Optimal external debt and default

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Abstract

This paper analyses whether sovereign default episodes can be seen as contingencies of optimal international lending contracts. The model considers a small open economy with capital accumulation and without commitment to repay debt. Taking first order approximations of Bellman equations, I derive analytical expressions for the equilibrium level of debt and the optimal debt contract. In this environment, debt relief generated by reasonable fluctuations in productivity is an order of magnitude below that generated by shocks to world interest rates. Debt relief prescribed by the model following the interest rate hikes of 1980-81 accounts for a substantial part of the debt forgiveness obtained by the main Latin American countries through the Brady agreements.

KEYWORDS: sovereign debt; default; capital flows; optimal contract; world interest rates.

JEL CLASSIFICATION: F3, F4, G1.

1 Introduction

Consider a small open economy that cannot commit to repay debt but faces output losses if it decides to default. Provided the domestic marginal productivity of capital is greater than the prevailing world interest rates, it is mutually beneficial for foreign creditors to lend and the domestic government to borrow. Creditors will lend up to the incentive compatible level of debt — beyond which greater debt triggers default. Moreover, if the economy is subject to shocks, the incentive compatible level of debt fluctuates with economic conditions. In this paper I study the optimal debt contract in this world; get results on the level of debt it prescribes and the size of optimal debt relief in response to different types of economic shocks; and compare the results to real world outcomes.

In the optimal contract, repayment is contingent on the state: the country is often paying a default premium on its borrowing and, when a negative shock occurs, its debt is reduced to an amount that is incentive compatible to repay. Such contingencies allow

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the country to exhaust its borrowing possibilities and therefore bring first-order gains to the world by transferring capital to an economy with higher marginal productivity.

It is important to understand the extent to which real world capital movements are mimicking an optimal arrangement. If observed sovereign default episodes can be seen as contingencies of optimal international lending contracts, then the inefficiencies of default are only arising from the costs of debt renegotiation.[1] Therefore, policies should aim at reducing such costs, either by facilitating the renegotiation process or by designing explicitly contingent contracts. Otherwise, restrictions on foreign borrowing or the imposition of capital controls may be welfare enhancing.[2]

I begin with a deterministic model that yields the main results concerning the level of debt and capital flows in this economy. Countries borrow to speed convergence to their steady state levels of capital and, as there is no uncertainty, there is no default in equilibrium. The debt-GDP ratio depends positively on the output cost of defaulting and growth prospects but negatively on world interest rates.

Once I move to a model with uncertainty, occasional debt relief is obtained as an equilibrium outcome. If the domestic marginal productivity of capital exceeds world interest rates, increasing borrowing yields first-order gains. Borrowing is maximized when the amount of debt equals its incentive compatible level *at all future states.* Debt reduction occurs when the economy switches to a state with a lower incentive compatible level of debt.

Debt relief predicted by the model depends positively on the magnitude of the shocks and the persistence of states, but ambiguously on the level of capital.[3] Importantly, the output cost of defaulting, a variable which is difficult to measure, has no significant effect on debt relief (as a fraction of the outstanding debt).

Debt reduction generated by reasonable fluctuations in productivity is an order of magnitude below that generated by shocks to world interest rates. This result builds on the observation that a change in interest rates from 1% to 2% doubles the cost of servicing debt while a 5% fall in productivity reduces by just 5% the cost of not repaying. The small effect of productivity fluctuations on debt default is consistent with empirical evidence (Tomz and Wright, 2006) although most of the recent literature on debt and default focuses on productivity shocks.

I then use data from the Latin American debt crisis of the 1980's to compare the

[1] Krueger (2002) argues that the absence of an orderly process for renegotiating sovereign debt has a number of costs because it delays or even inhibits agreements on a needed restructuring.

[2] Reinhart and Rogoff (2004) argue in favor of policies that limit the ability of poor countries to contract debt.

[3] In the numerical examples, debt relief depends negatively on the level of capital.

predictions of the model to the observed debt relief. I find that the increase in world interest rates at the beginning of the 1980's can individually account for more than half of the debt relief obtained by the main Latin American countries through the Brady agreements. If the marginal productivity of capital in Latin America was higher than in the developed world in the 1970's, then a substantial debt reduction following the increase in US interest rates would have been part of an optimal contract drawn up in the 1970's.[4]

The model builds on the literature of endogenous sovereign debt and default (Eaton and Gersovitz (1981), Arellano (2006)).[5] Three important features of this paper are worth stressing: (i) the analysis of the optimal debt contract; (ii) the focus on the effects of capital flows in terms of growth instead of risk sharing; and (iii) analytical results.

One of the key assumptions in this model is that if the country repudiates its debt, it is excluded from capital markets and incurs an output loss.[6] The assumption of an output loss is a simple way of modelling the implicit costs to a country that defaults. Debt repudiation inhibits foreign direct investment and undermines a country's capacity to obtain beneficial deals in multi-lateral organisations such as the WTO. In addition, creditors can threaten countries that might repudiate debt with sanctions such as the loss of access to short-term trade credit and seizure of assets.[7]

In reality, however, observed punishment for default is both tame and temporary. Following a negative shock, creditors have an incentive to renegotiate debt down to a level where it is incentive compatible for the country to repay. So contracts are contingent *de facto* even if not written as such. Indeed, default premia are regularly paid and partial defaults sometimes occur.

Instead of modelling the renegotiation process,[8] I derive the optimal contingent debt contracts.[9] All results are derived from the point of a view of a benevolent domestic

[4] The welfare implications of borrowing depend on the differences in marginal productivity of capital. Caselli and Feyrer (2006) find that the marginal productivity of capital was indeed higher in poor countries in the 1970's. In the context of the Latin American debt crisis, the task of measuring productivity of capital is complicated by the fact that a substantial part of the debt went to financing government investment, often in infrastructure, on which return is not easily measured.

[5] It is also related to the literature on consumers' and firms' debt. In particular, the optimal contract in this model bears resemblance to that in Albuquerque and Hopenhayn (2004).

[6] The output cost of debt repudiation, as modeled here, is present in Cohen and Sachs (1986), Bulow and Rogoff (1989), Arellano (2006) and Wright (2002), to name a few.

[7] See Bulow and Rogoff (1989) for a discussion of such costs.

[8] Renegotiation in models of sovereign debt is studied by Bulow and Rogoff (1989), Fernandez and Rosenthal (1990) and Yue (2005).

[9] Sovereign debt was also analysed as an (implicitly) contingent claim by Grossman and Van Huyck (1988), Atkinson (1991) and Calvo and Kaminsky (1991) among others. Grossman and Van Huyck show that an equilibrium in which "excusable" default is allowed without sanctions can be sustained. Alfaro and Kanczuk's (2005) quantitative analysis builds on Grossman and Van Huyck. Calvo and Kaminsky (1991) take the optimal contract approach to study whether the small default premium paid by Latin American countries in the 1970's would be compatible with large debt reductions in the 1980's.

government.[10] In the model, it is never optimal to renege on the contract, so the output cost is never paid in equilibrium, but premium rates on borrowing and occasional debt reduction are observed.

The results are normative benchmarks but such implicit contracts can be implemented in practice. In fact, the optimal contract yields the same results as the assumption that lenders have all the bargaining power — a usual implicit assumption in the literature — and they promptly agree to debt relief if it is optimal for them.

The study of external debt and default is closely related to the question of why capital does not flow from rich to poor countries. One proposed explanation is that the risk of default prevents larger capital inflows in emerging economies (Reinhart and Rogoff (2004) and Reinhart, Rogoff and Savastano (2003)). Alternative explanations emphasise differences in productivity (Lucas (1990)) and question whether the marginal productivity of capital is really higher in poor countries (Caselli and Feyrer (2006)). Yet, most of the recent work on debt and default building on Eaton and Gersovitz (1981) focuses on risk sharing.[11] Cohen and Sachs (1986) present a growth model in which debt is repaid only if it is incentive compatible to do so, but assume a linear production function and have no uncertainty.[12]

Models with endogenous decision of debt repayment are not very tractable analytically, so including capital accumulation in the model could make it even harder to obtain analytical solutions. It turns out that, for limiting cases of small costs of default and/or small fluctuations and/or small differences in the marginal productivity of capital, I can derive analytical solutions for the level of debt and debt relief. The method consists of taking first order approximations of the Bellman equations. The obtained formulae are simple and intuitive. I also solve the model numerically for non-limiting cases and show that the formulae are good approximations.

[10] In the standard Ramsey model, the central planner solution and the decentralised equilibrium are the same. However, that is not true without commitment to repay debt. The distinction between the central planner solution and the decentralised allocation is analysed by Kehoe and Perri (2004) and Jeske (2006). Kehoe and Perri (2004) show that the central planner solution can be decentralised if the central government is in charge of deciding about defaulting or not and taxes capital income to counteract an externality of capital accumulation. The logic behind Jeske's argument is similar and he finds that capital controls may be welfare improving.

[11] Arellano (2006), Aguiar and Gopinath (2006) and Yue (2005) study the relation between default risk and macroeconomic variables in endowment economies. In such models, the government is assumed to be substantially more impatient than the outside world, otherwise the equilibrium level of debt is too small and default is a rare phenomenon.

[12] Cohen and Sachs (1986) also analyse a numerical example with decreasing returns to capital which is basically the same as my deterministic model. Marcet and Marimon (1992), Kehoe and Perri (2004), Wright (2002) and Bai and Zhang (2005) also study economies with capital accumulation.

2 Deterministic Model

In this section, I consider a deterministic, discrete time model of an open economy that can borrow from abroad, but cannot commit to repay its debts.

The economy is populated by a continuum of infinitely lived agents whose preferences are aggregated to form the usual representative agent utility function:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

Where $\beta \in (0,1)$ is the subjective time discount factor, c_t is consumption at time t and $u(.)$ is the felicity function that satisfies the Inada conditions.

I model debt default costs as an instantaneous permanent fall in productivity and loss of access to capital markets. The permanent fall specifically captures the loss that a country suffers by taking an antagonistic position towards the rest of the world and never repaying its debts. In the model this is out-of-equilibrium behaviour, which corresponds to never observing such action in reality. For this reason, it is difficult to obtain an estimate of the costs; nevertheless I expect it to be a small fraction of GDP.

Output in the economy is a function of the level of capital, as labour is normalised to 1. In the deterministic case, the productivity parameter, A, is constant over time. I denote by γ the fraction of output lost due to default, so production is given by:

$$y_t = \begin{cases} A.f(k_t) \text{ , if no default} \\ A(1-\gamma).f(k_t) \text{ , if default} \end{cases}$$

There is a continuum of risk-neutral lenders that, in equilibrium, lend to the country as long as the expected return on their assets is not lower than the risk-free interest rate in international markets, r^*. The price of a bond that delivers one unit of the good next period with certainty, $(1+r^*)^{-1}$, will be denoted q^* and $q^* = \beta$. There is a maximum amount of debt the country can contract that prevents it from running Ponzi schemes but it is never reached in equilibrium.

The economy's flow budget constraint is then given by:

$$c_t + k_{t+1} = \begin{cases} A.f(k_t) + (1-\delta)k_t - d_t + q_t d_{t+1} \text{ , if no default} \\ A(1-\gamma).f(k_t) + (1-\delta)k_t \text{ , if default} \end{cases}$$

Where q_t is the (endogenously determined) price of debt and δ is the depreciation rate. The focus here is sovereign debt, so I analyse the central planner solution to maximise the representative agent's utility function subject to the economy's budget constraint.

In each period, the central planner chooses between repaying or defaulting. Each option yields a different value function and the planner chooses the maximum of the two:

$$V(k,d) = \max\{V_{pay}(k,d), V_{def}(k,\gamma)\}$$

Where:

$$\begin{aligned} V_{pay}(k,d) &= \max_{k',d'}\{u(Af(k)+(1-\delta)k-k'-d+qd')+\beta V(k',d')\} \\ V_{def}(k,\gamma) &= \max_{k'}\{u((1-\gamma)Af(k)+(1-\delta)k-k')+\beta V_{def}(k')\} \end{aligned}$$

I assume that decisions about k' and d' are made simultaneously and lenders can observe k' before taking their lending decisions (or condition their decisions on k'). As noted by Cohen and Sachs (1986), the country would otherwise have an incentive to borrow d' but then invest less, consume more and default on its debt.[13]

An equilibrium is a $\{k_t\}_{t=0}^{\infty}$, $\{d_t\}_{t=0}^{\infty}$ and $\{q_t\}_{t=0}^{\infty}$ such that the central planner maximises the value function $V(k,d)$ at every period and lenders are indifferent between the domestic and risk-free bond.

