggests that the Federal Reserve targets the growth rate of output as opposed to the output gap. The possibility of this variation is mentioned in Taylor (1993). The idea that the monetary authority responds to deviations from the growth rate of output is implicit in many macro-finance models including Ang and Piazzesi (2004) and Ang, Dong, and Piazzesi (2005b). Although some may question the idea of using forecasts of "experts" as opposed to the forecasts of the Fed (which are

only made available after a considerable lag), this assumption is a reasonable approximation, especially in light of the increased transparency in policy that has more closely aligned the expectations of the market with those of the central bank. In fact, Gavin and Mandal (2001) find that the mean Blue Chip forecasts of inflation and growth are good proxies for the expectations of the Fed policymakers.

A policy reaction function based purely on inflation and GDP growth expectations is not able to fully match the dynamic properties of yields. The short rate is very persistent and the macroeconomic forecasts do not sufficiently capture this persistence. Orphanides (2003) promotes the adoption of a growth rate targeting rule augmented with the lagged interest rate and illustrates a forecast-based version using survey data. We can express this variation of the rule as follows:

$$r_t = \rho_0 + \rho_{11} E_t^m(\bar{g}_{t+h}) + \rho_{12} E_t^m(\bar{\pi}_{t+h}) + \rho_{13} f_{t-\delta}. \tag{T2b}$$

In this paper, we take this interpretation a step further by replacing the lagged funds rate in Model T2b with information about the *forecasted* funds rate. The proposed policy reaction function is given as

$$r_t = \rho_0 + \rho_{11} E_t^m(\bar{g}_{t+h}) + \rho_{12} E_t^m(\bar{\pi}_{t+h}) + \rho_{13} E_t^m(\bar{f}_{t+h})^o \tag{T2}$$

where the first two factors are defined as in Model T1 and the third factor, $E_t^m(\bar{f}_{t+h})^o$, is the component of the fed funds rate forecast that is orthogonal to the first two factors. This factor is constructed by taking the residual of an OLS regression of the fed funds rate forecast on the first two factors and a constant. By orthogonalizing this third factor, the model gives forecasted real GDP growth and inflation the first opportunity to explain movements in bond yields. This orthogonal third factor is to be interpreted as the market's anticipation of future monetary policy that is free from the "endogenous" influence of expected inflation and output growth.[12] This factor goes beyond simply capturing the persistent nature of the short rate process, it essentially says that changes in anticipated monetary policy will have an effect on the current short rate. This can be motivated by the central bank's tendency to consider market expectations about the future path of the target interest rate.[13] Hence the third factor also has the interpretation as a forward-looking "smoothing" component that facilitates policy inertia in a manner that allows the central bank to accommodate and move toward a future target path for interest rates.

Although Model T2 can be expected to nicely capture movements in short-term interest rates, when embedded within a formal term structure model it will not adequately explain movements in long-term interest rates. If there is additional information in the forecasted slope of the yield curve, say, information about other omitted variables, or information about longer horizons outcomes than captured by the first 3 forecasted factors, then including such a variable may be important. Cochrane and Piazzesi (2002) suggest that the Federal Reserve responds to long term interest rates and to the slope of the yield curve. Thus, in the spirit of the "McCallum-based" models, we allow the Federal Reserve to respond to the forecasted yield spread by extending equation T2:

$$r_t = \rho_0 + \rho_{11} E_t^m(\bar{g}_{t+h}) + \rho_{12} E_t^m(\bar{\pi}_{t+h}) + \rho_{13} E_t^m(\bar{f}_{t+h})^o + \rho_{14} E_t^m(\bar{s}_{t+h})^o \tag{T3}$$

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[12]A possible interpretation is that the Taylor rule in the mind of the forecasters captures a *contemporaneous* reaction to inflation and output growth. Hence once the federal funds rate forecast is purged of the linear influence of the macro-variables, what is left is the market forecast of the endogenous component of policy, what we call the anticipated monetary policy factor.

[13]From the Wall Street Journal: "While the Fed doesn't necessarily follow the market's lead, a look at recent history suggests Fed Chairman Alan Greenspan doesn't like to disappoint the stock and bond markets. To wit: The fed-funds futures contract has correctly predicted the outcome of the Fed's rate decision for 35 consecutive meetings, going back to September 1996, according to Bianco Research LLC." The article further quotes James Bianco, "However, this Fed is painfully aware of the risk of not giving the market what it wants." Source: Waiting for Greenspan: Fed Hopes Lift Stocks By Gregory Zuckerman. Wall Street Journal. (Eastern edition). New York, N.Y.: Mar 20, 2001. pg. C.1

where first three factors are defined as in Model T2 and the fourth factor, $E_t^m(\bar{s}_{t+h})^o$, is the component of the 10-year yield forecast minus the federal funds rate forecast that is orthogonal to the first two factors. Our use of orthogonal 3rd and 4th factors provide both forecasted real GDP growth and inflation with the first opportunity to explain movements in bond yields. With a forward-looking spin, Model T3 integrates the reaction function proposed by Taylor (1993) with that of McCallum (1994), where the federal reserve reacts to future inflation, future output growth, a forward-looking "smoothing" component and the information in the future term spread. Table 3 summarizes the two classes of models as well as a few benchmark models that will be described later in the paper.

3.3   A Note on Monetary Policy Shocks

The models in this paper do not explicitly accommodate "monetary policy shocks" as defined by the literature reviewed in Christiano, Eichenbaum, and Evans (1999). Although the macro-term structure literature takes the short rate, r_t, as a proxy for the federal funds rate, f_t, technically these two concepts are not the same. The federal funds target rate is not a part of the US treasury yield curve and, reflecting an additional premium from the credit risk associated with the overnight rate, frequently offers a higher yield than observed at the short end of the yield curve. In a strict sense, a Taylor rule describes the dynamics of the instrument of policy, $f_t = \tilde{\rho}_0 + \tilde{\rho}_1 X_t + \epsilon_t^s$, where we have included a policy shock term in the equation. In contrast, policy rules used in the macro-finance literature describe the response of the shortest maturity interest rate in the model, $r_t = \rho_0 + \rho_1 X_t$. We formalize this relationship as follows

$$f_t = r_t + \nu_t + \epsilon_t^s \tag{1}$$

which says that fed funds rate is equal to the model-implied short rate, plus a time-varying premium term, ν_t, and a monetary policy shock, ϵ_t^s. We neither explicitly study nor model the premium or the policy shock terms in this paper. We only focus on modeling the short rate, and hence will refer to the equation characterizing r_t as a *policy reaction function* or a *monetary policy rule*. This is meaningful in that the short rate equation captures the reaction of the central bank to the observable factors in the model, and a monetary policy shock constitutes a deviation from this rule.

Christiano, Eichenbaum, and Evans (1999) provide possible interpretations on the meaning of shocks to monetary policy. One explanation is that "the Fed's desire to avoid the social costs of disappointing private agents' expectations can give rise to a exogenous source of variation in policy like that captured by ϵ_t^s. Specifically, shocks to private agents' expectations about Fed policy can be self-fulfilling and lead to exogenous variations in monetary policy." The anticipated monetary factor in our paper captures the market's perception of a *future* deviation from a *contemporaneous* policy rule and represents the Fed's accommodation of market expectations as well as any policy inertia. This factor and the slope factor in Model T3, though not policy shocks in this paper, explains a portion of what others, including Ang, Dong, and Piazzesi (2005b), refer to as a monetary policy shock.[14]

In the next section we embed a monetary policy rule into a complete no-arbitrage model of the term structure. Although not explicitly modeled, the presence of a monetary policy shock will have an impact on yields of all maturities. Hence, we will assume that all yields are measured with error, where one source of error is the impact of the monetary policy shock that is not embodied by the model.

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[14]Ang, Dong, and Piazzesi (2005b) express the short rate as a function of future inflation, output and a latent yield factor, which they interpret as a monetary policy shock capturing the Fed's current deviation from a forward-looking policy rule. They find a strong correlation between their monetary policy shock, the OLS Taylor rule residuals and the 3-month short rate. They express their forward looking rule as

$$r_t = \rho_0 + \rho_{11} E_t(g_{t+h}) + \rho_{12} E_t(\pi_{t+h}) + \epsilon_t^{MP,F} \tag{2}$$

where the final term is defined to be the monetary policy shock in a forward-looking Taylor rule.

4   Term Structure Model

In this paper a monetary policy rule is synonymous with the short rate equation in a dynamic term structure model. The role of the term structure model is an important one, it provides the formal framework that allows for the study of the dynamical properties of interest rates by imposing restrictions so as to eliminate arbitrage opportunities.

The characteristics of our data justify using a continuous-time modeling approach, departing from the recent trend toward discrete-time models within the macro-finance literature beginning with Ang and Piazzesi (2004). Most discrete-time studies use historical zero-coupon data constructed from bond prices on the last business day of either the month or the quarter. This is true of studies using the Fama-Bliss data set of zero-coupon yields. Blue Chip's financial survey is typically conducted around the 24th and 25th of each month and released a few days later on the 1st. A careful reader concerned with the intricate details of timing may question a model that combines factors observed at the beginning of the month with yields observed on the last business day of the month. A discretization may not be precise, even as a rough approximation, since the market sentiment may have shifted considerably over the course of that month. For example, the July issue of the Blue Chip Financial Forecasts has already gone to print before the end-of-month June yields are realized. Hence, the July forecast better reflects the information set at the end of June than the June forecast. However, using the July forecasts to model end of June yields is not the best option either. If the forecasts further anchor market expectations or color the Fed's perception, they would only do so after the publication of the survey on the 1st of the month. In addition, it is generally the case that discretizing the yield data using a single sample point makes the data vulnerable to high frequency effects that may have no links to underlying movements in the forecasts.

In light of these issues, we employ daily observations of yield data as opposed to only using end-of-month yields. A continuous-time specification of the model mitigates the timing issue and preserves the theoretically imposed relationship between yields and a contemporaneous vector of forecasts. It is assumed that forecasts are observed monthly (which they are) and yields measured continuously (approximated here by daily observations). To accommodate this structure we modify the standard framework and show that the expected average yield over a month is an affine function of the vector of forecasts observed on the first of the month. In addition, day-to-day variation in yields are averaged out, reducing the volatility associated with high frequency noise.

We work within the affine term structure setting characterized by Duffie and Kan (1996). Section 3.1 details the set-up for the standard Gaussian model, Section 3.2 enlarges the model to work with discretely observed forecasts and continuously observed yields, and Section 3.3 provides details on the estimation procedure.

4.1   Model Set-up

An arbitrage is defined as the existence of a zero investment trading strategy yielding a strictly positive payoff in some continency with no possibility of a negative payoff. It is well known that the absence of arbitrage is technically equivalent to the existence of an equivalent martingale measure Q.[15] A standard assumption in asset pricing is the existence of an instrument that pays the continuously compounded

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[15]Harrison and Kreps (1979) show that, under technical conditions, the absence of arbitrage is equivalent to the existence of an equivalent martingale measure Q used to price any security. Duffie (2001) provides results where this equivalence requires the absence of an "approximate arbitrage" and a bounded short rate process, a requirement which is satisfied as we will assume X_t to be bounded. More general technical conditions are treated in Delbaen and Schachermayer (1994). Elegant variation in the literature results in references to the physical measure $\mathcal{P}$ as the historical, actual, true or data-generating measure and the equivalent martingale measure Q as the risk-neutral or pricing measure. As is standard, we fix a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and the *usual* information filtration $\{\mathcal{F}_t\}$.

rate of return, r_t. The price process for any security is a Q-martingale after normalization by the value of continual reinvestment at the short rate r_t. Thus the time t value of a zero-coupon bond with n years to maturity is given as an expectation taken under Q

$$P_t(n) = E^Q \left[e^{-\int_t^{t+n} r_s ds} | \mathcal{F}_t \right] \tag{3}$$

where the principal amount has been normalized to one. The instantaneous short rate is modeled as a function of an underlying N-dimensional real-valued vector of state variables, X_t, which in our case is taken to be a vector of observable forecasts. We will assume that the short-rate process is an affine function of the state vector

$$r_t = \rho_0 + \rho_1 X_t. \tag{4}$$

This takes on the interpretation of a forward-looking monetary policy reaction function where the set of policy variables is given by the vector X_t. We further assume that the dynamics of X_t under the physical measure $\mathcal{P}$ is governed by the following stochastic differential equation

$$dX_t = \mathcal{K}\left(\theta - X_t\right) dt + \Sigma d\mathcal{B}_t \tag{5}$$

where $\mathcal{K}$ is an NxN matrix, Σ is a lower triangular NxN matrix, θ is an N-vector and $\mathcal{B}$ is an N-dimensional Brownian motion under $\mathcal{P}$. The matrix $\mathcal{K}$ governs the rate of mean reversion and the vector θ represents the central tendency or the long run mean.

The attitude of the market toward risk is captured by the *market price of risk* vector λ_t which is assumed to be affine in the vector of forecasts

$$\lambda_t = \lambda_0 + \lambda_1 X_t \tag{6}$$

where λ_0 is an N-vector and λ_1 an NxN matrix. The market prices of risk relate the volatility of bond returns (corresponding to each of the underlying forecasts) with the instantaneous expected excess return on the bond. The affine assumption on the market price of risk is the Gaussian case of the "essentially affine" model of Duffee (2002). Due to the flexibility it affords in allowing the risk prices to change signs over time, this specification of the market price of risk process has been adopted by all the papers in the recent macro-term structure literature. The conceptual innovation over existing models is that our market prices of risk are forward-looking and functions of expectations, thereby reflecting a relationship between risk-premia and perceptions about *future* economic outcomes.

Assuming the absence of arbitrage, the dynamics under the equivalent martingale measure Q is given by

$$dX_t = \tilde{\mathcal{K}}(\tilde{\theta} - X_t)dt + \Sigma d\tilde{\mathcal{B}}_t. \tag{7}$$

where $\tilde{\mathcal{K}} = \mathcal{K} + \Sigma\lambda_1$ is an NxN matrix, $\tilde{\theta} = \tilde{\mathcal{K}}^{-1}(\mathcal{K}\theta - \Sigma\lambda_0)$ is an N-vector and $\tilde{\mathcal{B}}$ is an N-dimensional Brownian motion under the equivalent martingale measure Q.[16]

Given the affine specifications of the short rate and the risk-neutral drift in (7), and since $\Sigma\Sigma'$ is trivially affine, we know from Duffie and Kan (1996) that the continuously compounded yield on a bond

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[16]This follows from Girsanov's theorem by which $d\tilde{\mathcal{B}}_t = \lambda_t dt + d\mathcal{B}_t$. Substituting this into equation (5) yields equation (7). Appendix B of Dai and Singleton (2002) shows that the affine specification of λ_t is enough to satisfy a known sufficient condition for Girsanov's theorem. It can be shown that the n-period ahead pricing kernel m_{t+n} takes on the

with maturity n is affine in X_t

$$Y_t(n) = A(n) + B(n)'X_t \tag{8}$$

where $A(n)$ is a scalar and $B(n)$ is an N-vector. These coefficients are determined by $A(n) = -a(n)/n$ and $B(n) = -b(n)/n$ where $a(n)$ and $b(n)$ solve a set of Riccatti equations with initial conditions

$$da(n)/dn = \tilde{\theta}'\tilde{\mathcal{K}}'b(n) + \frac{1}{2}b(n)'\Sigma\Sigma'b(n) - \rho_0 \qquad a(0) = 0 \tag{9}$$

$$db(n)/dn = -\tilde{\mathcal{K}}'b(n) - \rho_1 \qquad b(0) = 0. \tag{10}$$

4.2 Model adjustment for using average yields

In light of the discussion in the introduction to this section we propose a method that mitigates both issues of timing and the rather arbitrary use of end-of-month yields. The unit time interval is defined to be 1 year and a month is taken to be $\frac{1}{12}$ of that interval. Define $\bar{Y}_t(n) = 12\int_t^{t+\frac{1}{12}} Y_s(n)ds$ as the average yield over the month beginning at time t. To coincide with the forecast data, it is assumed that the factors, X_t, are only observed at the beginning of each month, remaining unobserved at all other times.

