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Abstract

Conventional fixed-income stress-testing methods often rely on polynomial loss approximations, whose accuracy deteriorates under scenarios of large yield shocks. We develop a simple yet accurate method for measuring shock-induced losses, introducing "proxy bonds" that match the duration and convexity of the original bond or a fixed-income portfolio. With virtually the same input information, the method delivers tight upper and lower bounds on bond and portfolio values even under extreme stress scenarios. The method also markedly outperforms conventional polynomial methods under stress scenarios involving non-parallel yield-curve shifts. We show how the precision of the new method lowers the data-disclosure requirements for stress testing, enabling reliable valuation based on aggregate portfolio characteristics.

Keywords: stress test, fixed income, interest rate risk, bonds, duration, convexity.

JEL classification: G11, G12, G24, G51.

Resumen

Los métodos convencionales de las pruebas de resistencia para los instrumentos de renta fija suelen basarse en aproximaciones polinómicas de las pérdidas, cuya precisión se deteriora cuando los escenarios contemplan grandes variaciones en los tipos de interés. En este documento se desarrolla un método simple pero preciso para medir las pérdidas inducidas por estos choques, introduciendo los denominados «bonos proxy», que preservan la duración y la convexidad del bono original o de una cartera de renta fija. Partiendo prácticamente de la misma información disponible, la metodología propuesta proporciona cotas superiores e inferiores muy ajustadas para la valoración de bonos y carteras, incluso bajo escenarios de movimientos extremos en los tipos de interés. Además, ofrece una precisión significativamente mayor que los métodos polinómicos convencionales en escenarios de estrés que implican desplazamientos no paralelos de la curva de rendimientos. Finalmente, se demuestra que la precisión de esta metodología permite reducir los requisitos de divulgación de datos necesarios para realizar las pruebas de resistencia, lo que posibilita valoraciones fiables basadas únicamente en las características agregadas de la cartera.

Palabras clave: prueba de resistencia, renta fija, riesgo de tipos de interés, bonos, duración, convexidad.

Códigos JEL: G11, G12, G24, G51.

1 Introduction

Although interest-rate risk is a key concern for institutional bond investors, detailed cash-flow information is not always available for full revaluation to the party conducting the stress test. In some settings, only aggregate portfolio characteristics are used, while security-level data is not employed due to confidentiality, reporting, or operational constraints. Even when detailed information is available, conducting stress tests at the granular level can be tedious for large portfolios.

For these reasons, investors and regulators routinely rely on approximation methods based on bond duration and convexity to evaluate changes in fixed-income portfolio values in response to potential shifts in the yield curve. For example, the Federal Reserve's stress-testing methodology relies on the duration-based approach when assessing potential losses on retail loans held for sale and certain types of fixed-income securities (Board of Governors of the Federal Reserve System, 2024). However, the usefulness of traditional methods is limited to short-term bonds or stress-test scenarios involving small interest rate changes. For long-term bonds or large shifts in rates, these methods produce severely inaccurate estimates (Fabozzi, 2021).

At the same time, regulatory stress tests do more than measure risk; they can also shape bank behavior. Banks adjust their lending and portfolio decisions to improve their stress-test performance, reducing exposures to assets that generate larger projected losses and reallocating toward assets treated more favorably by supervisory models. As a result, stress tests can affect credit supply, portfolio composition, and even systemic risk through common portfolio adjustments (Acharya et al., 2018; Brauning and Fillat, 2025; Shapiro and Zeng, 2024).

This channel implies that the choice of bond stress-testing methodology can affect bank asset allocation. For example, duration-based approaches can materially overstate the sensitivity of long-term fixed-income securities to positive interest-rate shocks (see Figure 1), potentially discouraging investment in long-term bonds and leading banks away from economically optimal asset allocations.

This paper develops methods that preserve the low informational requirements and operational simplicity of traditional approaches while delivering highly accurate valuations and reliable bounds for the post-shock bond and portfolio values. The proposed methods rely on virtually the same input data, yet markedly outperform conventional approximations under both large yield shocks and non-parallel yield-curve movements. Figure 1 shows how for a 30-year bond under a scenario of a parallel shift in the yield curve of 3 percentage points (pp)[1] the valuation error compared to the duration method can be reduced by 98.7%.

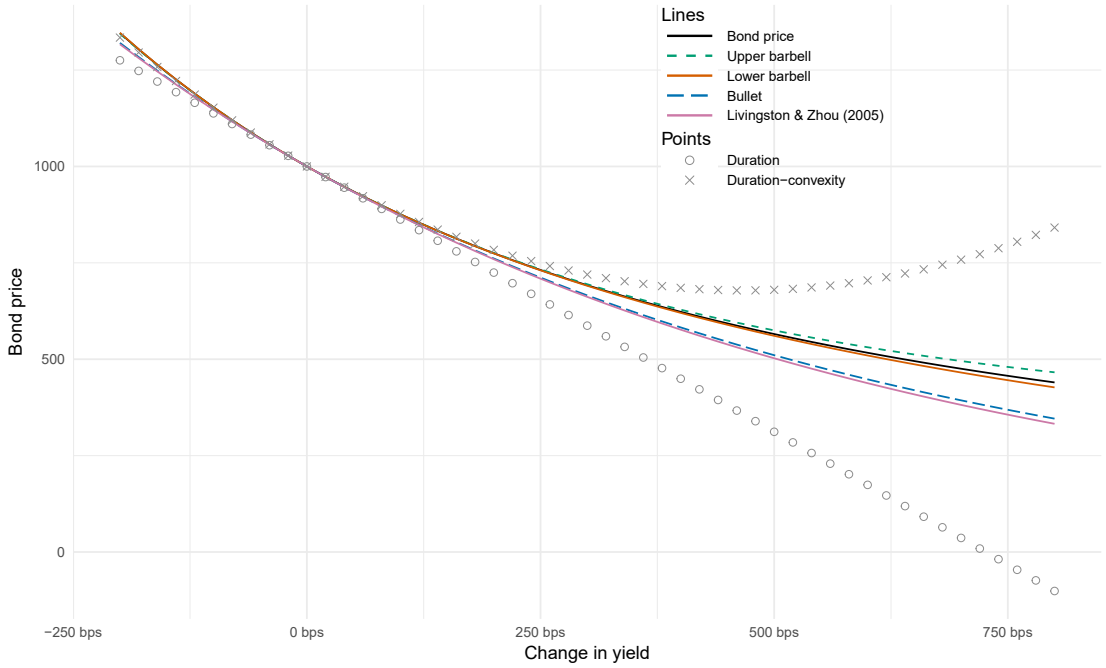
The central idea behind our approach is that the price of a regular bond—and, con-

[1]For reference, see Figure 2 for the rapid growth of interest rates in the U.S. reaching 4 pp in the post-COVID-19 period.

sequently, its price changes—can be closely replicated with the price of a fictitious or "proxy" bond with a simpler coupon structure that is constructed such that it matches the original bond's market price and key characteristics: duration, convexity, and maturity. The more information available on these core characteristics, the more accurate the method can be used for stress-testing purposes.

We derive closed-form solutions for constructing such fictitious bonds that allow to calculate the size and timing of one or two cash flows that constitute a proxy bond. Stress testing a portfolio of proxy bonds instead of the underlying bonds results in a more precise loss evaluation than traditional polynomial methods. Moreover, we document that two of the proposed methods—or proxy bonds—allow us to confine the true after-shock bond price in a tight interval.

Figure 1: Bond price response to changes of the discount rate



Notes: The figure plots the actual price of the bond and the price approximations according to six methods. Traditional approaches include duration and duration-convexity methods. The proxy-bond approach includes bullet, upper and lower barbells. The considered bond is a 30-year maturity bond that pays a 6% coupon annually, and is currently traded at par.

The derivation of the new methods is based on the observation that the price, duration, and convexity of a bond are proportional to the zeroth, first, and second moments of the discounted cash-flow measure. From a mathematical perspective, our approach can be interpreted as a truncated Hausdorff moment problem. Given a finite set of moments of the distribution of cash-flow present values in the time interval from today to the maturity date, we construct extremal cash-flow structures—consistent with those moments and delivering the highest and lowest possible post-shock bond prices[2] within the class of

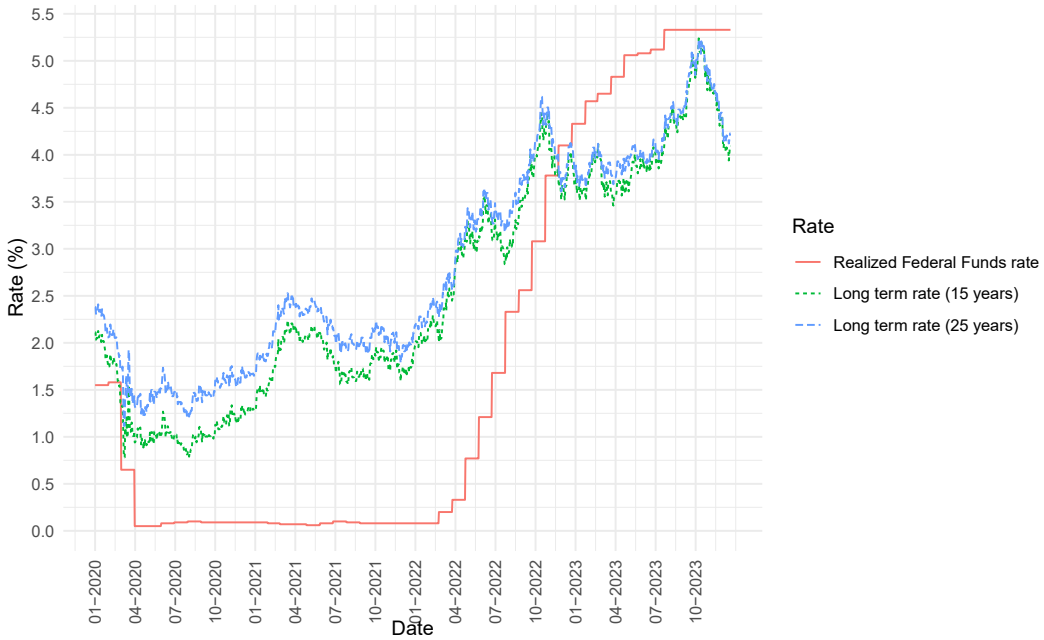
[2]Note that the post-shock bond price can be seen as the Laplace transform of the discounted cash-flow measure.

bonds satisfying the moment conditions. The proxy-bond methodology naturally extends beyond matching duration and convexity: in addition to providing closed-form solutions for the method up to the third moment, we formulate a general procedure for constructing extremal proxy bonds that match an arbitrary number of moments.

The improvements in widely used risk-management practices proposed in this paper come at a time when the vulnerability of the financial system to risks associated with long-term fixed-income instruments have been brought into focus. Shifts from low- to high-interest-rate environments, such as those observed in the post-COVID-19 period (Figure 2), pose significant challenges for the financial system. Prolonged periods of low interest rates encourage institutional investors and financial intermediaries to rebalance their fixed-income portfolios toward higher-yield assets, including those with longer maturities (ECB (2017), Schneider (2025)). When interest rate growth exceeds expectations, these exposures raise concerns about the resilience of financial institutions to the shock, as illustrated by the collapse of Silicon Valley Bank in March 2023, which was linked to losses on long-term bonds.

Thus, both the underestimation and the overestimation of the risks associated with long-term fixed-income instruments can have implications for financial stability. To address this challenge in settings where only aggregate bond or portfolio characteristics are available, we propose a cost-effective methodology that delivers highly accurate risk assessments even under extreme stress scenarios.

Figure 2: Evolution of interest rates in the U.S.



Notes: The figure plots rapid growth of the interest rates in the post-COVID-19 period.

Duration and convexity, which serve as inputs to bond approximation methods, are the coefficients of the Taylor expansion of the bond price with respect to the yield to

maturity (YtM).[3] Duration captures the local sensitivity of the bond price to changes in the YtM and gives rise to the first-order (linear) approximation, while convexity captures the curvature of the price-yield relationship and provides the second-order (quadratic) correction.

Unlike the traditional methods, our proposed methods are non-polynomial, which allows for a much higher precision for extreme scenarios. Figures 1 illustrates the relative performance of all methods. The first method proposed in this paper—the "bullet" approximation—relies solely on duration and approximates the price change of the original bond using the change in the price of a proxy zero-coupon bond, which we construct by matching the zeroth and first moments. The bullet method provides a superior approximation of bond prices compared to the linear duration approach. Its relative accuracy increases with both the magnitude of the interest rate shock and the duration of the original bond. Importantly, and of particular relevance for risk-averse investors and supervisory authorities, we prove that the bullet approximation is conservative (or lower bound), meaning it never exceeds the actual bond price for a given level of the after-shock YtM.

The second new method—the "upper barbell"—relies on both duration and convexity data and approximates the price change of the original bond with the change in the price of a proxy two-cash-flow bond, which we construct by matching the zeroth, first and second moments. It performs substantially better in terms of accuracy than the corresponding traditional duration–convexity approach. Similar to the duration–convexity method, the upper barbell approximation is anti-conservative for positive YtM shocks; that is, it exceeds the actual bond prices—or, put differently, serves as an upper bound for the price. Nonetheless, unlike the traditional method, it continues to produce economically meaningful results even for large interest rate changes, where the duration–convexity approximation may predict a reversal in the relationship between YtM and bond price due to its quadratic form, implying an implausible price increase when interest rates go beyond a certain point.

The third method we propose—the "lower barbell"—produces an exceptionally accurate and conservative bond price approximation for positive YtM shocks. In addition to bond duration and convexity, it requires data on the remaining maturity of the original bond. Like the upper barbell method, the lower barbell relies on a proxy two-cash-flow bond, but with a different timing of cash flows. To the best of our knowledge, within the literature seeking to improve bond price approximation techniques, the lower barbell method delivers the most precise bond price predictions across varying levels of YtM.

When it comes to negative YtM shocks, the two barbell approximations are flipped:

[3] YtM is the estimated total return generated by the bond if held to maturity, or equivalently, the discount rate that equates the present value of the bond's future cash flows to its current market price. The relationship between a bond's price and its YtM is inverse: a lower market price implies a higher YtM and, hence, a more profitable investment.

the upper barbell is conservative (a lower bound for the price), and the lower barbell is anti-conservative (an upper bound for the price). Irrespective of the direction of the interest rate shock—and regardless of which approximation is conservative and which one is anti-conservative—the true bond price is always confined between the two barbell approximations. Moreover, the range between the two barbell approximations is the tightest possible interval in which one can enclose the true bond price with the same input information.

The advantages of the new methods become particularly pronounced when applied to bond portfolios. We evaluate their performance under different levels of portfolio information disclosure, ranging from aggregate portfolio to position-level reporting of duration and convexity. Although more granular information improves precision, stress tests based on aggregate portfolio characteristics or a small number of maturity-based subportfolios allows for high accuracy.

Since both barbell methods rely on proxy bonds to evaluate the price of the original bond, an additional advantage is their applicability to scenarios involving non-parallel shifts in the yield curve.[4] We test the performance of the bond price approximation methods in response to changes in the yield curve of the U.S. Treasury securities for the period from 1985 to 2025. We document superior performance of the new methods compared to the traditional ones and show theoretically that small changes in the slope of the yield curve have a similar effect on the price of the original bond (or the market value of a portfolio) and on its corresponding proxy bond.

The use of the bond value approximation methods is not limited to industry and regulatory stress-testing exercises. In its collateral framework, the European Central Bank (ECB) employs an approximation method to determine the size of the haircuts[5] applied to bonds eligible as collateral in its credit operations (Adler et al., 2023). Recognizing the limitations of traditional approximation methods, the ECB employs the exponential duration method developed by Livingston and Zhou (2005).[6] This method suggests using a first-order approximation to the natural logarithm of bond price to capture the bond price sensitivity to yield changes. Like the traditional method, it requires only duration as an input; however, it performs better than the traditional duration method.[7]

The example of the ECB collateral framework demonstrates that there is demand for improved, yet operationally simple, approximation tools. Beyond the exponential

[4] In this context, our methodology may be viewed as related to approaches in the portfolio immunization literature; we discuss the comparison with this type of literature in Appendix E.

[5] A haircut is a discount applied to an asset's price when its value as collateral is being determined. It reflects the asset's risk and, hence, the value that can be recovered by the lender in case of the borrower defaults and the asset depreciates.

[6] Another example of application of the exponential duration method in bond price analysis and stress-testing can be found in Macías Fernández and de la Peña Leal (2024).

[7] The exponential duration method is marginally less precise than our bullet approximation and considerably less precise than either of our barbell approximations. In Section 5.1 we show that our bullet method is equivalent to the exponential duration method with continuously compounded yields.

duration method, limited progress has been made in this direction until now. Bajo et al. (2013) propose a discrete duration method, which offers a modest improvement over the exponential duration method for bonds that use annual compounding convention, but does not guarantee a conservative approximation of the bond price.

Each of the methods proposed in this paper outperforms the corresponding existing approach. As can be seen from the example in Figure 1, the bullet approximation outperforms the exponential duration method slightly. However, the upper barbell and lower barbell methods offer substantially more accurate evaluations. It suggests that with little to no effort investors and regulators can enhance their stress-testing frameworks to be able to obtain rapid and accurate bond price valuations even for long-term bonds and large interest rate shocks.[8]

The remainder of the paper is organized as follows. In Section 2 we briefly summarize the traditional approximation methods. In Section 3 we present the definitions and mathematical notation that is used throughout the paper. In Section 4 we present the new methods and their performance advantages compared to the traditional methods. In Section 5 we derive the new methods, i.e., the proxy bonds, and document their properties. In Sections 6 and 7 we discuss the application of the new methods to bond portfolios and examine their performance in approximating bond and bond portfolios price reactions to non-parallel changes of the yield curve using the data from the U.S. Treasury market. Section 8 concludes. The appendices provide a practical guide for the step-by-step application of the proxy-bond methods, extensions of the methodology to higher-order moments, mathematical proofs, and other supplementary material.

2 Traditional approximation methods in stress-testing

Bond price stress testing is widely used in practice as a forward–looking risk management tool that allows investors and regulators to assess how a bond or a portfolio would perform under adverse market conditions. By simulating market shocks, stress tests help identify vulnerabilities and quantify potential losses. A stress-test scenario or a "shock" for the yield curve refers to a hypothetical change in the term structure of interest rates that is used to revalue cash flows. These scenarios may be based on historical episodes or constructed as severe (tail) events; they can involve parallel shifts (uniform increases or decreases in yields across maturities), changes in slope, or more complex distortions of the curve.

Although these potential changes in the yield curve can materialize gradually, stress scenarios are typically implemented as if the changes occur instantaneously, i.e., the natural evolution of a fixed-income portfolio's value driven by coupon payments and the changes in discount rates due to progressive shortening of maturity are abstracted from

[8] Scenarios featuring large increases in interest rates are particularly relevant for developing economies.

for simplicity. Throughout the paper, we assume that the hypothesized shocks happen instantaneously as well.

For stress scenarios that imply small parallel shifts in the yield curve, the approximate change in the value of a bond or a bond portfolio can be inferred using methods derived from the Taylor expansion of the bond price function. In this section, we briefly summarize these traditionally used approximation methods.

The price of a bond is the present value of the cash flows it generates—that is, the sum of coupons and the face value of the bond discounted at the yield to maturity (YtM). For a straight bond, the price is estimated as follows:

$$P_0 = \sum_{i=1}^{I} \frac{\mathrm{CF}_i}{(1 + Y_0/f)^{f \cdot t_i}} \equiv \sum_{i=1}^{I} \mathrm{CF}_i e^{-y_0 t_i},$$

where CF_i is the i^{th} cash flow—coupons or face value of the bond—paid at time t_i, f is the frequency of cash flows per year, and Y_0 is the annualized YtM.[9]

If we take the first derivative of the bond price with respect to YtM and divide by minus the bond price P, we obtain the modified duration:

$$D = -\frac{1}{P(Y)} \cdot \left. \frac{dP(Y)}{dY} \right|_{Y=Y_0}.$$

Similarly, convexity is defined as follows:

$$C = \frac{1}{P(Y)} \cdot \left. \frac{d^2P(Y)}{dY^2} \right|_{Y=Y_0}.$$

So, the linear approximation (also known as the *duration method*) is given by the following formulas:

$$\frac{\Delta P}{P_0} \approx -D \times \Delta Y,$$

and thus,

$$P \approx P_0 \left(1 - D \times \Delta Y\right).$$

Similarly, the quadratic approximation (also known as the *duration–convexity method*) is given by the following formulas:

$$\frac{\Delta P}{P_0} \approx -D \times \Delta Y + \frac{1}{2} C \times \Delta Y^2,$$

and thus,

$$P \approx P_0 \left(1 - D \times \Delta Y + \frac{1}{2} C \times \Delta Y^2\right).$$

[9] Note that in Sections 4–7 we use the continuous YtM (y_0) instead of the discrete YtM (Y_0) for simplicity of exposition. This is purely a notational convenience and does not affect generality.