The following results hold in an equilibrium with no uncertainty:

1. $q = q^*$, a constant. As there is no uncertainty, $q = q^*$ if the country will repay and $q = 0$ otherwise. The choice $d' = 0$ is strictly better than any choice d' such that $q = 0$ because that yields the same amount of consumption today and more production next period (by avoiding the $\gamma Af(k)$ output loss). So, in equilibrium, I obtain the no-default condition: $V_{pay}(k,d) \geq V_{def}(k,\gamma)$.

2. $d^{\max}(k,\gamma)$ is the maximum level of incentive compatible debt and is increasing in γ. Since by differentiating the value function we obtain that V_{pay} is decreasing in d and V_{def} is decreasing in γ, $V_{pay}(k,d^{\max}) = V_{def}(k,\gamma)$. Thus, an increase in γ implies an increase in $d^{\max}(k,\gamma)$.

3. If k' is below the steady state level of capital, k^*, then $d' = d^{\max}$. In the steady state, $k' = k = k^*$ and the marginal productivity of capital, $mpk = Af'(k^*) - \delta$, equals the marginal cost of renting an extra unit of capital, r^*. In this case, the country has no incentive to change the level of its debt, because capital is at the optimal level and smooth consumption can be achieved by always choosing $d' = d$. In contrast, if $k < k^*$, $mpk > r^*$ and d cannot be smaller than $d^{\max}$, otherwise the no-default

[13] To see this, note that in the optimal plan $V_{pay}(k',d') = V_{def}(k',\gamma)$ and $u'(c) = \beta\frac{\partial V_{pay}}{\partial k'}(k',d')$. But $\frac{\partial V_{pay}}{\partial k'}(k',d') > \frac{\partial V_{def}}{\partial k'}(k')$, so if the country has already borrowed d' and hasn't committed to k', a marginal decrease in k' leads to an increase in today's utility that is bigger than the decrease in tomorrow's value. This moral hazard problem is studied by Atkeson (1991).

condition would not bind, so the country could borrow an extra unit at r^*, invest it and obtain a greater return than r^* next period.

4. If $\gamma = 0$, no debt can be sustained. If $\gamma = 0$, any positive discounted stream of repayment is worse than defaulting, so the maximum incentive compatible positive discounted stream of repayment is zero.[14]

5. If $\gamma = 1$, we obtain full commitment. If $\gamma = 1$ with no default, in the steady state, the country obtains consumption equal to $Af(k^*) - \delta k^* - r^* d^*$ which is positive because $d^* \leq k^*$, $Af'(k^*) - \delta = r^*$ and $f(k^*) > k^* f'(k^*)$. Hence the first best, full commitment equilibrium is incentive compatible.[15]

Using these results, I now derive the path of debt in the neighbourhood of $\gamma = 0$. I will detail two observations that are used to derive it.

First, consider k'_p and d' such that $V_{pay}(k'_p, d') = V_{def}(k'_p, \gamma)$ and $k'_p \leq k^*$. Then there exists some $k \leq k'_p$ and d such that the country is indifferent between "repaying and choosing (k'_p, d')" and "defaulting and choosing (k'_d)". The value functions will be equivalent in this case. The value functions are given by:

$$\begin{aligned} V_{pay}(k,d) &= \max_{k'_p, d'} \left\{ u(y + (1-\delta)k - k'_p - d + qd') + \beta V_{pay}(k'_p, d') \right\} \\ V_{def}(k,\gamma) &= \max_{k'_d} \left\{ u((1-\gamma)y + (1-\delta)k - k'_d) + \beta V_{def}(k'_d, \gamma) \right\} \end{aligned}$$

Second, from result 4, we know that the value functions when $\gamma = d = d' = 0$ are identical and have a common optimal level of capital:

$$\begin{aligned} V_0(k) &= \max_{k'} \left\{ u(y + (1-\delta)k - k') + \beta V_{def}(k', 0) \right\} \\ &= \max_{k'} \left\{ u(y + (1-\delta)k - k') + \beta V_{pay}(k', 0) \right\} \end{aligned}$$

Then, from result 3, we always have $d' = d^{\max}$ and therefore the no default condition will always bind: $V_{pay}(k, d^{\max}) = V_{def}(k, \gamma)$. Using the first observation, we rewrite these value functions in the form above. Then by taking a linear approximation of $V_{pay}(k,d)$ and $V_{def}(k,\gamma)$ around $V_0(k)$ and manipulating the linearised expressions, we can find d and γ that equate $V_{pay}(k,d)$ and $V_{def}(k,\gamma)$ without solving for the value functions. The expression for d turns out to be a good approximation for its true value if γ is not more than a few percentage points. This is because when $\gamma \to 0$, the optimal choice of capital

[14] This result is due to the absence of uncertainty in the model. It has already been shown in the literature that, with uncertainty, there may be debt in equilibrium even in the absence of output costs (Eaton and Gersovitz, 1981).

[15] Again, this is a result due to the absence of uncertainty in the model. It has already been shown in the literature that with uncertainty and incomplete asset markets, the full commitment equilibrium may not be the first best equilibrium because the possibility of default makes debt somewhat contingent (Zame, 1993).

is independent of the decision about defaulting — which allows us to get the analytical results. As γ moves away from 0, that is no longer true, however the impact on the value function of reoptimising the level of capital due to a 1% or 2% fall in productivity is very small, and so is its impact on the maximum incentive compatible level of debt.

The results do not depend on functional forms of the utility function or the production function, these would only have second order effects.

Proposition 1 *In this economy, for a very small γ:*

1. *In steady state:*
$$d^* = \frac{\gamma y^*}{1-q^*} \tag{1}$$

2. *For $y_t < y^*$:*
$$\frac{d_{t+1}}{y_t} = \frac{\gamma}{1-q} + \frac{\Delta d_{t+1}}{(1-q)y_t}$$
and:
$$d_t = q.d_{t+1} + (1-q)\frac{\gamma y_t}{1-q}$$

Proof: see appendix.

This result will allow us to determine the path for debt in the economy, given a path for output.

Part (1) of the proposition shows that in the steady state, the country keeps repaying its debt if the interest payment, $d^*(1-q^*)$, is not greater than the output loss, γy^*, due to default. The debt as proportion of GDP is, as a first order approximation, equal to $\gamma/(1-q^*)$. Positive debt with no uncertainty arises in equilibrium to finance convergence. If $\gamma = 1\%$ and $q^* = 0.98$ ($r^* \approx 2\%$), the debt-GDP ratio is 50%.

The level of debt is proportional to the output loss and inversely proportional to the risk-free interest rate. Note that a change in interest rate from 1% to 2% has the same impact on d as a decrease of 50% in GDP. Fluctuations in interest rates from 1% to 2% are much more common than a 50% fall in output.

Part (2) of the proposition shows that for $y_t < y^*$, the condition for default reduces to a comparison between output losses and resources paid to foreign agents *in the present period*. But the increase in debt is endogenously determined by considering that the country will be indifferent in the next period between repaying and defaulting — so, ultimately, debt at period t is obtained by backward induction from the steady state level of debt.

For $y_t < y^*$, the absolute level of debt is increasing over time because the present value of the output loss due to default is increasing as output rises to its steady state level. And, from the second equation in the proposition above, since the absolute level of debt is increasing, debt is a higher proportion of GDP. This is because positive capital inflows generate more incentive for the country to repay.[16]

The proposition also shows that in equilibrium, the country must experience net outflows of resources on the way to convergence. Debt is increasing (financial account is in surplus) but such increase is smaller than the interest paid in its debt. So, even though the current account is in deficit, the country is a net exporter of goods.

2.1 Numerical solution

In the numerical examples of this paper, specific utility and production functional forms are assumed as follows:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma} \quad , \quad f(k) = k^{\alpha}$$

I calibrate the specific parameters as follows: one period corresponds to one year. $A = 1$, $\alpha = 0.36$, $\beta = 1.02^{-1}$, $\sigma = 3$ and $\delta = 0.10$. The price of a riskless bond, q^* , equals β. The productivity loss in terms of default, $\gamma = 0.01$.

The numerical solution is obtained through value function iteration. The state space is discretised using grids for debt and capital but the planner can choose any point in the grid. From Figure 1, the numbers obtained in this solution are very similar to those from the analytical formulae using the path of y_t given by the numerical example.

Figure 1 also shows the behavior of capital in this economy, $\gamma = 0.01$, compared to the closed-economy case, $\gamma = 0$, and the full-commitment open-economy case, $\gamma = 1$. Without the possibility of default, the level of capital jumps to its steady state level and the marginal productivity of capital equals r^* in one period. The possibility of default makes convergence slower. Due to the initial capital inflow, the level of capital is higher in this economy than in the closed economy case until they converge. However, the closed economy slowly catches up, as the open economy will be experiencing net capital outflows (trade balance surpluses) during the whole history, as shown at Figure 1. Debt stabilises at 51% of GDP but reaches 60% of GDP at earlier stages.

A usual intuition is that financially open economies should converge faster to their steady states (Barro, Mankiw and Sala-i-Martin, 1995). In contrast, the equilibrium from this model shows that an indebted open economy would take *more* time to converge than a closed economy with the same level of capital. After the initial capital inflow, the

[16] This effect appears already in a numerical example in Cohen and Sachs (1986).

country experiences net outflows of resources i.e. a positive trade balance. In addition, a closed economy that opens to capital flows would not converge significantly faster but, on the way towards the steady state, would have higher output than if it remained closed. In order to experience faster convergence, emerging economies need trade deficits and, as Proposition 1 shows, that does not occur in equilibrium.

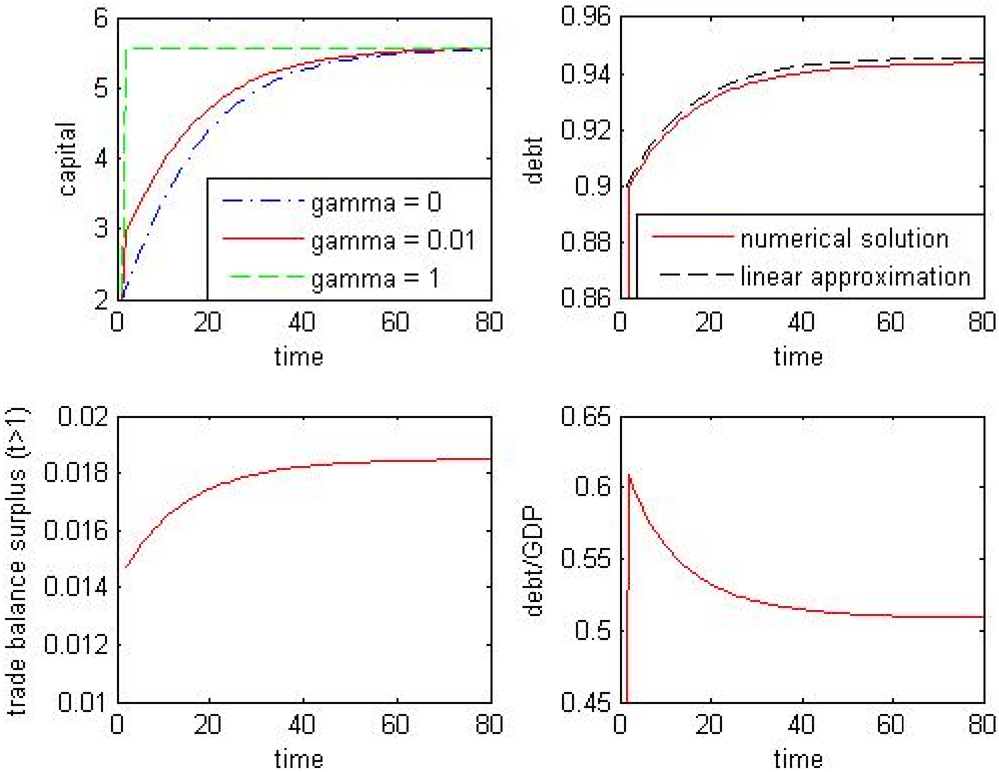

Figure 1: Deterministic model

The steady state level of debt as a first order approximation is given by γ/r^*, so $\gamma = 1\%$ and $r^* = 2\%$ yield a steady state level of debt equal to half of the country's GDP. Models based on risk sharing and endowment economies (Arellano (2006), Aguiar and Gopinath (2006)) predict lower levels of debt than we observe in reality, even when assuming very low values of β. If debt is issued to buy capital and finance convergence, substantially higher levels of debt are obtained.