The time t conditional expectation of the average yield over the monthly interval $[t, t+\frac{1}{12})$ is shown in the Appendix to be affine in X_t

$$E_t[\bar{Y}_t(n)] = A^*(n) + B^*(n)'X_t \tag{11}$$

where

$$A^*(n) \equiv A(n) + B(n)'[I - 12\mathcal{K}^{-1}(I - e^{-\frac{1}{12}\mathcal{K}})]\theta \tag{12}$$

$$B^*(n)' \equiv B(n)'[12\mathcal{K}^{-1}(I - e^{-\frac{1}{12}\mathcal{K}})]. \tag{13}$$

In this way the expected average yield is constructed to be a function of the vector of forecasts observed at the beginning of the month. This is done in a manner that preserves, for all t, the contemporaneous affine relationship between yields and the underlying vector of forecasts as given by (8).

4.3 Estimation of Observable Factor Models

We estimate the unknown parameters $\phi = \{\mathcal{K}, \Sigma, \theta, \rho_0, \rho_1, \lambda_0, \lambda_1\}$ of the model via maximum likelihood. For tractability we propose a 2 step procedure that produces consistent estimates of the parameters. In the first step we solve for the parameters, $\phi_1 = \{\mathcal{K}, \Sigma, \theta\}$, that govern the state process. In the second

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following form

$$m_{t+n} \equiv e^{-\int_t^{t+n} r_\tau d\tau} \frac{\xi_{t+n}}{\xi_t}$$

$$\frac{\xi_{t+n}}{\xi_t} = e^{-\frac{1}{2}\int_t^{t+n} \lambda_\tau' \lambda_\tau d\tau - \int_t^{t+n} \lambda_\tau' dB_\tau}$$

where $\lambda_t \in \mathbb{R}^N$ is the market price of risk process, $\xi_t \in \mathbb{R}^{++}$ and B is an N-dimensional Brownian motion under $\mathcal{P}$. Equation (3) can be expressed as $P_t(n) = E_t[m_{t+n}|\mathcal{F}_t] = E_t[m_{t+n}|X_t] = E_t[m_{t+n}]$, due to the Markovian dynamics of X_t.

step, we fix the parameters estimated in the first step and estimate the short rate and market prices of risk parameters, $\phi_2 = \{\rho_0, \rho_1, \lambda_0, \lambda_1\}$.

Let $s = t + \frac{1}{12}$. The conditional density function for our Gaussian state process (5) is multivariate normal

$$f(X_s|X_t) = (2\pi)^{-N/2}|\Omega_X|^{-\frac{1}{2}}e^{-\frac{1}{2}(X_s-E_t[X_s|X_t])'\Omega_X^{-1}(X_s-E_t[X_s|X_t])} \tag{14}$$

where the conditional mean is given by $E_t[X_s|X_t] = \theta + e^{-\frac{1}{12}\mathcal{K}}[X_t - \theta]$ and the conditional variance $\Omega_X \equiv V[X_s|X_t]$ is given in the Appendix.

Given n yields, $A = [A(1),\ldots,A(N)]'$ and $B = [B(1),\ldots,B(N)]$ depend on the parameters ϕ and are solutions to the Riccatti equations (9) and (10). A desire for parsimony notwithstanding, technical considerations limit the number of variables to include in X. The impact of relevant factors that are omitted from the model, which may include the influence of monetary policy shocks, implies that the model will not obtain a perfect fit. Further compounding the measurement error is the method used to generate constant horizon forecasts from the forecast data. Thus, from the perspective of the modeler, *all* yields are assumed to be measured with error. We stack the observed yields into a vector $\bar{Y}_t$ and define

$$\epsilon_t^Y \equiv \bar{Y}_t - A^* - B^{*'}X_t \tag{15}$$

where ϵ_t^Y is assumed distributed $N(0,\Omega_Y)$. It follows that the conditional density function is[17]

$$f(\bar{Y}_t|X_t) = (2\pi)^{-N/2}|\Omega_Y|^{-\frac{1}{2}}e^{-\frac{1}{2}(\epsilon_t^Y)'\Omega_Y^{-1}(\epsilon_t^Y)}. \tag{16}$$

Parameter estimates are obtained using a two-stage maximum likelihood procedure. The asymptotic covariance matrix for the estimates of ϕ_1 and ϕ_2 are denoted by Ω_1 and Ω_2, respectively. In the first stage, the state parameters $\phi_1 = [\mathcal{K}, \Sigma, \theta]$ are estimated by maximizing the log-likelihood function $\mathcal{L}_X = 1/M \sum_{n=1}^M log f(X_{\tau(n)}|X_{\tau(n-1)})$ where M is the number of observations in the sample minus one (as the first date in our sample coincides with the 0th observation) and $\tau(n)$ is the time of the n-th observation. We assume that the sample size is sufficiently large and compute asymptotic standard errors by estimating Ω_1 using the information matrix via the BHHH estimator.[18] In the second step, the short-rate and market price of risk parameters, $\phi_2 = [\rho_0, \rho_1, \lambda_0, \lambda_1]$, are estimated by maximizing the conditional log-likelihood function, $\mathcal{L}_{Y|X}=1/M \sum_{n=1}^M log f(\bar{Y}_{\tau(n)}|X_{\tau(n)})$, with the first stage estimates $\hat{\phi}_1$ assumed known and fixed. We concentrate the likelihood function $\mathcal{L}_{Y|X}$ and estimate Ω_Y from the sample covariance matrix of the residuals, ϵ_t^Y. Next we estimate Ω_2, the asymptotic covariance matrix for the second stage parameters via BHHH. It is not appropriate to use this estimate when computing standard errors for ϕ_2. Using a result from Murphy and Topel (1985) the correct formulation of the asymptotic covariance matrix for ϕ_2 is given by

$$\hat{\Omega}_2^* = \hat{\Omega}_2 + \hat{\Omega}_2 \left[\hat{C}\hat{\Omega}_1\hat{C}' - \hat{R}\hat{\Omega}_1\hat{C}' - \hat{C}\hat{\Omega}_1\hat{R}'\right]\hat{\Omega}_2 \tag{17}$$

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[17]The joint density function conditioned on month t information can be expressed as $f(\bar{Y}_s, X_s|\bar{Y}_t, X_t) = f(\bar{Y}_s|X_s)f(X_s|X_t)$. This follows from an application of Bayes' Rule $f(\bar{Y}_s, X_s|\bar{Y}_t, X_t) = f(\bar{Y}_s|X_s, X_t, \bar{Y}_t)f(X_s|X_t, \bar{Y}_t)$ and the observation that $f(\bar{Y}_s|X_s, X_t, \bar{Y}_t) = f(\bar{Y}_s|X_s)$ and $f(X_s|X_t, \bar{Y}_t) = f(X_s|X_t)$.

[18]The Berndt, Hall, Hall, and Hausman (1974) or BHHH estimate is also known as the outer product estimate. Computing standard errors using the Hessian method yields similar results.

where

$$\hat{C} = \frac{1}{M} \sum_{n=1}^{M} \frac{\partial log f\left(\bar{Y}_{\tau(n)}|X_{\tau(n)}\right)}{\partial \hat{\phi}_2} \frac{\partial log f\left(\bar{Y}_{\tau(n)}|X_{\tau(n)}\right)}{\partial \hat{\phi}_1'} \qquad (18)$$

$$\hat{R} = \frac{1}{M} \sum_{n=1}^{M} \frac{\partial log f\left(\bar{Y}_{\tau(n)}|X_{\tau(n)}\right)}{\partial \hat{\phi}_2} \frac{\partial log f\left(X_{\tau(n)}|X_{\tau(n-1)}\right)}{\partial \hat{\phi}_1'}. \qquad (19)$$

This correction is necessitated by our use of the estimate of ϕ_1 in estimating ϕ_2. Both $\hat{\phi}_1$ and $\hat{\phi}_2$ are consistent with asymptotic standard errors given by the square root of the diagonals of $\hat{\Omega}_1$ and $\hat{\Omega}_2^*$ respectively.

This two-step procedure may have several advantages. It is reasonable to argue that bond yields do not contribute information useful toward estimating the first stage parameters, ϕ_1, and thus the estimation is well suited for a two-step procedure. If one stage of the model is misspecified, then the entire set of estimates under the full information estimation would be inconsistent. Moreover, this procedure facilitates the estimation of models with a large number of parameters by bisecting the problem into two computationally tractable pieces. As the number of parameters increases, full information maximum likelihood quickly becomes infeasible. A two-step approach is employed in Ang and Piazzesi (2004) and Ang, Piazzesi, and Wei (2005c). Ang and Piazzesi (2004) resort to a two-step estimation after finding explosive parameters values when attempting to estimate the joint likelihood function.

The estimation results are very sensitive to choice of initial starting values. The dynamics of the continuous-time Gaussian state process can be discretized as a vector autoregression (VAR) and these estimates for $\mathcal{K}$, θ and Σ are used to initiate the first stage maximum likelihood estimates. As expected the maximum likelihood estimates only vary slightly from these VAR estimates. The second step of the estimation requires a more intricate procedure. We initialize ρ_0 and ρ_1 from the coefficients of an OLS regression of the 1 month yield on the factors. Note these initial values are not fixed in the estimation, they only serve to pin down reasonable starting values. The market prices of risk parameters are the most elusive and are very difficult to estimate. We select the initial values by randomly selecting 40,000 points from a sensible subset of the parameter space. The likelihood function conditioned on the initialized parameters is evaluated and the L points with the highest likelihood values are selected. We set L = 5 for this version of the paper. To avoid getting caught in a local optima, a random point in a medium sized neighborhood of each of the selected points is evaluated and if the likelihood increases, we move to that point, construct a new neighborhood and repeat this process 10,000 times. The L resulting intermediate estimates for ϕ_2 are used as initial starting values in L separate maximum likelihood estimations using a line-search algorithm. The entire procedure is repeated if none of the L parameter estimates coincide. The estimation results for the 2 classes of models are given in Table 2 and Table 3.

5 Empirical Results

5.1 Pricing Performance

Table 5 summarizes the ability of the different models to match historical movements in bond yields. The pricing errors, defined as the difference between model implied expected yields and realized averaged yields, are analyzed using both the mean absolute error (MAE) criterion and by examining the square-root of the mean squared errors (RMSE). The performance of the models are qualitatively similar across the two criteria. As expected, Model T1 which uses only factors derived from macro forecasts does a

Table 2: "McCallum-based" Models

Model M1: $r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h})$

$\boldsymbol{\theta}$			$\boldsymbol{\rho_0}$	$\boldsymbol{\rho_1}$			$\boldsymbol{\lambda_0}$	
2.12	3.40		-0.00	-0.10	0.95		2.50	-0.14
(1.01)	(1.47)		(0.00)	(0.02)	(0.01)		(0.2)	(0.1)
K				$\boldsymbol{\Sigma}$			$\boldsymbol{\lambda_1}$	
0.43	0.26			60.58			7.5	-33.4
(0.19)	(0.11)			(2.87)			(4.8)	(2.3)
-0.42	-0.10			-34.64	66.86		-26.7	-11.6
(0.20)	(0.13)			(4.28)	(2.96)		(2.7)	(0.9)

$\mathcal{L}_X = 9.775$ $\mathcal{L}_{Y|X} = 37.431$ $\mathcal{L}_{X,Y} = 47.206$

Model M2: $r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h}) + \rho_{13} E_t^m(\bar{g}_{t+h})$

$\boldsymbol{\theta}$			$\boldsymbol{\rho_0}$	$\boldsymbol{\rho_1}$			$\boldsymbol{\lambda_0}$		
1.63	3.36	3.15	0.01	-0.10	0.87	-0.19	11.93	-8.22	-2.93
(1.18)	(1.86)	(0.57)	(0.00)	(0.02)	(0.01)	(0.03)	(1.2)	(2.0)	(5.5)
K				$\boldsymbol{\Sigma}$			$\boldsymbol{\lambda_1}$		
0.48	0.70	1.68		53.17			17.8	-78.2	-303.1
(0.17)	(0.12)	(0.24)		(2.55)			(12.4)	(14.4)	(14.3)
-0.47	-0.52	-1.59		-20.71	64.09		-23.3	59.0	153.7
(0.17)	(0.15)	(0.37)		(4.38)	(3.20)		(11.0)	(28.9)	(4.5)
-0.16	0.48	1.60		-0.99	31.19	82.92	96.8	109.8	-204.4
(0.37)	(0.24)	(0.41)		(8.06)	(6.77)	(3.32)	(48.6)	(85.8)	(30.5)

$\mathcal{L}_X = 14.610$ $\mathcal{L}_{Y|X} = 37.772$ $\mathcal{L}_{X,Y} = 52.381$

Model M3: $r_t = \rho_0 + \rho_{11} E_t^m(\bar{s}_{t+h}) + \rho_{12} E_t^m(\bar{f}_{t+h}) + \rho_{13} E_t^m(\bar{\pi}_{t+h})$

$\boldsymbol{\theta}$			$\boldsymbol{\rho_0}$	$\boldsymbol{\rho_1}$			$\boldsymbol{\lambda_0}$		
1.82	3.59	2.20	0.00	-0.20	0.83	0.18	1.04	0.04	0.84
(1.69)	(1.66)	(0.88)	(0.00)	(0.03)	(0.04)	(0.06)	(0.4)	(0.2)	(0.7)
K				$\boldsymbol{\Sigma}$			$\boldsymbol{\lambda_1}$		
0.96	0.79	-0.92		60.48			-0.5	-30.5	37.7
(0.35)	(0.32)	(0.57)		(3.14)			(13.9)	(14.8)	(24.5)
-1.12	-0.81	1.22		-33.08	65.86		2.6	15.1	-64.4
(0.43)	(0.43)	(0.75)		(4.62)	(3.23)		(8.6)	(8.6)	(12.9)
-0.68	-0.58	0.95		0.29	15.40	33.55	73.9	42.5	-131.3
(0.32)	(0.28)	(0.40)		(3.09)	(4.07)	(0.92)	(18.4)	(17.8)	(31.0)

$\mathcal{L}_X = 15.344$ $\mathcal{L}_{Y|X} = 37.930$ $\mathcal{L}_{X,Y} = 53.274$

Note that θ has been multiplied by 100, and Σ, the lower-triangular volatility matrix, has been multiplied by 10,000 Two-step maximum likelihood estimates and asymptotic standard errors in parenthesis. $\mathcal{L}_X$ and $\mathcal{L}_{Y|X}$ denotes the log likelihood values for the first two steps. $\mathcal{L}_{X,Y}$ is the value of the joint likelihood function evaluated at the given estimates.