Duration and convexity are fundamental concepts for interest rate risk and asset-liability management. Thus, for example, duration is the underlying tool for portfolio immunization, or duration hedging, in asset-liability management practices, that is, bond portfolio construction in a way that its value is "immune" to small parallel shifts in the yield curve. One way to achieve it is the "barbell" trade, which involves obtaining the target bond duration exposure by balancing shorter- and longer-duration bonds. Such portfolio has higher convexity than the alternative "bullet" portfolio, and theoretically leads to profits in the case of a large parallel shift of the yield curve (Mele, 2022). We lean into this conventional terminology to name the proposed approximation methods in an intuitive manner.

3 Notation and definitions

In this section, we present notations that are used throughout this paper.

- t – the time span from now to the date until we receive a payment from a bond; each payment has its own t.
- f – the coupon frequency of a bond (number of coupon payments per year).
- μ_0 – the current dirty[10] market price of the original bond.
- Y_0 – the pre-shock YtM defined as the solution to the following equation:[11]

$$\mu_0 = \sum_i \mathrm{CF}_i (1 + Y_0/f)^{-f t_i},$$

  where $\{\mathrm{CF}_i\}$ are the cash flows of the original bond, and $\{t_i\}$ are their respective timings from the current date.

- y_0 – the pre-shock continuously-compounded YtM defined as the solution to the following equation:

$$\mu_0 = \sum_i \mathrm{CF}_i e^{-y_0 t_i},$$

  It is easy to see that $y_0 = f \log(1 + Y_0/f)$, and vice versa, $Y_0 = f \cdot \left(e^{y_0/f} - 1\right)$.

- Y – the after-shock YtM of the original bond.
- y – the after-shock continuously-compounded YtM of the original bond.

  Similar to y_0, we have $y = f \log(1 + Y/f)$, and vice versa, $Y = f \cdot \left(e^{y/f} - 1\right)$.

[10] Also known as the "invoice" price—that is, the price of the bond including the accrued part of the next coupon payment. The dirty price is the amount of money that changes hands during a trade.

[11] The compounding convention implied in the definition depends on the coupon frequency. This is the industry standard for a majority of trading venues and data providers.

- $\Delta Y = Y - Y_0$ – the shock to the discretely-compounded YtM.
- $\Delta y = y - y_0$ – the shock to the continuously-compounded YtM.
- $P(y)$ – the price of the original bond as a function of the after-shock continuously-compounded YtM.
- $P_{\text{bond X}}(y)$ – the price of bond X as a function of the after-shock continuously-compounded YtM.[12]
- $\text{PV}_{\text{bond X}}(\mathbb{R}_+) = P_{\text{bond X}}(y_0)$ – present value of all the future payments of bond X.[13]
- More generally, $\text{PV}_{\text{bond X}}(\cdot)$ denotes the σ-additive measure associated with the present value of the future payments of bond X. For any time interval Θ, we define $\text{PV}_{\text{bond X}}\{\Theta\}$ as the aggregate dollar value of all the present values of the payments of bond X that will happen within the interval Θ computed using the pre-shock YtM:

$$\text{PV}_{\text{bond X}}\{\Theta\} = \sum_{i=1}^{\#\text{ of payments of bond X}} \text{CF}_i^{\text{bond X}} e^{-y_0 t_i^{\text{bond X}}} \mathbb{1}_\Theta\left(t_i^{\text{bond X}}\right),$$

  where $\text{CF}_i^{\text{bond X}}$ are the payments of bond X, $t_i^{\text{bond X}}$ are their respective timings, and $\mathbb{1}$ denotes the indicator function.[14]
- We call the quantity $\mu_n^{\text{bond X}} = \int_t t^n d\text{PV}_{\text{bond X}}$, the n-th moment of bond X ($n \in \{0, 1, 2\}$). Thus, $\mu_0^{\text{bond X}}$—which is the current price—is the zeroth moment, $\mu_1^{\text{bond X}}$ is the first moment, and $\mu_2^{\text{bond X}}$ is the second moment.
- We call the quantity $\mu_{n/0}^{\text{bond X}} := \mu_n^{\text{bond X}}/\mu_0^{\text{bond X}}$, the normalized n-th moment of bond X.

  The normalized moment $\mu_{1/0}^{\text{bond X}}$ is the Macaulay duration of bond X. It is easy to see that

$$\mu_{1/0}^{\text{bond X}} = -P'_{\text{bond X}}(y_0)/P_{\text{bond X}}(y_0),$$

  so $\mu_{1/0}^{\text{bond X}}$ can be also called "continuous-time duration".

  The normalized moment $\mu_{2/0}^{\text{bond X}}$ is the "continuous-time convexity" of bond X. It is easy to see that

$$\mu_{2/0}^{\text{bond X}} = P''_{\text{bond X}}(y_0)/P_{\text{bond X}}(y_0).$$

[12] The symbol "bond X" will be replaced by "original bond," "bullet," "upper barbell," "lower barbell," and "average barbell" throughout the rest of this paper.

[13] The symbol $\mathbb{R}_+$ denotes the set of non-negative real numbers.

[14] We need this abstract notation, $\text{PV}_{\text{bond X}}$, in order to make our calculations succinct. To this end, in further text we avoid using summations, replacing them with corresponding Lebesgue integrals with respect to the measure $\text{PV}_{\text{bond X}}$.

If the bond is not specified in the superscript, then by default it is the "original bond."

- D – modified duration as reported in Bloomberg and Datastream (therefore, differentiated with respect to discrete YtM Y):

$$D_{\text{bond X}} = -\left.\frac{dP_{\text{bond X}}}{dY}\middle/ P_{\text{bond X}}\right|_{Y=Y_0}.$$

If the bond is not specified, then by default it is the "original bond."

- C – convexity as reported in Bloomberg and Datastream (therefore, differentiated with respect to discrete YtM Y):

$$C_{\text{bond X}} = \left.\frac{d^2P_{\text{bond X}}}{dY^2}\middle/ P_{\text{bond X}}\right|_{Y=Y_0}.$$

If the bond is not specified, then by default it is the "original bond."

- $\text{yc}_0(\cdot)$ – pre-shock continuously-compounded yield curve.
- $\text{yc}(\cdot)$ – after-shock continuously-compounded yield curve.
- $\Delta\text{yc}(\cdot)$ – shock to the continuously-compounded yield curve.
- Throughout the paper, uppercase symbols C_* denote cash flows. Meanwhile, lowercase symbols c_* refer to the corresponding present values discounted at the pre-shock YtMs.

Remark 3.1. It is easy to show that there is a link between conventional duration and convexity[15]—D and C—and the same parameters with continuous-time compounding[16]—$\mu_{1/0}$ and $\mu_{2/0}$:

$$\begin{aligned}\mu_{1/0} &= D \cdot (1 + Y_0/f),\\ \mu_{2/0} &= C \cdot (1 + Y_0/f)^2 - \mu_{1/0}/f\\ &= C \cdot (1 + Y_0/f)^2 - D \cdot (1 + Y_0/f)/f.\end{aligned}$$

For simplicity of exposition we use continuously-compounded durations and convexities throughout this paper.

Remark 3.2. For brevity, we use bond terminology throughout the paper. Nevertheless, the methods proposed in Sections 5.1, 5.2, and 5.3 are applicable to bond portfolios as well.[17] So, the "original bond" can be understood more generally as a portfolio of bonds.

[15] Based on the YtMs Y, compounded f times per year.
[16] Based on the YtMs y, compounded ∞ times per year.
[17] As we explain in Section 6.

One intricacy arises, however, when one applies our methods to a bond portfolio. Unlike individual bonds, bond portfolios do not have a single YtM. To apply our method, we assume that all continuously-compounded yields to maturity in a portfolio change in a parallel way: one common parameter for all bonds is the shock to the YtM Δy. In this setting, the after-shock market value of a portfolio can be written as follows:

$$\hat{P}(\Delta y) = \sum_{i}^{\text{\# of payments in the portfolio}} \text{PV}_i^{\text{portfolio}} e^{-t_i \Delta y} = \int_t e^{-t\Delta y} d\text{PV}_{\text{portfolio}},$$

where $\text{PV}_i^{\text{portfolio}}$ is the i-th cash flow of in the portfolio of bonds discounted at the current YtM of its corresponding bond, and

$$\text{PV}_{\text{portfolio}}(\Theta) := \sum_{i=1}^{\text{\# of payments in the portfolio}} \text{PV}_i^{\text{portfolio}} \mathbb{1}_{\Theta}\left(t_i^{\text{bond X}}\right)$$

for any time interval Θ.

This approach allows us to apply the bullet, the upper barbell, and the lower barbell approximations to portfolios of bonds as long as we know:

- The weighted average continuous-time duration (for the bullet method),
- The weighted average duration and convexity, both continuously-compounded (for the upper barbell method),
- The weighted average duration and convexity, both continuously-compounded, as well as a cap on the maximum remaining time to maturity (for the lower barbell method).

4 New methods and comparative performance

In this section, we introduce the new methods in more detail, provide instructions on the calculation and application of the methods, and demonstrate their superior performance compared to the traditional approximation methods.

The new methods are based on the idea that the price of a bond whose coupon schedule is unknown can be approximated with another bond with a simple payment structure of one or two payments. Figure 3 illustrates the payment structure of the proxy bonds next to that of the original bond. The bullet method involves matching a fictitious zero-coupon bond such that its duration coincides with the duration of the original bond. In Section 5.1, we demonstrate that such a bond pays amount C_b at time T_b given by

$$T_b = \mu_{1/0} \equiv D \cdot (1 + Y_0/f)$$

and

$$C_b = \mu_0 e^{y_0 T_b}.$$

Thus, the after-shock price of the proxy zero-coupon bond is calculated as follows:

$$P_{\text{bullet}}(y) = C_b \cdot e^{-yT_b} = \mu_0 \cdot e^{-(y-y_0)\mu_{1/0}},$$

where y is the after-shock YtM, μ_0 is the current dirty price of the original bond, and $\mu_{1/0}$ is the Macaulay duration of the original bond.

Further, in Section 5.2 we introduce a two-cash-flow proxy bond—we call it the upper barbell bond—that matches the duration and the convexity of the original bond with the first cash flow, C_0, received at time $t = 0$. We demonstrate that in order to match the duration and convexity, the upper barbell bond has to have the following payment structure.

Date	Present value	Payment
0	$c_0 := \mu_0 - c_{ub}$	$C_0 := c_0$
$T_{ub} := \mu_2/\mu_1$	$c_{ub} := \mu_1/T_{ub}$	$C_{ub} := c_{ub} e^{y_0 T_{ub}}$

where

$$\mu_1 = \underbrace{\mu_0}_{\text{Dirty price}} \cdot \underbrace{\mu_{1/0}}_{\text{Macaulay duration}}$$

and

$$\mu_2 = \underbrace{\mu_0}_{\text{Dirty price}} \cdot \underbrace{\mu_{2/0}}_{\text{Continuous-time convexity}}.$$ [18]

Importantly, we show that for positive shocks to the YtM, the price of the upper barbell bond is always above that of the original bond (Theorem 5.6).

Finally, in Section 5.3 we introduce the lower barbell bond—a two-cash-flow bond whose last payment occurs at the maturity date of the original bond. We choose all the other parameters so as to fit the duration and convexity and show that the payment structure must be the following:

Date	Present value	Payment
$t_{lb} := T + \breve{\mu}_2/\breve{\mu}_1$	$c_{lb} := \breve{\mu}_1^2/\breve{\mu}_2$	$C_{lb} := c_{lb} e^{y_0 t_{lb}}$
T	$c_T := \mu_0 - c_{lb}$	$C_T := c_T e^{y_0 T}$

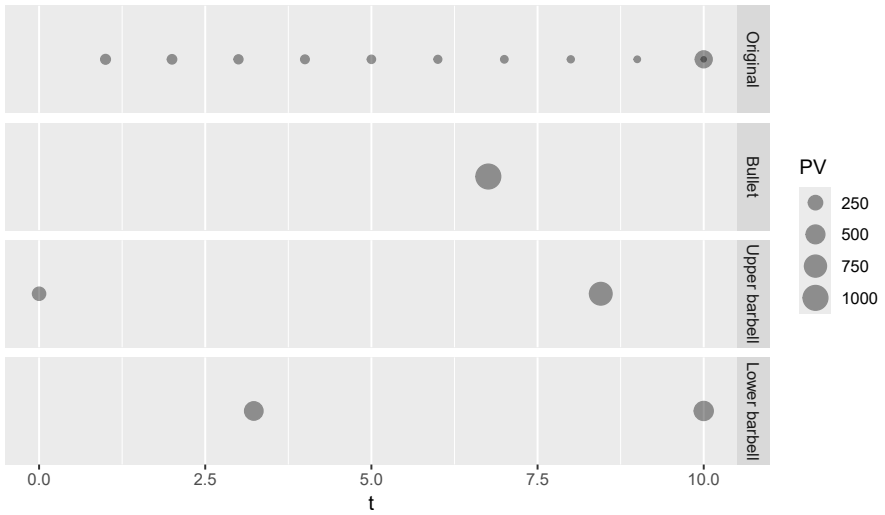
where T is the time to maturity of the original bond, the parameters $\breve{\mu}_1$ and $\breve{\mu}_2$ are the first and the second moments of the original bond about date $t = T$.[19] The moments $\breve{\mu}_1$ and $\breve{\mu}_2$ are closely related to μ_1 and μ_2:

$$\breve{\mu}_1 = \mu_1 - T\mu_0, \qquad \breve{\mu}_2 = \mu_2 - 2T\mu_1 + T^2\mu_0.$$

[18] The quantity $\mu_{2/0}$ is discussed in detail is Section 3.

[19] The formal definition is provided in Section 5.3.

Figure 3: Payment structure of proxy bonds



Notes: The figure illustrates the payment structure of the proxy (or fictitious) bonds. Each circle represent a cash flow, the circle size reflects the present value of the payment, and its position on the horizontal axis corresponds to the timing of the payment. The original bond pays one coupon payment per year and the nominal value in ten years. The proxy bond for the bullet approximation method is a zero-coupon bond that matures at t equal to the Macaulay duration of the original bond. The upper barbell proxy bond achieves the duration and convexity matching to the original bond by balancing (hence, the name) two cash flows, where the first one is always at time $t = 0$ and the second one is at a specific date in the future. The lower barbell proxy bond achieves the duration, convexity and maturity matching to the original bond by balancing two cash flows, where the second one is always at the maturity date of the original bond and the first one is at a specific date before maturity.

We show that for positive shocks to the YtM, the lower barbell approximation constitutes a lower bound for price of the original bond (Theorem 5.10).

Figure 4 considers four examples of bonds with varied maturity and coupon rates that are traded at par. For each bond, the figure plots its price at the varying YtM increment (or shock), and the approximated price according to all considered approximation methods. Figure 5 complements Figure 4 by depicting the size of the approximation errors of each method relative to the actual bond price over the varying YtM increment.

The widely known shortcomings of the traditional approximation methods are immediately obvious from the figure. The duration-based model—that is, the linear approximation of the price—produces a tolerable approximation for small changes in the YtM rate of up to 200 basis points (bps) for 10-year bonds and up to 100 bps for 30-year bonds. Beyond these margins, the linear approximation becomes noticeably imprecise.

A clearer picture emerges from Figure 5 depicting the quickly increasing approximation error. For the considered long-term bonds, the error reaches an economically significant size already at the 200 bps rate change level – for the 30-year bond with the 3% coupon rate, the errors climb to 8% at the 200 bps threshold. Since the relationship of the bond price to the YtM is non-linear, the duration approximation quickly deviates from the actual bond price and, eventually, falls below zero. The duration-convexity method—that is, the quadratic approximation—performs better than the linear method for shorter

maturity bonds and for relatively small YtM increments. However, for longer-term bonds it generates a meaningless prediction of a price growth after a certain point in the interest rate increase.

The new methods perform substantially better than the traditional ones. For 10-year bonds, our new approximations are almost indistinguishable from the actual bond price. For the 30-year bonds, small deviations from the bond price can be observed. If we zoom into the figure depicting the relative errors—that is, Figure 5, we can see that the bullet method—that closely matches the Livingston and Zhou (2005) exponential duration method—performs better than its counterpart duration method, but can generate larger error than the duration–convexity method for small changes in the YtM. Meanwhile, the upper and lower barbell methods closely follow the actual bond price, and generate minimal errors for the considered challenging cases of long-term bonds and large interest rate swings.

Importantly, our methods outperform the traditional ones for any interest rate changes $y - y_0$, not only in extreme scenarios. To summarize our analytical results, we formulate the following statement, in which we compare all the approximations plotted in Figures 4 and 5.

Summary statement. For non-trivial bonds, that is, bonds consisting of at least three distinct cash flows, and positive changes in the rates, $y - y_0 > 0$, the following holds:

$$P_l(y) < P_b(y) < P_{lb}(y) < P(y) < P_{ub}(y) < P_q(y), \tag{1}$$

where $P_l(y)$ and $P_q(y)$ denote the linear and quadratic approximations, respectively.

For non-trivial bonds and negative changes in the rates, $y - y_0 < 0$, the following holds:

$$P_l(y) < \left\{ \begin{array}{c} P_q(y) \\ P_b(y) \end{array} \right\} < P_{ub}(y) < P(y) < P_{lb}(y). \tag{2}$$

Proof. See Appendix G, Section G.1. □

Remark 4.1. As one can see from inequalities 1 and 2, the upper and lower barbells are the best bounds for the bond price among all the approximations considered in these inequalities. In fact, we show that the barbell approximations are the tightest bounds one can obtain knowing the price, duration, convexity, and maturity. In Section 6 we provide a more general statement about the bound tightness (which can be applied to portfolios of bonds) and formalize this result in Proposition 6.1 and Corollary 6.1.1.

Remark 4.2. As one can see from inequality 2, there is no universal comparison between the quadratic approximation and the bullet approximation for $y < y_0$. Both approximations are lower bounds for the bond price. Both are tighter than the linear approximation and looser than the upper barbell approximation. Which one performs better depends

on the magnitude of the shock, $y - y_0$. For smaller shocks, the quadratic approximation outperforms as it matches the second derivative, which makes it locally more precise. For larger shocks, the bullet approximation wins. This trade-off between traditional approximation techniques[20] can be seen in Figure 5, particularly pronounced in the panel with the 30-year 3%-coupon bond. This observation makes the upper barbell approximation particularly attractive because it outperforms both traditional methods irrespective of the shock magnitude.

Remark 4.3. The degree by which our approximations outperform traditional ones for small-magnitude shocks $y-y_0$ is hard to discern from Figures 4 and 5. In order to visually gauge the advantage of our methods for small shocks, we plot the ratios of approximation errors in Appendix D.

Remark 4.4. For the ease of notation, we consider the linear and quadratic approximations with respect to y, the continuously compounded rate. The ranking is preserved if $P_l(y)$ is replaced with $P_L(y)$, and $P_q(y)$ is replaced with $P_Q(y)$, where $P_L(f\log(1+Y/f))$ and $P_Q(f\log(1+Y/f))$ are linear and quadratic functions in Y. Numerically, the choice of Y versus y for the Taylor expansion does not change the approximation quality significantly for realistic values of y_0 and y.

Remark 4.5. For simplicity of exposition we do not include in our comparison the Livingston and Zhou (2005) and Bajo et al. (2013) approximations because they are numerically close to the bullet approximation. For details, see Appendix C.