3 The stochastic model

In order to study default, this section introduces uncertainty in the analysis. Two stochastic versions of the model are considered: (1) stochastic r^* and (2) stochastic A. There are 2 possible states, $s_t \in \{h, l\}$, and the probability of switching states is ψ: $\Pr(s_t = h|s_{t-1} = l) = \Pr(s_t = l|s_{t-1} = h) = \psi$, $\psi < 0.5$. As before, the country can issue only one-period uncontingent debt.

In a stochastic world, the incentive compatible level of debt fluctuates. If debt goes above its incentive compatible level, the debtor prefers not to repay it. But then, both creditors and debtors have incentives to renegotiate it. Here, I solve for the optimal contingent contract, which provides the normative benchmark: the level of debt is contingent on the state. The optimal contract can be implemented if creditors immediately reduce the level of debt to its incentive compatible level when it goes above it.

I follow the literature (Eaton and Gersovitz (1981), Arellano (2006)) and assume that the debtor fully repays debt provided it is incentive compatible to do so. As noted by Fernandez and Rosenthal (1990), we are assuming that all bargaining power lies with the creditors. In this model, creditors are always able to extract the maximum incentive compatible payment from the lenders, so the assumption about bargaining power is internally consistent.[17]

If the country wants to borrow as much as it can then, in order to expand its borrowing possibilities, it chooses to make debt in each of the future states as high as possible, respecting the incentive compatibility constraints. Thus the contingent contract has not only second moment benefits but, more importantly, brings first order efficiency gains by increasing the amount of resources transferred to the country with higher marginal productivity of capital.

3.1 Stochastic world interest rates

Here, I analyse the optimal debt contract for an economy with fluctuations in world interest rates, r^*, which leads to fluctuations in the price of risk-free debt, q^*. The technology level, A, is fixed. The contract specifies the country's debt at each possible state, and the country has the choice between honouring its debt and defaulting.

The price of a riskless bond in international markets is q^{*h} in the high state and q^{*l} in the low state, $q^{*h} > q^{*l}$. A risk-neutral creditor that lends q^*d' must get an expected

[17] Fernandez and Rosenthal (1990) analyse the bargaining process and show that, in some situations, the creditor indeed holds all the bargaining power. Hence, this assumption is justified, even though differences in our models, in particular their assumption of a single creditor, mean that their results cannot be automatically transferred to this setting.

repayment equal to d'. Denote by d^h and d^l the repayment conditional on high and low state, respectively, and $\Delta d = d^h - d^l$. If $s_{t-1} = h$, a country that borrowed $q^{*h}d'$ has debt d^h if $s_t = h$ and d^l if $s_t = l$ such that $d^h(1-\psi) + d^l\psi = d'$. Hence, $d^h = d' + \psi\Delta d$ and $d^l = d' - (1-\psi)\Delta d$. If $s_{t-1} = l$, a country borrowing $q^{*l}d'$ has debt d^h if $s_t = h$ and d^l if $s_t = l$ such that $d^l(1-\psi) + d^h\psi = d'$. Hence, $d^l = d' - \psi\Delta d$ and $d^h = d' + (1-\psi)\Delta d$. Thus the country is choosing, at every state, one extra variable, Δd. The value functions conditional on repayment are:

$$\begin{aligned} V^h_{pay}(k,d) &= \max_{k',d',\Delta d} \left\{ u(c) + \beta\left[(1-\psi)V^h(k', d' + \psi\Delta d) + \psi V^l(k', d' - (1-\psi)\,\Delta d)\right]\right\} \\ V^l_{pay}(k,d) &= \max_{k',d',\Delta d} \left\{ u(c) + \beta\left[(1-\psi)V^l(k', d' - \psi\Delta d) + \psi V^h(k', d' + (1-\psi)\,\Delta d)\right]\right\} \end{aligned}$$

where $c = Af(k) + (1-\delta)k - k' - d + q^i(k', d')d'$, $V^i(k,d) = \max\left\{V^i_{pay}(k,d), V^i_{def}(k,\gamma)\right\}$ and $i \in \{l, h\}$.

In case of default, the value function in both states is:

$$V_{def}(k,\gamma) = \max_{k'} \left\{u((1-\gamma)Af(k) + (1-\delta)k - k') + \beta V_{def}(k',\gamma)\right\}$$

It makes no difference whether foreign interest rates are low or high if the country is excluded from international financial markets.

If contracts can be written and enforced contingent on q^*, debt repudiation is never optimal, so d^h and d^l will always be incentive compatible. A contract specifying no debt in one of the states is strictly better than the case of repudiating debt at that state because, given that the expected payment must be the same, the debt payments in the alternative state will be identical so the only difference between the cases will be the loss in output when debt is repudiated.

The optimal Δd will depend on k, d and s. As in equilibrium there is no debt repudiation, $V^h = V^h_{pay}$ and $V^l = V^l_{pay}$. However, the optimal Δd depends on whether the borrowing constraint is binding or not, that is, whether the country would borrow more in the absence of commitment problems.

Proposition 2 *If the borrowing constraint is not binding, Δd is chosen to make*

$$\frac{\partial V^h_{pay}(k', d^h)}{\partial d} = \frac{\partial V^l_{pay}(k', d^l)}{\partial d}$$

Proof: see appendix.

Proposition 3 *If the borrowing constraint is binding, so that $mpk > r^{*i}$, and $q^{*h} - q^{*l}$ is arbitrarily small, then Δd is chosen to make $V^h_{pay}(k', d^h) = V^l_{pay}(k', d^l)$.*

Where $mpk = Af'(k) - \delta$ and $r^{*i} = 1/q^{*i} - 1$.

If the country's borrowing constraint is not binding, Δd is chosen to equate the marginal value functions in both states. When the constraint is binding, the optimal contract equates the value functions in both states in order to ensure there are no further gains from transferring debt across states. If $V^h_{pay}(k', d^h)$ and $V^l_{pay}(k', d^l)$ do not coincide, the country can always borrow more by transferring debt across states. In equilibrium, if the borrowing constraint is binding, $V^h_{pay}(k', d^h) = V^l_{pay}(k', d^l) = V_{def}(k', \gamma)$.

The binding borrowing constraint implies that the benefits of the optimal contract are not related to risk-sharing, as they would be if the country were not constrained. First-order benefits arise because transferring debt between states enables the country to borrow more. As its marginal productivity of capital is greater than r^*, world output is higher.

3.1.1 The value of Δd

For analytical convenience, I consider that $dq^* = q^{*h} - q^{*l}$ is sufficiently small and work with linear approximations. By taking the linear approximations, we are not considering the welfare effects of reoptimising k, d and Δd when the country changes state.

In addition, I temporarily consider an alternative process for q^* that I denote by the ξ-process, as opposed to the ψ-process that we described above. At time $t = 0$, $q^*_0 = q^{*\xi}$; from time $t = 1$ on, there is a constant probability at each period that q^* permanently goes to $\bar{q}$. So, for $t > 0$:

- if $q^*_{t-1} = q^{*\xi}$, $\Pr(q^*_t = q^{*\xi}) = \xi$ and $\Pr(q^*_t = \bar{q}) = 1 - \xi$;
- if $q^*_{t-1} = \bar{q}$, $q^*_t = \bar{q}$.

The value function at (k, d) if $q_t = q^{*\xi}$ is:

$$V^{\xi}(k, d, q^{*\xi}) = \max_{k', d', d'^{\xi}} \left\{ u\left(c\right) + \beta \left[(1-\xi) V^{\det}(k', d') + \xi V^{\xi}(k', d'^{\xi}, q^{*\xi}) \right] \right\}$$

where $c = Af(k) + (1-\delta)k - k' - d + q^{*\xi}(d'(1-\xi) + \xi d'^{\xi})$ and $V^{\det}$ is the value function in the model with no uncertainty.

The ξ-process and the ψ-process are related using the following lemma:

Lemma 4 *Define $\bar{q} = (q^{*h} + q^{*l})/2$ and denote by $V^h(k, d, q^{*h})$ the value function at (k, d, q^{*h}) in the case of the ψ-process. Then $V^h(k, d, q^{*h}) = V^{\xi}(k, d, q^{*h})$ if $\xi = 1 - 2\psi$.*

Proof: see appendix.

Compare the following two cases when q^* follows the ξ-process: (1) $q^* = q^{*\xi}$ and debt is d^{ξ}_0 and (2) $q^* = \bar{q}$ and debt is d_0. We want to find the values of d^{ξ}_0 and d_0 that make the country indifferent between both cases in order to determine Δd using proposition 3. By taking a linear approximation of $V^{\xi}(k, d^{\xi}, q^{*\xi})$ around $V^{\det}(k, d_0)$ and using the indifference condition that $V^{\xi}(k, d^{\xi}, q^{*\xi}) = V^{\det}(k, d_0)$, we get the following lemma:

Lemma 5 *The country's indifference between both states, $V^{\xi}(k, d^{\xi}, q^{*\xi}) = V^{\det}(k, d_0)$, implies:*

$$\begin{aligned} & u'(c_0)\left(d_0^{\xi} - d_0\right) \\ = & \sum_{t=0}^{\infty} (\beta\xi)^t u'(c_t)\left(q^{*\xi} - \bar{q}\right) d_{t+1} \\ & + \sum_{t=0}^{\infty} (\beta\xi)^t \left[u'(c_t)\bar{q} + \beta \frac{\partial V(k_{t+1}, d_{t+1})}{\partial d}\right] (d_{t+1}^{\xi} - d_{t+1}) \end{aligned}$$

where d_{t+1}^{ξ} is debt contracted at time t if $q_t^* = q^{*\xi}$ and d_{t+1} is debt contracted at time t if $q_t^* = \bar{q}$.

Proof: see appendix.

Suppose that $q^{*\xi} > \bar{q}$. The first line in the above expression shows the utility cost of having a higher debt. The second line shows the utility benefit of borrowing at a lower rate, and takes into account the probability of borrowing at a cheaper rate in future periods. The third term is the benefit of being able to borrow more due to the lower interest rates. If the borrowing constraint is binding, then

$$u'(c_t)\bar{q} > -\beta \frac{\partial V(k_{t+1}, d_{t+1})}{\partial d}$$

which means that the benefit of borrowing an extra unit this period is bigger than the cost of having an extra unit of debt next period.

From Lemma 5, we can write:

$$\frac{d_0^{\xi} - d_0}{d_0} > \sum_{t=0}^{\infty} (\beta\xi)^t \frac{u'(c_t)}{u'(c_0)} \left(q^{\xi} - \bar{q}\right) \frac{d_{t+1}}{d_0}$$

To simplify our understanding of this equation and to derive explicit analytical solutions, it is convenient to consider this expression in the neighbourhood of the steady state. In the numerical section, I will show that this is a reasonable approximation.

As k approaches k^*, this equation holds with equality because the borrowing constraint stops binding, and c_t and d_t approach their steady state, constant, values. Taking this limit:

$$\frac{d_0^{\xi} - d_0}{d_0} = \frac{q^{\xi} - \bar{q}}{1 - \beta\xi} \tag{2}$$

And that leads to the following proposition:

Proposition 6 *Consider a deterministic steady state around,* $\bar{k}$, $\bar{d}$ *and* $\bar{q}$, *such that* q^{*h} *and* q^{*l} *are close to* $\bar{q}$ *and* $\bar{q} = \left(q^{*h} + q^{*l}\right)/2$. *Then, a linear approximation around the steady state implies that* $V(\bar{k}, d^h, q^{*h}) = V\left(\bar{k}, d^l, q^{*l}\right)$ *when:*

$$\frac{d^h - d^l}{\bar{d}} = \frac{q^{*h} - q^{*l}}{1 - \beta(1 - 2\psi)} \tag{3}$$

where $\bar{d} = \left(d^h + d^l\right)/2$.

Proof: see appendix.

From Lemma 5 and Proposition 6, $\Delta d/\bar{d}$ depends on:

1. the magnitude of interest rate fluctuations;
2. the persistence of the interest rate process;
3. the current level of capital — and its marginal productivity.