Table 3: "Taylor-based" Models

Model T1: $r_t = \rho_0 + \rho_{11}E_t^m(\bar{g}_{t+h}) + \rho_{12}E_t^m(\bar{\pi}_{t+h})$

$\boldsymbol{\theta}$			$\boldsymbol{\rho_0}$	$\boldsymbol{\rho_1}$			$\boldsymbol{\lambda_0}$		
2.86	2.64		0.02	-0.63	1.31		11.36	-1.56	
(0.42)	(0.75)		(0.00)	(0.09)	(0.10)		(1.9)	(1.4)	

$\mathbf{K}$			$\boldsymbol{\Sigma}$			$\boldsymbol{\lambda_1}$		
1.24	0.90		88.51			-115.4	-146.7	
(0.43)	(0.35)		(2.28)			(6.6)	(23.7)	
-0.55	-0.16		-1.63	35.82		160.9	1.1	
(0.17)	(0.16)		(2.42)	(1.32)		(7.1)	(17.3)	

$\mathcal{L}_X = 10.050$ $\mathcal{L}_{Y|X} = 35.474$ $\mathcal{L}_{X,Y} = 45.524$

Model T2: $r_t = \rho_0 + \rho_{11}E_t^m(\bar{g}_{t+h}) + \rho_{12}E_t^m(\bar{\pi}_{t+h}) + \rho_{13}E_t^m(\bar{f}_{t+h})^o$

$\boldsymbol{\theta}$			$\boldsymbol{\rho_0}$	$\boldsymbol{\rho_1}$			$\boldsymbol{\lambda_0}$		
2.79	2.56	0.31	0.04	-1.01	1.18	0.92	1.75	-5.53	-7.98
(0.48)	(0.79)	(1.64)	(0.00)	(0.03)	(0.03)	(0.02)	(3.8)	(0.7)	(1.6)

$\mathbf{K}$			$\boldsymbol{\Sigma}$			$\boldsymbol{\lambda_1}$		
1.25	0.91	0.48	88.79			-223.8	139.3	163.0
(0.54)	(0.41)	(0.25)	(3.05)			(141.9)	(144.4)	(142.9)
-0.55	-0.16	-0.16	-1.37	35.55		134.9	54.8	67.7
(0.19)	(0.17)	(0.10)	(2.56)	(1.46)		(20.4)	(39.2)	(38.0)
0.41	0.39	0.43	102.65	-26.18	60.28	206.3	38.1	17.7
(0.83)	(0.62)	(0.38)	(6.69)	(5.42)	(3.15)	(44.5)	(38.8)	(32.7)

$\mathcal{L}_X = 15.005$ $\mathcal{L}_{Y|X} = 37.034$ $\mathcal{L}_{X,Y} = 52.039$

Model T3: $r_t = \rho_0 + \rho_{11}E_t^m(\bar{g}_{t+h}) + \rho_{12}E_t^m(\bar{\pi}_{t+h}) + \rho_{13}E_t^m(\bar{f}_{t+h})^o + \rho_{14}E_t^m(\bar{s}_{t+h})^o$

$\boldsymbol{\theta}$				$\boldsymbol{\rho_0}$	$\boldsymbol{\rho_1}$				$\boldsymbol{\lambda_0}$			
3.20	1.93	0.12	-0.27	0.04	-1.18	1.08	0.72	-0.17	-15.41	-0.41	-3.14	3.27
(0.66)	(1.27)	(2.20)	(1.86)	(0.00)	(0.09)	(0.07)	(0.06)	(0.06)	(11.6)	(5.1)	(6.1)	(2.7)

$\mathbf{K}$				$\boldsymbol{\Sigma}$				$\boldsymbol{\lambda_1}$			
1.22	0.87	-0.14	-0.80	87.54				491.1	305.2	83.1	-121.8
(0.54)	(0.43)	(0.64)	(0.75)	(3.60)				(274.4)	(219.8)	(152.3)	(194.8)
-0.54	-0.16	-0.58	-0.54	-0.92	35.07			138.5	-132.9	-79.3	-7.0
(0.19)	(0.18)	(0.29)	(0.34)	(2.91)	(1.64)			(168.0)	(101.6)	(59.9)	(104.0)
0.36	0.33	-0.09	-0.69	99.83	-27.25	60.02		-11.6	5.9	45.3	1.0
(0.82)	(0.65)	(0.89)	(1.00)	(7.35)	(6.26)	(3.58)		(166.5)	(105.5)	(53.3)	(79.7)
0.51	0.27	0.97	1.27	-69.21	6.11	-21.24	47.78	-101.1	20.4	-3.1	-24.1
(0.51)	(0.45)	(0.60)	(0.70)	(4.83)	(5.22)	(3.56)	(2.25)	(60.6)	(53.4)	(47.1)	(57.0)

$\mathcal{L}_X = 20.230$ $\mathcal{L}_{Y|X} = 38.075$ $\mathcal{L}_{X,Y} = 58.305$

Note that θ has been multiplied by 100, and Σ, the lower-triangular volatility matrix, has been multiplied by 10,000 Two-step maximum likelihood estimates and asymptotic standard errors in parenthesis. $\mathcal{L}_X$ and $\mathcal{L}_{Y|X}$ denotes the log likelihood values for the first two steps. $\mathcal{L}_{X,Y}$ is the value of the joint likelihood function evaluated at the given estimates.

Table 4: Pricing Errors: MAE and RMSE

	Model	3	6	12	Maturity 36	60	84	120	Mean
MAE									
	M1	18.78	18.52	24.16	31.46	29.26	28.58	26.10	25.26
	M2	16.91	18.66	24.19	32.06	28.52	27.98	23.09	24.49
	M3	18.18	18.58	22.86	29.41	27.89	28.71	26.07	24.53
	T1	114.25	118.78	115.32	92.35	72.21	64.24	55.80	90.42
	T2	17.01	17.94	24.13	42.39	48.10	50.02	49.65	35.61
	T3	17.34	18.04	22.50	29.46	27.69	28.31	23.77	23.87
RMSE									
	M1	24.11	23.58	30.75	39.07	35.83	34.76	31.47	31.37
	M2	21.89	24.02	30.68	39.57	35.32	34.86	28.86	30.74
	M3	23.52	23.62	29.10	36.31	34.38	35.17	31.68	30.54
	T1	135.32	139.39	134.09	111.32	87.65	77.47	66.07	107.33
	T2	21.96	23.95	32.66	52.50	59.36	60.82	60.55	44.54
	T3	21.93	23.76	29.32	37.08	34.54	35.28	29.83	30.25

For each model the Mean Absolute Errors (MAE) and Root Mean Squared Errors (RMSE) are given in basis points. The mean of the pricing errors across all maturities is given in the last column.

Table 5: Benchmark Comparison: MAE

	Model	3	6	12	Maturity 36	60	84	120	Mean
MAE									
	M1	18.78	18.52	24.16	31.46	29.26	28.58	26.10	25.26
	M1b	21.95	30.13	36.51	36.23	26.21	22.87	18.69	27.51
	M3	18.18	18.58	22.86	29.41	27.89	28.71	26.07	24.53
	M3c	18.38	18.49	22.83	29.33	27.78	28.52	26.06	24.48
	M3b	21.04	28.12	34.37	34.63	25.65	23.04	20.07	26.70
	T2	17.01	17.94	24.13	42.39	48.10	50.02	49.65	35.61
	T2b	20.97	27.41	36.09	51.04	53.23	53.09	51.31	41.88

For each model the Mean Absolute Errors (MAE) are given in basis points. The benchmark models are Models M1b, M3b and T2b. Model M3c is a version of Model M3 using only the CPI forecasts. The mean of the pricing errors across all maturities is given by the last column.

poor job of matching the yield curve. Models M2 and M3 are nearly evenly matched. Model M3 is slightly superior to Model M2 for matching the middle part of the yield curve, whereas Model M2 fares slightly better on the short and long ends of the yield curve. Based on a Taylor-rule argument the federal funds rate forecast already incorporates much of the information available in expected inflation and output. However, as evidenced from comparing the results of Models M2 and M3 to those of Model M1, macroeconomic forecasts may contain subtle information not present in the funds rate forecast. The inability of Model T2 to compete with Models M1, M2 and M3 at fitting the long end of the yield curve suggests that an explicit "slope" factor is crucial for matching the dynamics of long maturity yields. The presence of a slope factor in Model T3 contributes to a dramatic improvement, when compared with the ability of Model T2 to fit the long end of the yield curve.

A model embodying expectations is useful if it outperforms the corresponding benchmark model without expectations. Historical data on real GDP and the corresponding deflator are released quarterly, only CPI is available at a monthly frequency. Due to this constraint, only Models M1 and M3 are compared with their benchmark models estimated using historical data. This is just a sketch of the comparisons, and real GDP growth is notably absent, but it does lend motivation to the idea of using forward-looking factors. Table 5.1 summaries the MAE and RMSE comparisons. The first 2 lines compare the fit of Model M1 with that of the corresponding benchmark model using only historical factors, Model M1b. The forward-looking model outperforms the benchmark model overall based on the mean MAE criteria, however the benchmark model does fare better at fitting the long end of the yield curve. This may partially be due to our choice of a 3-month forecast horizon which was based on a decision to achieve a balanced fit. We also compare a version of Model M3 that uses only CPI in defining the inflation factor, Model M3c, with the corresponding benchmark model using historical CPI data, Model M3b.[19] Once again, based on the mean MAE criteria, the forward-looking model is the better performer overall, while the benchmark model appears to have an edge at fitting the long end of the yield curve. Finally, we compare Model T2 with a version that uses the lagged fed funds rate in lieu of the funds rate forecast, Model T2b. Note that the model using the forecasted fed funds rate clearly affords a superior fit across all maturities and suggests the presence of significant information in the federal funds rate forecast that is not present in the lagged funds rate. Finally, we estimate (not reported) the "essentially affine" $\mathcal{A}_0(2)$ model of Duffee (2002) and Dai and Singleton (2002). The latent variable model fares better across all horizons and this advantage is especially pronounced for long maturity yields. Latent factor models will outperform observable factor models at fitting historical yields and this is to be expected as these models will, by construction, find the optimal set of factors that best match yield dynamics.

5.2   Market Prices of Risk

Conventional wisdom asserts that risk premia on bonds is counter-cyclical, that is recessions are accompanied by upward sloping yield curves as investors are adverse to taking on risk during bad times. A conceptual innovation in this paper is to take this idea one step further by acknowledging that investors' risk tolerances are altered when they *think* bad times are coming. In this paper, we link the market prices of risk, and hence time variation in risk premia to *expectations* about the future state of the economy.

The market price of risk vector λ_t is related to the instantaneous expected excess return on an n-period bond as follows

$$e_t(n) \equiv E_t\left[\frac{1}{P_t(n)}\frac{dP_t(n)}{dt} - r_t\right] = b(n)'\Sigma\lambda_t \tag{20}$$

where $\lambda_t = \lambda_0 + \lambda_1 X_t$, and the expectation is taken under the data-generating measure $\mathcal{P}$. See the

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[19]Here we used percentage change over the previous 12 months in the CPI as the benchmark level of inflation.
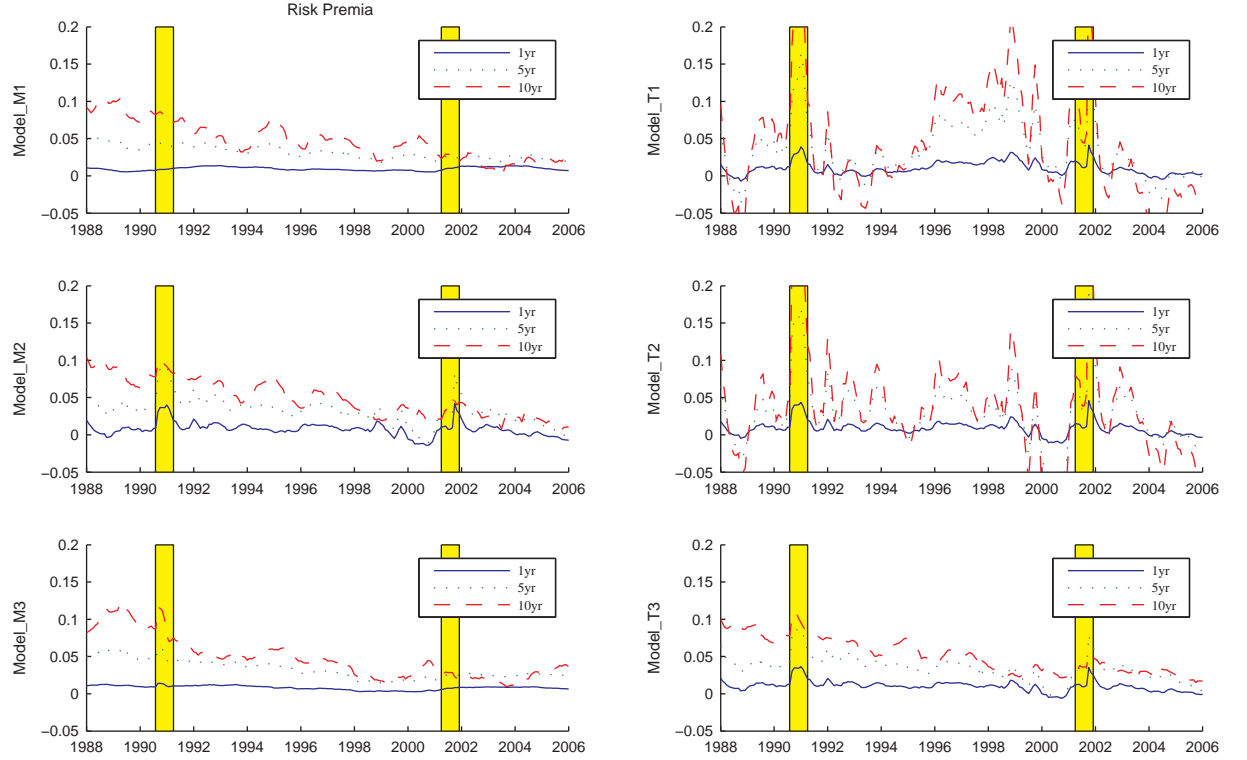
Figure 4: Plots of model-implied instantaneous excess returns as a measure of risk premia for bonds with maturities of 1 year, 5 years and 10 years.

Appendix for more details and intuition on this relationship. The vector $b(n)'\Sigma$ is the 1xN volatility matrix of the return process where the k-th element corresponds to the volatility associated with the k-th Brownian motion. Hence, the k-th element of the market price of risk vector λ_t is the "price" of risk as measured by the instantaneous expected excess return per each unit of volatility corresponding to the k-th Brownian motion driving the return process.

The dynamical properties of the market prices of risk are governed by the parameters in the λ_1 matrix. For Model M1, 3 out of the 4 market price of risk parameters in λ_1 are significant so that, in the absence of macroeconomic variables, both the slope and the federal funds factor appear to drive time variation in market risk premia. For Model M2, the estimates of the third column of λ_1 are significant and relatively large in magnitude, suggesting that market risk premia are highly sensitive to forecasted levels of GDP growth. Contrasting this with Model M1, it appears that the introduction of the GDP growth factor changes the time varying nature of the market prices of risk and diminishes the significance of the first factor in explaining excess returns. For Model M3, the relative magnitudes of the values in the third column of λ_1 are large in magnitude, suggesting that the market prices of risk are highly sensitive to the inflation factor. However, the introduction of the inflation factor results in a much more balanced contribution of significant factors driving market risk premia relative to the estimates of Model M2. Unlike real GDP growth, inflation does not possess the dominating effect of diminishing the significance of the first factor as all factors play a significant role in driving movements in the market prices of risk.

For Model T1, notice that the two significant elements of λ_1 are both associated with first factor, hence real output forecasts are the primary drivers of time variation in the market risk premia. Note

from the Model T2 estimates that GDP growth appears to be the dominant factor driving time variation in the market risk premia. However, the other factors do appear to be a significant drivers as well. The current estimates of Model T3 show several significant elements of λ_1. This model is over-parameterized and thus we withhold any conclusion at this time until the model parameters can be estimated with greater precision.
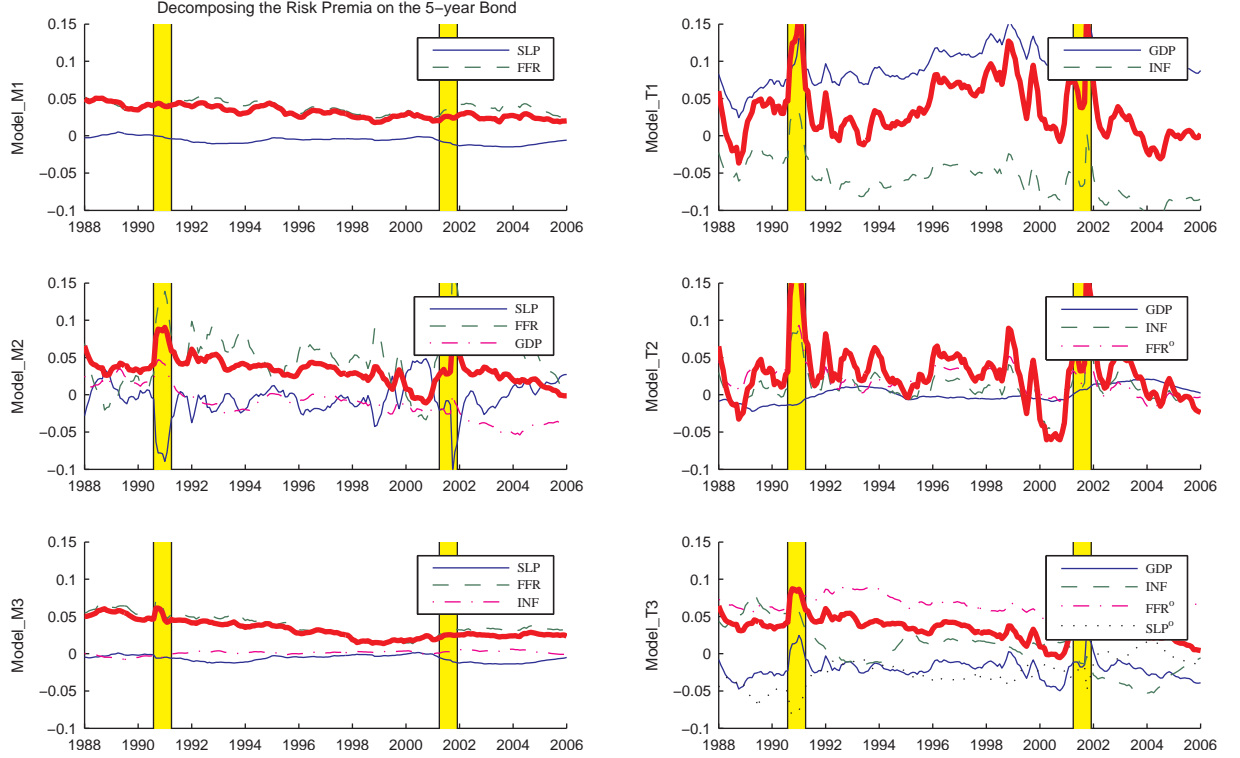

Figure 5: For each model, the figure shows the contribution of each factor to the overall expected (instantaneous) excess return of the 5-year bond.