5 Construction of the proxy bonds

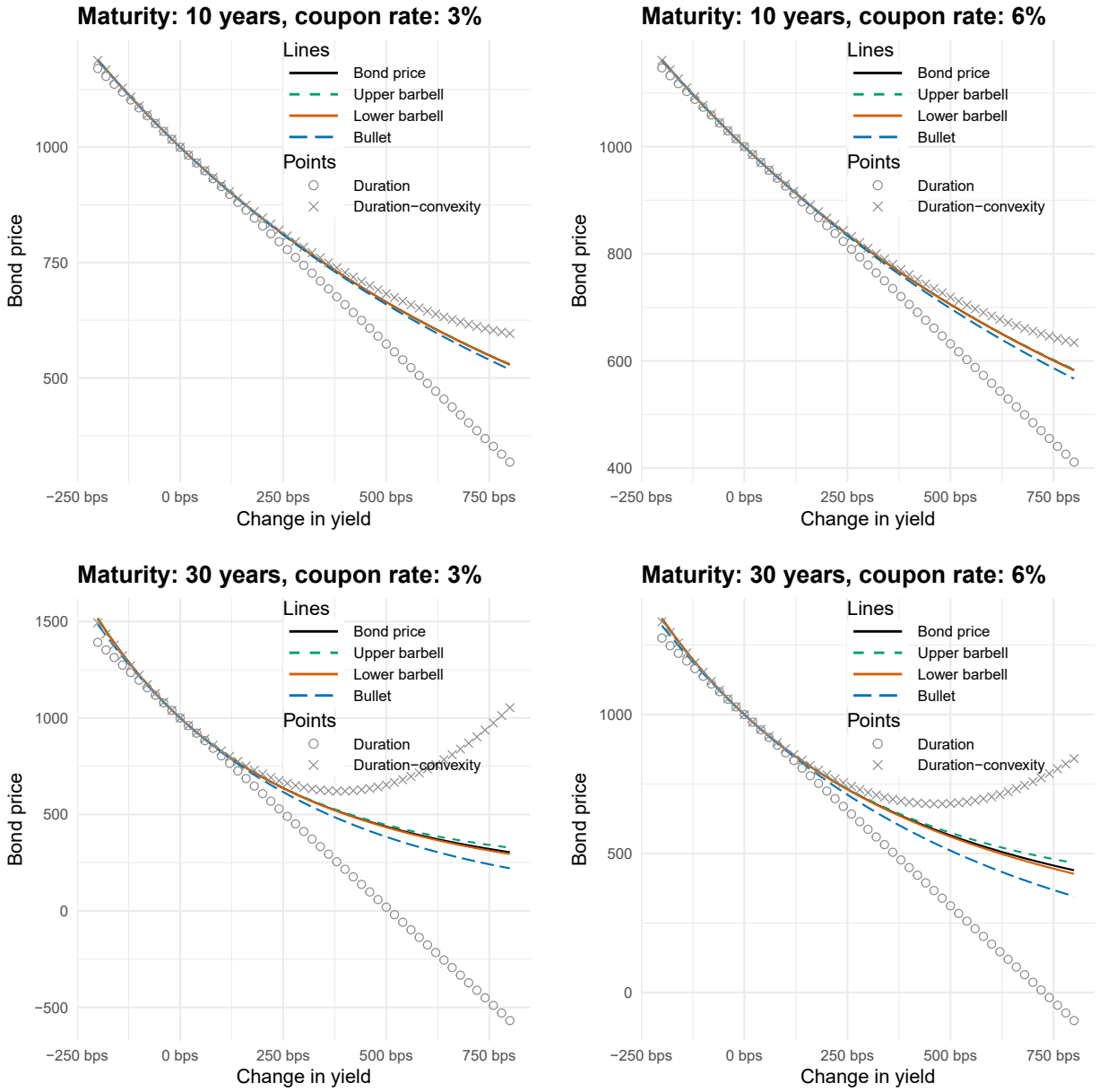
Construction of the proxy bonds can be viewed as a truncated Hausdorff moment problem. Given the first three moments of the discounted cash-flow measure on a bounded interval, (μ_0, μ_1, μ_2), we seek an atomic measure made of the fewest possible atoms that reproduces these moments. To solve this problem we need at least two atoms. Moreover, the extremal solutions—the measures generating the highest and lowest possible post-shock bond prices and the third moment—happen to be two-atom measures. The proxy bonds that come from these measures are named the upper and lower barbells.

The potential of the proxy bond approach is not limited to two moments; it can be extended to higher order moments. We discuss the possibility of such an extension in practice to the third moment μ_3 in Appendix B.1 and the general application of the proxy bond methodology to any number of moments in Appendix B.2.[21]

[20] Recall that the bullet approximation almost coincides with that of Livingston and Zhou (2005).

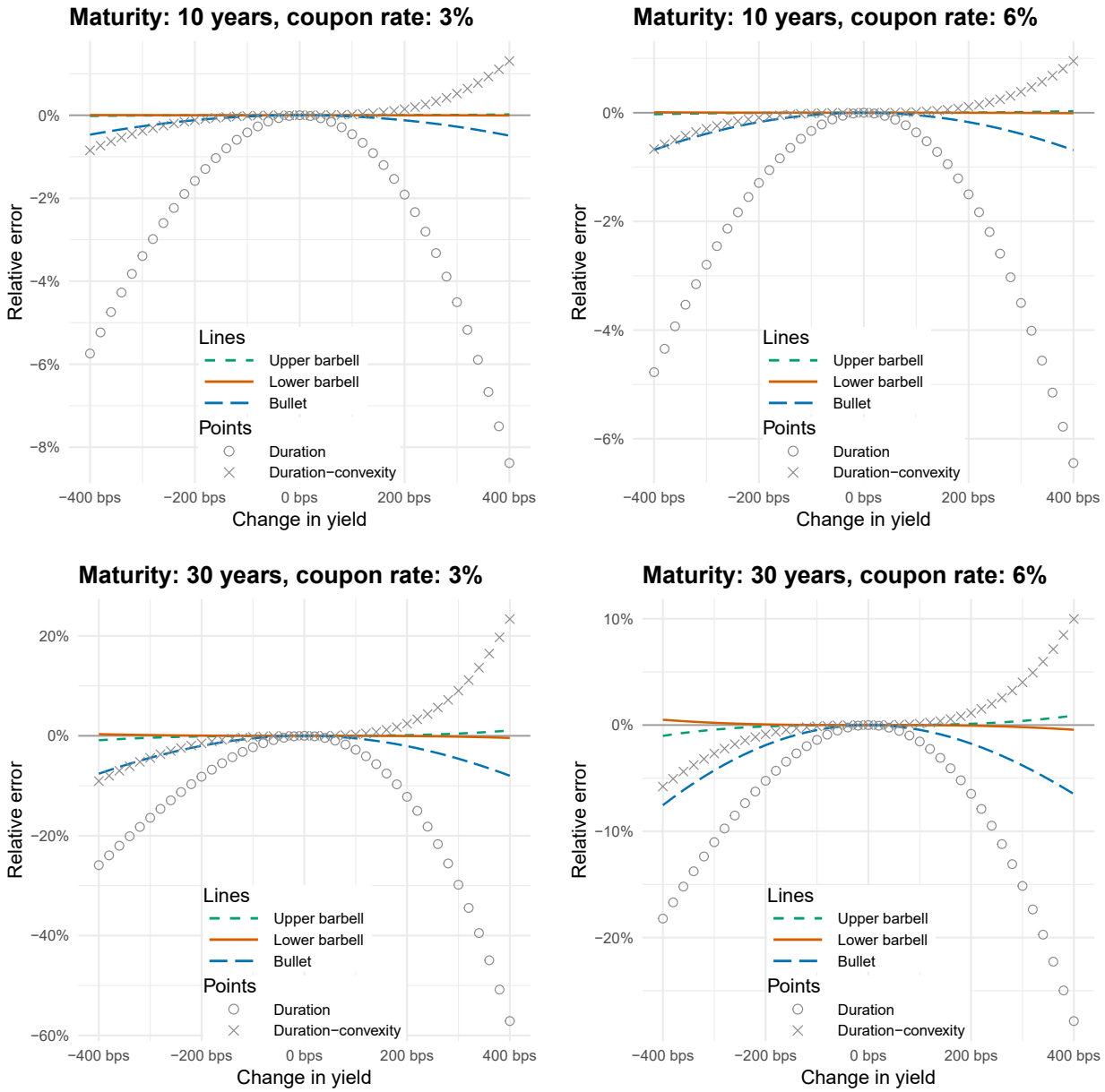
[21] For brevity, the exposition in Appendix B partially relies on proofs and results from Sections 5-7.

Figure 4: Bond prices and their approximations



Notes: The figures plot the actual price of the bond and the price approximations according to all approximation methods. The characteristics of the considered bonds are indicated on the top of each panel. The bonds pay a single annual coupon at the indicated rate and are traded at par before the YtM changes.

Figure 5: Approximation errors relative to actual bond price



Notes: The figures plot the errors of the approximation methods relative to the actual bond price. The characteristics of the considered bonds are indicated on top of each panel. The bonds pay a single annual coupon at the indicated rate and are traded at par before the YtM changes. The pictures clearly indicate that our approximation methods are significantly more precise than the traditional methods. Superiority of our method for small changes in the YtM is shown in Figure A.3.

5.1 Bullet proxy bond

We start by introducing the idea of approximating a bond with a proxy bond from a simple example of a bullet bond. We choose a fictitious bond in such a way that its price behaves similarly to that of the original bond. In this section, we consider the most parsimonious universe for the proxy bonds—zero-coupon bonds.

Suppose the proxy zero-coupon bond matures at time T_b, paying C_b. We choose T_b and C_b so as to fit the respective prices and the durations of the two bonds:

$$P(y_0) = P_{\text{bullet}}(y_0),$$

$$P'(y_0) = P'_{\text{bullet}}(y_0).$$

This can be rewritten in terms of moment-matching conditions as follows:

$$\mu_0 = C_b e^{-y_0 T_b},$$

$$\mu_1 = C_b T_b e^{-y_0 T_b}.$$

This allows us to find that

$$T_b = \mu_{1/0}$$

and

$$C_b = \mu_0 e^{y_0 T_b} = \mu_0 e^{y_0 \mu_{1/0}}.$$

Thus, the approximated after-shock bond price is as follows:

$$P_b(y) = C_b e^{-y T_b}. \tag{3}$$

Remark 5.1. It is possible to compute the bullet approximation in a simpler way—specifically, from the present value of the proxy bond cash flow, $c_b = \mu_0$, and the YtM shock, Δy:

$$\hat{P}_b(\Delta y) = c_b e^{-\Delta y T_b} = \mu_0 e^{-\mu_{1/0} \cdot \Delta y}. \tag{4}$$

Theorem 5.1. The bullet approximation is conservative—put differently, the approximation is a lower bound of the true bond price:

$$P_{\text{bullet}}(y) \leq P(y), \tag{5}$$

where the equality is attained if and only if at least one of the two following conditions are satisfied:

- The original bond is a zero-coupon bond (in this case the bullet approximation is tautological).
- $y = y_0$—that is, the YtM shock, Δy, is null.

Proof. See Appendix G, Section G.2. □

The bullet approximation is closely related to the exponential duration method introduced by Livingston and Zhou (2005). The difference between the two methods comes from the fact that the exponential duration method uses discrete YtM, whereas the bullet method employs continuous compounding. Put differently, Livingston and Zhou (2005) linearize the price–YtM relationship in $(\log P, Y)$ coordinates, whereas we linearize it in log–log coordinates—$(\log P, \log(1+Y/f))$. In Figure 1, we have shown that the Livingston and Zhou's method underperforms compared to our bullet method. The reason is that, due to its functional form, Livingston and Zhou's approximation has a slightly smaller convexity relative to that of the original bond . More details about the comparison of the bullet method to existing methods can be found in Appendix C.

5.2 Upper barbell proxy bond

In this section, we propose an augmented model in which a bond is approximated using a portfolio consisting of cash and a zero-coupon bond. We call this portfolio the "fictitious bond" (the first payment of the fictitious bond is paid at time $t = 0$). The extended approach gives us an extra degree of freedom compared to the bullet approach (Section 5.1). This extra degree of freedom allows us to tailor the parameters of the fictitious bond and match the current price of the original bond as well as its duration and convexity. Unlike the duration–convexity approach (quadratic function), the approach with a fictitious bond results in a more sensible approximating function: the price of the fictitious bond is a monotonically decreasing function of the YtM, just like the price of the original bond.

We introduce the following notation: T_{ub} is the timing of the future cash flow of the fictitious bond, C_{ub} is the size of this future cash flow, and finally, C_0 is the size of the cash flow that the fictitious bond pays at time $t = 0$. The present values of C_{ub} and C_0 discounted at the current YtM are denoted as c_{ub}, and c_0, respectively. Therefore, we have $c_0 = C_0$ and $c_{ub} = C_{ub}e^{-y_0 T_{ub}}$.

The current continuously compounded YtM for both the original bond and the fictitious bond are equal to y_0. Thus, to ensure that the price of the fictitious bond matches that of the original bond, we need the following:

$$\mu_0 \equiv P(y_0) = C_0 + C_{ub}e^{-y_0 T_{ub}}.$$

Provided that the two prices match, in order to match duration and convexity it is nec-

essary and sufficient that:[22]

$$\mu_1 \equiv P(y_0) \cdot \mu_{1/0} = C_{ub} T_{ub} e^{-y_0 T_{ub}}$$

and

$$\mu_2 = C_{ub} T_{ub}^2 e^{-y_0 T_{ub}}.$$

Overall, we have three equations with three unknowns (T_{ub}, C_{ub}, and C_0). For the sake of simplicity, we recast them in terms of T_{ub}, c_{ub}, and c_0 as follows:

$$\mu_0 = c_0 + c_{ub},$$

$$\mu_1 = c_{ub} T_{ub},$$

and

$$\mu_2 = c_{ub} T_{ub}^2.$$

Combining the last two equations, we obtain:

$$T_{ub} = \frac{\mu_2}{\mu_1},$$

The duration-matching condition—that is, the second equation—allows us to find c_{ub}, expressing it via T_{ub}:

$$c_{ub} = \frac{\mu_1}{T_{ub}} = \frac{\mu_1^2}{\mu_2}.$$

Finally, the price-matching condition allows us to find c_0 expressing it via T_{ub} and c_{ub}:[23]

$$c_0 = \mu_0 - c_{ub} = \mu_0 - \frac{\mu_1^2}{\mu_2} = \mu_0 \cdot \left(1 - \frac{\mu_{1/0}^2}{\mu_{2/0}}\right) = \mu_0 \cdot \frac{\mu_{2/0} - \mu_{1/0}^2}{\mu_{2/0}}.$$

The following formulas for C_0 and C_{ub}, and $P_{ub}(y)$ accomplish the approximation task:

$$C_0 = c_0, \tag{6}$$

$$C_{ub} = c_{ub} e^{y_0 t_{ub}}, \tag{7}$$

and

$$P_{ub}(y) = C_0 + C_{ub} e^{-y t_{ub}}. \tag{8}$$

Remark 5.2. It is possible to compute the upper barbell approximation in a simpler way—specifically, from the present values of the proxy bond cash flows, c_0 and c_{ub}, and

[22] Due to the fact that y is a continuously compounded rate, the ratio of the bond price first derivative over the bond price $\frac{\mu_1}{\mu_0}$ is the Macaulay duration.

[23] Recall that notation $\mu_{a/b}$ stands for $\frac{\mu_a}{\mu_b}$

the YtM shock, Δy:

$$\hat{P}_{ub}(\Delta y) = c_0 + c_{ub}e^{-\Delta y t_{ub}}, \tag{9}$$

which serves as an approximation of the original bond's price,

$$\hat{P}(\Delta y) := \int_t e^{-t\Delta y} d\mathrm{PV}_{\text{original bond}}. \tag{10}$$

Proposition 5.2. 1. T_{ub} cannot be smaller than the Macaulay duration of the original bond but cannot exceed its maturity:

$$\mu_{1/0} \leq T_{ub} \leq T.$$

2. Both cash flows of the proxy bond are non-negative:

$$c_0 = C_0 \geq 0,$$

$$c_{ub} \geq 0, \quad C_{ub} \geq 0,$$

3. The timing of the second payment of the proxy bond is strictly positive:

$$T_{ub} > 0.$$

Proof. See Appendix G, Section G.3. □

Lemma 5.3. The third derivative of the price of the original bond with respect to the continuously compounded yield to maturity for $y = y_0$ is negative and greater than or equal in absolute value to that of the proxy bond. The third derivative of the proxy bond is also negative. The equality is attained if and only if the original is a zero-coupon bond (or a combination of cash and a zero-coupon bond), in which case the proxy bond coincides with the original bond.

$$P'''(y_0) < 0,$$

$$P'''_{ub}(y_0) < 0,$$

$$|P'''(y_0)| \geq |P'''_{ub}(y_0)|.$$

Proof. See Appendix G, Section G.4. □

Theorem 5.4. If the cash flow stream of the original bond for $t > 0$ consists of at least two distinct payments at two distinct dates, then

- In some right neighborhood $(y_0, y_0 + \varepsilon)$, the approximation is an upper bound for the bond price:

$$P(y) < P_{ub}(y) \text{ for any } y \in (y_0, y_0 + \varepsilon).$$

- In some left neighborhood $(y_0 - \varepsilon, y_0)$, the approximation is a lower bound for the bond price:

$$P(y) > P_{ub}(y) \text{ for any } y \in (y_0 - \varepsilon, y_0).$$

Otherwise, the proxy bond coincides with the original bond, and therefore, the approximating error is null for any y.

Proof. The proof relies on Lemma 5.3. For details, see Appendix G, Section G.5. □

Thus, the approximation is anti-conservative for small positive changes of the yield to maturity and conservative for small negative changes. In the remainder of this section, we show that the approximation constitutes an upper (lower) bound for all $y > y_0$ ($y < y_0$). The key, and perhaps most elegant, step of the proof is the extension of these inequalities from local neighborhoods of y_0 to the entire real line. We show this by establishing that the approximating function $P_{ub}(\cdot)$ and the true bond price function $P(\cdot)$ intersect only once, namely at $y = y_0$. Surprisingly, this follows from a generalized version of a classical algebraic result about real roots of polynomials—namely, Descartes' rule of signs.

Lemma 5.5 (Generalized Descartes' rule of signs)**.** Let

$$f(x) = \sum_{i=1}^{N} a_i x^{\alpha_i}$$

be a polynomial with non-zero real coefficients $\{a_i\}$ and real exponents $\{\alpha_i\}$ with $\alpha_1 < ... < \alpha_N$.

Then, the number of distinct strictly positive roots (or "zeros"[24]) counting their order[25]—$Z(f)$—is less than or equal to the number of changes of signs of the polynomial coefficients—$V(f)$.[26] Moreover, the difference between the two characteristics is an even number:

$$V(f) - Z(f) \equiv 0 \,(\text{mod } 2).$$

Proof. Our proof is inspired by Wang (2004). However, it differs from Wang's approach in that we ensure its validity for the generalized form of Descartes' rule, where the exponents $\{\alpha_i\}$ are not restricted to natural numbers. See Appendix G, Section G.6. □

Theorem 5.6. If the cash flow stream of the original bond for $t > 0$ consists of at least two distinct payments at two distinct dates, then

[24] "Zero" is mathematical jargon commonly used in mathematical texts and papers. In this context, the term "zero" is used interchangeably with terms "root" and "solution."

[25] By definition, a root x_0 has the n-th order if $f(x_0) = f'(x_0) = ... = f^{(n-1)}(x_0) = 0$, and $f^{(n)}(x_0) \neq 0$.

[26] The number of changes of signs of the coefficients can be formally defined as number of cases such that $a_i a_{i+1} < 0$:

$$V(f) := \#\{i | a_i a_{i+1} < 0\}.$$

- $P(y) < P_{ub}(y)$ for any $y > y_0$,
- $P(y) > P_{ub}(y)$ for any $y < y_0$.

Otherwise, the proxy bond coincides with the original bond, and therefore, the approximating error is null for any y.

Proof. The function:

$$\mathcal{D}(y) = P_{ub}(y) - P(y)$$

can be represented as a real-exponent polynomial, $\mathcal{F}(x)$, where $x := e^{-(y-y_0)}$. The exponents in $\mathcal{F}(x)$ correspond to the dates of the expected cash flow receipts, and the coefficients of $\mathcal{F}(x)$ correspond to the amounts of these cash flow discounted at the continuously-compounded rate of y_0.

Let $\{K_i\}$ be the cash flow sequence associated with the portfolio that consists of two positions: a long position (one unit) of the proxy bond and a short position (one unit) of the original bond. Let k_i be their respective present values discounted at the pre-shock YtM: $k_i = K_i e^{-y_0 t_i}$.

Then,

$$\mathcal{D}(y) = \sum K_i e^{-y t_i} = \sum k_i \left(e^{-(y-y_0)}\right)^{t_i} =: \mathcal{F}(x).$$

By construction of the long-short portfolio, the signs of the coefficients $\{k_i\}$ of the polynomial $\mathcal{F}$ change at most three times:

1. The junior coefficient is positive because it corresponds to the first cash flow of the proxy bond. It happens at time $t = 0$.
2. Next, there may be negative coefficients corresponding to the original bond's cash flows with expected dates before the second payment of the proxy bond.
3. Next, there is a positive coefficient that corresponds to the second payment of the proxy bond.
4. Finally, there might be negative coefficients corresponding to the original bond's cash flows with expected dates after the second payment of the proxy bond.