Close to the steady state, the key variables for determining Δd are the size of interest rate fluctuations, $q^{*h} - q^{*l}$, and the persistence of the interest rate process. In the i.i.d. case, $\psi = 0.5$, and $d^h - d^l = \bar{d}\left(q^{*h} - q^{*l}\right)$, that is, the debt in the low state has to decrease to exactly compensate the smaller borrowing. In the other extreme, $\psi \to 0$, $d^h - d^l = \bar{d}\left(q^{*h} - q^{*l}\right)/(1 - \beta)$, that is, the debt reduction in the low state must be much greater to compensate for the expected future loss brought on by the fall in q^*. Hence, higher persistence implies higher difference between d^h and d^l.

The lower is the level of capital, the greater are the marginal productivity of capital and the difference between $u'(c)\bar{q}$ and $-\beta\frac{\partial V(k',d')}{\partial d}$, which contribute to increase Δd: a switch to the low state that prevents the country from borrowing is more punitive when capital is lower. A lower level of capital also implies lower consumption and, therefore, higher marginal utilities, so present consumption is more important and higher costs of borrowing in the future are less relevant, which induces a decrease in Δd. Lastly, a higher ratio between future and present debt increases the importance of future costs of borrowing, which induces an increase in Δd. Thus the overall effect cannot be deduced from the formula. In the numerical examples, Δd is slightly decreasing in k, implying the effect of the borrowing constraint predominates.

The output cost γ has no effects on $\Delta d/d$ in the limiting case of small fluctuations, it is just important to determine the *level* of debt.

The analysis has focused on the two-state case, but the same insights apply if we consider more general processes. The next proposition considers the case of an autoregressive process for q^*.

Proposition 7 *Suppose that q^* follows an AR(1) process:*

$$q^*_{t+1} - \bar{q} = \zeta(q^*_t - \bar{q}) + \varepsilon_{t+1}$$

and $Var(\varepsilon_t)$ is arbitrarily small. If the economy is close to its steady state ($k \simeq k^*$), $V(k, d^1, q^{*1}) = V(k, d^2, q^{*2})$ for any $\{q^{*1}, q^{*2}\}$ close to $\bar{q}$ and $\{d^1, d^2\}$ when:

$$\frac{d^1 - d^2}{\bar{d}} = \frac{q^{*1} - q^{*2}}{1 - \beta\zeta}$$

where $\bar{d}$ is the level of debt in the deterministic model when $q = \bar{q}$.

Proof: see appendix.

3.2 Stochastic technology

In this section, I consider fixing the world interest rates at r^* and allowing for fluctuations in A. Productivity is A^h in the high state and A^l in the low state, $A^h > A^l$. A risk-neutral creditor that lends q^*d' must get an expected repayment equal to d'. Denote by d^h and d^l the repayment conditional on high and low state, respectively, and $\Delta d = d^h - d^l$. The value functions conditional on repayment are:

$$\begin{aligned}
V^h_{pay}(k, d) &= \max_{k', d', \Delta d} \left\{u(c^h) + \beta\left[(1-\psi)V^h(k', d' + \psi\Delta d) + \psi V^l(k', d' - (1-\psi)\Delta d)\right]\right\} \\
V^l_{pay}(k, d) &= \max_{k', d', \Delta d} \left\{u(c^l) + \beta\left[\psi V^h(k', d' + (1-\psi)\Delta d) + (1-\psi)V^l(k', d' - \psi\Delta d)\right]\right\}
\end{aligned}$$

where $c^i = A^i f(k) + (1-\delta)k - k' - d + q(k', d')d'$, $V^i(k, d) = \max\left\{V^i_{pay}(k, d), V^i_{def}(k, \gamma)\right\}$ and i denotes the state. In case of default, the value functions are:

$$\begin{aligned}
V^h_{def}(k, \gamma) &= \max_{k'} \left\{u((1-\gamma)A^h f(k) + (1-\delta)k - k' + \beta\left[(1-\psi)V^h_{def}(k', \gamma) + \psi V^l_{def}(k', \gamma)\right]\right\} \\
V^l_{def}(k, \gamma) &= \max_{k'} \left\{u((1-\gamma)A^l f(k) + (1-\delta)k - k' + \beta\left[(1-\psi)V^l_{def}(k', \gamma) + \psi V^h_{def}(k', \gamma)\right]\right\}
\end{aligned}$$

As before, there is no debt repudiation in equilibrium and, if the country's borrowing constraint is binding, it chooses to borrow as much as it can by limiting debt at each possible future state just enough to make it incentive compatible. This result is stated in the following proposition:

Proposition 8 *If the borrowing constraint is binding, so that $mpk > r^{*i}$, and $A^h - A^l$ is arbitrarily small, then Δd is chosen to make $V^h_{pay}(k', d^h) = V^h_{def}(k', \gamma)$ and $V^l_{pay}(k', d^l) = V^l_{def}(k', \gamma)$.*

Proof: see appendix.

The proof is analogous to the proof of Proposition 3.

As before, we need to obtain an expression for Δd. The analogy to Proposition 6 for the case of stochastic technology requires the additional assumption that γ is arbitrarily small and yields the following result:

Proposition 9 *Consider a deterministic steady state, $\{\bar{k}, \bar{d}, \bar{A}\}$, such that A^h and A^l are close to $\bar{A} = \left(A^h + A^l\right)/2$. Then, for arbitrarily small γ, a linear approximation around the steady state implies that $V^h_{pay}(k', d^h) = V^h_{def}(k', \gamma)$ and $V^l_{pay}(k', d^l) = V^l_{def}(k', \gamma)$ when:*

$$\frac{d^h - d^l}{\bar{d}} = \frac{(1-q^*)}{1-\beta(1-2\psi)} \frac{A^h - A^l}{\bar{A}} \tag{4}$$

where $\bar{d} = \left(d^h + d^l\right)/2$.

Proof: see appendix.

As before, larger fluctuations and more persistent states imply higher debt relief and γ has no first order impact on $\Delta d/\bar{d}$.

3.3 Contrasting stochastic q^* and stochastic A

In order to contrast debt relief in the cases of stochastic interest rates and technology, we need to contrast the numerators of Equations 3 and 4, as the denominator is the same. The key distinction is that in Equation 3 we have the difference between interest rates in both states, while in Equation 4 we have the difference in productivity multiplied by $(1-q^*)$, the present discounted interest rate. A reasonable range for the numerator of Equation 3 (stochastic interest rates) is between 2% and 4%. On the other hand, a reasonable range for productivity flucutations is 2-6%, which in combination with a range of interest rates from 1% to 3% gives a range for the numerator of Equation 4 (stochastic technology) of 0.02% to 0.2%. This is at least one order of magnitude below what we get from fluctuations in world interest rates.

Fluctuations in q^* alter the cost of servicing debt. Shocks to A change the present value of losses due to default. Both these changes affect the incentive compatible level of debt and the amount of debt relief depends on how the incentive compatible level of debt is affected by fluctuations in A and q^*. There is a great distinction in quantitative effects of shocks to A and q^* because an increase in world interest rates from 1% to 2% doubles the cost of servicing debt while a 5% fall in productivity reduces by 5% the loss due to default. The intuition for the difference in debt relief from Equations 3 and 4 is similar to the reasons for the distinct impacts of A and q^* in the incentive compatible level of debt in the deterministic model (Equation 1 and Proposition 1).

4 Contrasting the model with data

In this section, I constrast the predictions of the model with data from the Latin American debt crisis of the 1980's. I compute the optimal debt relief from the model's analytical approximations. I subsequently check the accuracy of these approximations using the same data in numerical simulations. My results show that the interest rate shock at the beginning of the 1980's can account for a large part of the observed debt relief.

4.1 The Latin American debt crisis of the 1980's

The Latin American crisis is particularly suited to evaluate the model's predictions for two reasons: (i) there is one important and easily measured shock, the strong increase in US interest rates, that impacted many countries and (ii) given the nature of the loans at that time, there was a clear distinction between foreign and domestic creditors. In other crises, e.g., Argentina 2001-2, borrowing occurred mainly through bonds and it would be difficult to default only on foreigners (Broner, Martin and Ventura, 2006).

External shocks were important factors in the Latin American debt crisis of the 1980's. As noted by Diaz-Alejandro (1984), countries with different policies and distinct economies ended up in similar crises in the beginning of the 1980's, facing problems that in 1979 would have been considered unlikely. One key external shock is the increase in US real interest rates, shown in Figure 2 (from Dotsey et al, 2003). In contrast to the 1970's when real interest rates were around 0%, in the 1980's they were around 4%.[18]

In the beginning of the 1980's, the prices of Latin American bonds in secondary markets went down, capital flows to those economies dried or reverted and the fast process of economic growth of the 1970's stopped. After countless IMF missions, several debt reschedules and some attempts of debt renegotiation (including the Baker Plan), came the Brady agreements, starting in 1989. In the period between 1989 and 1994, most of the main Latin American countries got some debt relief. Table 1 shows the debt relief following the Brady Plan agreements as a percentage of the outstanding long term debt in the main Latin American countries. With the exception of Venezuela, at 20%, the other four countries are around 30%.[19]

[18] There are other cases in which interest rate increases in the US contributed to crises in other countries. The sharp increase in US interest rates in 1994, for example, is sometimes mentioned as one of the factors that led Mexico close to defaulting in December 1994.

[19] As noted by Cline (1995), the initial approach for dealing with the problem of debt overhang was aimed both at reducing debt and providing new loans, but "for practical purposes the Brady Plan has been all forgiveness and no new money" (Cline, 1995, page 236). Indeed, according to the model, if the amount of debt exceeds its incentive compatible level, new money will not be made available.

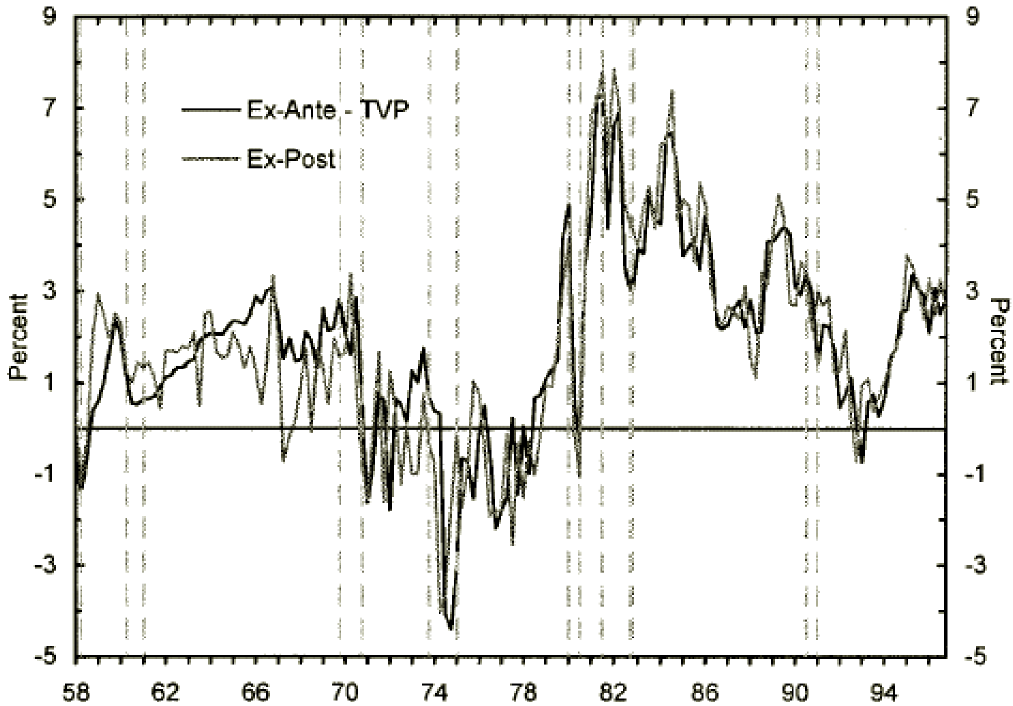

Figure 2: US real interest rates

As the Brady agreements did not cover all forms of external debt, the figures in Table 1 should be seen as upper bounds (Cline, 1995).

In the model, the optimal contract prescribes automatic instantaneous debt relief in order to compensate for unexpected increases in interest rates. In reality, debt relief came ten years later, and what happened in those ten years had some influence on the final agreement. Despite the delay, it is worth comparing the debt relief prescribed by the model and the one observed in reality because the Brady agreements were in fact the main relief solution to the crisis.