Figure 4 plots the model-generated instantaneous excess returns, $e_t(n)$, for yields of 3 different maturities - 1, 5 and 10-years. Note that excess returns on both the 5 and 10-year bonds appear much smaller in the latter part of the sample than in the earlier part. Recalling that Models T1 and T2 do not have an explicit slope factor responsible for moving the long end of the yield curve independent of the short end, it clear from the plots that the these two models show excess return patterns that are wild and inconsistent with the other models. This reveals the inadequate nature of these two models at capturing the level of expected excess returns on long term bonds

Deeper insights into the nature of risk-premia can be obtained by breaking apart and individually plotting the time-series of each term that comprises the total expected excess return. To illustrate, Figure 5 decomposes $e_t(5)$, the expected excess return for the 5-year bond, into N terms where each term is the product of the volatility associated with the i-th risk factor and the i-th market price of risk. The sum of these individual terms equal $e_t(5)$ and is shown by the thick line in each of the graphs. We saw in teh previous section that Models T1 an T2 don't capture movements at the long end of the yield curve and thus will also not provide an accurate description of the underlying risk premia. Focusing on the other 4 Models, note that decomposing the total excess return given by the thick line into its constituent parts offers a fascinating insight into the inner workings of the models in the paper.
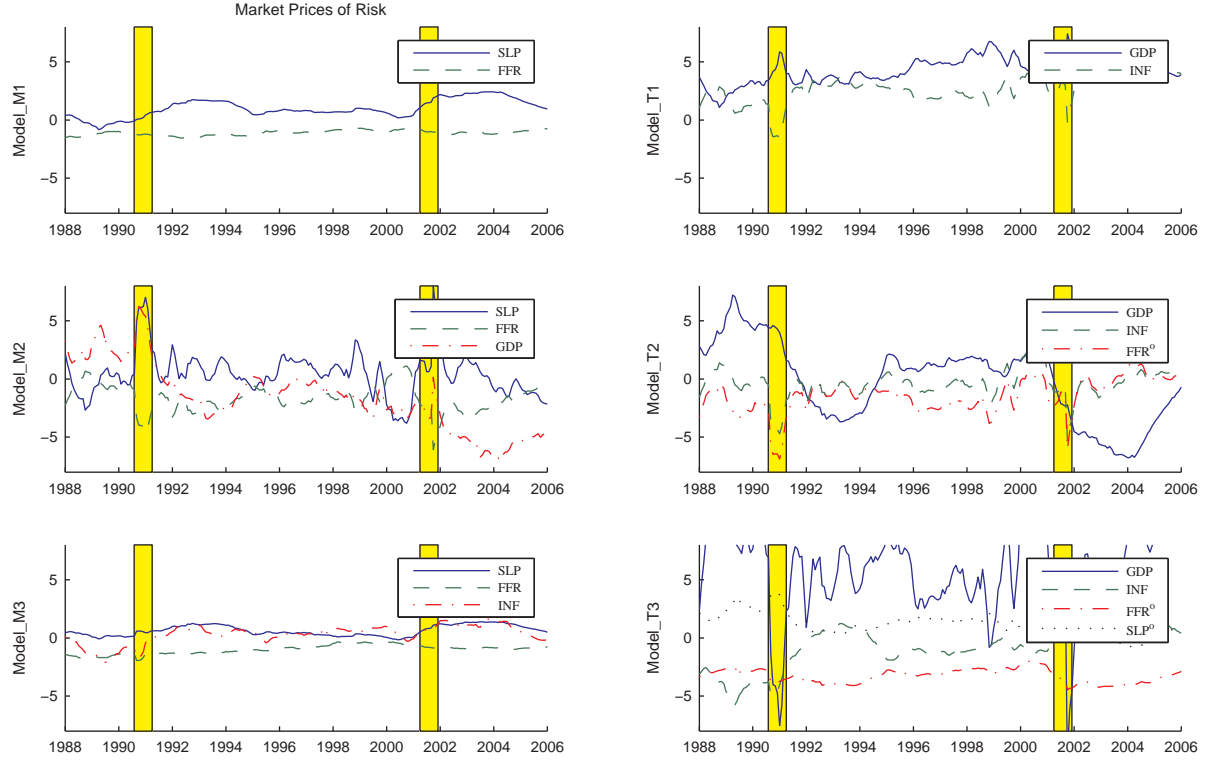
Figure 6: For each model, the figures shows time-varying market prices of risk associated with each of the model risk factors.

One of the main conclusions of the paper is that forecasted GDP growth plays a dominant role in explaining time-variation in expected excess returns. What is particularly salient from Figure 5 is that the introduction of forecasted GDP growth as a factor *does* in fact introduce variation in the individual terms comprising the expected excess return, however the impact on the total risk premia is dampened as the individual effects offset one another. To illustrate, note that the fluctuations in the components of Model M2 exhibit much more volatility than in Model M1, however this volatility is only partially reflected in the total excess return given by the thick line. Recall that Models M2, T1, T2 and T3 all incorporate forecasted GDP growth as a factor, whereas models M1 and M3 do not. What is intriguing is that in all the models where GDP growth is present, the individual components of expected excess returns exhibit large movements within the yellow bands which represent NBER recessions. Although in models M2 and T3 these differences cancel out somewhat, the total expected excess returns do increase noticeably during the NBER recessions.

We explore this further by examining the market prices of risk associated with each of the risk factors. Figure 6 plots the time-path of the estimated market prices of risk, λ_t. Since we are working within a Gaussian framework, volatilities are constant and variations in expected excess returns are only driven by variations in the market prices of risk. When comparing model M1 with model M2, note that the introduction of forecasted GDP growth induces significant additional variations in the risk prices. More generally, note that when forecasted GDP growth is present in Models M2, T1, T2 and T3, the magnitudes in the levels of the market prices of risk increase sharply during the NBER recessions. This suggests that the market price for exposure to a unit of volatility is dramatically different during recessions than during other times and that the difference is linked to perceptions about future GDP growth. One interpretation is that investors are much more sensitive to shocks during recessions than at

Table 6: Yield Equation Estimates - Factor Loadings and Initial Responses

Maturity	ρ	3	6	12	36	60	84	120
Model:M1								
Factor Weights								
A	-0.000	-0.001	-0.001	-0.001	-0.003	-0.003	-0.003	-0.003
B_1	-0.099	-0.022	0.050	0.181	0.561	0.790	0.931	1.051
B_2	0.950	0.958	0.965	0.978	1.012	1.027	1.032	1.027
Initial Response								
Factor1	-0.642	-0.569	-0.502	-0.379	-0.018	0.202	0.341	0.464
Factor2	0.950	0.958	0.965	0.978	1.012	1.027	1.032	1.027
Model M2								
Factor Weights								
A	0.009	0.006	0.004	0.001	-0.003	-0.001	0.002	0.004
B_1	-0.098	-0.009	0.075	0.228	0.653	0.875	0.990	1.081
B_2	0.870	0.907	0.937	0.983	1.042	1.025	1.000	0.978
B_3	-0.195	-0.187	-0.181	-0.172	-0.169	-0.183	-0.197	-0.212
Initial Response								
Factor1	-0.434	-0.358	-0.286	-0.152	0.251	0.479	0.604	0.704
Factor2	0.776	0.816	0.849	0.899	0.960	0.936	0.904	0.875
Factor3	-0.195	-0.187	-0.181	-0.172	-0.169	-0.183	-0.197	-0.212
Model M3								
Factor Weights								
A	0.002	-0.000	-0.003	-0.006	-0.011	-0.009	-0.007	-0.004
B_1	-0.196	-0.056	0.071	0.290	0.806	0.992	1.044	1.056
B_2	0.826	0.907	0.976	1.084	1.246	1.203	1.114	1.007
B_3	0.183	0.094	0.018	-0.101	-0.278	-0.220	-0.103	0.058
Initial Response								
Factor1	-0.648	-0.551	-0.462	-0.304	0.123	0.333	0.434	0.506
Factor2	0.869	0.929	0.981	1.061	1.181	1.151	1.090	1.021
Factor3	0.183	0.094	0.018	-0.101	-0.278	-0.220	-0.103	0.058
Model:T1								
Factor Weights								
A	0.017	0.018	0.019	0.022	0.029	0.033	0.036	0.038
B_1	-0.626	-0.617	-0.608	-0.591	-0.531	-0.484	-0.446	-0.405
B_2	1.313	1.304	1.295	1.278	1.224	1.190	1.173	1.175
Initial Response								
Factor1	-0.650	-0.641	-0.632	-0.614	-0.554	-0.505	-0.468	-0.427
Factor2	1.313	1.304	1.295	1.278	1.224	1.190	1.173	1.175
Model T2								
Factor Weights								
A	0.038	0.037	0.036	0.034	0.032	0.031	0.030	0.028
B_1	-1.006	-0.976	-0.944	-0.875	-0.615	-0.429	-0.308	-0.199
B_2	1.180	1.224	1.259	1.311	1.383	1.390	1.393	1.411
B_3	0.919	0.915	0.903	0.863	0.641	0.450	0.318	0.196
Initial Response								
Factor1	0.039	0.063	0.080	0.103	0.105	0.069	0.038	0.006
Factor2	0.503	0.550	0.595	0.675	0.911	1.058	1.159	1.267
Factor3	0.919	0.915	0.903	0.863	0.641	0.450	0.318	0.196
ModelT3								
Factor Weights								
A	0.043	0.036	0.033	0.031	0.028	0.027	0.027	0.026
B_1	-1.182	-0.981	-0.898	-0.815	-0.584	-0.406	-0.292	-0.195
B_2	1.075	1.229	1.304	1.381	1.459	1.453	1.442	1.440
B_3	0.719	0.809	0.882	0.986	1.094	1.038	0.974	0.917
B_4	-0.171	-0.111	-0.013	0.186	0.676	0.869	0.952	1.015
Initial Response								
Factor1	-0.239	0.017	0.104	0.148	0.114	0.075	0.051	0.032
Factor2	0.487	0.582	0.617	0.647	0.727	0.798	0.851	0.905
Factor3	0.779	0.848	0.886	0.920	0.855	0.730	0.637	0.557
Factor4	-0.171	-0.111	-0.013	0.186	0.676	0.869	0.952	1.015

The table lists for each model and maturity, n, the intercept term, $A(n)$, and the coefficients $B(n)$ on the yield equation $Y(n) = A + B(n)'X$. The initial responses of yields are those under Cholesky orthogonalized innovations, and as such the response of the last factor in the model is equal to its factor weight.
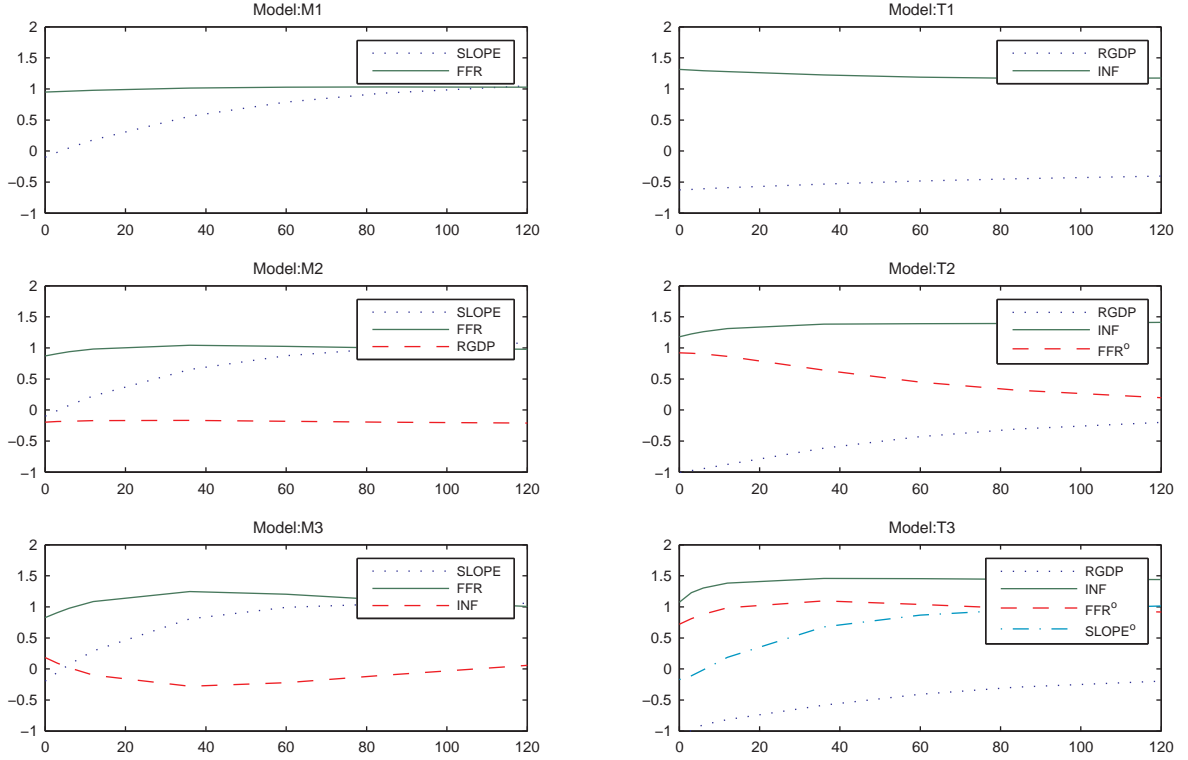
Figure 7: The first column plots the term structure of factor loadings for Models M1, M2 and M3. The second column plots the term structure of factor loadings for models T1, T2 and T3. The different "level", "slope" and "curvature" effects of the different factors across the models are visible. The factor loadings are given by the coefficients of the yield equation, $B(n)$, for each factor and maturity. Time to maturity n in months is on the x-axis.

other times. This finding is economically sensible, in that investors' aversion to risk might be different when they suspect that bad times lie ahead than when they expect the opposite. To my knowledge, this is the first term structure model to explicitly link market risk prices to the expected state of the economy and by doing so to suggest that macroeconomic expectations, especially expectations about GDP growth are strongly linked to the composition of excess returns. The figures also suggest that Model M1, a 2 factor model using only yield based forecasts, is not able to capture the variation in excess returns during recessions, highlighting an important role played by macroeconomic forecasts in these models.

The evidence of a relationship between expected GDP growth and the market prices of risk is a key contribution of the paper, as it reaffirms, within the context of a dynamic term structure model, the intuitive link between market risk premia and the expected path of the business cycle.[20]

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[20]The general finding that macroeconomic variables are important for driving time variation in the market prices of risk echoes the findings of Ang and Piazzesi (2004) and Ang, Dong, and Piazzesi (2005b). The more specific result that GDP growth is an important driver of time variation in risk premia has a partial analogue in consumption based models of asset pricing such as Duffie and Zame (1989), where risk premia are linked to uncertainty about the growth rate of aggregate endowment. This finding is also supported by Ang, Dong, and Piazzesi (2005b) who conclude that real GDP growth accounts for over half of the time-variation in excess returns. The weaker impact of expected inflation partially echoes the evidence in Duffee (2004), who concludes that market prices of risk are largely unaffected by inflation.

5.3 Factor Loadings and Initial Response Functions

How do changes in market expectations impact the shape of the term structure? Table 6 summarizes how yields of various maturities load on the individual factors as given by the coefficients on the yield equation, $Y_t(n) = A(n) + B(n)'X_t$. The first column of this table also gives the response of the instantaneous short rate, $r_t = \rho_0 + \rho_1 X_t$, to innovations in the factors. Each element of $B(n)$ captures the contemporaneous response of the yield curve to a 1-unit (or 1% when divided by 100) change in the associated state variable, holding all other variables constant. Interpreting these loadings as instantaneous responses would be fine if the factors were contemporaneously uncorrelated. However, the factors are contemporaneously correlated, and this correlation is expressed through the Cholesky lower-triangular form of the volatility matrix Σ. This correlation structure means that an innovation to a factor results in an innovation to all subsequent factors in the system. For each maturity n, we can normalize each element of $B(n)'\Sigma$ by dividing by the corresponding diagonal element of Σ. Each element of the resulting 1xN vector gives the model-implied instantaneous response of yields to a 1% innovation in the corresponding factor. To illustrate, a 1% shock to the first factor in a 2 factor model results in a $(B_1(n)\Sigma_{11}+B_2(n)\Sigma_{22})/\Sigma_{11}$ percent model-implied instantaneous response in the n-maturity yield. Table 6 also summarizes these model-implied initial responses.