At this step, we do not claim that all four items are present. We allow the possibility that item 2 or item 4 is missing. The only statement that we are making so far is that the signs of $\mathcal{F}$'s coefficients do not change more than thrice. Since they do not switch more than thrice, the polynomial $\mathcal{F}$ cannot have more than three positive roots, counting their order.

At the same time, we know that the maximum of three roots, counting multiplicity,

is exhausted: $x = 1$ is a third-order root.[27] Thus, $\mathcal{F}$ cannot have roots different from $x = 1$, which in turn means that the $\mathcal{D}(y)$ turns into 0 only when $y = y_0$. Hence, function $\mathcal{D}(y)$ cannot change sign in the area $y > y_0$ and in the area $y < y_0$. In conjunction with Theorem 5.4, this yields the statement of the current theorem. □

Remark 5.3. The intuition of the last proof goes along the lines of capital budgeting theory (Brealey et al., 2011), chapter 5. The long-short portfolio composed of the proxy bond and the original bond can be viewed as an investment project with alternating cash flows (inflows and outflows). Given that the cash flows of this project flip sign up to three times, the project may have up to three internal rates of return (IRRs). Unlike the capital budgeting literature, however, we allow for non-integer timing and treat IRRs more carefully, taking their order into consideration. If an NPV profile touches the horizontal axis such that $\text{NPV}' = 0$, then we count it as two IRRs. If $\text{NPV} = \text{NPV}' = \text{NPV}'' = 0$, then it is considered as three IRRs. In our case we have three IRRs, all equal to $e^{y_0} - 1$. Thus, there is no room for other IRRs, and the NPV profile of the long-short portfolio does not cross the horizontal axis for discount rates different from $e^{y_0} - 1$.

5.3 Lower barbell proxy bond

In this section, we propose a model similar to the one in Section 5.2. Specifically, we introduce another proxy bond. This bond is composed of two payments in the future. Unlike in Section 5.2, we posit that the remaining time to maturity of the proxy bond should coincide with that of the original bond. The proxy bond is composed of two cash flows: C_{lb} at time t_{lb} and C_T at time T. We choose C_{lb}, t_{lb}, and C_T to match the price, the duration, and the convexity of the original bond. We use lower-case letters to denote the present values of the proxy bond's cash flows computed using the current yield to maturity, y_0: $c_{lb} = C_{lb}e^{-y_0 t_{lb}}$ and $c_T = C_T e^{-y_0 T}$.

In terms of t_{lb}, c_{lb}, and c_T, the matching conditions for the price, duration, and convexity look as follows:

$$\mu_0 = c_{lb} + c_T, \tag{11}$$

$$\mu_1 = c_{lb}t_{lb} + c_T T, \tag{12}$$

and

$$\mu_2 = c_{lb}t_{lb}^2 + c_T T^2. \tag{13}$$

Intuitively, the approach is the same as in Section 5.2 but flipped: in Section 5.2 one of the payments of the proxy bond is *before* all the payments of the original bond (at time

[27] Recall that the discrepancy of the approximation has the order of magnitude proportional to Δy^3—or even smaller—because the approximating function matches the second derivative of the bond price. On the other hand, the order of magnitude of Δy coincides with that of $x - 1$. Therefore, $\mathcal{F}(1) = F'(1) = F''(1) = 0$, which means—by definition—that $x = 1$ is a root of at least third order.

$t = 0$). Meanwhile, in the present section one of the payments of the proxy bond is *after or at the same time as* each payment of the original bond (at time $t = T$).

In both approaches, a physical analogy can illustrate the idea. We think of bond cash flows as weight plates placed along a straight line. The plates' masses are equal to the cash flow present values (for $y = y_0$), and their positions along the straight line correspond to their timing.

Thus, choosing the cash flow amounts and their dates intuitively corresponds to choosing two weight plates—that is their masses—and placing them along the straight line so that the following characteristics match with those of the original bond:

1. total mass (price),
2. location of the center of gravity relative to the origin of the coordinate system at $t = 0$ (duration),
3. moment of inertia related to pivoting around $t = 0$ (convexity).

To exploit the symmetry between the two models (the "upper barbell" and the "lower barbell"), we recast the moment-matching conditions for moments about date $t = T$. So instead of matching conventional moments, based on powers of t, we match moments about $t = T$—that is, moments based on powers of $t - T$. Within the physical framework, this corresponds to matching the center of gravity in coordinate system with the origin at $t = T$, and matching the moment of inertia around $t = T$. This approach allows to simplify the equations by getting rid of the terms related to the cash flow occurring at time $t = T$, and arrive at a system of equations almost identical to that in Section 5.2. Intuitively, we are matching

1. total mass (price),
2. location of the center of gravity relative to the origin of the coordinate system at $t = T$ (duration minus T),
3. moment of inertia related to pivoting around $t = T$.

Matching the three moments about $t = T$ is necessary and sufficient for matching the moments about $t = 0$ because for any given bond—say, "bond X", the set of moments about $t = T$ ($\breve{\mu}_0^{\text{bond X}}, \breve{\mu}_1^{\text{bond X}}, \breve{\mu}_2^{\text{bond X}}$) can be computed from the corresponding moments about $t = 0$ ($\mu_0^{\text{bond X}}, \mu_1^{\text{bond X}}, \mu_2^{\text{bond X}}$) and vice-versa.

Indeed, we define the first moment of bond X around $t = T$ as $\breve{\mu}_1^{\text{bond X}} := \int (t - T) d\text{PV}_{\text{bond X}}$ and show that it is closely linked to the conventional first moment, $\mu_1^{\text{bond X}} \equiv \int t d\text{PV}_{\text{bond X}}$:[28]

$$\breve{\mu}_1^{\text{bond X}} = \mu_1^{\text{bond X}} - T\mu_0^{\text{bond X}} \text{ and vice versa, } \mu_1^{\text{bond X}} = \breve{\mu}_1^{\text{bond X}} + T\mu_0^{\text{bond X}}$$

[28] $\breve{\mu}_1^{\text{bond X}} = \int (t - T) d\text{PV}_{\text{bond X}} = \int t d\text{PV}_{\text{bond X}} - T \int d\text{PV}_{\text{bond X}} = \mu_1^{\text{bond X}} - T\mu_0^{\text{bond X}}$.

Similarly, we define the second moment of bond X about $t = T$ as $\breve{\mu}_2^{\text{bond X}} := \int (t - T)^2 d\text{PV}_{\text{bond X}}$ and show that it is closely related to the conventional second moment of bond X:[29]

$$\breve{\mu}_2^{\text{bond X}} = \mu_2^{\text{bond X}} + T^2 \mu_0^{\text{bond X}} - 2T\mu_1^{\text{bond X}},$$

and vice versa,

$$\mu_2^{\text{bond X}} = \breve{\mu}_2^{\text{bond X}} + T^2 \mu_0^{\text{bond X}} + 2T\breve{\mu}_1^{\text{bond X}}.$$

Consequently, to match the three conventional moments of the original bond,

$$\mu_0 = \mu_0^{lb}, \quad \mu_1 = \mu_1^{lb}, \text{ and } \mu_2 = \mu_2^{lb},$$

it is necessary and sufficient to match the three moments about $t = T$:

$$\mu_0 = \mu_0^{lb}, \quad \breve{\mu}_1 = \breve{\mu}_1^{lb}, \text{ and } \breve{\mu}_2 = \breve{\mu}_2^{lb}.$$

Finally, these moment-matching conditions (about $t = T$) yield the following equations for the parameters of the lower barbell bond:

$$\mu_0 = c_{lb} + c_T, \tag{14}$$

$$\breve{\mu}_1 = c_{lb}(t_{lb} - T), \tag{15}$$

and

$$\breve{\mu}_2 = c_{lb}(t_{lb} - T)^2. \tag{16}$$

For bonds having non-zero payments before T (that is, non-ZCB), combining equations 15 and 16, we obtain that

$$t_{lb} - T = \frac{\breve{\mu}_2}{\breve{\mu}_1}, \tag{17}$$

and thus we have that[30]

$$t_{lb} = T + \frac{\breve{\mu}_2}{\breve{\mu}_1} \leq T. \tag{18}$$

Next, we find c_{lb} from equations 15 and 17:

$$c_{lb} = \frac{\breve{\mu}_1}{t_{lb} - T} = \frac{\breve{\mu}_1^2}{\breve{\mu}_2}. \tag{19}$$

Finally, we find the present value of the second cash flow of the proxy bond, c_T, from

[29] $\breve{\mu}_2^{\text{bond X}} = \int (t - T)^2 d\text{PV}_{\text{bond X}} = \int t^2 d\text{PV}_{\text{bond X}} + T^2 \int d\text{PV}_{\text{bond X}} - 2T \int t d\text{PV}_{\text{bond X}} = \mu_2^{\text{bond X}} + T^2 \mu_0^{\text{bond X}} - 2T\mu_1^{\text{bond X}}$.

[30] Note that $\breve{\mu}_2 > 0$, and $\breve{\mu}_1 < 0$.

equations 14 and 19:

$$c_T = \mu_0 - c_{lb} = \mu_0 - \frac{\breve{\mu}_1^2}{\breve{\mu}_2} = \mu_0 \cdot \left(1 - \frac{\breve{\mu}_{1/0}^2}{\breve{\mu}_{2/0}}\right) = \mu_0 \cdot \frac{\breve{\mu}_{2/0} - \breve{\mu}_{1/0}^2}{\breve{\mu}_{2/0}}.$$

The following formulas for C_{lb} and C_T, and $P_{lb}(y)$ accomplish the solution:

$$C_{lb} = c_{lb} e^{y_0 t_{lb}}, \tag{20}$$

$$C_T = c_T e^{y_0 T}, \tag{21}$$

and

$$P_{lb}(y) = C_{lb} e^{-y t_{lb}} + C_T e^{-yT}. \tag{22}$$

Remark 5.4. It is possible to compute the lower barbell approximation in a simpler way—specifically, from the present values of the proxy bond cash flows, c_{lb} and c_T, and the YtM shock, Δy:

$$\hat{P}_{lb}(\Delta y) = c_{lb} e^{-\Delta y t_{lb}} + c_T e^{-\Delta y T}. \tag{23}$$

Proposition 5.7.

1. t_{lb} is non-negative but cannot exceed the Macaulay duration of the original bond:

$$0 \leq t_{lb} \leq \mu_{1/0}.$$

2. Both cash payments of the proxy bond are non-negative:

$$c_{lb} \geq 0, \quad C_{lb} \geq 0,$$

$$c_T \geq 0, \quad C_T \geq 0,$$

3. The timing of the first payment of the proxy bond is strictly less than the time to maturity T:

$$t_{lb} < T.$$

Proof. Analogous to the proof of Proposition 5.2. □

In the remainder of this section, we establish the bounding property of the lower barbell bond in a manner entirely analogous to that of Section 5.2. Specifically, Lemma 5.8 establishes an inequality between the third derivatives of the original bond and the lower barbell bond. Theorem 5.9 then shows that the bounding property holds locally. Finally, Theorem 5.10 extends this result to the case of large shocks Δy.

Lemma 5.8. The third derivative of the price of the original bond with respect to the continuously compounded yield to maturity for $y = y_0$ is negative and less than or equal in absolute value to that of the proxy bond. The third derivative of the proxy bond is

also negative. The equality is attained if and only if the original bond has at most one payment before T, in which case the proxy bond coincides with the original bond.

$$P'''(y_0) < 0,$$

$$P'''_{lb}(y_0) < 0,$$

$$|P'''(y_0)| \leq |P'''_{lb}(y_0)|.$$

Proof. See Appendix G, Section G.7. □

Theorem 5.9. If the cash flow stream of the original bond contains at least two distinct payments at two distinct dates, both before T, then

- In some right neighborhood $(y_0, y_0 + \varepsilon)$, the approximation is a lower bound for the bond price:
$$P(y) > P_{lb}(y) \text{ for any } y \in (y_0, y_0 + \varepsilon).$$
- In some left neighborhood $(y_0 - \varepsilon, y_0)$, the approximation is an upper bound for the bond price:
$$P(y) < P_{lb}(y) \text{ for any } y \in (y_0 - \varepsilon, y_0).$$

Otherwise, the proxy bond coincides with the original bond, and therefore, the approximating error is null for any y.

Proof. The proof is analogous to that of Theorem 5.4. The local bounds follow from the identities $P(y_0) = P_{lb}(y_0), P'(y_0) = P'_{lb}(y_0), P''(y_0) = P''_{lb}(y_0)$, which hold by construction, together with the inequality $P'''(y_0) \geq P'''_{lb}(y_0)$, established in Lemma 5.8. □

Thus, the approximation is conservative for small positive changes of the yield to maturity and anti-conservative for small negative changes. Finally, much like in Section 5.2 the generalized Descartes' rule of signs (Lemma 5.5) allows us to establish global properties of the lower barbell approximation: it is lower (upper) bound for the bond price not only in a small neighborhood, but for any rate $y \neq y_0$.

Theorem 5.10. If the cash flow stream of the original bond contains at least two distinct payments at two distinct dates, both before T, then

- $P(y) > P_{lb}(y)$ for any $y > y_0$,
- $P(y) < P_{lb}(y)$ for any $y < y_0$.

Otherwise, the proxy bond coincides with the original bond, and therefore, the approximating error is null for any y.

Proof. The proof is analogous to that of Theorem 5.6. The Descartes rule of signs (Lemma 5.5) allows to extrapolate the inequalities of Theorem 5.9 (applicable in a neighborhood of y_0) to the entire real line—that is any $\Delta y \in \mathbb{R}$. □

Thus, the properties of the lower barbell approximation are the opposite to those of the upper barbell approximation, presented in Section 5.2. The true price of the original bond is always between the two approximations: the upper barbell approximation and the lower barbell approximation.

6 Applying proxy bonds to portfolios

In this section, we apply our barbell methods to approximate changes of the value of a bond portfolio. The use cases considered in this section are particularly relevant for banking supervision and risk management, as well as for insurance companies and asset managers. In many settings, providing security-level portfolio holdings may be impractical, costly, or undesirable because the information is commercially sensitive, proprietary, or simply unavailable to external parties conducting the stress test. Our approach circumvents these limitations by requiring only aggregate portfolio characteristics, such as weighted-average duration and convexity.

We rely on a simplifying assumption that continuously compounded YtMs of all the bonds in the portfolio move in a synchronous manner—that is, the parallel shift assumption. Although in practice the changes can be asynchronous, one can interpret the projected parallel shift as a uniform cap on the discount rate change, which results in a conservative estimate for portfolio losses. For instance, if an analyst makes the assumption that none of the YtMs moves by more than 500 basis points, then a 500-bps parallel shift conservatively measures the worst-case scenario portfolio losses.

In practice, analysts rely on bond YtMs, that is, they apply the same discount rate to all cash flows of a given bond, but discount rates across the portfolio differ. In a more general case, one can even use a distinct discount rate for each cash flow stemming from a portfolio of bonds. Suppose that the portfolio generates cash flows $K_1, ..., K_N$ paid on dates $\tau_1, ..., \tau_N$ and that the corresponding discount rates are $y_0^1, ..., y_0^N$. Then, the corresponding value of the portfolio before shock is as follows:

$$\hat{P}(0) = \sum_{i=1}^{N} K_i e^{-\tau_i y_0^i},$$

while the after-shock portfolio value (under the assumption that all discount rates increase

synchronously by Δy) is as follows:

$$\hat{P}(\Delta y) = \sum_{i=1}^{N} K_i e^{-\tau_i (y_0^i + \Delta y)}.$$

Denoting the pre-shock present values of the cash flows as k_i ($k_i := K_i e^{-\tau_i y_0^i}$), we rewrite the after-shock portfolio value as follows.

$$\hat{P}(\Delta y) = \sum_{i=1}^{N} k_i e^{-\tau_i \Delta y} \equiv \int_t e^{-t\Delta y} d\mathrm{PV}_{\text{original portfolio}}. \tag{24}$$

Thus, the task of approximating the portfolio value is mathematically analogous to that of approximating the bond price, as equation 24 is analogous to equation 10. Intuitively, the approximation relies on the time distribution of present values of the future cash flows, and the fact that the initial discount rates differ across maturities/future payments does not prevent us from applying the same methodology to the portfolio of bonds as long as the shock Δy is the same for all payments. Hence, all our bound theorems remain intact for portfolio approximations as long as the rates move synchronously and the portfolio weights are non-negative:[31] Theorems 5.1, 5.6, and 5.10.

We demonstrate that our methods retain a high degree of precision even when the available information is limited to aggregate portfolio characteristics. Moreover, we consider three reporting protocols that vary in levels of information disclosure, ranging from aggregate portfolio statistics to full transparency, and examine their comparative performance in terms of stress-testing accuracy.

We introduce two agents: a *Portfolio manager* and a *Risk manager*. The *Portfolio manager* is a person or an institution making the trading decisions—and thus, having detailed information about portfolio constituents. Specifically, the *Portfolio manager* knows the following characteristics for each bond: current YtM, dirty price, coupon frequency, maturity date, duration, convexity, and the number of units currently held. Meanwhile, the *Risk manager* is a person or institution overseeing the risk of the bond portfolio held by the *Portfolio manager* and running a stress test based on limited information provided by the *Portfolio manager*.

We analyze the following three protocols of how the *Portfolio manager* and the *Risk manager* interact and apply our methods:

1. *Portfolio aggregation:* The *Portfolio manager* aggregates the following parameters across the entire portfolio: current market value, weighted average Macaulay duration, weighted average convexity (continuously compounded), and the maximum remaining time to maturity (in years) and reports the aggregate characteristics to

[31] To deal with long-short portfolios, one needs to build proxy bonds separately for long positions and for short positions.

the *Risk manager*. The *Risk manager*, in turn, applies the barbell methods to the entire bond portfolio.

2. *Stratified aggregation:* The *Portfolio manager* and the *Risk manager* agree on a set of maturity brackets (intervals): $(0, \Theta_1], (\Theta_1, \Theta_2], ..., (\Theta_{J-1}, \Theta_J]$.

   (a) The *Portfolio manager* classifies the portfolio of bonds into subportfolios based on the remaining time to maturity: the i-th bond falls into the j-th subportfolio if

   $$T_i \in (\Theta_{j-1}, \Theta_j]. \tag{25}$$

   (b) For each subportfolio, the *Portfolio manager* aggregates the following parameters:

      i. current market value,

      $$\mu_0^{\mathrm{SP}_j} = \sum_i^{T_i \in (\Theta_{j-1}, \Theta_j]} q_i \mu_0^i, \tag{26}$$

      where μ_0^i denotes the dirty price of the i-th bond, and q_i denotes the number of units of bond i in the portfolio.

      ii. weighted average Macaulay duration,

      $$\mu_{1/0}^{\mathrm{SP}_j} = \frac{\sum_i^{T_i \in (\Theta_{j-1}, \Theta_j]} q_i \mu_0^i \mu_{1/0}^i}{\sum_i^{T_i \in (\Theta_{j-1}, \Theta_j]} q_i \mu_0^i}, \tag{27}$$

      where $\mu_{1/0}^i$ denotes the Macaulay duration of the i-th bond.

      iii. weighted average convexity (continuously compounded)

      $$\mu_{2/0}^{\mathrm{SP}_j} = \frac{\sum_i^{T_i \in (\Theta_{j-1}, \Theta_j]} q_i \mu_0^i \mu_{2/0}^i}{\sum_i^{T_i \in (\Theta_{j-1}, \Theta_j]} q_i \mu_0^i}, \tag{28}$$

      where $\mu_{2/0}^i$ denotes the continuously compounded convexity of the i-th bond.

   (c) The *Portfolio manager* reports the subportfolio parameters to the *Risk manager* for all non-empty subportfolios: $\left(\mu_0^{\mathrm{SP}_j}, \mu_{1/0}^{\mathrm{SP}_j}, \mu_{2/0}^{\mathrm{SP}_j}\right)$.