4.2 Debt relief according to the model

If the borrowing constraints of the Latin American countries were binding in 1979, then the interest rate rise at the beginning of the 1980's would have brought debt, d, above its incentive compatible level. In this section, I compute the optimal debt reduction prescribed by the model and compare it to the data.

Consider the model calibrated to represent the 1970's and 1980's. Suppose that world interest rates may be either 0% or 4% a year and that each state lasts for an average of 10

Table 1: Debt relief – Brady plan agreements, Cline (1995)

Venezuela	20%
Brazil	28%
Argentina	29%
Mexico	30%
Uruguay	31%

years: $q^{*h} = 1.00$, $q^{*l} = 1.04^{-1}$, $\beta = 1.02^{-1}$ and $\psi = 0.10$. Equation 3 yields the optimal debt relief given a switch from the high state to the low:

$$\frac{\Delta d}{\bar{d}} = \frac{1.00 - 1.04^{-1}}{1 - 1.02^{-1}(1 - 2 \times 0.10)} = 0.178$$

That implies a spread over treasury of 1.8%[20] when the state is high but debt relief of 18% when the state switches to low and interest rates jump from 0% to 4%. The result holds for any $\gamma > 0$.

If $\beta = (q^h + q^l)/2$ then $\Delta d/\bar{d}$ is not significantly affected by the length of debt contracts. For example, if interest rates switch between 0% and 4% and $\beta = (1.02)^{-1}$, one-year contracts and $\psi = 0.1$ yield $\left(d^h - d^l\right)/\bar{d} = 0.1783$. If, instead, there are five-year contracts and $\psi = 0.5$ we obtain $\left(d^h - d^l\right)/\bar{d} = 0.1781$.

This result is robust to other interest rates processes. Using the auto-regressive process and assuming a half-life of 3 years for the interest rate increase, the AR-1 coefficient, ζ, would be 0.79. A jump in real interest rates from 1% a year to 6% a year would then imply even greater debt relief: $\Delta d/\bar{d} = 22.5\%$.

The decrease in the level of debt predicted by the model in response to an interest rate increase of the magnitude observed in the data exceeds half the debt relief of the Brady agreements.

On the other hand, a negative productivity shock would not generate results of similar magnitude. A huge 10% reduction in productivity, assuming an average persistence of 10 years ($\psi = 0.10$) and world interest rates of 2% a year would imply debt reduction slightly *below 1%* according to Equation 4.

[20] From the definition $d^h = d' + \psi \Delta d$. Data from 1973-80 show a spread over treasury of 1.4% for Brazil/Argentina and 1.1% for Mexico (Calvo and Kaminsky, 1991).

4.3 Numerical results

In this section, the accuracy of the analytical approximations used in Section 4.2 is tested using numerical simulations. The main results from this section are:

1. The formula in Proposition 6 is a good approximation for $\Delta d/d'$ as long as the borrowing constraint is binding in both states of the economy, $mpk > r^{*l}$;

2. If the borrowing constraint is binding and the level of capital is substantially below its steady state level, then there are large gains from debt contracts designed to make $V^h(k', d^h) = V^l(k', d^l)$;

3. Regarding the Latin American debt crisis, if the marginal productivity of capital in those countries was not lower than $r^{*l} = 4\%$, the analytical results are good approximations. Therefore, the sharp interest rate rise at the beginning of the 1980's would imply debt relief of more than half of the observed reduction obtained through the Brady agreements. However, lower marginal productivity of capital (combined with low adjustment costs for capital) reduces the debt relief prescribed by the model.

I want to obtain the values of $\Delta d/d'$ that make $V^h(k', d^h) = V^l(k', d^l)$ at every state (k, d). The numerical solution is obtained through value function iteration. The state space is discretised using grids for debt and capital but the planner can choose any point in the grid. At the beginning of each iteration, Δd is calculated to make $V^h(k', d^h) = V^l(k', d^l)$.

I use the same stylisation of the 1970's and 1980's to calibrate the model: one period correspond to one year, $\alpha = 0.36$, $\beta = 1.02^{-1}$, $\gamma = 0.01$, $\sigma = 3$ and $\delta = 0.10$. $A = 1$, and q^* fluctuates around β: $q^{*l} = 1.04^{-1}$ and $q^{*h} = 1.00$. I constrain $k' - k$ to lie in some interval — adjusment costs for capital are zero in that interval and infinity outside it. Figure 3 shows $\Delta d/d'$ as a function of the marginal productivity of capital if the borrowing constraint is binding and the state is high in two situations: (a) $k' - k \in (-0.10k, 0.10k)$ and (b) $k' - k \in (-0.05k, 0.10k)$.

The main results are as follows:

- For $mpk > 0.04 = r^{*l}$, the linear approximation works well: $\Delta d/d'$ is around 0.17 and gradually increasing in mpk. The possibility of borrowing an additional unit in the high state is worth slightly more to countries with high mpk.

- For mpk below $r^{*l} = 4\%$, $\Delta d/d'$ is considerably smaller. For lower values of mpk, when the state shifts to low, interest rates are higher than mpk, so the country sells

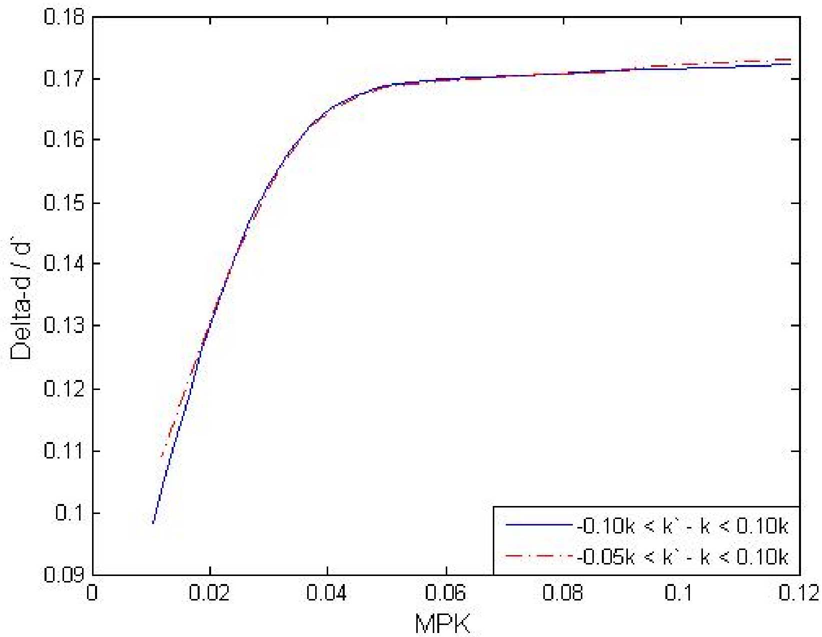

Figure 3: Debt relief

capital and could even end up buying high-interest-rate foreign bonds. This sounds unrealistic because adjustment costs for capital would prevent such rapid capital movement. Indeed, the optimal $\Delta d/d'$ for lower values of mpk are sensitive to the assumptions on adjusment costs. Figure 3 shows that when mpk is at its lowest value, $k' = k$, then $\Delta d/d'$ is around 0.10 with adjustment costs given by (a) and around 0.11 in case (b).

The optimality of $V^l(k', d^l) = V^h(k', d^h)$ is conditional on a binding borrowing constraint. If the constrint is not binding, then the condition becomes $\frac{\partial V^h_{pay}(k',d^h)}{\partial d} = \frac{\partial V^l_{pay}(k',d^l)}{\partial d}$ and the optimal contract in this case may prescribe very different values of Δd. The numerical simulations show that for an unconstrained country, with high levels of k, imposing debt contracts that imply $V^l(k', d^l) = V^h(k', d^h)$ at all states may be worse than trading unconditional bonds.

If the country is constrained, then there are substantial potential gains from debt contracts designed to make $V^h(k', d^h) = V^l(k', d^l)$. With adjustment costs given by (a), a country with $k = 2$ that holds its maximum incentive compatible level of debt and has access only to unconditional bonds would be indifferent between receiving a donation equal to 8.3% of its initial capital level and gaining access to those contingent contracts.

5 Policy discussion

According to the model in this paper, productivity fluctuations do not lead to significant default, but fluctuations in world interest rates can generate debt relief that accounts for a large proportion of what we observe in the data. In particular, the increase in US interest rates in the beginning the 1980's may account for more than half of the debt relief obtained by the main Latin American countries through the Brady agreements. Debt relief in this particular episode of sovereign default is not far from what an optimal contract would prescribe.

However, such debt relief came with a 10-year delay. By analysing the optimal contract, this paper does not consider the costly bargaining process that follows the announcement of a sovereign default. Some recent policy prescriptions focus on such bargaining costs: the IMF's sovereign debt restructuring mechanims (SDRM) is an important example (Krueger (2002)). The collective action clauses (CAC's) aim at allowing creditors to quickly reduce the level of debt when it goes above its incentive compatible level. That is a way to implement the optimal contract.

Even if contingencies are implicitly considered, given the high costs of debt renegotiations and defaults, it would be desirable to make such connection explicit in order to save the heavy retaliation and bargaining costs — the recent dramatic fall in Argentinean GDP after its default is just one example of such costs. At least in the case of the Latin American debt crisis of the 1980's, having debt explicitly contingent on world interest rates would have avoided 10 years of costly bargaining.[21]

Many of the problems related to GDP-indexed bonds do not apply to contracts contingent on world interest rates:

- there is no moral hazard: the country's 'effort' does not affect its debt;
- there are no major measurement problems, danger of misreporting, data revisions, lag in data announcements, we only need to estimate expected inflation in the relevant developed countries;
- while countries' GDP's are positively correlated, r^* and y^* are negatively correlated. As shown by Neumeyer and Perri (2005), interest rates and output are positively correlated in developed countries but negatively correlated in emerging markets.[22]

[21] Following the Latin American debt crisis of the 80's, many suggestions for indexing debt to some real economic variables have been made. Some popular variables are GDP (Athanasoulis and Shiller (2001) and Borenzstein and Mauro (2004) are recent references) and commodities prices (Kletzer et al (1992)). Much of the discussion was about moral hazard issues (Krugman, 1988): indexing debt to GDP would reduce incentives for governments to make effort.

[22] Given these benefits from this type of contingent contract, the question of why it has not been implemented has yet to

If default is never optimal, then limiting debt to "prudent" thresholds is a sensible policy presciption (Reinhart and Rogoff (2004)). However, in a world with large fluctuations in the incentive compatible level of debt, occasional debt relief is part of the optimal arrangement. Therefore, international lending contracts should allow for contingencies.

A Proofs

While proving the propositions in this paper, I often use the following method and I refer to it as a first order Taylor approximation. Consider the following Value function:

$$V(x,y) = \max_{a,b,c}\{u(x,y,a,b,c) + \beta V(x'(a,b,c), y'(a,b,c))\}$$

So the values of a, b, c are chosen, which determine the next period's x, y (denoted as a convention by x', y'). Take the function that is to be maximized:

$$f(x,y,a,b,c) = u(x,y,a,b,c) + \beta V(x'(a,b,c), y'(a,b,c))$$

Denote by a^*, b^* and c^* the maximizing values of $V(x,y)$ and by $\tilde{a}^*$, $\tilde{b}^*$ and $\tilde{c}^*$ the maximizing values of $V(\tilde{x},\tilde{y})$. Then $V(x,y) = f(x,y,a^*,b^*,c^*)$ and $V(\tilde{x},\tilde{y}) = f(\tilde{x},\tilde{y},\tilde{a}^*,\tilde{b}^*,\tilde{c}^*)$. Now, if (x,y) and $(\tilde{x},\tilde{y})$ are sufficiently close, we can take the approximation of the maximand function with respect to the variables to be chosen:

$$\begin{aligned} f(x,y,a^*,b^*,c^*) &\approx f(\tilde{x},\tilde{y},\tilde{a}^*,\tilde{b}^*,\tilde{c}^*) + \sum_{z=x,y,a,b,c} \frac{\partial f(\tilde{x},\tilde{y},\tilde{a}^*,\tilde{b}^*,\tilde{c}^*)}{\partial z}(z^* - \tilde{z}^*) \\ &= V(\tilde{x},\tilde{y}) + \sum_{z=x,y,a,b,c} \left(\frac{\partial u(\tilde{x},\tilde{y},\tilde{a}^*,\tilde{b}^*,\tilde{c}^*)}{\partial z} + \beta \frac{\partial V(\tilde{x}',\tilde{y}')}{\partial z} \right)(z^* - \tilde{z}^*) \end{aligned}$$

If there is a binding constraint on the possible values of a variable, then its maximized value will be determined by the constraint. Otherwise, the envelope theorem applies and the derivative of f with respect to that variable will be zero.