5.3.1 Factor Loadings

The first column of Figure 7 plots the term structure of factor loadings for Models M1, M2 and M3 as a function of time to maturity. The interpretation of the factors in Model M1 are intuitive, the first factor behaves like a "slope" factor - a positive shock to this factor increases the slope of the yield curve by affecting long rates positively. This impact gradually decreases as maturity of the bond decreases, becoming near zero at the short end of the curve. The second factor, the forecasted funds rate, acts as a "level" factor by impacting all yields nearly identically. In Model M2, the introduction of forecasted real GDP growth has an almost negligible negative "level" across all maturities. In Model M3, the introduction of forecasted inflation possess a subtle curvature component with slightly negative loadings over the middle and slightly positive loadings over the short and long ends of the curve.

The second column of Figure 7 plots the term structure of factor loadings for Models T1, T2 and T3. In all the plots, the inflation forecast affects all yields equally and plays the role of a "level" factor. The interpretation of inflation as a "level" factor is supported by Dewachter and Lyrio (2004), Dewachter, Lyrio, and Maes (2005) and Bekaert, Cho, and Moreno (2005). Rudebusch and Wu (2003) state that their "level" factor should be interpreted as perceived future inflation by private agents. Why does expected inflation have a level effect? For now we only mention that it is due to a market adjustment for risk, and take up this question later in the paper. From the plot depicting the Model T1 loadings, it is clear that none of the macro factors are able to adequately capture the slope of the yield curve as both factors have strong level effects across the term structure.

The addition of an anticipated monetary policy factor in Model T2 enables the model to capture a "negative slope" effect. By moving short yields more than long yields, the component of the federal funds rate forecast that is orthogonal to forecasts of output growth and inflation has a negative impact on the slope of the yield curve. This finding agrees with Knez, Litterman, and Scheinkman (1994), Evans and Marshall (1998), Wu (2001b), Wu (2001a) and Bekaert, Cho, and Moreno (2005) who suggest a relationship between monetary policy innovations and the slope factor. In addition, it appears that this third factor acts to amplify the slope component inherent in GDP growth forecasts. This may appear reasonable since the slope of the yield curve is considered a leading indicator of future GDP growth, a relationship that is well known in the literature. However, the direction of the relationship shown here is perhaps counterintuitive, we return to examine this in more detail later in the paper. Although this model offers superior performance at the short end of the yield curve when compared to Model T1, the
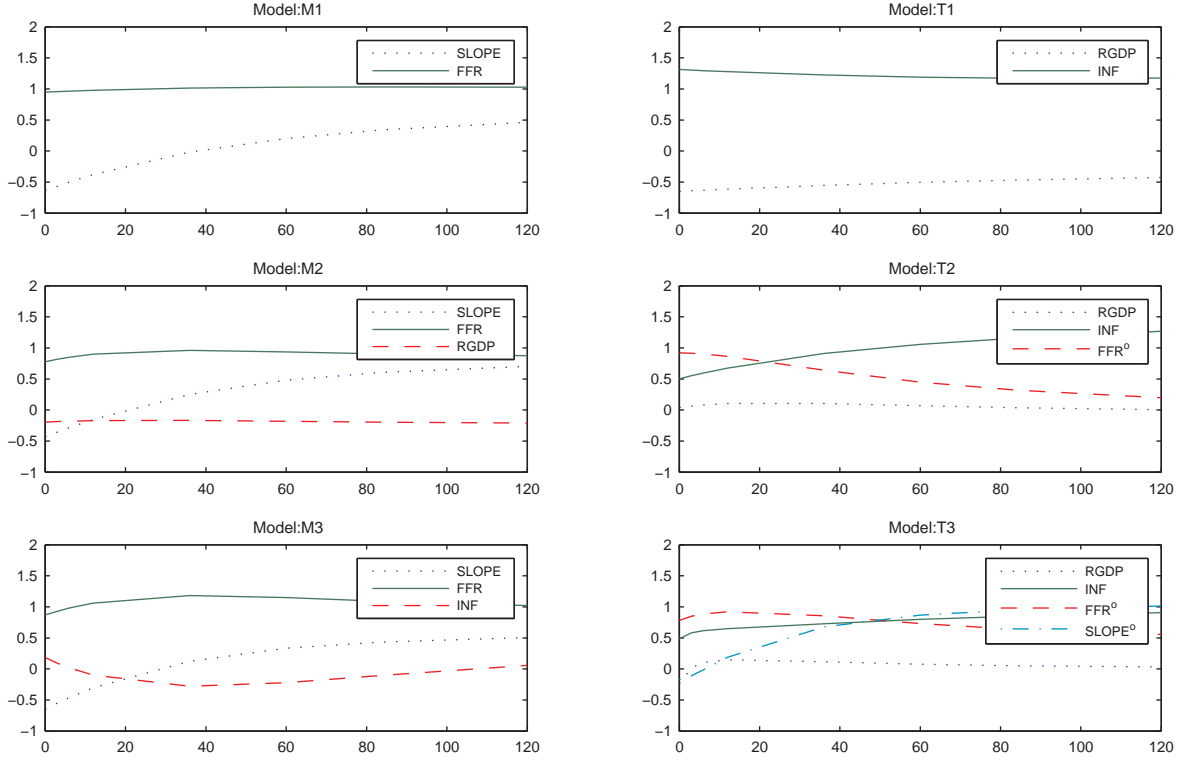
Figure 8: The first column plots the term structure of initial responses for Models M1, M2 and M3. The second column plots the term structure of initial responses for Models T1, T2 and T3. The initial response is the model implied reaction of yields to a 1% innovation in the factors. Time to maturity n in months is on the x-axis.

lack of an explicit "slope" factor, that is one that moves long yields more than short yields, means a less than satisfactory performance when matching the dynamics of long-term yields. Model T3 augments Model T2 by introducing the forecasted slope of the yield curve. The impact of this fourth factor is immediately apparent from the bottom right panel of Figure 7, where this factor shows very strong slope effects until around 40 months.

5.3.2 Initial Response Functions

Figure 8 plots the initial response of the yield curve to a 1-unit (or 1% when divided by 100) innovation in each of the factors. Recall that the lower triangular specification of Σ means that an innovation to the first variable has an immediate impact on all variables that follow. Hence, only the response of the last factor in the model will be identical to the factor weights displayed in Figure 7. Although there are many subtle differences between Figure 7 and Figure 8, the most interesting differences are seen when comparing the factor loadings to the initial responses of GDP growth forecasts in Models T2 and T3. In Figure 7, an innovation to this factor creates a slope effect by lowering the short end of the curve. In contrast, Figure 8 shows that an innovation to this factor has virtually no contemporaneous impact on yields! We will return to this fact later in the paper.

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5.4 Impulse Response Functions

Having gained some insight into the initial reaction of the yield curve to changes in each forecasted variable, we would like to further ask how the impact of these changes unfolds over time. For a 1% innovation to each forecast used as a factor we study 1) the response of the entire system of forecasts, 2) the response of the entire system of yields, and 3) the response of the short rate process under both the physical and risk-neutral pricing measures.

5.4.1 Impulse Response of Forecasts

Figure 9 plots the impulse response functions depicting the dynamic response of the model factors to a 1% innovation in each of the factors. Hence the initial response of a factor to its own shock will always be 1%.

The top panel depicts the impulse response functions of the first class of "McCallum-based" models. In all the plots note that the response of the fed funds rate forecast and the slope forecast mirror each other by exhibiting opposite response patterns. This is intuitive in that an increase in the slope forecast is often accompanied by a decrease in the fed funds rate forecast and visa versa. The market anticipates that as the slope shock dissipates, short rates rise. As shown by the plots in the first row, macroeconomic forecasts respond only ever so slightly to a slope shock. The second row depicts the response of a 100 basis point shock to the fed funds forecast. In all three plots, target rate expectations exhibit a hump-shaped response to its own shock, whereby reflecting perceptions about a gradual increase in interest rates and the smoothing/policy inertia behavior of the monetary authority. The peaks in the response are more dramatic for Models M2 and M3 than for Model M1. Romer and Romer (2000) assert that the Federal Reserve has additional information about the future of real output and inflation, thus a contractionary policy move will serve as a signal and lead to a revision in macroeconomic expectations. From the Model M2 plot, note that the initial response of forecasted GDP growth to a forecasted fed funds rate shock is positive and then decreases as the response of the expected funds rate increases. This suggests that a positive innovation to the fed funds forecast is accompanied by some information about higher than expected GDP growth. Subsequently, as the path of the expected funds rate rises expected GDP growth declines, reflecting the belief that the Fed will be successful at slowing down the economy. Focusing now on the Model M3 plot, note that the initial response of the inflation forecast to an innovation in the target rate forecast is slightly positive, gradually increasing over time before fading back to zero. The interpretation here is that a positive innovation to the fed funds forecast is accompanied by some information about higher than expected inflation leading to a slight initial revision about the subsequent path of future inflation. The bottom row of the top panel shows the response of real output and inflation forecasts to their own shocks are strongly mean reverting. An innovation to the macroeconomic forecast represent a shock that does not have a contemporaneous impact on the two yield-based forecasts in the model, a property characteristic of the final variable in the system. To illustrate we ask the reader to think about a negative shock to these forecasts. The Model M2 plot suggests that following a *negative* innovation in GDP growth expectations, growth expectations subsequently increase while the expected path of the fed funds rate decreases. This suggests a belief that the central bank may accommodate growth expectations. The Model M3 plot of the third row suggests that a *negative* innovation in inflation is followed by a period of increasing inflation, and that the market expects the Federal Reserve to intensify its reaction to anticipated price pressures. Such a scenario might arise during a period of rapid expansion, where overzealous inflation forecasts are revised downward, yet inflation is expected to rise rapidly in the future, prompting the central bank to continue a contractionary policy path.

The lower panel of Figure 9 displays the impulse response functions of the second class of "Taylor-based" models. Recall that in these models the 3rd and 4th factors have been purged of the linear

influence of first 2 factors, and play the roles of anticipated monetary policy and orthogonal slope factors, respectively. Thus the interpretation of these factors differ from the forecasts from which they were derived. The top row of the lower panel plots the response of expected GDP growth to its own shock which is marked by a high degree of mean reversion. The response of forecasted inflation is initially flat yet increases and peaks slightly after several months, indicating a perceived lead-lag relationship between real output and inflation.[21] Models T2 and T3 suggest a strong response of the anticipated monetary policy factor to an innovation in forecasted output. The second row of the figure depicts the impulse response of a shock to forecasted inflation. The own response of the inflation forecast dissipates gradually over time, while the response of GDP growth dips for a few months before rising back to zero. The anticipated monetary policy factor in Models T2 and T3 exhibits a strong yet opposite response that negatively mirrors the response of inflation to its own shock. The 3rd row of the lower panel depicts a relatively muted response of the macroeconomic variables to an anticipated monetary policy shock. Yet the subtle response patterns can be interpreted as the market's perception about the degree to which the Fed is able to meet its policy goals. If the market perceives the Fed to be slightly more effective at controlling real output growth than at fighting inflation then both the flat (perhaps slightly positive) response path of inflation and the slightly negative response path of real output would be expected.

5.4.2 Impulse Response of Yields

Figure 10 shows the impulse response of yields of various maturities to a 1% innovation in each of the factors. The intercepts of the response functions on the y-axis show the initial response of yields which is also detailed in tabular form in Table 6. Consistent with our intuition, a positive shock to the slope forecast initially shifts short yields downward and long yields upward, increasing the term spread. Over time the responses are hump-shaped and the term spread gradually diminishes. A shock to the fed funds forecast has an immediate level effect and contemporaneously shifts all yields by approximately 1%. The subsequent peak and average levels of the responses are higher for short yields than for long yields, leading to a transient period of declining slope. The third row of the upper panel exhibits a period of declining slope in response to a rise in forecasted GDP growth and a longer period of increasing slope in response to a rise in forecasted inflation.

The lower panel of Figure 10 plots the impulse response of yields for the second class of models. A positive shock to forecasted real output results in a negligible response of yields of all maturities, subsequently however, interest rates rise before eventually falling back to zero suggesting a delayed, yet positive reaction by the monetary authority to real output expectations. In Models T2 and T3, the initial responses to a revision in forecasted inflation are strongly positive and affects long yields more than short yields. The third row depicts the response of yields to an innovation in the anticipated monetary policy factor. Short yields react more than long yields, and this decrease in the slope is maintained as the response path decays. As expected, the bottom row shows a strong initial response of long yields to an othogonalized expected slope shock. The initial response of short rates is muted. However, in accordance with the view that higher long term interest rates signal higher future short rates, the response of path of short rates increases rapidly thereafter.

5.4.3 Short Rate Path Under Q and the Sensitivity of Long Term Interst Rates

Many macroeconomic models fail to reproduce the sensitivity of the long end of the yield curve to macroeconomic or monetary policy shocks, and we stress that this is due to the failure of these models