   (d) The *Risk manager* applies the barbell methods to each of the subportfolios. Note that in order to apply the lower barbell method in strict sense, the *Risk manager* needs to know the "time to maturity" of each subportfolio. Given that this information is not reported by the *Portfolio manager*, the *Risk manager* uses maturity caps instead of actual maximum time to maturity. So, for j-th subportfolio, the lower barbell method is applied as if the time to maturity was Θ_j—that is, the upper limit of the corresponding maturity bracket.

i. Upper barbell approximation for each subportfolio

The future payment date	$T_{ub}^{\mathrm{SP}_j} := \mu_{2/0}^{\mathrm{SP}_j} / \mu_{1/0}^{\mathrm{SP}_j}$
The PV of the future payment	$c_{ub}^{\mathrm{SP}_j} := \mu_0^{\mathrm{SP}_j} \mu_{1/0}^{\mathrm{SP}_j} / T_{ub}^{\mathrm{SP}_j}$
The size of the payment at time $t = 0$	$c_0^{\mathrm{SP}_j} := \mu_0^{\mathrm{SP}_j} - c_{ub}^{\mathrm{SP}_j}$
The upper barbell approximation	$p_{ub}^{\mathrm{SP}_j} := c_0^{\mathrm{SP}_j} + c_{ub}^{\mathrm{SP}_j} e^{-T_{ub}^{\mathrm{SP}_j} \Delta y}$

ii. Lower barbell approximation for each subportfolio

The 1st normalized moment about Θ_j	$\breve{\mu}_{1/0}^{\mathrm{SP}_j} := \mu_{1/0}^{\mathrm{SP}_j} - \Theta_j$
The 2nd normalized moment about Θ_j	$\breve{\mu}_{2/0}^{\mathrm{SP}_j} := \mu_{2/0}^{\mathrm{SP}_j} + \Theta_j^2 - 2\Theta_j \mu_{1/0}^{\mathrm{SP}_j}$
The timing of the first payment	$t_{lb}^{\mathrm{SP}_j} := \Theta_j + \breve{\mu}_{2/0}^{\mathrm{SP}_j} / \breve{\mu}_{1/0}^{\mathrm{SP}_j}$
The PV of the payment at $t_{lb}^{\mathrm{SP}_j}$	$c_{lb}^{\mathrm{SP}_j} := \mu_0^{\mathrm{SP}_j} \left(\breve{\mu}_{1/0}^{\mathrm{SP}_j}\right)^2 \Big/ \breve{\mu}_{2/0}^{\mathrm{SP}_j}$
The PV of the payment at Θ_j	$c_{\Theta_j}^{\mathrm{SP}_j} := \mu_0^{\mathrm{SP}_j} - c_{lb}^{\mathrm{SP}_j}$
The lower barbell approximation	$p_{lb}^{\mathrm{SP}_j} := c_{lb}^{\mathrm{SP}_j} e^{-t_{lb}^{\mathrm{SP}_j} \Delta y} + c_{\Theta_j}^{\mathrm{SP}_j} e^{-\Theta_j \Delta y}$

(e) The *Risk manager* aggregates the subportfolio value approximations to obtain approximations of the value of the whole portfolio:

$$p_{ub}^{\Theta} := \sum_{j:\mu_0^{\mathrm{SP}_j} > 0} p_{ub}^{\mathrm{SP}_j} \tag{29}$$

and

$$p_{lb}^{\Theta} := \sum_{j:\mu_0^{\mathrm{SP}_j} > 0} p_{lb}^{\mathrm{SP}_j}. \tag{30}$$

The true bond price always lies within the interval between the two approximations. The finer the grid of maturity thresholds Θ_j, the tighter the interval.

3. *Full transparency:* The *Portfolio manager* discloses the bond-wise information. The *Risk manager* applies the barbell methods bond-wise and aggregates the after-shock approximations to arrive at the after-shock portfolio value.

Naturally, the more information the *Risk manager* has, the more precisely she or he can estimate potential portfolio losses. Irrespective of which of the three protocols is used, the *Risk manager* is in a position to calculate the upper and lower bounds for the bond prices. With more information, the bounds become tighter.

In order to illustrate the performance of our methods, we randomly generate a portfolio of fifty bonds with maturities ranging from zero to thirty years and the weighted average maturity of 15.6 years and the coupon rate of 7.6%.[32] We execute the three protocols: bond-wise, portfolio-wise, and stratified. The stratified protocol is executed using three maturity brackets:

[32] The code and details of the simulated portfolio can be found in Appendix I.

- (0 years, 10 years];
- (10 years, 20 years];
- (20 years, 30 years].

In Figure 6, we plot the value of the portfolio as a function of the interest rate shock, Δy, under the assumption that the YtMs of all bonds in the portfolio change by the same amount. We plot all our approximations: the upper and lower barbells and the bullet methods applied bond-wise, portfolio-wise, and in the stratified way. For the sake of comparison, we also plot the traditional approximations: linear and quadratic. In Figure 7, we plot the relative errors of all these approximation methods and show that our barbell methods significantly outperform the traditional methods even when applied to the entire portfolio.

6.1 Tightness of barbell bounds

The bounds stemming from the upper and lower barbell bonds are the best possible bounds one can obtain the information available to the *Risk manager* (price, duration, convexity, and a maturity cap). This result is somewhat tautological because both barbell approximations themselves stem from the bonds with the same price, duration, and convexity as the original bond. Moreover, both proxy bonds satisfy the maturity cap. Thus, someone who knows only the price, duration, convexity, and maturity cap cannot rule out the possibility that the true bond coincides with either of the two proxy bonds. This statement is formalized in the following proposition. The corollary that follows extends the proof to the setting in which the *Risk manager* knows the precise maturity.

Proposition 6.1 (Upper and lower barbell approximations are the best possible bounds)**.** Suppose X is a bond or bond portfolio. Suppose the *Risk manager* only knows its price (or market value) μ_0^X, duration $\mu_{1/0}^X$, and convexity $\mu_{2/0}^X$, and the fact that X does not have cash flows after $\overline{T}_X$. The risk manager is unaware of the specific payments of X and can imagine any cash flow sequence with positive present values $\{c_i\}$ and corresponding dates $\{t_i\}$ such that

$$\sum c_i = \mu_0^X, \tag{31}$$

$$\sum c_i t_i = \mu_0^X \mu_{1/0}^X, \tag{32}$$

$$\sum c_i t_i^2 = \mu_0^X \mu_{2/0}^X, \tag{33}$$

and

$$t_i \in [0, \overline{T}_X] \text{ for any } i. \tag{34}$$

Suppose $\hat{P}^X(\Delta y)$ is the true bond price (or portfolio value) after a synchronous change in discount rates by the value of Δy

Next, suppose $\hat{P}^X_{ub}(\Delta y)$ and $\hat{P}^X_{lb}(\Delta y)$ are the upper- and lower barbell approximations as defined by formulas UB4 and LB6 in the Section A.2.1.2.2 of Appendix A with the following values of input parameters:

$$\mu_0^{\text{Prt}} = \mu_0^X,$$

$$\mu_{1/0}^{\text{Prt}} = \mu_{1/0}^X,$$

$$\mu_{2/0}^{\text{Prt}} = \mu_{2/0}^X,$$

and

$$T^{\text{Prt}} = \overline{T}_X.$$

Then, from the perspective of the *Risk manager*, who considers a synchronous change in discount rates of Δy, the true price, $\hat{P}^X(\Delta y)$, can be anywhere in the interval:

$$\left[\min(\hat{P}^X_{ub}(\Delta y), \hat{P}^X_{lb}(\Delta y)), \max(\hat{P}^X_{ub}(\Delta y), \hat{P}^X_{lb}(\Delta y))\right]$$

Formally, this theorem can be restated as follows. Denote the set of all possible cash flow streams consistent with moments μ_0^X, $\mu_{1/0}^X$, and $\mu_{2/0}^X$ and the maturity cap $\overline{T}_X$ with symbol $\mathbf{X}$—that is, all possible pairs of sequences $(\{c_i\}, \{t_i\})$ that satisfy equations 31, 32, 33, and 34. Then the theorem statement can be written as the following equation:

$$\{\hat{P}^x(\Delta y)\}_{x\in\mathbf{X}} = \left[\min(\hat{P}^X_{ub}(\Delta y), \hat{P}^X_{lb}(\Delta y)), \max(\hat{P}^X_{ub}(\Delta y), \hat{P}^X_{lb}(\Delta y))\right],$$

that is, the true prices of the bonds in class $\mathbf{X}$ cover the entire interval between the two barbell approximations, and thus, the approximations are the best available bonds given the information set $\{\mu_0^X, \mu_{1/0}^X, \mu_{2/0}^X, \overline{T}_X\}$

Proof. See Appendix G, Section G.8. □

Attaining these bounds in practice relies on the assumption of an idealized complete market for zero-coupon bonds. Another interpretation of Proposition 6.1 is that the *Risk manager* abstracts from the information about the zero coupon bonds available for trading an conservatively allows for any maturities between 0 and $\overline{T}_X$.

Corollary 6.1.1. If the *Risk manager* in Proposition 6.1 knows the exact maturity of a bond/bond portfolio—instead of a maturity cap $\overline{T}_X$, the upper and lower (based on the known maturity) barbell approximations are the best bounds.

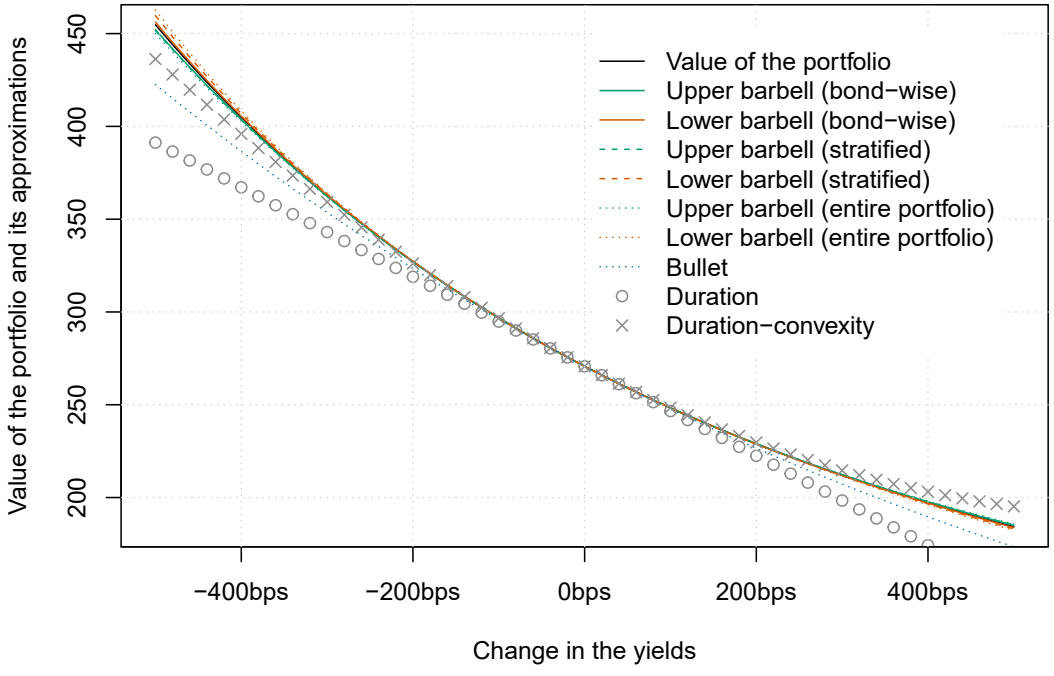
Proof. Even though the *Risk manager*, who knows the exact maturity of a bond (or a portfolio) can rule out the possibility that the portfolio exactly coincides with the upper barbell proxy bond, we show that one can slightly modify the upper barbell bond, preserving its duration and convexity and artificially increasing its maturity. Thus, the

Risk manager cannot rule out the possibility of dealing with a bond arbitrarily similar to the upper barbell. For details, see Appendix G, Section G.1. □

6.2 Average barbell

In addition to the upper and lower barbells, we introduce the *average barbell*, which is a 50-50 combination of the two barbells (thus, a proxy bond composed of four cash flows). The average barbell approximation is the arithmetic mean between the upper and lower barbell approximations; it is the midpoint of the admissible interval in which the original bond price is confined.[33] In Figure 8, we show that the relative error of the average barbell approximation does not exceed 0.38% for interest rate shocks not exceeding 500 basis points even when we apply the proxy technique to the entire portfolio (4 cash flows instead of 4261). In the region of positive shocks to the yields ($\Delta y \in [0, 5\%]$), where the portfolio value falls compared to the pre-shock state, the errors do not exceed 0.12%. It is remarkable that the bond-wise approximation does not produce the best result. For individual bonds, the approximation errors of the average barbell method are typically positive for $\Delta y > 0$ (a slight overpricing) due to the fact that the lower barbell usually matches the true bond price better than the upper barbell.

Figure 6: Portfolio value and its approximations



[33] Thus, per se the average barbell approximation does not have bounding properties.

Figure 7: Approximation errors relative to actual portfolio value

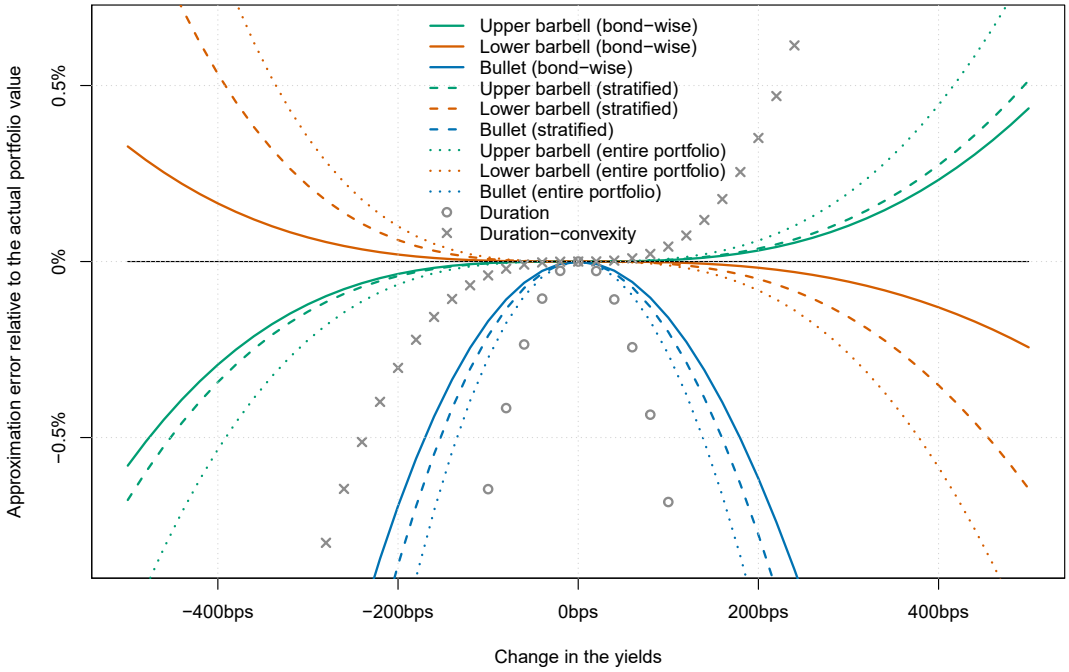
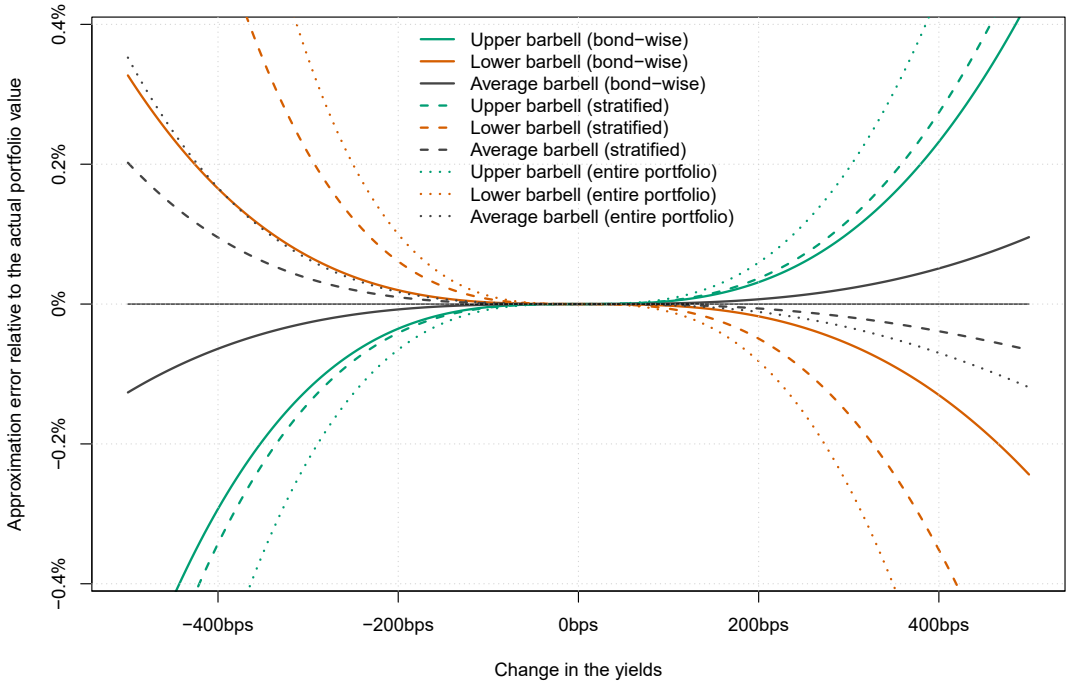


Figure 8: Approximation errors relative to actual portfolio value: focus on the average barbell approximation



7 Proxy bonds and scenarios of non-parallel shocks to yield curve

In this section, we evaluate the performance of the bullet and barbell methods in approximating bond—and bond portfolio—value changes based on the historical dynamics of the U.S. Treasury zero-coupon yield curve. The challenge arises from the complex dynamics of the yield curve, which includes non-parallel shifts of various shapes. Put differently, we test whether the bullet and barbell proxy bonds can work as a proxy of true bond prices when we apply realistic stress-test scenarios and discount bond cash-flows using historical shapes of the zero-coupon yield curve. Given that the cash-flows of the true and the proxy bonds differ, their prices do not respond identically to non-parallel shifts of the curve; however, we find that i) the approximation methods perform well on an aggregate portfolio level, ii) average barbell, the 50-50 combination of the barbell approximations, substantially outperforms its alternatives.

The practical application of this approach is as follows. A risk manager can simulate realistic non-parallel yield curve shock scenarios over a given horizon (e.g., three months) and apply them to a portfolio consisting of average barbell bonds, each representing a proxy bond for an original bond or bond (sub)portfolio. The advantage of this approach is that the risk manager does not need to know the coupon schedule of the bonds in the original portfolio. It is enough to know their main characteristics: the YtM, the Macaulay duration, the convexity, the coupon frequency, and the remaining time to maturity—or even coarser information, i.e., aggregate (sub)portfolio parameters.

For our analysis, we rely on the data reported by the Federal Reserve for the period 1985-2025.[34] The data contain the parameters for the Nelson-Siegel-Svensson (NSS) model which allows reconstructing the historical evolution of the US Treasury yield (Nelson and Siegel (1987) and Svensson (1994)). We perform the analysis on the same simulated portfolio as in Section 6 at the quarterly frequency (90 days) applying half-year (180 days) rolling-window yield curve changes to the bond portfolio. Note that for the purpose of this stress-testing exercise we assume that such changes in the curve can potentially occur over a short period and do not adjust the bond value and duration for the passage of time. The applied NSS model is as follows:

$$\mathrm{yc}_{\mathrm{start},i}(t) = \beta_{0,\mathrm{date}_0+90i} + \beta_{1,\mathrm{date}_0+90i} \cdot \mathrm{NS}_{\mathrm{slope}}\left(\frac{t}{\tau_{1,\mathrm{date}_0+90i}}\right) + \tag{35}$$

$$+\beta_{2,\mathrm{date}_0+90i} \cdot \mathrm{NS}_{\mathrm{hump}}\left(\frac{t}{\tau_{1,\mathrm{date}_0+90i}}\right) + \beta_{3,\mathrm{date}_0+90i} \cdot \mathrm{NS}_{\mathrm{hump}}\left(\frac{t}{\tau_{2,\mathrm{date}_0+90i}}\right)$$

and

$$\mathrm{yc}_{\mathrm{end},i}(t) = \beta_{0,\mathrm{date}_0+90i+180} + \beta_{1,\mathrm{date}_0+90i+180} \cdot \mathrm{NS}_{\mathrm{slope}}\left(\frac{t}{\tau_{1,\mathrm{date}_0+90i+180}}\right) + \tag{36}$$

[34] https://www.federalreserve.gov/data/nominal-yield-curve.htm

$$+\beta_{2,\text{date}_0+90i+180}\text{NS}_{\text{hump}}\left(\frac{t}{\tau_{1,\text{date}_0+90i+180}}\right)+\beta_{3,\text{date}_0+90i+180}\text{NS}_{\text{hump}}\left(\frac{t}{\tau_{2,\text{date}_0+90i+180}}\right),$$

where

- $\beta_{0,\ldots}$, $\beta_{1,\ldots}$, $\beta_{2,\ldots}$, $\beta_{3,\ldots}$, $\tau_{1,\ldots}$, and $\tau_{2,\ldots}$ are the fitted historical values of the Nelson-Siegel-Svensson parameters;
- The subscripts date_0+90i and $\text{date}_0+90i+180$ indicate the dates at which we take the parameters—the first date, date_0, is November 25, 1985, and the subsequent dates are calculated by adding a number of days to this starting point ("$+n$" means "n day later").[35]
- i is a non-negative integer, indexing all our observations;
- $\text{NS}_{\text{slope}}(\cdot)$ is a monotonically decreasing function, introduced by Nelson and Siegel (1987) that resembles the empirically observed combination of factors level-minus-slope as in Litterman and Scheinkman (1991):

$$\text{NS}_{\text{slope}}(\xi)=\frac{1-e^{-\xi}}{\xi}; \tag{37}$$

- $\text{NS}_{\text{hump}}(\cdot)$ is a hump-shaped function, introduced by Nelson and Siegel (1987) that resembles the empirically observed curvature factor in Litterman and Scheinkman (1991):

$$\text{NS}_{\text{hump}}(\xi)=\frac{1-e^{-\xi}}{\xi}-e^{-\xi}. \tag{38}$$

For each index $i\geq 0$—and the corresponding pre-shock yield curve, taken from date date_0+90i, and the after-shock yield curve, taken from date $\text{date}_0+90i+180$—we compute the true initial price of each one of the K bonds[36] in our simulated portfolio using the initial yield curve.

$$p_{i,\text{start}}^{\text{Bond}_k}=\sum_j \text{CF}_j^{\text{Bond}_k}\exp\left(-t_j^{\text{Bond}_k}\cdot \text{yc}_{\text{start},i}\left(t_j^{\text{Bond}_k}\right)\right). \tag{39}$$

Next, we compute the implied YtM, $y_{i,\text{start}}^{\text{Bond}_k}$—that is, the solution of the following

[35] If a date corresponds to a non-trading day, we take the first available parameter vector afterwards.