This method is different from the general first order Taylor approximation: the original function $V(x,y)$ is not a function of the variables with respect to which the approximation is done. This is why the maximand function has to be defined.

be answered. As argued by Borensztein and Mauro (2004), there may be many obstacles to innovations in sovereign debt, for example: policy makers are usually considered to have short horizons and preferences different than the agents; there may be fixed costs to introduce a different kind of debt — and so a free rider problem and, perhaps, an equilibrium in which no-one pays the fixed cost. Last, the heavy bargaining costs are paid not only by creditors and debtors but also by international organisations (e.g., the IMF), which reduces incentives for creditors and debtors to write contingent contracts.

A.1 Proposition 1

Proof. The value functions in case of repayment and in case of default are maximized at (k'_p, d'_p) and (k'_d) respectively, which means that:

$$\begin{aligned} V_{pay}(k,d) &= u(y+(1-\delta)k-k'_p-d+qd'_p)+\beta V_{pay}(k'_p,d'_p) \\ V_{def}(k,\gamma) &= u(y+(1-\delta)k-k'_d-\gamma y)+\beta V_{def}(k'_d,\gamma) \end{aligned}$$

When $d=d'=0$ and $\gamma=0$, the value functions are identical in the two cases, $V_0(k)$. It is maximized by choosing $k'=k'_o$.

Consider (k,d,γ) such that the country is indifferent between repaying and choosing (k'_p,d') or defaulting and choosing (k'_d), which means that $V_{pay}(k,d)=V_{def}(k,\gamma)$. Approximate the functions that are to be maximized,

$$\begin{aligned} f_{pay}(d,k',d') &= u(y+(1-\delta)k-k'-d+qd')+\beta V_{pay}(k',d') \\ f_{def}(k',\gamma) &= u(y+(1-\delta)k-k'-\gamma y)+\beta V_{def}(k',\gamma) \end{aligned}$$

around $V_0(k)$. The first order Taylor approximation of these functions around $V_0(k)$ with respect to (k',d,d') or (k',γ) respectively yields

$$\begin{aligned} f_{pay}(d,k',d') &= V_{pay}(k,0)+\frac{\partial f_{pay}(0,k'_o,0)}{\partial d}(d-0)+\frac{\partial f_{pay}(0,k'_o,0)}{\partial d'}(d'-0) \\ &\quad +\frac{\partial f_{pay}(0,k'_o,0)}{\partial k'}(k'-k'_o)+O_{k,p} \\ &= V_0(k)+u'(c_o)(-d+qd')+\beta\frac{\partial V_{pay}(k'_o,0)}{\partial d'}d'+O_{k,p} \end{aligned}$$

$$\begin{aligned} f_{def}(k',\gamma) &= V_{def}(k,0)+\frac{\partial f_{def}(k'_o,0)}{\partial \gamma}(\gamma-0)+\frac{\partial f_{def}(k'_o,0)}{\partial k'}(k'-k'_o)+O_{k,d} \\ &= V_0(k)-u'(c_o)\gamma y+\beta\frac{\partial V_{def}(k'_o,0)}{\partial \gamma}\gamma+O_{k,d} \end{aligned}$$

Where I used that since $\frac{\partial V_{pay}}{\partial k'}=\frac{\partial V_{def}}{\partial k'}=0$ due to the Envelope condition for the unconstrained maximization of V_{pay} and V_{def} with respect to k', their evaluation $\frac{\partial V_{pay}(k,0)}{\partial k'}=\frac{\partial V_{def}(k,0)}{\partial k'}=0$ as well. The optimal consumption with no borrowing, no punishment is denoted by $c_o=y+(1-\delta)k-k'_o$. Furthermore, $\lim_{(d,d',k')\to(0,0,k'_o)}\frac{O_{k,p}}{\|d,d',k'\|^2}=0$ and $\lim_{(\gamma,k')\to(0,k'_o)}\frac{O_{k,p}}{\|\gamma,k'\|^2}=0$.

Note that $V_{pay}(k,d)=V_{def}(k,\gamma)\iff f_{pay}(d,k'_p,d')=f_{def}(k'_d,\gamma)$. Using the first order Taylor expansions at points (d,k'_p,d') and (k'_d,γ), we get:

$$u'(c_o)(-d+qd'+\gamma y)+\beta\left(\frac{\partial V_{pay}(k'_o,0)}{\partial d'}d'-\frac{\partial V_{def}(k'_o,0)}{\partial \gamma}\gamma\right)+(O_{k,p}-O_{k,d})=0.$$

The last part is to show that $\frac{\partial V_{pay}(k'_o,0)}{\partial d'}d'-\frac{\partial V_{def}(k'_o,0)}{\partial \gamma}\gamma$ is approximately zero.

If $k < k^*$ then the country borrows the maximum level of incentive compatible debt. This debt level makes the country indifferent between repaying and defaulting at (k'_p, d'):

$$V_{pay}(k'_p, d') = V_{def}(k'_p, \gamma)$$

First order Taylor approximations of $V_{pay}(k'_p, d')$ and $V_{def}(k'_p, \gamma)$ around $V_o(k'_o)$ with respect to only k', d' and γ yield:

$$\begin{aligned} V_{pay}(k'_p, d') &= V_0(k'_p) + \frac{\partial V_{pay}(k'_o, 0)}{\partial k'}.(k'_p - k'_o) + \frac{\partial V_{pay}(k'_o, 0)}{\partial d'}.d' + O_{k'}(d'^2) \\ V_{def}(k'_p, \gamma) &= V_0(k'_p) + \frac{\partial V_{def}(k'_o, 0)}{\partial k'}.(k'_p - k'_o) + \frac{\partial V_{def}(k'_o, 0)}{\partial \gamma}.\gamma + O_{k'}(\gamma^2) \end{aligned}$$

$V_{pay}(k'_p, d') = V_{def}(k'_p, \gamma)$ implies:

$$\frac{\partial V_{pay}(k'_p, 0)}{\partial d'}.d' - \frac{\partial V_{def}(k'_p, 0)}{\partial \gamma}.\gamma \approx 0$$

Using this last equation the difference of the original Taylor expansions simplifies to:

$$u'(c_o)\left(-d + qd' + \gamma y\right) + (O_{k,p} - O_{k,d}) \approx 0$$

$u'(c_o) \neq 0$, and $(O_{k,p} - O_{k,d})$ is very small near $(d, d', k') = \left(0, 0, k'_p\right)$ and $(\gamma, k') = (0, k'_d)$ imply that

$$-d + qd' + \gamma y = 0 \Longleftrightarrow d = qd' + \gamma y$$

Which yields the second part of the claim. In steady state, $d = d'$, and we get the first part of the claim. ■

A.2 Proposition 2

Proof. Suppose $s = h$. If the borrowing constraint is not binding, the first order condition with respect to Δd yields:

$$\beta(1-\psi)\frac{\partial V^h_{pay}(k', d' + \psi\Delta d)}{\partial d}\psi - \beta\psi\frac{\partial V^l_{pay}(k', d' - (1-\psi)\Delta d)}{\partial d}(1-\psi) = 0$$

which yields the claim. If $s = l$, a similar expression is obtained. ■

A.3 Proposition 3

Proof. Suppose $s = h$ and $k', d', \Delta d$ are such that $V^h_{pay}(k', d^h) > V^l_{pay}(k', d^l)$. If the borrowing constraint is binding, the country borrows up to the incentive compatible level (and there is no debt repudiation in equilibrium), so $V^l_{pay}(k', d^l) = V_{def}(k')$. By increasing Δd by dD, V^l_{pay} increases and V^h_{pay} decreases. The borrowing constraint is no longer binding, so the country can increase k' by some dk' and d' by dk'/q^{*h} while still respecting the borrowing constraint.

Consumption in the present period is unchanged. $V^l(k', d^l)$ changes by

$$u'(c'_L).\left[dk'.(mpk - r^{*h}) + (1-\psi).dD\right]$$

and $V^h(k', d^h)$ changes by

$$u'(c'_H).\left[dk'.\left(mpk - r^{*h}\right) - \psi.dD\right]$$

where c'_L and c'_H are consumption next period in the low and high state, respectively.

So the change in $V^h(k, d)$ equals:

$$\beta.\left\{\left[\psi u'(c'_L) + (1-\psi)u'(c'_H)\right] dk'(mpk - r^{*h}) + \psi(1-\psi)dD\left[u'(c'_L) - u'(c'_H)\right]\right\}$$

Now, if $q^{*h} - q^{*l}$ is small enough, $(u'(c'_L) - u'(c'_H))$ is small and the change in $V^h(k, d)$ is positive because $mpk > r^{*h}$.

Similar expressions can be derived if $V^l_{pay}(k', d^l) > V^h_{pay}(k', d^h)$ and/or if $s = l$. ■

A.4 Lemma 4

Before proving lemma 4, we need an auxiliary result:

Lemma 10 *Consider the model with 2 states, h and l, and probability of changing state equal to ψ. Define $\bar{q} = (q^h + q^l)/2$. In a first order approximation,*

$$V(k, d, \bar{q}) = \left[V(k, d, q^h) + V(k, d, q^l)\right] \div 2$$

Proof. *Proof: Comes directly from a Taylor expansion of $V(\bar{k}, \bar{d}, q^h)$ and $V(\bar{k}, \bar{d}, q^l)$ around $V(\bar{k}, \bar{d}, \bar{q})$.* ■

We are ready to prove lemma 4.

Proof. First, we need to show that, close to the deteministic steady state $(k = \bar{k}, d = \bar{d})$, $V(\bar{k}, \bar{d}, q^h) = V^\xi(\bar{k}, \bar{d}, q^h)$ if $\xi = 1 - 2\psi$.

$$V^\xi(k, d, q^h) = \max_{k', d', \Delta d}\left\{\begin{array}{c} u(Af(k) + (1-\delta)k - k' - d + q^h d') \\ +\beta\left[(1-\xi)V^{\det}(k', d' - \xi\Delta d) + \xi V^\xi(k', d' + (1-\xi)\Delta d, q^h)\right] \end{array}\right\}$$

Near the deterministic steady state, choosing the optimal (d', k') instead of $(\bar{d}, \bar{k})$ has only second order effect on the value function V^ξ. As a first order approximation, we can write:

$$\begin{aligned} V^\xi(\overline{k}, \overline{d}, q^h) &= u(c^h) + \beta\left[(1-\xi)V^{\det}(\overline{k}, \overline{d} - \xi\Delta d) + \xi V^\xi(\overline{k}, \overline{d} + (1-\xi)\Delta d, q^h)\right] \\ V^{\det}\left(\overline{k}, \overline{d}\right) &= u(\overline{c}) + \beta V^{\det}(\overline{k}, \overline{d}) \end{aligned}$$

where $c_h = Af(\overline{k}) - \delta\overline{k} - \overline{d}(1 - q^h)$ and $\overline{c} = Af(\overline{k}) - \delta\overline{k} - \overline{d}(1 - \overline{q})$.