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[21]This lead-lag realtionship between the macro-forecasts may explain the negative coeficient on GDP growth in the policy reaction function, having implications for a forward-looking monetary authority choosing to target expected inflation while accomodating expected output growth. We return to this later in the paper.
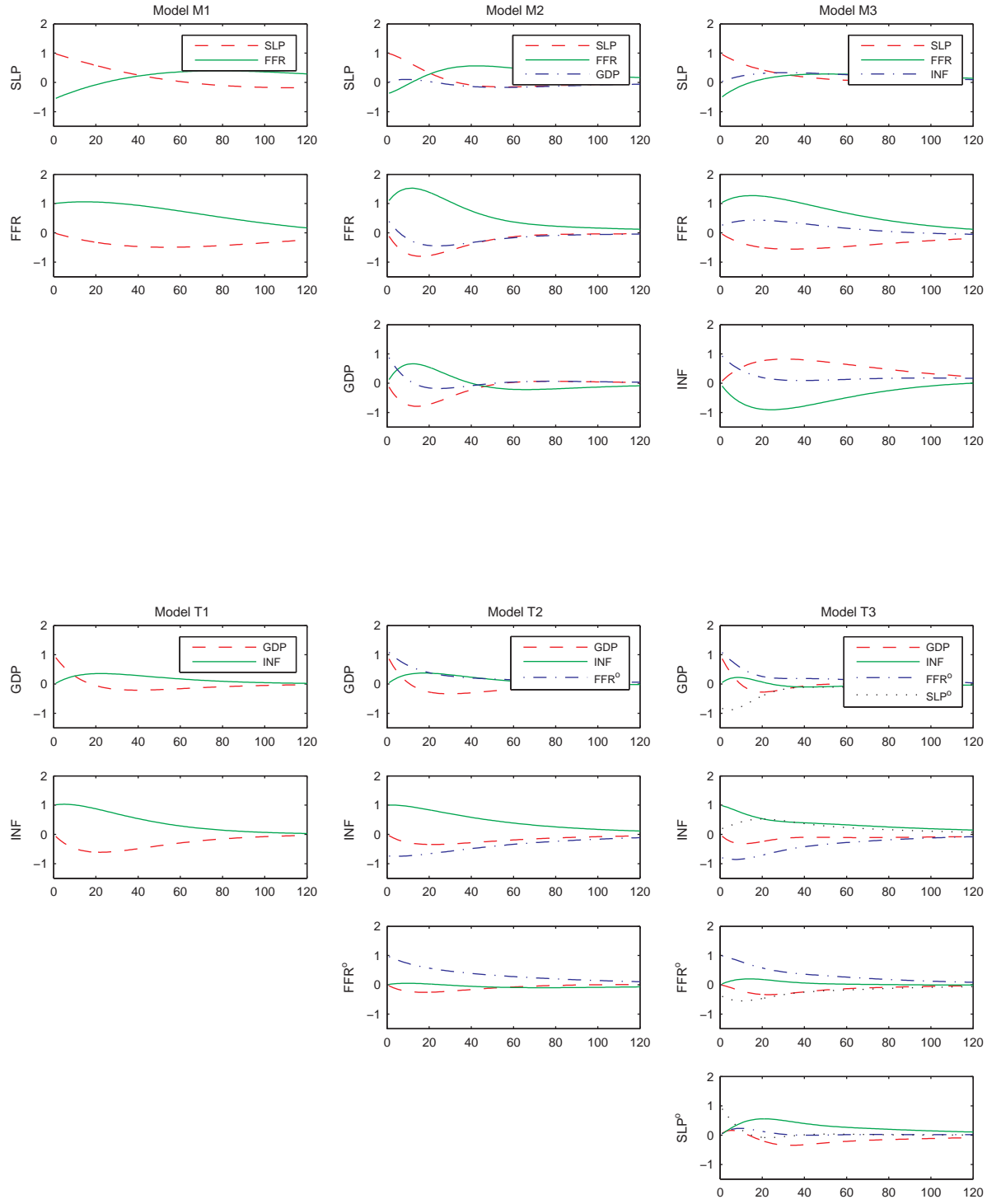
Figure 9: Impulse response functions. The plots show the impulse response for each of the model factors to a 1% innovation in each factor under the actual probability measure $\mathcal{P}$. The lower triangular specification of Σ is consistent with a Cholesky orthogonalization.
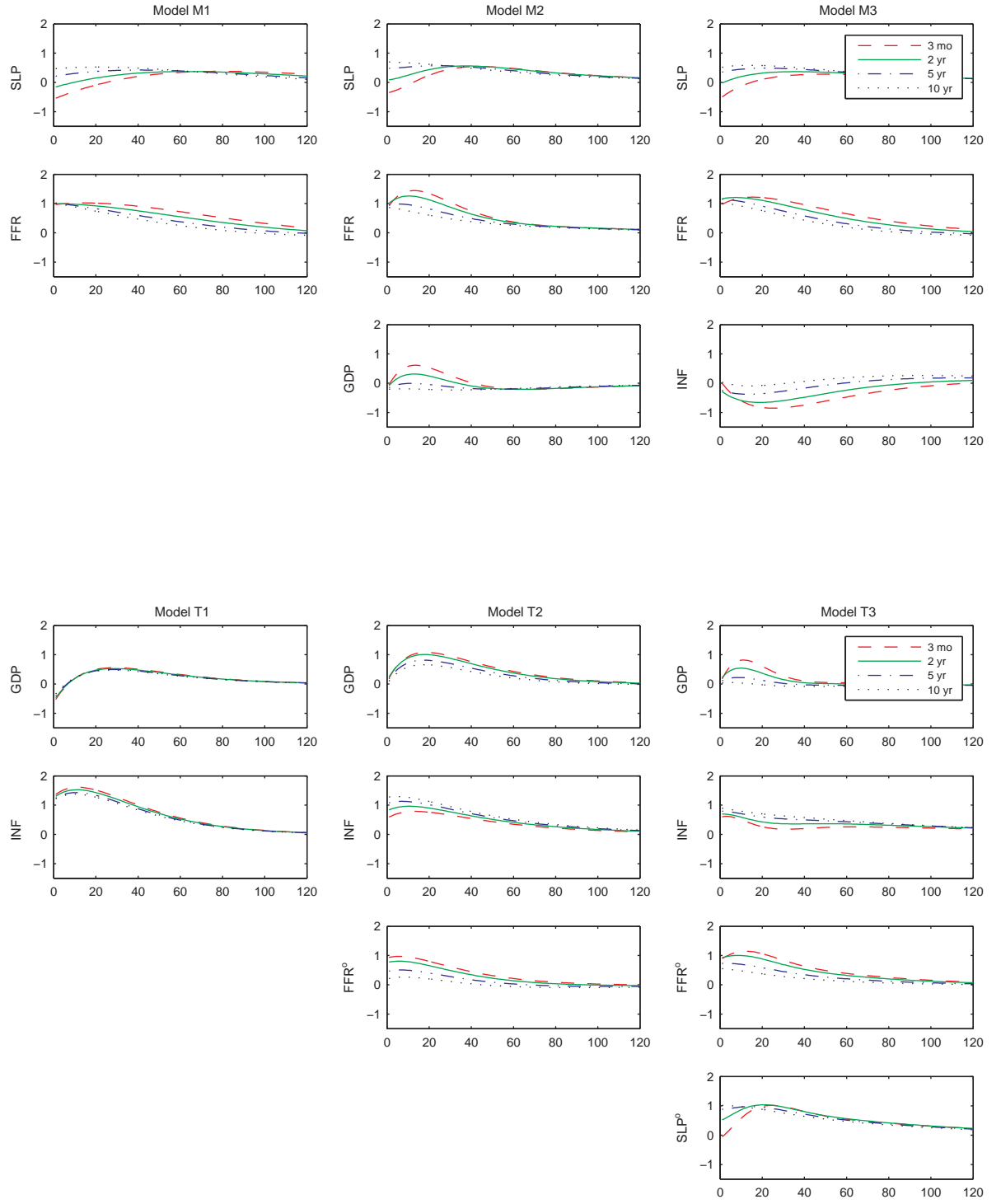
Figure 10: Impulse response functions. The plots show the impulse response for the 3 month, 2 year, 5 year and 10 year yields to a 1% innovation in each model factor under the actual probability measure $\mathcal{P}$. The lower triangular specification of Σ is consistent with a Cholesky orthogonalization.
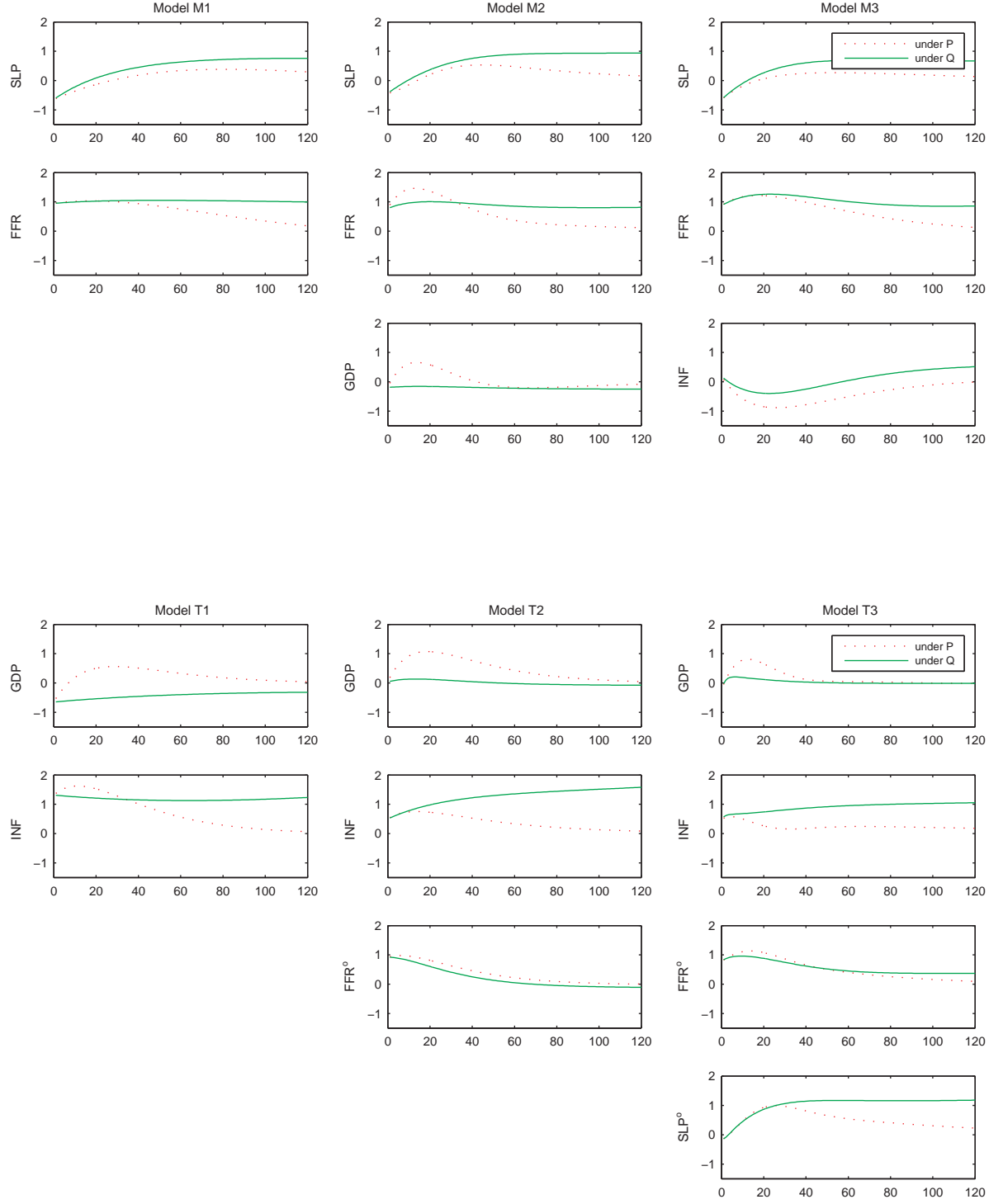
Figure 11: Impulse response functions. The plots show the impulse response for the instantaneous short rate r_t to a 1% innovation in each model factor under both the equivalent martingale measure Q and the actual, data generating probability measure $\mathcal{P}$. The lower triangular specification of Σ is consistent with a Cholesky orthogonalization.

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to adequately address the impact of risk-premia on long bond prices. It is known from the empirical term structure literature that the existence of a highly persistent, or even explosive factor under the equivalent martingale measure Q is responsible for the sensitivity of long term interest rates to shocks in the factors. Under the physical measure $\mathcal{P}$, we saw in Figure 9 that the response of all shocks have almost completely faded after 10 years, leaving them unable to explain the strong movements at the long end of the yield curve as seen in Figures 7 and 8. Although studying the dynamics of the response under the physical, data generating measure is important and interesting, insights on matters of pricing can only be obtained by studying these response patterns under the measure relavant for pricing, Q. The effect of risk-adjustment on pricing is best seen by plotting the path of the short rate process under both probability measures. From equation (3) it follows that

$$Y_t(n) = \frac{1}{n} E^Q \left[\int_t^{t+n} r_s ds \right] - \frac{1}{2} V^Q(n). \tag{21}$$

where $V^Q(n)$ is the variance term resulting from a convexity correction associated with taking the expectation of an exponential Gaussian random variable. When computing the impulse response function, we shutdown the convexity term[22] and examine the response of the first term, the expected integrated path of the short rate under the risk-neutral measure Q.

Figure 11 plots the impulse response functions of the instantaneous short rate r_t to a 1% shock under both the risk-neutral pricing and the data generating measures. Since long yields depend on the sum of expected short rates under Q, these plots shed light on the initial response of yields as previously seen in Figure 8. The upper panel of Figure 11 shows the response of the short rate in Models M1, M2 and M3. The response of the short rate to a shock in the forecasted slope factor is increasing and more persistent under Q than under $\mathcal{P}$, this explains the ability of this factor to impact long rates more than short rates. The response of the short rate to a fed funds forecast shock is relatively flat, yet more persistent under Q than under $\mathcal{P}$, explaining the origin of the "level" factor. The third row of the top panel exhibits the response of the short rate to macroeconomic forecast revisions. The response to expected real output changes is flat and nearly zero under Q, explaining the lack of an initial response of yields to forecasted output growth as seen in the Model M2 plot of Figure 8. The responses to inflation forecasts are slightly negative at the short end and positive at the long end under Q, and this explains the mild "curvature" effect in the Model M3 plot of the same figure.

The bottom panel of the figure plots the response paths of the short rate in Models T1, T2 and T3. The response pattern to a shock in the GDP forecast is relatively flat under Q, which explains the flat initial responses of Model T2 and Model T3 yields in Figure 8. The response to a shock in the inflation forecast is much more persistent under Q than under $\mathcal{P}$. Thus most of the persistence and hence much of the dynamics of the long end of the yield curve is being driven by expected inflation. As can be seen from the third row, a shock the anticipated monetary policy factor is nearly the same under P as under Q and explains its role as a negative "slope" factor. Finally, the delayed, yet strong and persistent response path resulting from a shock to the orthogonal slope factor illuminates the ability of this factor to actuate movements in long rates while leaving short yields relatively unchanged.

Most macroeconomic models predict a transient effect of monetary policy shocks on long-term interest rates. Evans and Marshall (2002) find that the response of short term rates to a one-standard deviation shock to the federal funds rate should die out almost completely after 12 months. This highlights a puzzling empirical fact that long rates exhibit a strong sensitivity to target rate movements. Roley and Sellon (1995), Mehra (1996) and Gurkaynak, Sack, and Swanson (2005b) explore the sensitivity of long-term interest rates to policy shocks. The strong sensitivity of long rates is puzzling when viewed under the high degree of mean reversion of inflation under the historical, data generating

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[22]An impulse-response function shocks one of the factors then shuts down all uncertainty and observes the response path as the system decays back to the long run mean.

measure $\mathcal{P}$. To reconcile this issue, Orphanides and Williams (2003) and Dewachter and Lyrio (2006) posit that the additional necessary persistence of inflationary expectations is generated by the learning behavior by the agents, implying a deviation from rational expectations under the data generating measure. Yet, the response of long yields is not puzzling when linked to the response of the path of the short rate under the risk-neutral measure used for pricing. Piazzesi (2005) notes that the strong response of long yields to monetary policy shocks is due to these shocks dying out slowly over time under the risk-neutral measure. From Figure 11 we see that for the first class of models a shock to the fed funds rate forecast dies out much more slowly under the risk neutral measure Q than under the actual measure $\mathcal{P}$. For the second class of models depicted in the lower panel of Figure 11, shocks to the anticipated monetary policy factor die out at the same rate under both probability measures, while shocks to expected GDP growth follow a much flatter response path under Q than under $\mathcal{P}$. Note however that the persistent impact of a shock to expected inflation under the risk-neutral measure Q will have a significant influence on long term interest rates. This suggests that the sensitivity of interest rates to a shift in monetary policy expectations is primarily due to the increased persistence of the expected path of inflation resulting from an adjustment for risk!