[36] In this section, we use the term "bond" generally. If we literally apply our methodology bond-wise, then we have fifty bonds ($K=50$). For the stratified version with three subportfolios, we say that we have three bonds, and $K=3$. Finally, if we apply our methodology using aggregated date for the entire portfolio, we say that we have one bond, and $K=1$. In this section, the formulas do not include bond quantities (q) because the p's already denote "assets under management" rather than the price per unit. The formulas are unchanged irrespective of the aggregation level.

equation:[37]

$$p_{i,\text{start}}^{\text{Bond}_k} = \sum_j \text{CF}_j^{\text{Bond}_k} \exp\left(-t_j^{\text{Bond}_k} \cdot y_{i,\text{start}}^{\text{Bond}_k}\right). \tag{40}$$

Next, we introduce the present value of each cash flow (at a constant discount rate $y_{i,\text{start}}^{\text{Bond}_k}$):

$$\text{PV}_{i,j,\text{start}}^{\text{Bond}_k} := \text{CF}_j^{\text{Bond}_k} \exp\left(-t_j^{\text{Bond}_k} \cdot y_{i,\text{start}}^{\text{Bond}_k}\right), \tag{41}$$

which allows us to compute the moments of the bonds: Macaulay duration and continuously discounted convexity.[38]

Knowing the moments, we build proxy bonds for each bond in our portfolio:

- $\text{Bullet}_{k,i}$,
- $\text{Upper Barbell}_{k,i}$,
- $\text{Lower Barbell}_{k,i}$,
- $\text{Average Barbell}_{k,i}$.

We choose the timings and the sizes of the proxy bonds' cash flows to match the moments (price, duration, and convexity). We denote the present values and the dates of future cash flows as follows:

- $\text{PV}^{\text{Bullet}_{k,i}}$ and $t^{\text{Bullet}_{k,i}}$ for the Bullet proxy bond,
- $\text{PV}_j^{\text{Upper Barbell}_{k,i}}$ and $t_j^{\text{Upper Barbell}_{k,i}}$ for the Upper Barbell proxy bond ($j = 1, 2$),
- $\text{PV}_j^{\text{Lower Barbell}_{k,i}}$ and $t_j^{\text{Lower Barbell}_{k,i}}$ for the Lower Barbell proxy bond ($j = 1, 2$),
- $\text{PV}_j^{\text{Av.Barbell}_{k,i}}$ and $t_j^{\text{Av.Barbell}_{k,i}}$ for the Average Barbell proxy bond ($j = 1, 2, 3, 4$),[39]

The true return on each bond is computed as follows:

$$r_{i,\text{true}}^{\text{Bond}_k} = \frac{p_{i,\text{end, true}}^{\text{Bond}_k}}{p_{i,\text{start}}^{\text{Bond}_k}} - 1, \tag{42}$$

where

$$p_{i,\text{end true}}^{\text{Bond}_k} = \sum_j \text{CF}_j^{\text{Bond}_k} \exp\left(-t_j^{\text{Bond}_k} \cdot \text{yc}_{\text{end},i}\left(t_j^{\text{Bond}_k}\right)\right). \tag{43}$$

[37] We aim to reconstruct the computations of a portfolio manager and a risk manager would have made, had they been limited to bond data and our approximation methodology. The implied YtM we compute is the YtM a portfolio manager would have observed in Bloomberg on date $\text{date}_0 + 90i$.

[38] Note that duration and convexity depend on the choice of the initial yield curve; we therefore recompute the moments in each simulated scenario.

[39] Note that the average barbell bond consists of four cash flows. The cash flows indexed as $j = 1, 2$ come from the upper barbell proxy bond, the other two cash flows—$j = 3, 4$—come from the lower barbell proxy bond.

Next, the true portfolio return is computed as the weighed average of the true bond returns:

$$r_{i,\text{true}}^{\text{Portfolio}} = \frac{\sum_k p_{i,\text{start}}^{\text{Bond}_k} r_{i,\text{true}}^{\text{Bond}_k}}{\sum_k p_{i,\text{start}}^{\text{Bond}_k}} = \frac{\sum_k p_{i,\text{end true}}^{\text{Bond}_k}}{\sum_k p_{i,\text{start}}^{\text{Bond}_k}} - 1. \tag{44}$$

In order to compute the returns on proxy bonds, we distort the present value of their future payments as follows:[40]

$$p_{i,\text{end}}^{\text{Av.Barbell}_{k,i}} = \sum_{j=1}^{4} \text{PV}_j^{\text{Av.Barbell}_{k,i}} \exp\left(-t_j^{\text{Av.Barbell}_{k,i}} \cdot \Delta\text{yc}_i\left(t_j^{\text{Av.Barbell}_{k,i}}\right)\right), \tag{45}$$

where

$$\Delta\text{yc}_i(m) = \text{yc}_{\text{end},i}(m) - \text{yc}_{\text{start},i}(m). \tag{46}$$

Finally, the average barbell approximation of the k-th bond return is computed as follows:

$$r_i^{\text{Av.Barbell}_{k,i}} = \frac{p_{i,\text{end}}^{\text{Av.Barbell}_{k,i}}}{p_{i,\text{start}}^{\text{Bond}_k}} - 1. \tag{47}$$

Next, the approximation of portfolio return is computed as the weighed average of the approximated bond returns:

$$r_i^{\text{Portfolio of } K \text{ Av.Barbells}} = \frac{\sum_k p_{i,\text{start}}^{\text{Bond}_k} r_i^{\text{Av.Barbell}_{k,i}}}{\sum_k p_{i,\text{start}}^{\text{Bond}_k}} = \frac{\sum_k p_{i,\text{end}}^{\text{Av.Barbell}_{k,i}}}{\sum_k p_{i,\text{start}}^{\text{Bond}_k}} - 1. \tag{48}$$

Finally, we compare the average barbell approximation with the bullet approximation of the portfolio returns, which is computed as follows:

$$r_i^{\text{Bullet}_{k,i}} = \frac{p_{i,\text{end}}^{\text{Bullet}_{k,i}}}{p_{i,\text{start}}^{\text{Bond}_k}} - 1, \tag{49}$$

where

$$p_{i,\text{end}}^{\text{Bullet}_{k,i}} = \text{PV}^{\text{Bullet}_{k,i}} \exp\left(-t^{\text{Bullet}_{k,i}} \cdot \Delta\text{yc}_i\left(t^{\text{Bullet}_{k,i}}\right)\right). \tag{50}$$

Next, the approximation of portfolio return is computed as the weighed average of the approximated bond returns:

$$r_i^{\text{Portfolio of } K \text{ Bullets}} = \frac{\sum_k p_{i,\text{start}}^{\text{Bond}_k} r_i^{\text{Bullet}_k}}{\sum_k p_{i,\text{start}}^{\text{Bond}_k}} = \frac{\sum_k p_{i,\text{end}}^{\text{Bullet}_{k,i}}}{\sum_k p_{i,\text{start}}^{\text{Bond}_k}} - 1. \tag{51}$$

[40] We rely on the present values of the proxy bonds' cash flows rather then cash flows themselves. This approach is an extension of the approach we use in Sections 5.1, 5.2, and 5.3, see Remarks 5.1, 5.2, and 5.4. Moreover, this formula is also applicable when the YtM of the original bond is unknown. Likewise, this approach will be applied to (sub)portfolios of bonds that have different YtMs, and the risk manager only knows the average (sub)portfolio duration and convexity. In this framework, it is impossible to compute the cash flows of the proxy bonds; so using the present value is our only option.

Finally, we also plot the portfolio return approximations based on the duration method, that is, the linear approximation of the bond prices. For simplicity, we linearize in y as opposed to Y. This approach yields results similar to the bullet method, as changes in the yield curve over a six-month horizon are typically modest. We take the yield changes that correspond to the duration of each bond, and compute the approximate bond return as the negative of the yield shock, $\Delta\text{yc}_i\left(\mu_{1/0}^{\text{Bond}_k}\left(y_{i,\text{start}}^{\text{Bond}_k}\right)\right)$, multiplied by the Macaulay duration of the corresponding bond given its YtM before shock, $\mu_{1/0}^{\text{Bond}_k}\left(y_{i,\text{start}}^{\text{Bond}_k}\right)$:

$$r_i^{\text{Duration}_{k,i}} = -\Delta\text{yc}_i\left(\mu_{1/0}^{\text{Bond}_k}\left(y_{i,\text{start}}^{\text{Bond}_k}\right)\right) \times \mu_{1/0}^{\text{Bond}_k}\left(y_{i,\text{start}}^{\text{Bond}_k}\right). \tag{52}$$

We compute the duration–convexity approximation similarly:

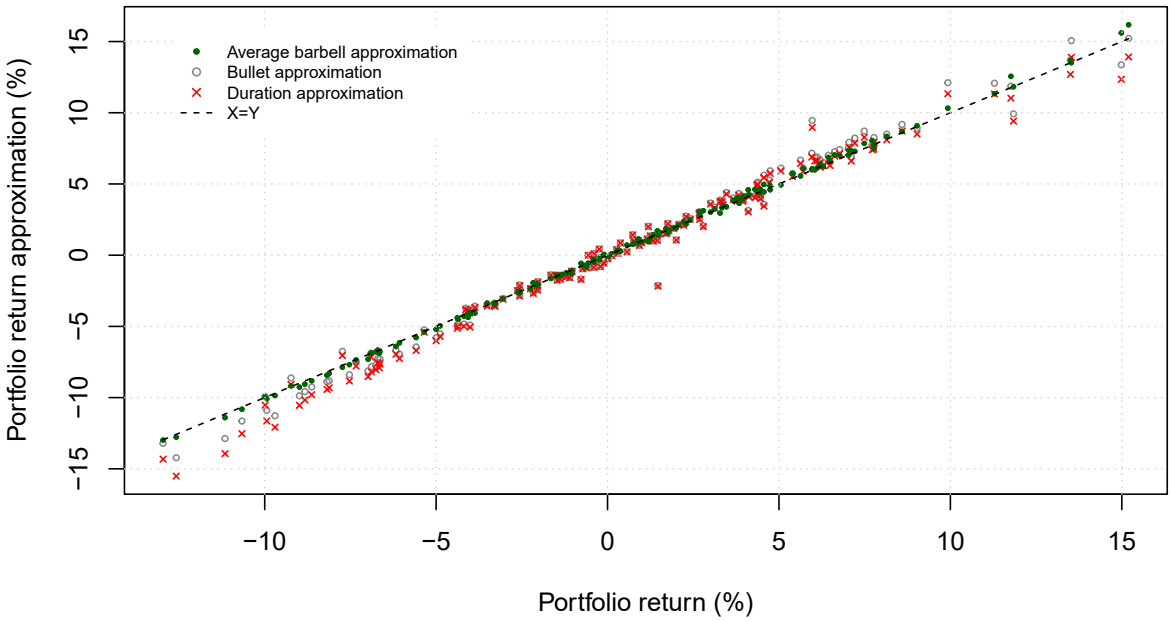
$$\begin{aligned} r_i^{\text{Duration–Convexity}_{k,i}} = &- \Delta\text{yc}_i\left(\mu_{1/0}^{\text{Bond}_k}\left(y_{i,\text{start}}^{\text{Bond}_k}\right)\right) \times \mu_{1/0}^{\text{Bond}_k}\left(y_{i,\text{start}}^{\text{Bond}_k}\right) \\ &+ \frac{1}{2}\Delta\text{yc}_i\left(\mu_{1/0}^{\text{Bond}_k}\left(y_{i,\text{start}}^{\text{Bond}_k}\right)\right)^2 \times \mu_{2/0}^{\text{Bond}_k}\left(y_{i,\text{start}}^{\text{Bond}_k}\right). \end{aligned} \tag{53}$$

The aggregation of the returns to the portfolio level is done in the same way as for the bullet and average barbell approximation, that is, as weighted average.

Figure 9 depicts the performance of the bond-wise portfolio return approximations.[41] The markers plot the return on the original portfolio (X-axis) against the return on the proxy portfolio (Y-axis). The figure demonstrates that the return on the portfolio of average barbell bonds very closely approximates the return on the original portfolio, as the vast majority of green circles lie on the 45-degree line $X = Y$, while the traditional duration approximation frequently deviates from the correct portfolio returns.

[41] Comparison of our methods with the duration–convexity approximation applied bond-wise is provided in Figure A.4.

Figure 9: Approximation of portfolio returns using proxy bonds for each bond.



Further, we repeat the same procedure, except that the bullet, barbell, and duration-based approaches are applied to subportfolios of bonds grouped by maturity brackets, as described in Section 6. We classify the bonds into three subportfolios according to the following maturity brackets:

- (0 years, 10 years];
- (10 years, 20 years];
- (20 years, 30 years].

Figure 10 summarizes the three types of portfolio return approximation: stratified average barbell, stratified bullet, and stratified duration approaches.[42] We observe that the average barbell approximation performs well, generally producing more accurate prediction than the duration method.

Finally, Figure 11 summarizes the three types of portfolio return approximation: average barbell for the entire portfolio, bullet for the entire portfolio, and duration approach for the entire portfolio.[43] We observe that the average barbell approximation performs well, generally yielding more accurate prediction than the duration method.

We complement the graphical evidence of the methods' performance with a quantitative summary of the average error size of each method and portfolio aggregation level. Figure 12 reports these error estimates. We use two measures of average error size: the root of the mean squared error and the mean absolute value of the error. Irrespective of

[42] Comparison of our methods with the duration–convexity approximation applied to subportfolios is provided in Figure A.5.

[43] Comparison of our methods with the duration–convexity approximation applied to the entire portfolio is provided in Figure A.6.

Figure 10: Approximation of portfolio returns using proxy bonds for subportfolios.

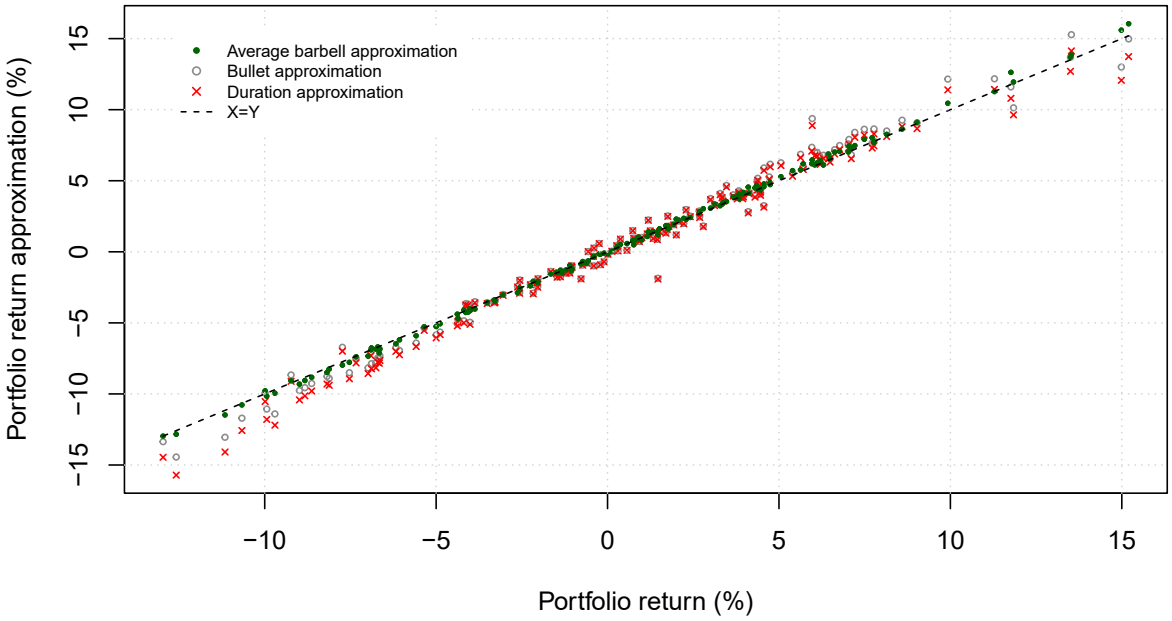
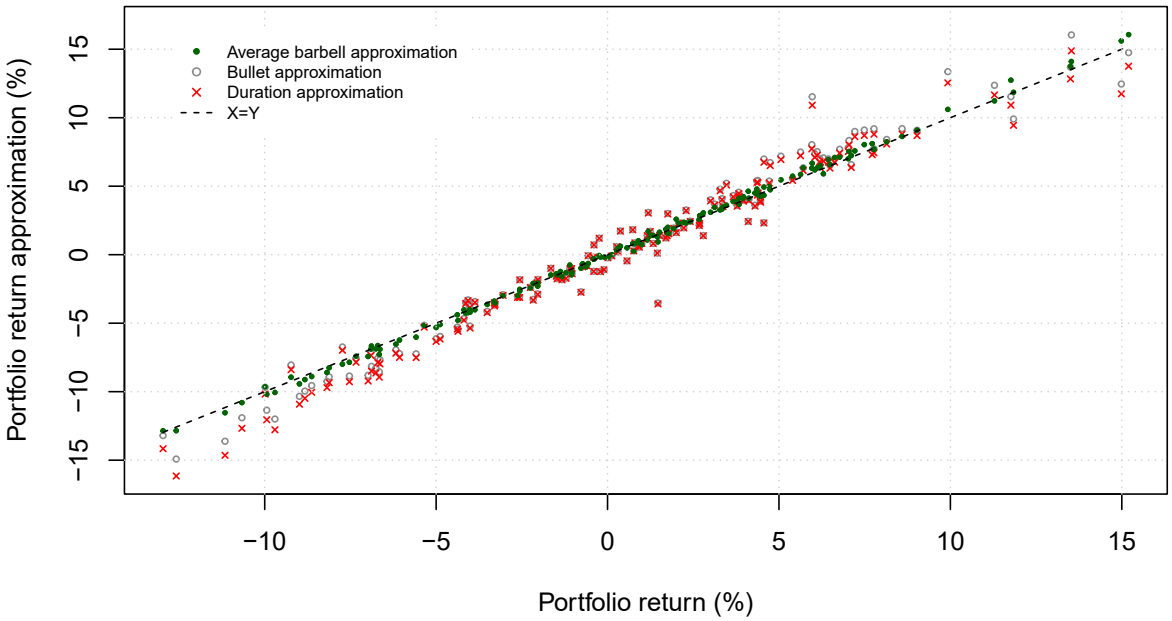
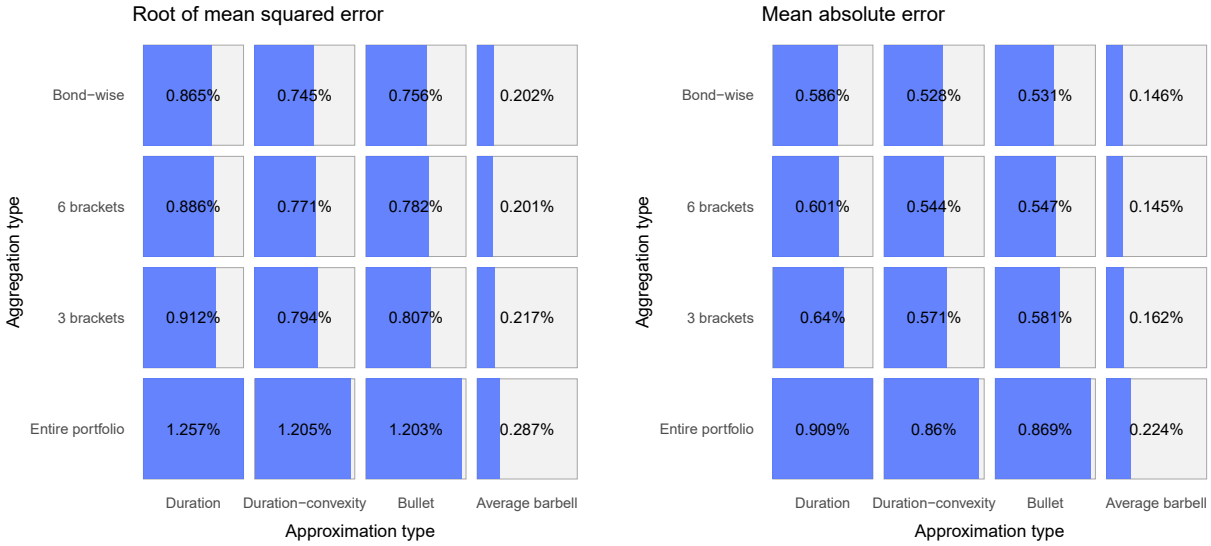


Figure 11: Approximation of portfolio returns using proxy bonds for the entire portfolio.



the error measure, we document that the average barbell method significantly outperforms the traditional methods and the bullet method. This result is robust with respect to the level of information transparency: irrespective of whether the *Portfolio manager* reports to the *Risk manager* information on individual bonds or at some level of aggregation, the average barbell method is superior to the other methods. Additionally, Figure 12 suggests that information transparency matters for the precision of the calculations: if the *Portfolio manager* reports the aggregate data on at least three subportfolios, this yields a noticeable improvement compared to the protocol in which the *Portfolio manager* reports only aggregate portfolio data. Thus, the main practical implication of this quantitative analysis is that the oversight system should use the average barbell method and rely on stratified aggregations with at least three subportfolios. The additional benefits of splitting the portfolio into more granular subportfolios are small to such an extent that more detailed reporting seems unnecessary.[44,45]

Figure 12: Measures of error size by method and portfolio aggregation level.