Taking a first order Taylor approximation of $f^{\xi}(\overline{k},\overline{d},q^h,\Delta d)=V^{\xi}(\overline{k},\overline{d},q^h)$ around $V^{\det}\left(\overline{k},\overline{d}\right)$ ($q^h=\overline{q}$ and $\Delta d=0$) with respect to $q,\Delta d$, we get:

$$\begin{aligned}
V^{\xi}(\overline{k},\overline{d},q^h) &\approx u\left(\overline{c}\right)+u'(\overline{c})(q^h-\overline{q})\overline{d}+\beta\left(V^{\det}(\overline{k},\overline{d})+(1-\xi)\frac{\partial V^{\det}(\overline{k},\overline{d})}{\partial d}(-\xi)(\Delta d-0)\right)\\
&\quad+\beta\xi\left(\frac{\partial V^{\xi}(\overline{k},\overline{d},\overline{q})}{\partial d}(1-\xi)(\Delta d-0)+\frac{\partial V^{\xi}(\overline{k},\overline{d},\overline{q})}{\partial q}(q^h-\overline{q})\right)\\
&= u\left(\overline{c}\right)+u'(\overline{c})(q^h-\overline{q})\overline{d}+\beta V^{\det}(\overline{k},\overline{d})+\beta(q^h-\overline{q})\xi\frac{\partial V^{\xi}(\overline{k},\overline{d},\overline{q})}{\partial q}
\end{aligned}$$

As a simple first order Taylor approximation, $\dfrac{\partial V^{\xi}(\overline{k},\overline{d},\overline{q})}{\partial q}(q^h-\overline{q})=V^{\xi}(\overline{k},\overline{d},q^h)-V^{\xi}(\overline{k},\overline{d},\overline{q})$, so the above approximation can be written as:

$$V^{\xi}(\overline{k},\overline{d},q^h)=u\left(\overline{c}\right)+u'(\overline{c})(q^h-\overline{q})\overline{d}+\beta V^{\det}(\overline{k},\overline{d})+\beta\xi\left[V^{\xi}(\overline{k},\overline{d},q^h)-V^{\xi}(\overline{k},\overline{d},\overline{q})\right]$$

Since $V^{\xi}(\overline{k},\overline{d},\overline{q})=V^{\det}(\overline{k},\overline{d})=u\left(\overline{c}\right)+\beta V^{\det}(\overline{k},\overline{d})$,

$$V^{\xi}(\overline{k},\overline{d},q^h)-V^{\det}(\overline{k},\overline{d})=u'(\overline{c})(q^h-\overline{q})\overline{d}+\beta\xi\left[V^{\xi}(\overline{k},\overline{d},q^h)-V^{\det}(\overline{k},\overline{d})\right]$$

so

$$V^{\xi}(\overline{k},\overline{d},q^h)-V^{\det}(\overline{k},\overline{d})=\frac{u'(\overline{c})(q^h-\overline{q})\overline{d}}{1-\beta\xi}\tag{5}$$

Now, note that, as a first order approximation:

$$\begin{aligned}
V(\bar{k},\bar{d},q^h)-V^{\det}(\bar{k},\bar{d}) &= u'(\bar{c})\left(q^h-\bar{q}\right)\bar{d}+\beta\left[(1-\psi)V(\bar{k},\bar{d},q^h)+\psi V(\bar{k},\bar{d},q^l)-V^{\det}(\bar{k},\bar{d})\right]\\
&= u'(\bar{c})\left(q^h-\bar{q}\right)\bar{d}+\beta(1-2\psi)\left[V(\bar{k},\bar{d},q^h)-V^{\det}(\bar{k},\bar{d})\right]
\end{aligned}$$

the last equality follows from lemma 10. Then:

$$V(\bar{k},\bar{d},q^h)-V^{\det}(\bar{k},\bar{d})=\frac{u'(\bar{c})\left(q^h-\bar{q}\right)\bar{d}}{1-\beta(1-2\psi)}\tag{6}$$

If $\xi=1-2\psi$, Equations (5) and (6) imply that $V(\bar{k},\bar{d},q^h)=V^{\xi}(\bar{k},\bar{d},q^h)$.

Now, we complete the proof by induction. Away from the steady state, we have:

$$V^{\xi}(k,d,q^h)=u(Af(k)+(1-\delta)k-k'-d+q^hd)+\beta\left[\xi V^{\xi}(k',d',q^h)+(1-\xi)V^{\det}(k',d')\right]$$

and

$$V(k,d,q^h)=u(Af(k)+(1-\delta)k-k'-d+q^hd)+\beta\left[(1-\psi)V(k',d',q^h)+\psi V^{\det}(k',d')\right]$$

If $V^{\xi}(k',d',q^h)=V(k',d',q^h)$, then, using lemma 10, we can write:

$$V(k,d,q^h)=u(Af(k)+(1-\delta)k-k'-d+q^hd)+\beta\left[\xi V^{\xi}(k',d',q^h)+(1-\xi)V^{\det}(k',d')\right]$$

and thus $V^{\xi}(k,d,q^h)=V(k,d,q^h)$. ■

A.5 Lemma 5

Proof. Subscripts denote time (k_0 is capital at time 0). The superscript ξ for $t>0$ means that at time $t-1$, when the variable (k or d) has been chosen, $q^* = q^{*\xi}$.

$$V^{\xi}(k_0, d_0^{\xi}, q^{\xi}) = \max_{k_1^{\xi}, d_1^{\xi}, \Delta d} \left\{ \begin{array}{c} u(Af(k_o) + (1-\delta)k_o - k_1^{\xi} - d_0^{\xi} + q^{*\xi} d_1^{\xi}) \\ +\beta[(1-\xi)V^{\det}(k_1^{\xi}, d_1^{\xi} - \xi\Delta d) + \xi V^{\xi}(k_1^{\xi}, d_1^{\xi} + (1-\xi)\Delta d, q^{*\xi})] \end{array} \right\}$$

again define the function to be maximized:

$$\begin{aligned} f(k_0, d_0^{\xi}, q^{\xi}, k_1^{\xi}, d_1^{\xi}, \Delta d) &= u(Af(k_o) + (1-\delta)k_o - k_1^{\xi} - d_0^{\xi} + q^{*\xi} d_1^{\xi}) \\ &\quad +\beta[(1-\xi)V^{\det}(k_1^{\xi}, d_1^{\xi} - \xi\Delta d) + \xi V^{\xi}(k_1^{\xi}, d_1^{\xi} + (1-\xi)\Delta d, q^{*\xi})] \end{aligned}$$

and take a Taylor approximation with respect to $k_1, d_0, d_1, q, \Delta d$ around $q = \overline{q}$, $d = d_0$, $\Delta d = 0$ that is when $f() = V^{\det}(k_0, d_0)$.

$$\begin{aligned} & V^{\xi}(k_0, d_0^{\xi}, q^{*\xi}) = f(k_0, d_0^{\xi}, q^{*\xi}, k_1^{\xi}, d_1^{\xi}, \Delta d) \\ \approx\ & u(Af(k_o) + (1-\delta)k_o - k_1 - d_0 + \overline{q} d_1) + \beta\left[(1-\xi)V^{\det}(k_1, d_1) + \xi V^{\xi}(k_1, d_1, \overline{q})\right] \\ & + \frac{\partial f(k_0, d_0, \overline{q}, k_1, d_1, 0)}{\partial k_1}(k_1^{\xi} - k_1) + \frac{\partial f(k_0, d_0, \overline{q}, k_1, d_1, 0)}{\partial d_0}(d_0^{\xi} - d_0) \\ & + \frac{\partial f(k_0, d_0, \overline{q}, k_1, d_1, 0)}{\partial d_1}(d_1^{\xi} - d_1) + \frac{\partial f(k_0, d_0, \overline{q}, k_1, d_1, 0)}{\partial q}(q^{*\xi} - \overline{q}) \\ & + \frac{\partial f(k_0, d_0, \overline{q}, k_1, d_1, 0)}{\partial \Delta d}(\Delta d - 0) \\ =\ & V^{\det}(k_0, d_0) + \beta\xi[V^{\xi}(k_1, d_1, \overline{q}) - V^{\det}(k_1, d_1)] + u'(c_0)[-(d_0^{\xi} - d_0) + \overline{q}(d_1^{\xi} - d_1) + d_1(q^{*\xi} - \overline{q})] \\ & + \beta\frac{\partial V^{\det}(k_1, d_1)}{\partial d_1}\left\{(1-\xi)[(d_1^{\xi} - d_1) - \xi(\Delta d - 0)]\right\} \\ & + \beta\frac{\partial V^{\xi}(k_1, d_1, \overline{q})}{\partial d_1}\left\{\xi[(d_1^{\xi} - d_1) + (1-\xi)(\Delta d - 0)]\right\} + \beta\xi\frac{\partial V^{\xi}(k_1, d_1, \overline{q})}{\partial q}(q^{*\xi} - \overline{q}) \end{aligned}$$

Where I used that $V^{\det}(k_0, d_0) = u(Af(k_o) + (1-\delta)k_o - k_1 - d_0 + \overline{q} d_1) + \beta V^{\det}(k_1, d_1)$.

Note that as a first order Taylor approximation:

$$\begin{aligned} V^{\xi}(k_1, d_1, q^{*\xi}) &= V^{\xi}(k_1, d_1, \overline{q}) + \beta\left[(1-\xi)\frac{\partial V^{\det}(k_1, d_1)}{\partial q} + \xi\frac{\partial V^{\xi}(k_1, d_1, \overline{q})}{\partial q}\right](q^{*\xi} - \overline{q}) \\ &= V^{\xi}(k_1, d_1, \overline{q}) + \beta\xi\frac{\partial V^{\xi}(k_1, d_1, \overline{q})}{\partial q}(q^{*\xi} - \overline{q}) \end{aligned}$$

When $q = \overline{q}$, $V^{\det}(k_1, d_1) = V^{\xi}(k_1, d_1, \overline{q})$ and $\dfrac{\partial V^{\det}(k_1, d_1)}{\partial d_1} = \dfrac{\partial V^{\xi}(k_1, d_1, \overline{q})}{\partial d_1}$. Using these last equations I get:

$$\begin{aligned} V^{\xi}(k_0, d_0^{\xi}, q^{*\xi}) - V^{\det}(k_0, d_0) &= u'(c_0)[-(d_0^{\xi} - d_0) + \overline{q}(d_1^{\xi} - d_1) + d_1(q^{*\xi} - \overline{q})] \\ &\quad + \beta\frac{\partial V^{\xi}(k_1, d_1, \overline{q})}{\partial d_1}(d_1^{\xi} - d_1) + \beta\xi[V^{\xi}(k_1, d_1, q^{*\xi}) - V^{\det}(k_1, d_1)]. \end{aligned}$$

Analogously, for $t > 0$ and starting with equal initial debts at both states, we get:

$$\begin{aligned} V^{\xi}(k_t, d_t, q^{*\xi}) - V^{\det}(k_t, d_t) &= u'(c_t)[\overline{q}(d^{\xi}_{t+1} - d_{t+1}) + d_{t+1}(q^{*\xi} - \overline{q})] \\ &\quad + \beta \frac{\partial V^{\xi}(k_{t+1}, d_{t+1}, \overline{q})}{\partial d_{t+1}}(d^{\xi}_{t+1} - d_{t+1}) \\ &\quad + \beta\xi[V^{\xi}(k_{t+1}, d_{t+1}, q^{*\xi}) - V^{\det}(k_{t+1}, d_{t+1})] \end{aligned}$$

Recursive substitution leads to

$$\begin{aligned} &V^{\xi}(k_0, d^{\xi}_0, q^{*\xi}) - V^{\det}(k_0, d_0) = -u'(c_0)(d^{\xi}_0 - d_0) \\ &+ \sum_{t=0}^{\infty} (\beta\xi)^t \left[u'(c_t)[\overline{q}(d^{\xi}_{t+1} - d_{t+1}) + d_{t+1}(q^{*\xi} - \overline{q})] + \beta \frac{\partial V^{\det}(k_{t+1}, d_{t+1})}{\partial d_{t+1}}(d^{\xi}_{t+1} - d_{t+1}) \right] \end{aligned}$$

Imposing $V^{\xi}(k_0, d^{\xi}_0, q^{*\xi}) = V^{\det}(k_0, d_0)$ we get the claim. ■

A.6 Proposition 6

Proof. Using Lemma 4 and Equation 2:

$$\begin{aligned} V(\bar{k}, d^h, q^{*h}) &= V(\bar{k}, \bar{d}, \bar{q}) \Rightarrow \\ V^{\xi}(\bar{k}, d^h, q^{*h}) &= V(\bar{k}, \bar{d}, \bar{q}) \Rightarrow \\ \frac{d^h - \bar{d}}{\bar{d}} &= \frac{q^{*h} - \bar{q}}{1 - \beta(1 - 2\psi)} \end{aligned}$$

.Analogously, $V(\bar{k}, d^l, q^{*l}) = V(\bar{k}, \bar{d}, \bar{q}) \Rightarrow$

$$\frac{d^l - \bar{d}}{\bar{d}} = \frac{q^{*l} - \bar{q}}{1 - \beta(1 - 2\psi)}$$

. Using both equations, we get the claim. ■

A.7 Proposition 7

Proof. Consider a Taylor approximation of $V^{AR}(\overline{k}, d^i, q^{*i})$ around the deterministic steady state $(V^{\det}(\overline{k}, \overline{d}, \overline{q}))$. The borrowing constraint is not binding, so choosing the optimal (d', k') instead of $(\overline{k}, \overline{d})$ has only second order effects on the value function V^{AR}. As a first order approximation, we can write:

$$\begin{aligned} V^{AR}(\overline{k}, d^i, q^{*i}) &= E\left(\sum_{t=0}^{\infty} \beta^t u\left(Af(k_t) + (1-\delta)k_t - k_{t+1} - d^i_t + q^{*i} d^i_{t+1}\right) \right) \\ &\approx \sum_{t=0}^{\infty} \beta^t u(\overline{c}) + E\left(\sum_{t=0}^{\infty} \beta^t u'(\overline{c})\overline{d}(q^{*i}_t - \overline{q}) \right) - u'(\overline{c})(d^i_0 - \overline{d}) \end{aligned}$$

Where $\overline{c} = Af(\overline{k}) - \delta\overline{k} - (1 - \overline{q})\overline{d}$.