5.5 Forward-looking Taylor Rule Estimates

Ang, Dong, and Piazzesi (2005b) estimate no-arbitrage Taylor rules by interpreting the 3-month model short rate in a discrete-time model as the target interest rate. A study using GDP data is limited to working at a quarterly frequency as the data is only available every 3 months and this constraint justifies their adoption of the 3-month interest rate as a proxy for the policy instrument. Our use of forecast data that is observed at a monthly frequency results in forward-looking policy rules that are "operational" at a frequency that more closely resembles the current pattern of policy actions. The continuous-time formulation of our model provides an additional key advantage - the instantaneous short rate is a more natural and intuitively appealing proxy for the premium-adjusted overnight fed funds rate.[23]

Although all the models in this paper can be implemented as policy rules, Models T2 and T3 have a particularly interesting interpretation as a variant of a forward-looking Taylor rule augmented with additional anticipatory policy factors. From Table 3 we extract the model-implied estimates for our forward-looking policy rules (standard errors in parenthesis)

$$r_t = .038 - 1.01 E_t^m[\bar{g}_{t+h}] + 1.18 E_t^m[\bar{\pi}_{t+h}] + .92 E_t^m[\bar{f}_{t+h}]^o \tag{22}$$
$$\quad\; (.002)\;\; (.033) \qquad\qquad (.028) \qquad\qquad (.026)$$

$$r_t = .043 - 1.18 E_t^m[\bar{g}_{t+h}] + 1.08 E_t^m[\bar{\pi}_{t+h}] + .72 E_t^m[\bar{f}_{t+h}]^o - .17 E_t^m[\bar{s}_{t+h}]^o \tag{23}$$
$$\quad\; (.000)\;\; (.009) \qquad\qquad (.07) \qquad\qquad (.06) \qquad\qquad (.08)$$

For comparison, we replace the dependent variable in Model T3 with the target rate and obtain the following OLS estimates

$$f_t = .042 - 1.17 E_t^m[\bar{g}_{t+h}] + 1.29 E_t^m[\bar{\pi}_{t+h}] + .83 E_t^m[\bar{f}_{t+h}]^o - .17 E_t^m[\bar{s}_{t+h}]^o. \tag{24}$$
$$\quad\; (.001)\;\; (.025) \qquad\qquad (.022) \qquad\qquad (.029) \qquad\qquad (.033)$$

The two estimation results are qualitatively similar. Based on the standard errors the estimated coefficients are all highly significant. The r-square for the above "Model T3 OLS regression" is 0.990, the "Model T2 OLS regression" is 0.988 and the "Model T1 OLS regression" 0.676. Recall that our state variable X_t is constructed using expected output growth, expected inflation, the component of the expected target rate that is orthogonal to the first 2 factors, and the component of the slope forecast that

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[23]To be exact, recall the earlier discussion on the fed funds rate equaling the short rate (the policy reaction function) plus an overnight premium.
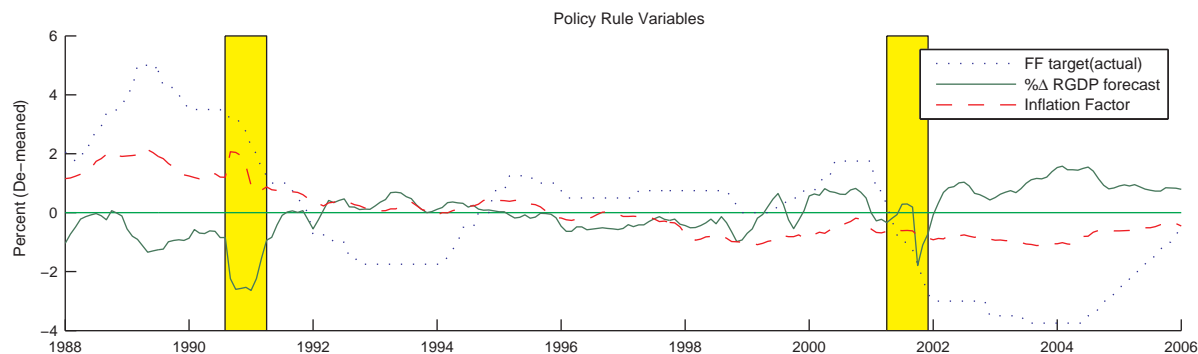
Figure 12: The plot shows as the shaded regions the last 2 NBER recessions, July 1990 to March 1991 and March 2001 to November 2001, along with a time-series of the fed funds target rate, forecasted real GDP growth and forecasted inflation. All series have been de-meaned to illustrate the relationship between the series.

is also orthogonal to the first 2 factors. An innovation to the third factor has a natural interpretation as a change in anticipated monetary policy - it is the component of an expected target rate move that is free from anticipatory movements in the underlying macro forecasts. From equations (22), a 100 basis point change in the anticipated component of monetary policy results in an approximate 90 basis point change in the current short rate. Shocks to inflation have a positive impact while shocks to real output expectations have a negative impact on the short rate. Consistent with standard intuition, high inflationary expectations prompt the Federal Reserve to react by taking a tighter policy stance as seen from the coefficient on expected inflation. This is in harmony with what we call the "Forward-looking Taylor Principle."[24] Both Models T2 and T3 indicate a more than one-to-one negative reaction, -1.01 and -1.19 respectively, to a change in expected real GDP growth. The negative impact of higher output forecasts on the short rate is puzzling at first as one might expect the Federal Reserve to raise rates in response to higher growth expectations. From Figure 12, it is apparent that when the market is expecting higher output growth 3 to 6 months off in the future then chances are we are currently at a business cycle trough or in the early stages of an expansion. Real GDP growth forecasts tend to increase during the later stages of recessions and throughout expansions when the markets are still anticipating higher output growth in the future. To accommodate these expectations the Fed continues to lower interest rates thereby stimulating and fueling the expansion of the economy. In addition, note in Figure 12 that except for the period leading up to the most recent recession, periods of higher than average target rates correspond to periods of lower than average real GDP growth. One explanation is that when the level of GDP is below trend, then the markets expect higher growth rates in the future. Yet because the market is below trend, this coincides with a central bank lowering interest rates to stimulate the economy.

Although this analysis provides insights on how yields react to a change in one variable while holding the others constant, the forecasts in the model are correlated, thus we look to the initial response function on the instantaneous short rate for insights on the actual behavior of the model implied policy rule. The initial response of the short rate is given in the first column of Table 6. The correlation structure of the innovations results in a model implied reaction that is very different than the negative one-to-one reaction implied by the coefficients. In both Model T2 and Model T3, a 1% change in the real GDP growth forecast induces a small initial reaction in the short rate. Impulse response analysis shows that this is followed by a delayed yet positive response path that peaks approximately a year after the initial

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[24]The Taylor Principle states that a stabilizing monetary policy should react more than one-to-one to changes in inflation. This is also consistent with a variant of the Fisher effect that incorporates the effect of taxes.

shock. Thus the Fed exhibits a delayed response to changes in growth rate expectations. The Model T2 and Model T3 initial responses to a change in the inflation forecast is also much smaller than what is suggested by the yield coefficients. We have learned that a central bank following the above rules will, in general, not contemporaneously react to changes in real growth expectations, since these changes are correlated with other variables in a way that balances the net reaction.

5.6   The Impact of Changes to Anticipated Policy on Yields

A policy move has two main contributing effects on the shape of the term structure of interest rates. First, an unanticipated change in the federal funds rate moves the short end of the curve, causing yields of all maturities to adjust as market participants eliminate arbitrage opportunities in the market. Second, target rate moves may change the market's expectations about the key variables influencing bond yields and subsequently alter perceptions about the future path of the short rate. This in turn changes the configuration of yields of all maturities. Federal Reserve announcements effectively augment and amplify this second effect by influencing expectations directly.

Piazzesi (2005) illuminates the first effect within a dynamic term structure model by studying the impact of unanticipated changes in monetary policy on the entire yield curve. Other studies also examine the relationship between the yield curve and federal funds rate changes, but without the assumption that yields are arbitrage-free. Cook and Hahn (1989) investigate the response of interest rates to changes in the federal funds target rate using data from 1974 to 1979. Evans and Marshall (1998) address the question using vector autoregressions and estimate the yield curve's response to Federal Reserve actions. Kuttner (2001) estimates the impact of monetary policy actions on bonds by separating the anticipated and unanticipated components of policy actions using federal funds futures prices. Roley and Sellon (1998) take a different approach and examine the effect of yields when anticipated policy actions do not occur. Cochrane and Piazzesi (2002) use differences in the 1-month Eurodollar rate to proxy for the unanticipated component of monetary policy. Romer and Romer (2000) study the impact of inflationary forecasts on the long-term behavior of interest rates in response to monetary policy shifts.

Recently, as evidenced by the ability of the fed funds futures contract to accurately anticipate target rate moves with precision, the usefulness of extracting the unanticipated component of monetary policy is waning. Thus, this paper stresses the importance of measuring the response of yields to changes in the anticipated component of monetary policy. Model T2 provides a framework that allows us to study the second effect, that is, the yield curve's reaction to changes in monetary policy expectations. Just as Romer and Romer (2004) use the Federal Reserve's internal forecasts to purge changes in the federal funds target of the influence of inflation and real activity, we use macroeconomic forecasts to purge the target rate forecast of anticipated "endogenous" influences. The loadings on this resulting factor capture the instantaneous response of yields to changes in anticipated monetary policy. Table 5.6 lists for each maturity the instantaneous response and compares this with results found in other studies. Naturally, these studies measure different variables and were conducted over different sample periods and hence are not directly comparable. However, it does provide the reader with a benchmark to judge the magnitude of the reactions. For instance, a 100 basis point change in the anticipated policy factor results in an instantaneous change of 91.7 basis points in the 1 month yield, a 86.9 basis point change for the 1 year yield, a 76.5 basis point change for the 2 year yield and a 46.4 basis point change in the 5 year yield. These results are notably higher when compared with Piazzesi (2005) who report a response of 90 basis points for the 1 month yield, 60 basis points for the 1 year yield, 41 basis points for the 2 year yield and 19 basis points for the 5 year yield. However, unanticipated shocks appears to have a stronger impact on the long end of the curve, as shown by the results of Kuttner (2001) and Cochrane and Piazzesi (2002). Similar results are obtained from the estimates of Model T3. The response of yields to changes in the anticipated policy factor fade gradually over time as was seen in Figure 10.

Table 7: Impact of Anticipated or Unanticipated Changes in Monetary Policy

	1	3	6	12	Maturity 24	36	60	84	120
Model T2	91.9	91.5	90.3	86.3	75.5	64.1	44.9	31.8	19.6
Piazzesi	90			60	41		19		
Kuttner		79.1	71.6	71.6	61.4		48.1		31.5
Cook &Hahn		26.8	21.9	19.8	18.2		10.4		4.3
Cochrane & Piazzesi		62		72		67			52

Reported are reactions to changes in anticipated monetary policy implied by Model T2, responses to policy shocks as reported by Piazzesi (2005), responses to the unanticipated component of policy movements computed by Kuttner (2001), results of Cook and Hahn's original regression to extended data as reported by Kuttner (2001), and responses to changes in the one month eurodollar rate as given in Cochrane and Piazzesi (2002).

5.7   Implications for Policy

To understand the reaction of the yield curve to both policy moves and federal reserve announcements, it is first essential to study the pivotal role of expectations in determining the arbitrage-free relationships across bond prices. This paper takes a first step toward providing a framework for assessing the impact of market expectations on yields of all maturities in a manner that is consistent with the absence of arbitrage. Standard macro-finance models are limited in their use as policy instruments as the central bank has virtually no control over the observable economic quantities used as factors. Given that policy statements and announcements can impact asset prices by shifting expectations, forecast-based models that translate observable expectations into prices may allow central banks to more effectively conduct monetary policy. Many have argued that the effectiveness of the policy rule can be augmented through the role of market expectations, see for example Taylor (2001), Gurkaynak, Sack, and Swanson (2005a), and Kohn and Sack (2003). By understanding the reaction of yields to changes in market expectations, the Federal Reserve could augment and amplify interest rate targets by issuing statements that are consistent with its policy objective.

In an era of increased transparency, this suggests that policy statements simply serve as a guide for anchoring market expectations by revealing the central bank's stance on the future path of both monetary policy and the macroeconomy. A stronger implication, which is beyond the scope of this paper, involves the idea of the central bank better managing expectations so as to impact longer-maturity yields and hence the behavior of consumption and investment across agents with different planning horizons. This raises a new set of questions. How should the central bank optimally fine tune the behavior of interest rates at this level? In light of the Lucas critique, to what extent could the monetary authority influence expectations without affecting the stability of the parameters? A deeper foray into these issues could furnish central bankers with an essential means for managing the entire yield curve thereby facilitating the achievement of policy goals. This paper proposes a tool to begin studying these relationships, and leaves these deeper questions for future research.

By building forecast-based models of the yield curve, market traders too could assess the impact of both open market and "open mouth" operations on yields within an arbitrage-free framework. In practice, the survey data used in the paper may naturally be augmented or replaced by more refined, higher frequency measures of market sentiment. This too is the topic of future research.

6 Conclusion

The term structure of interest rates reacts to forward-looking information about the future path of the economy, and is often linked to predictions about future short rates, output and inflation. However, the forward-looking components in standard macro-finance models are estimated and based on historical information (current macroeconomic quantities are based on averages over the past period). As such they do not explicitly respond to shifts in market sentiment due to news, information or Federal Reserve announcements. This paper contributes to the literature by explicitly incorporating forward-looking information (from observable survey-based expectations) into a dynamic term structure model, thus formally linking yields of all maturities to a set of underlying forecasts. By only incorporating observable forecasts as factors, we are able to explore the limits of a model's ability to explain yield movements absent the supporting role of latent variables. Benchmark comparisons of our forward-looking model with corresponding models using only historical data suggest that the performance of a class of observable factor models can be enhanced by incorporating information imbedded in forecasts.

From a modeling perspective, our specification affords several advantages over extant models. First, we allow full feedback between all factors in the model. This is an improvement over many previous macro-finance models of the yield curve that limit the feedback between different subsets of factors. Second, we allow all factors to affect the market prices of risk. This is also an improvement in flexibility over many existing models of the yield curve, where the coefficient matrix, λ_1 is often restricted. Third, our continuous-time specification allows us to fit our model to a mixture of monthly observed forecasts and daily observed yields averaged over the month. This bypasses the rather arbitrary convention of using yield data from the last business day of the month. This also mitigates the intricate timing issues inherent in working with forecast data of this nature.

This paper expands our understanding of the relationship between expectations, interest rates and monetary policy by generating key insights into a forward-looking policy function. A positive coefficient on the inflation factor is consistent with the usual interpretation of the central bank working to counteract inflationary expectations. The negative coefficient on the GDP growth factor suggests that the Federal Reserve accommodates GDP growth expectations, however due to the correlation structure of the factors, the actual contemporaneous response of the target rate is extremely small. Nevertheless, impulse response analysis suggests that the monetary authority responds in a gradual and delayed manner to expected GDP growth. We also capture the impact on the term structure to changes in the anticipated component of future monetary policy. The focus on measuring responses to anticipated movements is important in light of the increased transparency of monetary policy and the ability of the market to anticipate policy moves. We find that this factor contributes by way of a negative "slope" factor that moves short maturity yields more than long yields. Yet our results indicate that the presence of an explicit "slope" factor is essential for fitting the dynamics of long yields. This paper also makes explicit a relationship between time varying risk-premia and expectations. We find that expected GDP growth is an important driver of time-variation in excess returns as its appearance in a model tends to dominate the role of the other factors driving the market prices of risk. The persistence of a shock to inflation under the risk-neutral pricing measure explains the sensitivity of long yields to an innovation in monetary policy expectations, in effect shedding light on a longstanding puzzle from the Macro-literature.

We conclude by noting that models of the type suggested in this paper may allow a forward-looking central bank to more effectively manage the entire yield curve. The financial markets are highly sensitive to shifts in expectations and policy makers are aware of this. By augmenting policy with credible announcements that are consistent with their policy objectives, the monetary authority can influence yields of all maturities. This paper has provided a new framework for assessing and examining these relationships in a manner consistent with the absence of arbitrage. Future research on explicitly integrating observable forecasts into arbitrage-free models of the term structure will provide traders,

market participants and policy makers with a powerful set of tools for understanding the links between expectations, bond yields and monetary policy.

A   Extracting zero-coupon yields

Our data set of historical zero-coupon yields are derived from Constant Maturity Treasury(CMT) securities computed by the U.S Treasury and published in the Federal Reserve Statistical Release H.15 report. The CMT yields are interpolated from the daily yield curve that is based on the closing market bid yields on actively traded Treasury securities in the over-the-counter market. The yields from these traded securities are calculated from composites of quotations obtained by the Federal Reserve Bank of New York. Yields on on-the-run securities trade close to par and these are designated as knot points in an interpolation algorithm. The interpolation algorithm used by the Treasury is referred to as a "quasi-cubic Hermite splines" method, and follows a piecewise interpolation algorithm developed in Akima (1970). The inclusion of off-the-run securities that are not necessarily trading at par means that the treasury curve is not a par yield curve in the strict sense. We view the difference as insignificant for our purpose and, at the suggestion of the US Treasury, will simply take the CMT yields as yields on par bonds.[25] All H.15 yields are annualized using a 360-day year (or bank interest) and reflect semi-annual compounding.

From this data, we extract implied continuously compounded zero-coupon yields for use in our model. We transform the raw data and back out an implied zero-coupon curve by employing the interpolation method used by the US Treasury. First we interpolate according to Akima (1970) using the known CMT yields as knot points. From this curve it is fairly straight forward to extract the implied zero curve by first extracting the yield on a 1.5 year zero and using that to compute the yield on a 2 year zero, then using these two yields to compute the yield on the 2.5 year zero and so on. Recall that the 3 month, 6 month and 1 year treasury bills do not pay a coupon and hence require no adjustment. Coupons on treasuries of longer maturities are paid semi-annually. All H.15 release yields are semi-annually compounded yields.