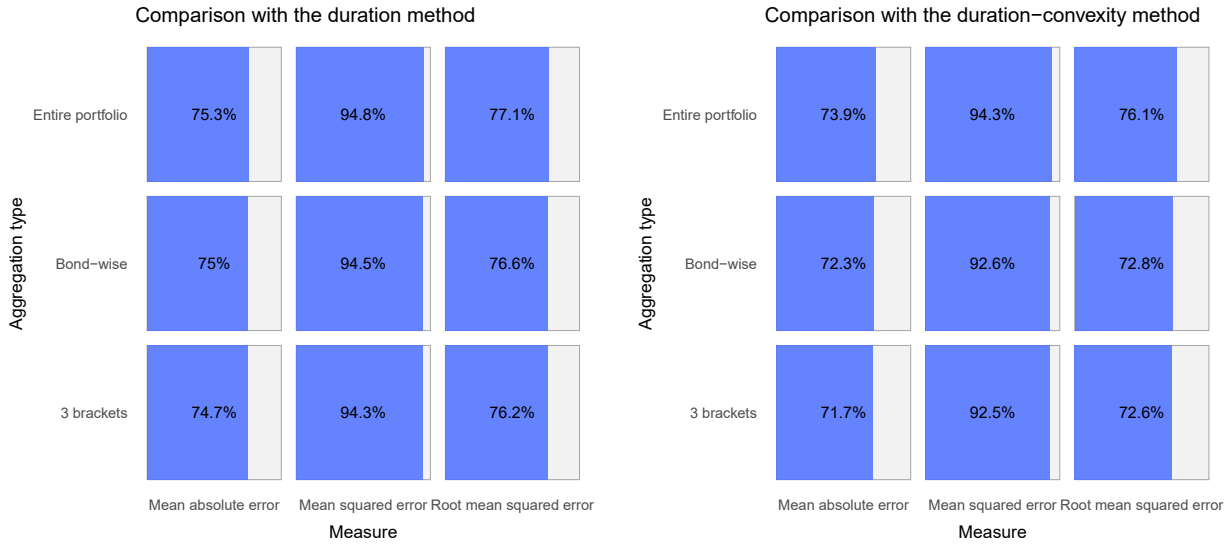
Note: The figure reports the approximation methods' errors, i.e., the deviation of the approximated portfolio value in a stress scenario from the actual portfolio value. Two measures are reported. For the left panel, the errors of a method in each scenario are squared, then the average of the squared errors is calculated; finally, a square root of the average value is calculated and reported in the respective cell. The total area of a cell corresponds to the largest error, i.e., that of the duration method applied to the entire bond portfolio. Hence, the colored area of other cells represent the relative size of the errors of the respective method compared to the worst method. The right panel analogously reports the average of the absolute values of the errors across scenarios for each method and levels of portfolio aggregations.

Figure 13 explicitly reports the percentages of reduction in average approximation errors. Specifically, we compare the performance of the average barbell method to the

[44] In Figures A.7 and A.8, we additionally analyze the most problematic stress-test scenarios in our sample. In Figure A.7, we find the cases in which each method generates the biggest possible errors (separately, positive and negative) and report the fifth and ninety-fifth percentiles in the sample distributions of the approximation errors. This additional analysis corroborates our main practical implications.

[45] Figures A.9, A.10, A.11 provide additional graphical comparison of the performance of the average barbell method applied to various levels of bond portfolio aggregation.

Figure 13: The accuracy improvement of the average barbell method compared to traditional methods



Note: The figure reports the accuracy improvement that the average barbel method achieves in our empirical exercise compared to traditional methods. The measure of improvement is the error reduction percentage calculated according to equation 54. The reported percentage value in each cell constitutes the reduction in the average error size resulting from the application of the average barbell method compared to the duration method (left panel) or the duration–convexity method (right panel).

duration method and the duration–convexity method by calculating the error reduction according to the following equation:

$$\text{Average Barbell Error} = \text{Old Method Error} \times (1 - \text{Error Reduction Percentage}) \quad (54)$$

The percentage of error reduction of the average barbell method compared to the traditional methods in our empirical exercise varies between 72% and 95% depending on the level of portfolio aggregation and the measure of average accuracy.

In Appendix H, we explain theoretically why our barbell method results in a good approximation for scenarios of moderate affine shocks to the yield curve (adding a small spread and a linear function of maturity). These two movements constitute the majority of empirically observable yield changes because they approximately correspond the first two principal components—that is, the level and the slope—documented by Litterman and Scheinkman (1991) and Fabozzi (2021), chapter 33. Our barbell methods can produce larger approximation errors when the yield curve changes are

- highly non-affine: $\Delta y(t)$ cannot be well approximated by a function $a(t) = a_0 + a_1 t$,
- affine $\Delta y(t) = a_0 + a_1 t$, when $|a_1|$ is relatively large.

Also, our methods can potentially produce larger approximation errors when the initial term structure, $y_0(t)$, is extremely steep.

8 Conclusion

We propose three methods—the bullet, upper barbell, and lower barbell approaches—that substantially enhance the precision of bond valuation for the purpose of stress testing. The underlying idea behind the methods is that the price of a standard bond—and hence its sensitivity to interest-rate movements—can be closely replicated by that of a fictitious or "proxy" bond with a simplified coupon structure calibrated to match the original bond's market price and key characteristics. These methods require input information similar to that of the traditional approximation methods that are based on the Taylor expansion of the bond price function: duration, convexity, and maturity. Similarly, the implementation of the new methods involves simple arithmetic operations, and we provide a step-by-step instruction on their application in Appendix A. We demonstrate that our methods deliver significantly higher accuracy than the traditional approaches, particularly for long-term bonds and extreme interest rate swing scenarios.

We document the properties of the new methods, including the fact that, under the scenario of the parallel shift of the yield curve, the true bond price always lies between the upper and lower barbell approximations. This effectively confines the post-shock bond price within a narrow interval without requiring knowledge of its coupon schedule. We demonstrate how the methods can be efficiently applied to entire bond portfolios using aggregated characteristics—namely, weighted average duration and weighted average convexity. This feature enables, for instance, regulators to estimate fixed-income portfolio risk with high precision without the information on individual securities, as long as they receive reports with aggregate portfolio characteristics. Moreover, we propose and test the methods' performance in application to bond portfolios stratified into subportfolios by the securities' maturity brackets. This approach to portfolio aggregation further improves the precision of portfolio valuation under stress scenarios.

We conduct an empirical test of the developed methods and examine their performance in predicting bond portfolio valuation changes in response to historical changes in the yield curve of the U.S. Treasury market. The purpose of this exercise is to test the methods' performance in application to scenarios of the actual complex shifts in the yield curve. For a simulated portfolio of bonds with an average duration of nine years, we find that, compared to the traditional approximation methods, the average barbell method——a combination of the upper and lower barbell——achieves substantial error reduction. Specifically, the reduction ranges between 72% and 95%, depending on the level of portfolio aggregation and the chosen error measure.

The reduction in dimensionality that is at the core of the new methods can be further employed in order to streamline the computation of Value-at-Risk or expected shortfall for fixed-income portfolios. In summary, the methods proposed in this paper offer a cost-effective solution to improve industry and regulatory practices in stress testing of bonds and bond portfolios.

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A Practical Manual

A.1 Applying to individual bonds

A.1.1 Input data

Input name	Notation	Bloomberg code*
Current dirty price	μ_0	PX_DIRTY_MID
Current Macaulay duration	$\mu_{1/0}$	DUR_MID
Current convexity	C	CNVX_MID
Coupon frequency	f	CPN_FREQ
Current YtM	Y_0	YLD_YTM_MID
Maturity date	Date_T	MATURITY
Today date	Date_0	Excel: =TODAY()
YtM shock (continuous compounding)	Δy	NA: user defined

*In MS Excel, the formula is the following: =@BDP(CUSIP&“ ”&“corp”,Bloomberg code)

A.1.2 Bullet method

Step	Quantity	Formula
1.	Bullet approximation (after-shock bond price)	$p_{\text{bullet}} := \mu_0 \cdot e^{-\mu_{1/0}\Delta y}$

A.1.3 Upper barbell method

Step	Quantity	Formula
1.	Current continuous-time YtM	$y_0 := f\log(1 + Y_0/f)$
2.	Current continuous-time convexity	$\mu_{2/0} := C \cdot (1 + Y_0/f)^2 - \mu_{1/0}/f$
3.	Timing of the future payment	$T_{ub} := \mu_{2/0}/\mu_{1/0}$
4.	Present value of the future payment	$c_{ub} := \mu_0\mu_{1/0}/T_{ub}$
5.	Size of the payment at time $t = 0$	$c_0 := \mu_0 - c_{ub}$
6.	Up.barbell approx. (after-shock price)	$p_{ub} := c_0 + c_{ub}e^{-T_{ub}\Delta y}$

A.1.4 Lower barbell method

Step	Quantity	Formula
1.	Time to maturity in years	$T := \frac{\text{Date}_T - \text{Date}_0}{365.25}$
2.	1st normalized moment about $t = T$: $\frac{\int_t (t-T)d\text{PV}}{\mu_0}$	$\breve{\mu}_{1/0} := \mu_{1/0} - T$
3.	2nd normalized moment about $t = T$: $\frac{\int_t (t-T)^2 d\text{PV}}{\mu_0}$	$\breve{\mu}_{2/0} := \mu_{2/0} + T^2 - 2T\mu_{1/0}$
4.	The timing of the first payment	$t_{lb} := T + \breve{\mu}_{2/0}/\breve{\mu}_{1/0}$
5.	Present value of the payment at $t = t_{lb}$	$c_{lb} := \mu_0\breve{\mu}_{1/0}^2/\breve{\mu}_{2/0}$
6.	Present value of the payment at $t = T$	$c_T := \mu_0 - c_{lb}$
7.	Low.barbell approx. (after-shock price)	$p_{lb} := c_{lb}e^{-t_{lb}\Delta y} + c_T e^{-T\Delta y}$

A.2 Applying to bonds portfolios

A.2.1 Aggregation at the portfolio level

A.2.1.1 Portfolio manager's side The *Portfolio manager* knows the characteristics of individual bonds. He reports to the *Risk manager* the overall characteristics of the portfolio: market value, duration and convexity.

A.2.1.1.1 Portfolio manager's inputs For each bond in the portfolio, the *Portfolio manager* knows following parameters.[46]

Input name	Notation	Bloomberg code
Current dirty price	μ_0^i	PX_DIRTY_MID
Number of units of the i-th bond in the portfolio	q^i	
Current Macaulay duration	$\mu_{1/0}^i$	DUR_MID
Current convexity	C^i	CNVX_MID
Coupon frequency	f^i	CPN_FREQ
Current YtM	Y_0^i	YLD_YTM_MID
Maturity date	Date_T^i	MATURITY

A.2.1.1.2 Portfolio manager's aggregation

Step	Quantity	Formula
1.	Portfolio market value	$\mu_0^{\text{Prt}} := \sum_{i=1}^I q^i \mu_0^i$
2.	Portfolio Macaulay duration	$\mu_{1/0}^{\text{Prt}} := \sum_{i=1}^I q^i \mu_{1/0}^i \mu_0^i / \mu_0^{\text{Prt}}$
3.	The i-th bond continuous-time YtM	$y_0^i := f^i \log(1 + Y_0^i / f^i)$
4.	The i-th bond continuous-time convexity	$\mu_{2/0}^i := C^i \cdot (1 + Y_0^i / f^i)^2 - \mu_{1/0}^i / f^i$
5.	Portfolio continuous-time convexity	$\mu_{2/0}^{\text{Prt}} := \sum_{i=1}^I q^i \mu_{2/0}^i \mu_0^i / \mu_0^{\text{Prt}}$
6.	Maximum time to maturity	$T^{\text{Prt}} := \frac{\max_i \left(\text{Date}_T^i\right) - \text{Date}_0}{365.25}$

A.2.1.2 Risk manager's side The *Risk manager* receives the aggregated portfolio characteristics from the *Portfolio manager*, and calculates approximation based on them.

A.2.1.2.1 Risk manager's inputs

Quantity	Notation
Portfolio market value	μ_0^{Prt}
Portfolio Macaulay duration	$\mu_{1/0}^{\text{Prt}}$
Portfolio continuous-time convexity	$\mu_{2/0}^{\text{Prt}}$
Maximum time to maturity	T^{Prt}

[46] For i-th bond, $i \in 1, ..., I$.

A.2.1.2.2 Risk manager's approximations

Step	Quantity	Formula
Bullet	**Bullet approximation**	$p^{\text{Prt}}_{\text{bullet}} := \mu^{\text{Prt}}_0 \cdot e^{-\mu^{\text{Prt}}_{1/0}\Delta y}$
UB1	Timing of the future payment	$T^{\text{Prt}}_{ub} := \frac{\mu^{\text{Prt}}_{2/0}}{\mu^{\text{Prt}}_{1/0}}$
UB2	Present value of the future payment	$c^{\text{Prt}}_{ub} := \frac{\mu^{\text{Prt}}_0 \mu^{\text{Prt}}_{1/0}}{T^{\text{Prt}}_{ub}}$
UB3	Size of the payment at time $t = 0$	$c^{\text{Prt}}_0 := \mu^{\text{Prt}}_0 - c^{\text{Prt}}_{ub}$
UB4	**Upper barbell approximation**	$p^{\text{Prt}}_{ub} := c^{\text{Prt}}_0 + c^{\text{Prt}}_{ub} e^{-T^{\text{Prt}}_{ub}\Delta y}$
LB1	1st normalized moment about $t = T$	$\breve{\mu}^{\text{Prt}}_{1/0} := \mu^{\text{Prt}}_{1/0} - T^{\text{Prt}}$
LB2	2nd normalized moment about $t = T$	$\breve{\mu}^{\text{Prt}}_{2/0} := \mu^{\text{Prt}}_{2/0} + \left(T^{\text{Prt}}\right)^2 - 2T^{\text{Prt}}\mu^{\text{Prt}}_{1/0}$
LB3	The timing of the first payment	$t^{\text{Prt}}_{lb} := T^{\text{Prt}} + \breve{\mu}^{\text{Prt}}_{2/0}/\breve{\mu}^{\text{Prt}}_{1/0}$
LB4	Present value of the payment at $t = t^{\text{Prt}}_{lb}$	$c^{\text{Prt}}_{lb} := \mu^{\text{Prt}}_0 \left(\breve{\mu}^{\text{Prt}}_{1/0}\right)^2 / \breve{\mu}^{\text{Prt}}_{2/0}$
LB5	Present value of the payment at $t = T^{\text{Prt}}$	$c_{T^{\text{Prt}}} := \mu^{\text{Prt}}_0 - c^{\text{Prt}}_{lb}$
LB6	**Lower barbell approximation**	$p^{\text{Prt}}_{lb} := c^{\text{Prt}}_{lb} e^{-t^{\text{Prt}}_{lb}\Delta y} + c_{T^{\text{Prt}}} e^{-T^{\text{Prt}}\Delta y}$
AB	**Average barbell approximation**	$p^{\text{Prt}}_{ab} := \left(p^{\text{Prt}}_{lb} + p^{\text{Prt}}_{ub}\right)/2$

A.2.2 Stratified aggregation

A.2.2.1 Portfolio manager's side

A.2.2.1.1 Portfolio manager's inputs For each bond in the portfolio, the *Portfolio manager* knows the same parameters as described in paragraph A.2.1.1.1. In addition to this information, the *Portfolio manager* knows the set of maturity hurdle that has been previously coordinated with the *Risk manager*:

$$\Theta_1, ..., \Theta_J.$$

The *Portfolio manager* aggregates the information about bonds based on maturity brackets $\{(\Theta_{j-1}, \Theta_j]\}_{j\in 1,...,J}$, where $\Theta_0 := 0$.