Looking at the middle part of this expression,

$$
\begin{aligned}
& E\left(\sum_{t=0}^{\infty}\beta^t u'(\overline{c})\overline{d}(q_t^{*i}-\overline{q})\right) \\
= \quad & u'(\overline{c})\overline{d}\left[(q_0^{*i}-\overline{q}) + E\left(\beta\left(\zeta(q_0^{*i}-\overline{q})+\varepsilon_1\right)+\beta^2\left(\zeta\left(\zeta(q_0^{*i}-\overline{q})+\varepsilon_1\right)+\varepsilon_2\right)+...\right)\right] \\
= \quad & u'(\overline{c})\overline{d}\left[(q_0^{*i}-\overline{q})\sum_{t=0}^{\infty}(\beta\zeta)^t + E\left(\sum_{t=1}^{\infty}\beta^t\varepsilon_t\frac{1}{1-\beta\zeta}\right)\right] \\
= \quad & \sum_{t=0}^{\infty}(\beta\zeta)^t u'(\overline{c})\overline{d}\left(q_0^{*i}-\overline{q}\right) = u'(\overline{c})\overline{d}\left(q_0^{*i}-\overline{q}\right)\frac{1}{1-\beta\zeta}
\end{aligned}
$$

So:

$$
V^{AR}(k,d^i,q^{*i}) \approx \sum_{t=0}^{\infty}\beta^t u(\overline{c}) + u'(\overline{c})\overline{d}\left(q_0^{*i}-\overline{q}\right)\frac{1}{1-\beta\zeta} - u'(\overline{c})(d_0^i-\overline{d})
$$

Now imposing $V^{AR}(k,d^1,q^{*1}) = V^{AR}(k,d^2,q^{*2})$, we get:

$$
\begin{aligned}
& \sum_{t=0}^{\infty}\beta^t u(\overline{c}) + u'(\overline{c})\overline{d}\left(q^{*1}-\overline{q}\right)\frac{1}{1-\beta\zeta} - u'(\overline{c})(d^1-\overline{d}) \\
= \quad & \sum_{t=0}^{\infty}\beta^t u(\overline{c}) + u'(\overline{c})\overline{d}\left(q^{*2}-\overline{q}\right)\frac{1}{1-\beta\zeta} - u'(\overline{c})(d^2-\overline{d}) \\
\Rightarrow \quad & (d^2-\overline{d})-(d^1-\overline{d}) = \frac{1}{1-\beta\zeta}\overline{d}\left(\left(q^{*2}-\overline{q}\right)-\left(q^{*1}-\overline{q}\right)\right) \\
\Rightarrow \quad & d^2-d^1 = \frac{1}{1-\beta\zeta}\overline{d}(q^{*2}-q^{*1}) \\
\Rightarrow \quad & \frac{d^2-d^1}{\overline{d}} = \frac{q^{*2}-q^{*1}}{1-\beta\zeta}
\end{aligned}
$$

■

A.8 Proposition 8

Proof. Suppose $s=h$ and $k', d', \Delta d$ are such that $V^h_{pay}(k',d^h) > V^h_{def}(k',\gamma)$. If the borrowing constraint is binding, the country borrows up to the incentive compatible level (and there is no debt repudiation in equilibrium), so $V^l_{pay}(k',d^l) = V_{def}(k',\gamma)$. By increasing Δd by dD, V^l_{pay} increases and V^h_{pay} decreases. The borrowing constraint is no longer binding, so the country can increase k' by some dk' and d' by dk'/q^* while still respecting the borrowing constraint.

Consumption in the present period is unchanged. $V^l(k',d^l)$ changes by

$$
u'(c'_L).\left[dk'.(A^l f'(k')-\delta k'-r^*)+(1-\psi).dD\right]
$$

and $V^h(k',d^h)$ changes by

$$
u'(c'_H).\left[dk'.\left(A^h f'(k')-\delta k'-r^*\right)-\psi.dD\right]
$$

where c'_L and c'_H are consumption next period in the low and high state, respectively.

So the change in $V^h(k,d)$ equals β times:

$$dk'\left[\psi u'(c'_L)(A^l f'(k') - \delta k' - r^*) + (1-\psi)u'(c'_H)(A^h f'(k') - \delta k' - r^*)\right] + \psi(1-\psi)dD\left[u'(c'_L) - u'(c'_H)\right]$$

Now, if $A^h - A^l$ is small enough, $(u'(c'_L) - u'(c'_H))$ is small and the change in $V^h(k,d)$ is positive because the marginal productivity of capital is larger than r^*.

Similar expressions can be derived if $V^l_{pay}(k', d^l) > V^l_{def}(k', d^h)$ and/or if $s = l$. ∎

A.9 Proposition 9

Proof. Consider that productivity follows the ξ-process, so that $A_0 = A^\xi$ and for $t > 0$:

- if $A_{t-1} = A^\xi$, $\Pr(A = A^\xi) = \xi$ and $\Pr(A = \bar{A}) = 1 - \xi$;
- if $A_{t-1} = \bar{A}$, $A_t = \bar{A}$.

The value function at (k,d) if $A_t = A^\xi$ is:

$$V^\xi(k,d,A^\xi) = \max_{k',d',d'^\xi}\left\{u(c) + \beta\left[(1-\xi)V^{\det}(k',d') + \xi V^\xi(k', d'^\xi, A^\xi)\right]\right\}$$

where $c = A^\xi f(k) + (1-\delta)k - k' - d + q^*(d'(1-\xi) + \xi d'^\xi)$ and $V^{\det}$ is the value function in the model with no uncertainty.

A Taylor approximation of $V^\xi(\overline{k}, \overline{d}, A^\xi)$ around the deterministic steady state $(V^{\det}(\overline{k}, \overline{d}))$ yields:

$$V^\xi_{pay}(\overline{k},\overline{d},A^\xi) = u(\overline{c}) + u'(\overline{c})(A^\xi - \bar{A})f(k) + \beta(1-\xi)V^{\det}(\overline{k},\overline{d}) + \beta\xi V^\xi(\overline{k},\overline{d},A^\xi)$$

where $\overline{c} = \bar{A}f(\overline{k}) - \delta\overline{k} - (1-q^*)\overline{d}$.

So:

$$V^\xi_{pay}(\overline{k},\overline{d},A^\xi) - V^{\det}(\overline{k},\overline{d}) = u'(\overline{c})(A^\xi - \bar{A})f(k) + \beta\xi\left[V^\xi_{pay}(\overline{k},\overline{d},A^\xi) - V^{\det}(\overline{k},\overline{d})\right]$$

which yields:

$$V^\xi_{pay}(\overline{k},\overline{d},A^\xi) = V^{\det}(\overline{k},\overline{d}) + \frac{u'(\overline{c})(A^\xi - \bar{A})f(k)}{1-\beta\xi}$$

If d^ξ is close to $\overline{d}$, $V^\xi_{pay}(\overline{k}, d^\xi, A^\xi)$ can be written as:

$$V^\xi_{pay}(\overline{k},d^\xi,A^\xi) = V^{\det}(\overline{k},\overline{d}) + \frac{u'(\overline{c})(A^\xi - \bar{A})f(k)}{1-\beta\xi} - u'(\overline{c})\left(d^\xi - \overline{d}\right) \tag{7}$$

Out of the equilibrium path, the value function conditional on default is:

$$V^\xi_{def}(k,\gamma,A^\xi) = \max_{k'}\left\{u(c) + \beta\left[(1-\xi)V^{\det}_{def}(k',\gamma) + \xi V^\xi_{def}(k',\gamma,A^\xi)\right]\right\}$$

where $c = (1-\gamma)A^\xi f(k) + (1-\delta)k - k'$ and $V^{\det}_{def}$ is the value function in the model with no uncertainty if the country decides to default.

A Taylor approximation of $V^{\xi}_{def}(\overline{k},\gamma,A^{\xi})$ around the deterministic steady state $(V^{\det}_{def}(\overline{k},\gamma))$ yields:

$$V^{\xi}_{def}(\overline{k},\gamma,A^{\xi}) = u(\overline{c}_d) + u'(\overline{c}_d)(A^{\xi}-\bar{A})(1-\gamma)f(k) + \beta(1-\xi)V^{\det}_{def}(\overline{k},\gamma) + \beta\xi V^{\xi}_{def}(\overline{k},\gamma,A^{\xi})$$

where $\overline{c}_d = (1-\gamma)\bar{A}f(\overline{k}) - \delta\overline{k}$, which yields:

$$V^{\xi}_{def}(\overline{k},\gamma,A^{\xi}) = V^{\det}_{def}(\overline{k},\gamma) + \frac{u'(\overline{c}_d)(1-\gamma)(A^{\xi}-\bar{A})f(k)}{1-\beta\xi} \tag{8}$$

Using an argument similar to Lemma 4, if $\xi = 1-2\psi$ and fluctuations of technology are small, $V^h(k,d,A^h) = V^{\xi}(k,d,A^h)$ and an argument similar to Proposition 8 shows that if the country is constrained, in the optimal contract, d^{ξ} and $\overline{d}$ are such that $V^{\xi}_{pay}(k',d^{\xi}) = V^{\xi}_{def}(k',\gamma)$ and $V^{\det}_{pay}(k',\overline{d}) = V^{\det}_{def}(k',\gamma)$. We want to know the values of d^{ξ} and $\overline{d}$ that make such equalities hold when we are close to the deterministic steady state.

Using Equations 7 and 8, $V^{\det}_{pay}(\overline{k},\overline{d}) = V^{\det}_{def}(\overline{k},\gamma)$ imply:

$$\begin{aligned}
V^{\xi}_{pay}(\overline{k},d^{\xi},A^{\xi}) &= V^{\xi}_{def}(\overline{k},\gamma,A^{\xi}) - \frac{u'(\overline{c}_d)(1-\gamma)(A^{\xi}-\bar{A})f(k)}{1-\beta\xi} \\
&\quad + \frac{u'(\overline{c})(A^{\xi}-\bar{A})f(k)}{1-\beta\xi} - u'(\overline{c})\left(d^{\xi}-\overline{d}\right) \\
&= V^{\xi}_{def}(\overline{k},\gamma,A^{\xi}) + \frac{\left[\gamma u'(\overline{c}) + u'(\overline{c}) - u'(\overline{c}_d)\right](A^{\xi}-\bar{A})f(k)}{1-\beta\xi} - u'(\overline{c})\left(d^{\xi}-\overline{d}\right)
\end{aligned}$$

From Proposition 1:

$$V^{\det}_{pay}(\overline{k},\overline{d}) = V^{\det}_{def}(\overline{k},\gamma) \Rightarrow \overline{d} = \frac{\gamma A f(\overline{k})}{1-q^*}$$

which implies $\overline{c}_d = \overline{c}$.

So $V^{\xi}_{pay}(\overline{k},d^{\xi},A^{\xi}) = V^{\xi}_{def}(\overline{k},\gamma,A^{\xi})$ if:

$$\begin{aligned}
\frac{(1-q^*)\overline{d}}{\bar{A}f(\overline{k})}u'(\overline{c})\frac{(A^{\xi}-\bar{A})f(k)}{1-\beta\xi} &= u'(\overline{c})\left(d^{\xi}-\overline{d}\right) \Rightarrow \\
\frac{(1-q^*)\overline{d}}{1-\beta\xi}\frac{(A^{\xi}-\bar{A})}{\bar{A}} &= d^{\xi}-\overline{d}
\end{aligned}$$

Using $\xi = 1-2\psi$ and substituting $\left(A^{\xi},d^{\xi}\right)$ for $\left(A^h,d^h\right)$ and $\left(A^l,d^l\right)$ we get 2 equations that relate debt and productivity at each of the two states. Combining both equations, we get Equation 4. ■

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