We illustrate the above procedure with an example. In January of 1983, the H.15 release figures for the 6 month and 1 year (zero-coupon) yields are $y_{.5}^s = 8.33$ and $y_1^s = 8.62$ percent, where the s superscript denotes the yield is semi-annually compounded. The interpolated value for the 1.5 year treasury yield is calculated to be 9.33 percent. Since a note of this maturity pays a coupon, we need to make an adjustment. Recall that for a par bond, the coupon rate is equal to its yield and its price is equal to par. Therefore, we can extract the implied yield on a 1.5 year zero by solving for $y_{1.5}^s$ in the following equation:

$$1 = \frac{.0933/2}{(1 + .0833/2)} + \frac{.0933/2}{(1 + .0862/2)^2} + \frac{1 + .0933/2}{(1 + y_{1.5}^s/2)^3} \tag{25}$$

If we knew the yield to maturity on a 1.5 year zero, we could plug it into the above equation and the equality would obtain, if this were not the case one could construct an arbitrage using the 3 underlying zero coupon bonds. Thus the solution to this equation of $y_{1.5}^s = 9.01\%$ is the yield on a 1.5 year zero. Similarly, one can iterate forward and extract the yield on a 2 year zero, and in this fashion construct the entire zero-coupon yield curve. We convert all semi-annually compounded yields to continuously compounded yields, y_n, using the conversion $y_n = -\frac{1}{n}log((1 + y_n^s/2)^{-2n})$. This is done for each business day and the average over the month is computed for use in the model.

B   Conditional Moments

The conditional moments for a Gaussian process are well known from Duan and Simonato (1999), Fisher and Gilles (1996a) and Fisher and Gilles (1996b), see Duffee (2002) for an alternate derivation.

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[25]We thank Fred Adams of the Office of Debt Management at the Department of the Treasury for extensive insights on the construction of the Constant Maturity Treasury (CMT) series contained in the Federal Reserve Statistical H.15 report.

B.1 Conditional Mean

Write the stochastic differential equation in integration notation $X_s = X_t + \int_t^s \mathcal{K}(\theta - X_v)dv + \int_t^s \Sigma d\mathcal{B}_t$ and take expectations conditional on X_t. Recall that the second term has zero expectation. Thus,

$$E\left[X_s|X_t\right] = X_t + E\left[\int_t^s \mathcal{K}(\theta - X_v)dv|X_t\right] \tag{26}$$

$$= X_t + \int_t^s \mathcal{K}(\theta - E\left[X_v|X_t\right])dv \tag{27}$$

It follows that the conditional expectation is the solution to the following ordinary differential equation

$$\frac{dE\left[X_s|X_t\right]}{ds} = \mathcal{K}[\theta - E[X_s|X_t]] \tag{28}$$

with initial condition $E[X_t|X_t] = X_t$. Let $\Psi(s) = E[X_s|X_t]$ then we can write $\Psi'(s) + \mathcal{K}\Psi(s) = \mathcal{K}\theta$ and by multiplying both sides by $e^{\mathcal{K}(s)}$ we have $\frac{d[e^{\mathcal{K}(s)}\Psi(s)]}{ds} = e^{\mathcal{K}(s)}\Psi'(s) - e^{\mathcal{K}(s)}\mathcal{K}\Psi(s) = e^{\mathcal{K}(s)}\mathcal{K}\theta$. By integrating we obtain $e^{\mathcal{K}(s)}\Psi(s) = \int e^{\mathcal{K}(s)}ds\mathcal{K}\theta + c$, where c is a constant of integration.

Assume $\mathcal{K}$ is diagonalizable. Let U be the matrix with the eigenvectors of $\mathcal{K}$ in its columns, then if U is linearly independent, we can write $e^{\mathcal{K}(h)} = Ue^{\Lambda h}U^{-1}$ where Λ is a diagonal matrix with the eigenvalues of $\mathcal{K}$ on the diagonal. Thus, $\int e^{\mathcal{K}(s)}ds = U\int e^{\Lambda(s)}dsU^{-1}$ which follows from pulling the constant matrices out of the integration. Integrating element-by-element along the diagonal matrix $e^{\Lambda(s)}$, yields $\int e^{\Lambda_{ii}(s)}ds = \frac{1}{\Lambda_{ii}}e^{\Lambda_{ii}(s)}$ which are the diagonal elements of the diagonal matrix, $\Lambda^{-1}e^{\Lambda(s)}$. Hence, $\int e^{\mathcal{K}(s)}ds = U\Lambda^{-1}e^{\Lambda(s)}U^{-1} = U\Lambda^{-1}U^{-1}Ue^{\Lambda(s)}U^{-1} = \mathcal{K}^{-1}e^{\mathcal{K}} = e^{\mathcal{K}}\mathcal{K}^{-1}$. Thus we can write $e^{\mathcal{K}(s)}\Psi(s) = e^{\mathcal{K}(s)}\theta + c$. Using the initial condition, $\Psi(t) = X_t$ we have $c = e^{\mathcal{K}(t)}[X_t - \theta]$, hence $e^{\mathcal{K}(s)}\Psi(s) = e^{\mathcal{K}(s)}\theta + e^{\mathcal{K}(t)}[X_t - \theta]$ yielding the solution

$$E[X_s|X_t] = \theta + e^{-\mathcal{K}(s-t)}(X_t - \theta) \tag{29}$$

where $s - t = \frac{1}{12}$ given monthly observation intervals.

B.2 Conditional Variance

We follow Fisher and Gilles (1996a) and Fisher and Gilles (1996b) and derive the variance $V_t(X_s)$ of the state vector X_s conditioned on X_t. Since the conditional mean under the $\mathcal{P}$ measure is given by

$$E[X_s|X_t] = \theta + e^{-\mathcal{K}(s-t)}(X_t - \theta) \tag{30}$$

and

$$\frac{\partial E[X_s|X_t]}{\partial t} = e^{-\mathcal{K}(s-t)}\mathcal{K}[X(t) - \theta], \ \frac{\partial E[X_s|X_t]}{\partial X_t} = e^{-\mathcal{K}(s-t)}\mathcal{K}, \ \frac{\partial^2 E[X_s|X_t]}{\partial X_t^2} = 0 \tag{31}$$

we have from Ito's Lemma

$$dE[X_s|X_t] = [e^{-\mathcal{K}(s-t)}\mathcal{K}(\theta - X_t) - e^{-\mathcal{K}(s-t)}\mathcal{K}(X_t - \theta)]dt + e^{-\mathcal{K}(s-t)}\Sigma d\mathcal{B}_t \tag{32}$$

$$= e^{-\mathcal{K}(s-t)}\Sigma dB_t. \tag{33}$$

Rewriting in integration notation

$$X_s = E[X_s|X_s] = E[X_s|X_t] + \int_t^s dE[X_s|X_v] = E[X_s|X_t] + \int_t^s e^{-\mathcal{K}(s-v)}\Sigma dB_v \tag{34}$$

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and taking the conditional variance of both sides, we obtain

$$V_t[X_s] = V_t \left[\int_t^s e^{-\mathcal{K}(s-v)} \Sigma d\mathcal{B}_v \right] \tag{35}$$

$$= E \left[\int_t^s e^{-\mathcal{K}(s-v)} \Sigma \Sigma' e^{-\mathcal{K}(s-v)'} dv | X_t \right] \tag{36}$$

$$= \int_t^s e^{-\mathcal{K}(s-v)} \Sigma \Sigma' e^{-\mathcal{K}(s-v)'} dv \tag{37}$$

where the second equality follows from an extension of the Ito isometry to multi-dimensions.[26]

Left in this form the solution is not very useful, so let U be the matrix with the eigenvectors of $\mathcal{K}$ in its columns, then if U is linearly independent, we can write $e^{-\mathcal{K}(s-t)} = Ue^{-\Lambda(s-t)}U^{-1}$ where Λ is a diagonal matrix with the eigenvalues of $\mathcal{K}$ on the diagonal. Then we can write

$$V_t[X_s] = \int_t^s Ue^{-\Lambda(s-v)}U^{-1} \Sigma \Sigma' U^{-1\prime} e^{-\Lambda(s-v)'} U' dv = U\Psi(s,t) \, U' \tag{38}$$

where we have defined

$$\Psi(s,t) \equiv \int_t^s e^{-\Lambda(s-v)} U^{-1} \Sigma \Sigma' U^{-1\prime} e^{-\Lambda(s-v)'} dv. \tag{39}$$

Since the off-diagonal elements of $e^{-\Lambda(s-v)}$ are zero, we can write the (i,j)-th element of $\Psi(s,t)$ as

$$\Psi(s,t)_{i,j} = \int_t^s e^{-\Lambda_{ii}(s-v)} (U^{-1} \Sigma \Sigma' U^{-1\prime})_{ij} e^{-\Lambda_{jj}(s-v)} dv \tag{40}$$

$$= (U^{-1} \Sigma \Sigma' U^{-1\prime})_{ij} e^{-(\Lambda_{ii}+\Lambda_{jj})s} \int_t^s e^{(\Lambda_{ii}+\Lambda_{jj})v} dv \tag{41}$$

Hence, if $\Lambda_{ii} + \Lambda_{jj} \neq 0$ we have

$$\Psi(s,t)_{i,j} = (U^{-1} \Sigma \Sigma' U^{-1\prime})_{ij} \frac{(1 - e^{-(\Lambda_{ii}+\Lambda_{jj})(s-t)})}{\Lambda_{ii} + \Lambda_{jj}} \tag{42}$$

otherwise

$$\Psi(s,t)_{i,j} = (U^{-1} \Sigma \Sigma' U^{-1\prime})_{ij} (s-t). \tag{43}$$

where $s - t = \frac{1}{12}$ given monthly observation intervals.

C   Adjusting for Average Yields

Define $\bar{Y}_t(n) = 12 \int_t^{t+\frac{1}{12}} Y_s(n) ds$ as the average yield and $\bar{X}_t = 12 \int_t^{t+\frac{1}{12}} X_s ds$ as the unobserved average value of X_t over the month beginning at time t. The h-period ahead conditional expectation at time t is given by $E_t[X_{t+h}] = \theta + e^{-\mathcal{K}h}[X_t - \theta]$. Let U be the matrix with the eigenvectors of $\mathcal{K}$ in its columns, then if U is linearly independent, we can write $U\Lambda U^{-1} = \mathcal{K}$ and $e^{-\mathcal{K}(h)} = Ue^{-\Lambda h}U^{-1}$ where Λ is a diagonal matrix with

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[26]See for instance Oksendal (1998) for a proof of the 1-dimensional case, which extends in a somewhat straightforward manner to multi-dimensional settings leading to the matrix version that is implied here.

the eigenvalues of $\mathcal{K}$ on the diagonal. The time t conditional expectation of $\bar{X}_t$ can now be written

$$E_t\left[12\int_t^{t+\frac{1}{12}} X_s ds\right] = 12\int_t^{t+\frac{1}{12}} \theta + e^{-\mathcal{K}(s-t)}[X_t - \theta]ds \tag{44}$$

$$= \theta + 12U\int_t^{t+\frac{1}{12}} e^{-\Lambda(s-t)}ds\ U^{-1}[X_t - \theta] \tag{45}$$

$$= \theta - 12U\Lambda^{-1}(e^{-\frac{1}{12}\Lambda} - I)U^{-1}[X_t - \theta] \tag{46}$$

$$= \theta - 12U\Lambda^{-1}U^{-1}(Ue^{-\frac{1}{12}\Lambda}U^{-1} - UU^{-1})[X_t - \theta] \tag{47}$$

$$= \theta + 12\mathcal{K}^{-1}(I - e^{-\frac{1}{12}\mathcal{K}})[X_t - \theta]. \tag{48}$$

The first line follows from assuming the conditions for Fubini's lemma admitting the interchange of integrals and expectations, the second line is a substitution using the diagonalizability assumption, the third line follows from evaluating the integral in obtaining a diagonal matrix whose ith diagonal element is given by $-\frac{1}{\Lambda_{ii}}\left(e^{-\frac{1}{12}\Lambda_{ii}} - 1\right)$, the fourth line inserts the identity matrix, $U^{-1}U$, and the final line uses the relation $U\Lambda^{-1}U^{-1} = \mathcal{K}^{-1}$. It follows that the time t conditional expectation of the average yield over the monthly interval $[t, t + \frac{1}{12})$ is affine in X_t

$$E_t[\bar{Y}_t(n)] = A(n) + B(n)'E_t[\bar{X}_t] \tag{49}$$

$$= A(n) + B(n)'[\theta + 12\mathcal{K}^{-1}(I - e^{-\frac{1}{12}\mathcal{K}})[X_t - \theta]] \tag{50}$$

$$= A^*(n) + B^*(n)'X_t. \tag{51}$$

where $A^*(n) \equiv A(n) + B(n)'[I - 12\mathcal{K}^{-1}(I - e^{-\frac{1}{12}\mathcal{K}})]\theta$ and $B^*(n)' \equiv B(n)'[12\mathcal{K}^{-1}(I - e^{-\frac{1}{12}\mathcal{K}})]$.

D   Market Prices of Risk

Given our affine setting bond yields are exponential affine functions, $P(X_t, n) = e^{a(n)+b(n)'X_t}$, so we can invoke Ito's Lemma

$$dP(X_t, n) = \mathcal{D}P(X_t, n)dt + P_X(X_t, n)\Sigma dB_t \tag{52}$$

where $\mathcal{D}P(X_t, n) = P_X(X_t, n)\mathcal{K}(\theta - X_t)$ and $P_X(X_t, n) = P(X_t, n)b(n)'$ is the partial derivative of price with respect to X. Taking expectations under the physical measure $\mathcal{P}$ yields

$$E_t[\frac{dP(X_t, n)}{dt}] = \mathcal{D}P(X_t, n). \tag{53}$$

Changing measures using Girsanov's Theorem, we express the return process under Q as

$$\frac{dP(X_t, n)}{P(X_t, n)} = [\frac{\mathcal{D}P(X_t, n) - P_X(X_t, n)\Sigma\lambda_t}{P(X_t, n)}]dt + \frac{P_X(X_t, n)}{P(X_t, n)}\Sigma dB_t^Q. \tag{54}$$

Since Q is an equivalent martingale measure, the market price of risk process must be such that the drift of the return process of the bond under Q equals the risk-free return, r_t. Thus, for all n, λ_t must satisfy

$$\frac{1}{P(X_t, n)}[\mathcal{D}P(X_t, n) - P_X(X_t, n)\Sigma\lambda_t] = r_t \tag{55}$$

or

$$\frac{1}{P(X_t, n)}\mathcal{D}P(X_t, n) - r_t = b(n)'\Sigma\lambda_t \tag{56}$$

Substituting for $\mathcal{D}P(X_t, n)$ using (53), we see that the instantaneous expected excess return takes the form given in Duffee (2002) and Dai and Singleton (2002)

$$e_t(n) \equiv E_t\left[\frac{1}{P(X_t, n)}\frac{dP(X_t, n)}{dt}\right] - r_t = b(n)'\Sigma\lambda_t. \tag{57}$$

The lower triangular assumption on Σ places certain restrictions on the interpretations of the Brownian motions and their associated market prices of risk.[27] To provide some intuition, take for instance a 3 factor model with $X_t = \{g_t, \pi_t, f_t\}$ where the three factors are GDP growth, inflation and the federal funds rate. We can write the stochastic differential equation (5) as

$$dg_t = \mathcal{K}_1(\theta - X_t)dt + \Sigma_{11}d\mathcal{B}_{1t} \tag{58}$$
$$d\pi_t = \mathcal{K}_2(\theta - X_t)dt + \Sigma_{21}d\mathcal{B}_{1t} + \Sigma_{22}d\mathcal{B}_{2t} \tag{59}$$
$$df_t = \mathcal{K}_3(\theta - X_t)dt + \Sigma_{31}d\mathcal{B}_{1t} + \Sigma_{32}d\mathcal{B}_{2t} + \Sigma_{33}d\mathcal{B}_{3t}. \tag{60}$$

where $\mathcal{K}_i$ denotes the i-th row of $\mathcal{K}$ and Σ_{ij} the (i,j)-th element of the volatility matrix Σ. We interpret the first Brownian motion as the source of risk associated with the first factor, the second Brownian motion as the source of risk associated with the second factor and the third Brownian motion as the source of risk associated with the third factor. A lower triangular volatility matrix says that a shock to first factor has an instantaneous impact on both the 2nd and 3rd factors, whereas a shock to the 2nd factor has an instantaneous impact on the 3rd factor. In our example above, the ordering of the variables is consistent with an economic story in that a shock to real output has an impact on inflation and a shock to inflation has an impact on the fed funds rate. In this setting, the first market price of risk governs the instantaneous excess returns associated with shocks to forecasted GDP. Similarly, the second and third market prices of risk govern the instantaneous excess returns associated with shocks to forecasted inflation and the target rate, respectively.

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[27]This lower triangular assumption, albeit somewhat arbitrary, is motivated by a Cholesky decomposition and attributes any common component of the innovations to the first variable in the system, and so on. The order of the decomposition here and in the "Taylor-based" models in the paper is consistent with, among others, Ang, Dong, and Piazzesi (2005b).

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