A.2.2.1.2 Portfolio manager's aggregation

Step	Quantity	Formula
1.	i-th bond's time to maturity	$T_i := \frac{\text{Date}^i_T - \text{Date}_0}{365.25}$
2.	j-th subportfolio market value	$\mu^{\text{SP}_j}_0 := \sum_{i:T_i\in(\Theta_{j-1},\Theta_j]} q^i \mu^i_0$
3.	j-th subportfolio Macaulay duration	$\mu^{\text{SP}_j}_{1/0} := \sum_{i:T_i\in(\Theta_{j-1},\Theta_j]} q^i \mu^i_{1/0}\mu^i_0/\mu^{\text{SP}_j}_0$
4.	i-th bond continuous-time YtM	$y^i_0 := f^i \log(1 + Y^i_0/f^i)$
5.	i-th bond continuous-time convexity	$\mu^i_{2/0} := (1 + Y^i_0/f^i)^2 C^i - \mu^i_{1/0}/f^i$
6.	j-th subportfolio continuous-time convexity	$\mu^{\text{SP}_j}_{2/0} := \sum_{i:T_i\in(\Theta_{j-1},\Theta_j]} q^i \mu^i_{2/0}\mu^i_0/\mu^{\text{SP}_j}_0$

A.2.2.2 Risk manager's side

A.2.2.2.1 Risk manager's inputs

Quantity	Notation
j-th subportfolio market value	$\mu_0^{\mathrm{SP}_j}$
j-th subportfolio Macaulay duration	$\mu_{1/0}^{\mathrm{SP}_j}$
j-th subportfolio continuous-time convexity	$\mu_{2/0}^{\mathrm{SP}_j}$
Maturity thresholds	Θ_j

A.2.2.2.2 Risk manager's approximation

Bullet approximation

Step	Quantity	Formula
Bullet1	j-th subportfolio bullet approximation	$p_{\mathrm{bullet}}^{\mathrm{SP}_j} := \mu_0^{\mathrm{SP}_j} \cdot e^{-\mu_{1/0}^{\mathrm{SP}_j}\Delta y}$
Bullet2	**Portfolio bullet approximation**	$p_{\mathrm{bullet}}^{\mathrm{Stratified}_\Theta} := \sum_{j:\mu_0^{\mathrm{SP}_j}\neq 0} p_{\mathrm{bullet}}^{\mathrm{SP}_j}$

Upper barbell approximation

Step	Quantity	Formula
UB1	Timing of the future payment	$T_{ub}^{\mathrm{SP}_j} := \frac{\mu_{2/0}^{\mathrm{SP}_j}}{\mu_{1/0}^{\mathrm{SP}_j}}$
UB2	Present value of the future payment: upper barbell proxy bond for the j-th subportfolio	$c_{ub}^{\mathrm{SP}_j} := \frac{\mu_0^{\mathrm{SP}_j}\mu_{1/0}^{\mathrm{SP}_j}}{T_{ub}^{\mathrm{SP}_j}}$
UB3	Size of the payment at time $t = 0$: upper barbell proxy bond for the j-th subportfolio	$c_0^{\mathrm{SP}_j} := \mu_0^{\mathrm{SP}_j} - c_{ub}^{\mathrm{SP}_j}$
UB4	Upper barbell approximation for the j-th subportfolio	$p_{ub}^{\mathrm{SP}_j} := c_0^{\mathrm{SP}_j} + c_{ub}^{\mathrm{SP}_j} e^{-T_{ub}^{\mathrm{SP}_j}\Delta y}$
UB5	**Portfolio upper barbell approximation**	$p_{ub}^{\mathrm{Stratified}_\Theta} := \sum_{j:\mu_0^{\mathrm{SP}_j}\neq 0} p_{ub}^{\mathrm{SP}_j}$

Lower barbell approximation

Step	Quantity	Formula
LB1	1st normalized moment of the j-th subportfolio about $t = \Theta^j$	$\breve{\mu}_{1/0}^{\mathrm{SP}_j} := \mu_{1/0}^{\mathrm{SP}_j} - \Theta^j$
LB2	2nd normalized moment of the j-th subportfolio about $t = \Theta^j$	$\breve{\mu}_{2/0}^{\mathrm{SP}_j} := \mu_{2/0}^{\mathrm{SP}_j} + \left(\Theta^j\right)^2 - 2\Theta^j \mu_{1/0}^{\mathrm{SP}_j}$
LB3	The timing of the first payment: lower barbell proxy bond for the j-th subportfolio	$t_{lb}^{\mathrm{SP}_j} := \Theta^j + \breve{\mu}_{2/0}^{\mathrm{SP}_j} / \breve{\mu}_{1/0}^{\mathrm{SP}_j}$
LB4	Present value of the payment at $t = t_{lb}^{\mathrm{SP}_j}$: lower barbell proxy bond for the j-th subportfolio	$c_{lb}^{\mathrm{SP}_j} := \mu_0^{\mathrm{SP}_j} \left(\breve{\mu}_{1/0}^{\mathrm{SP}_j}\right)^2 / \breve{\mu}_{2/0}^{\mathrm{SP}_j}$
LB5	Present value of the payment at $t = \Theta_j$: lower barbell proxy bond for the j-th subportfolio	$c_{\Theta^j} := \mu_0^{\mathrm{SP}_j} - c_{lb}^{\mathrm{SP}_j}$
LB6	Lower barbell approximation for the j-th subportfolio	$p_{lb}^{\mathrm{SP}_j} := c_{lb}^{\mathrm{SP}_j} e^{-t_{lb}^{\mathrm{SP}_j} \Delta y} + c_{\Theta^j} e^{-\Theta^j \Delta y}$
LB7	**Portfolio lower barbell approximation**	$p_{lb}^{\mathrm{Stratified}_\Theta} := \sum_{j:\mu_0^{\mathrm{SP}_j} \neq 0} p_{lb}^{\mathrm{SP}_j}$

Average barbell approximation

Step	Quantity	Formula
AB1	Average barbell approximation for the j-th subportfolio	$p_{ab}^{\mathrm{SP}_j} := \left(p_{lb}^{\mathrm{SP}_j} + p_{ub}^{\mathrm{SP}_j}\right)/2$
AB2	**Portfolio average barbell approximation**	$p_{ab}^{\mathrm{Stratified}_\Theta} := \left(p_{lb}^{\mathrm{Stratified}_\Theta} + p_{ub}^{\mathrm{Stratified}_\Theta}\right)/2$

A.3 Approximating price reaction to non-parallel shifts

Aggregation and calculation of proxy bond parameters is unchanged. The only difference concerns the very last stage: calculating the after-shock proxy bond prices: one needs to replace the maturity-uniform shock, Δy, with

$$\Delta \mathrm{yc}(t) = \mathrm{yc}_{\text{after shock}}(t) - \mathrm{yc}_{\text{before shock}}(t),$$

where t needs to be replaced by with the timing of the corresponding cash flow of the proxy bond. For example, step "LB6" in paragraph A.2.2.2.2 becomes more general, as follows:

$$p_{lb}^{\mathrm{SP}_j} := c_{lb}^{\mathrm{SP}_j} \exp\left(-t_{lb}^{\mathrm{SP}_j} \Delta \mathrm{yc}\left(t_{lb}^{\mathrm{SP}_j}\right)\right) + c_{\Theta^j} \exp\left(-\Theta^j \Delta \mathrm{yc}\left(\Theta^j\right)\right).$$

B Matching higher-order moments

B.1 Matching the third moment

In this section, we assume that the *Risk manager* knows also the third moment of the original bond/portfolio, μ_3, and creates proxy bonds that match it. We introduce two more proxy bonds: the "balanced barbell" and the "triple barbell". For parallel of synchronous shifts, this translates into matching the third derivative of the bond price/portfolio value with respect to the interest rate shock, Δy.

B.1.1 Balanced barbell

The balanced barbell is composed of two payments at some dates between date $t = 0$, and the date of the last cash flow in the fixed income portfolio. Both dates and cash flow present values are calibrated to match the parameters of the original bond/portfolio, namely, the first four moments: μ_0, μ_1, μ_2, and μ_3. We denote the balanced barbell's payment dates and the corresponding present values as $t_{1,2}$ and $c_{1,2}$, respectively.

Theorem B.1. Suppose the original bond/portfolio has at least two distinct payments. Then, the following holds:

1. The moment matching writes as $c_1 t_1^i + c_2 t_2^i = \mu_i$, $i \in 0, 1, 2, 3$ ($t^0 := 1$ even if $t = 0$).

2. The moment matching conditions yield a unique solution:

$$t_{1,2} = \frac{e_1 \pm \sqrt{e_1^2 - 4e_2}}{2}, \qquad c_1 = \frac{\mu_1 - \mu_0 t_2}{t_1 - t_2}, \qquad c_2 = \frac{\mu_0 t_1 - \mu_1}{t_1 - t_2}, \tag{55}$$

   where $e_1 = \frac{\mu_0\mu_3 - \mu_1\mu_2}{\Delta}$, $e_2 = \frac{\mu_1\mu_3 - \mu_2^2}{\Delta}$, and $\Delta = \mu_0\mu_2 - \mu_1^2$.

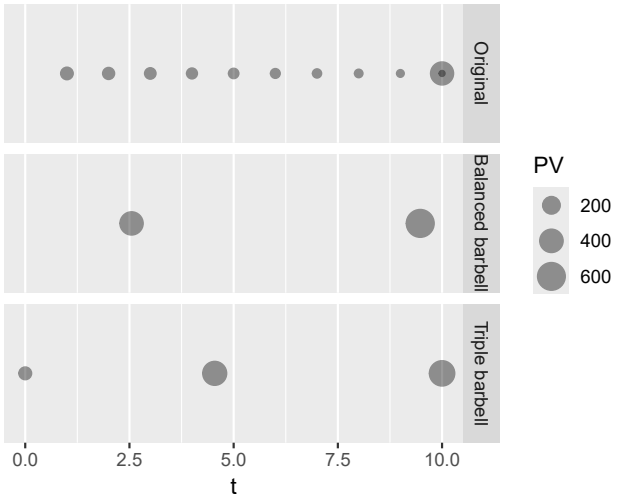
3. The solution satisfies: $0 \leq t_{1,2} \leq T$ and $c_{1,2} > 0$.

4. For parallel/synchronous interest rate shocks, the balanced barbell constitutes a lower bound for the true bond price/portfolio value (irrespective of the shock sign).

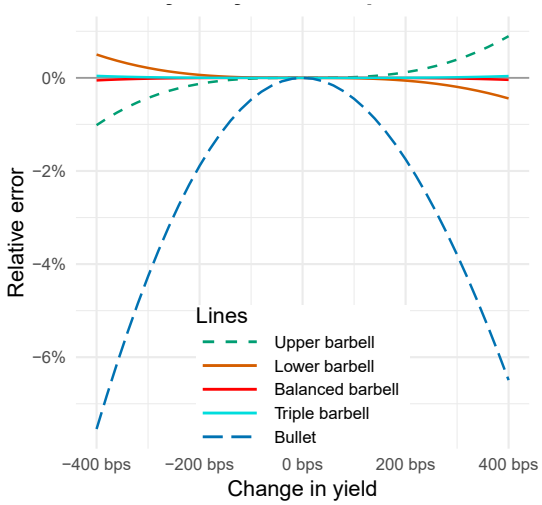
Proof. The first point is trivial. The second point follows from the theory of orthogonal polynomials and quadratures (Gautschi, 2004). For the sake of self-sufficiency, we provide a proof in Appendix G, Section G.10.

The proof of positivity of cash flows, the fact that both dates are between 0 and T, and the fact that the balanced barbell approximation constitutes a lower bound relies on our generalized version of the Descartes rule of signs (Lemma 5.5). The details are provided in Appendix G, Section G.10. □

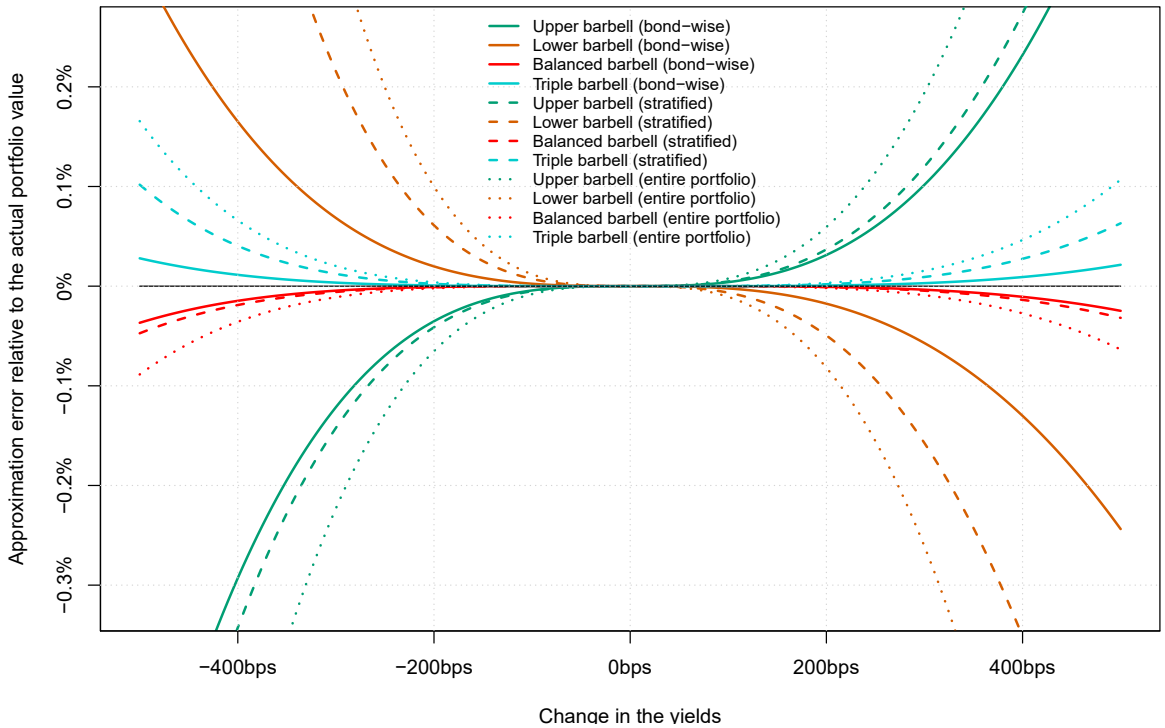
Figure A.1: Cash flow structure and performance of approximations with the third moment: parallel shocks



(a) Cash flow schedule of the proxy bonds.



(b) Bond price reaction to YtM shocks. A 30-year par bond with 6% coupon rate.



(c) Performance of proxy bond methods at the portfolio level: scenario of parallel shock to discount rates.

B.1.2 Triple barbell

The triple barbell proxy bond is composed of three payments with present values γ_0, γ_τ, and γ_T at dates 0, τ, and T. The date T is a given date that represents the maturity of the original bond, the longest maturity of the fixed income portfolio, or a maturity cap know to the *Risk manager*. Conversely, the date τ is unknown.

Theorem B.2. Suppose the original bond/portfolio has at least three distinct payments and all its cash flows are between 0 and T. Then, the following holds:

1. The moment matching writes as $\gamma_0 0^i + \gamma_\tau \tau^i + \gamma_T T = \mu_i$, $i \in 0, 1, 2, 3$ ($0^0 := 1$).
2. The moment matching conditions yield a unique solution:

$$\tau = \frac{\mu_3 - \mu_2 T}{\mu_2 - \mu_1 T}, \quad \gamma_\tau = \frac{\mu_1 T - \mu_2}{\tau/(T-\tau)}, \quad \gamma_T = \frac{\mu_2 - \tau\mu_1}{T(T-\tau)}, \quad \gamma_0 = \mu_0 - \gamma_\tau - \gamma_T. \tag{56}$$

3. The solution satisfies: $0 \leq \tau \leq T$ and $\gamma_{0,\tau,T} > 0$.
4. For parallel/synchronous interest rate shocks, the triple barbell constitutes an upper bound for the true bond price/portfolio value (irrespective of the shock sign).

Proof. The first point is trivial. The proof of the second point is provided in Appendix G, Section G.11.

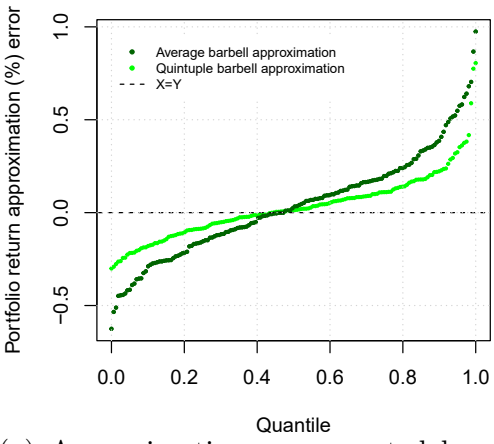
The proof of positivity of cash flows, the fact that date τ is between 0 and T, and the fact that the triple barbell approximation constitutes an upper bound relies on our generalized version of the Descartes rule of signs (Lemma 5.5). The details are provided in Appendix G, Section G.11. □

The inclusion of the third moment into the *Risk manager*'s information set can render the evaluation of parallel shock scenarios virtually flawless (see figure A.1 (b)). To evaluate the performance of our μ_3-matched methods under non-parallel scenarios historically observed in the US, we rely on a 50-50 combination of the balanced barbell and triple barbell, which we call the "quintuple barbell" (since it has five cash flows). Introducing the quintuple barbell allows to reduce the pricing errors by 38% compared to the performance of the average barbell (see figure A.2), which we analyze in Section 7.

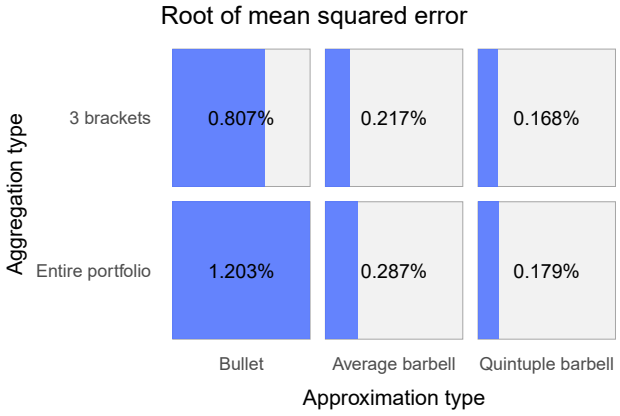
B.2 General case

Now, suppose that the *Risk manager* knows $\mu_0, ..., \mu_N, N = 1, 2, 3, ...$ and the fact that all the payments are within the interval $[0, T]$, and creates proxy bonds that match the moments while making sure the payment schedule remains within $[0, T]$. For parallel or synchronous shifts, the moment-matching conditions translate into matching all derivatives of the bond price/portfolio value with respect to the interest rate shock, Δy up to

Figure A.2: Performance of approximations with the third moment: non-parallel shocks



(a) Approximation errors sorted by size. Both methods are applied using aggregate portfolio data.



(b) Overall performance of our methods.

Notes: The details of non-parallel shock test are presented in Section 7. The figures report the comparative performance of the proxy bond approach with three moments—average barbell—and four moments—quintuple barbell.

the N-th order.

$$\hat{P}^{(i)}(\Delta y)|_{\Delta y=0} = \hat{P}^{(i)}_{\text{proxy}}(\Delta y)|_{\Delta y=0} \text{ for } i = 0, 1, ..., N.$$

Simply put, the construction of the proxy bonds boils down to building a quadrature (Gautschi, 2004) based on a judiciously chosen measure and—in most cases—adjusting the quadrature by adding one or two corner nodes at $t = 0$ and/or at $t = T$. A detailed procedure how to build the proxy bonds is provided in the following theorem.

Theorem B.3. One can construct the following uniquely determined proxy bonds constituting lower and upper bounds for the original bond price/portfolio value after a parallel/synchronous change of the discount rates.

1. For even numbers $N \geq 2$, if the original bond has at least $N/2 + 1$ distinct cash flows on $(0, T)$, then the two proxy bonds are built in the following way:

   (a) The "upper proxy bond" is built in the following fashion:

      i. Construct a $\frac{N}{2}$-node quadrature of the measure m_t defined via $dm_t = t \times d\text{PV}_{\text{original bond}}$. This quadrature's nodes are the roots of the orthogonal polynomial of degree $N/2$ under measure m_t. This quadrature matches moments $\mu_0^{m_t}, ..., \mu_{N-1}^{m_t}$ of the measure m_t.

      ii. Accomplish the quadrature by adding an extra node at $t = 0$. Choose the weight c_0 so as to match the moment μ_0, that is, the bond price.

   (b) The "lower proxy bond" is built in the following fashion:

i. Construct a $\frac{N}{2}$-node quadrature of the measure m_{T-t} defined via $dm_{T-t} = (T-t) \times d\text{PV}_{\text{original bond}}$. This quadrature's nodes are the roots of the orthogonal polynomial of degree $N/2$ under measure m_{T-t}. This quadrature matches moments $\mu_0^{m_{T-t}}, ..., \mu_{N-1}^{m_{T-t}}$ of measure m_{T-t}.

ii. Accomplish the quadrature by adding an extra node at $t = T$. Choose the weight c_T so as to match the moment μ_0, that is, the bond price.

For $\Delta y > 0$ and even numbers N, the upper proxy bond constitutes an upper bound, and the lower proxy bond constitutes the lower bound. For $\Delta y < 0$, the bounds flip.

2. For odd numbers $N \geq 1$,[47] if the original bond has at least $(N+3)/2$ distinct cash flows on $(0, T)$, then the two proxy bonds are built in the following way:

   (a) The "upper proxy bond" is built in the following fashion:

      i. Construct a $\frac{N-1}{2}$-node quadrature of the measure $m_{(T-t)t}$ defined via $dm_{(T-t)t} = (T-t)t \times d\text{PV}_{\text{original bond}}$. This quadrature's nodes are the roots of the orthogonal polynomial of degree $(N-1)/2$ under measure $m_{(T-t)t}$. This quadrature matches moments $\mu_0^{m_{(T-t)t}}, ..., \mu_{N-2}^{m_{(T-t)t}}$ of measure $m_{(T-t)t}$.

      ii. Accomplish the quadrature by adding two extra nodes: at $t = 0$ and at $t = T$. Choose the weight c_T so as to match the moment μ_1. Choose the weight c_0 so as to match the moment $\breve{\mu}_1$.

   (b) To construct the "lower proxy bond", simply build the $(N+1)/2$ quadrature for the measure $\text{PV}_{\text{original bond}}$.

   For odd numbers N, the upper proxy bond constitutes an upper bound, and the lower proxy bond constitutes the lower bound irrespective of the sign of Δy.

Proof. The proof relies of the theory of orthogonal polynomials and quadratures (Gautschi, 2004) and the generalized Descartes' rule of signs (Lemma 5.5). The details are provided in Appendix G, Section G.12 □

Remark B.1. Theorem B.3 encompasses the specific moment-matching problems considered throughout this paper.

- $N = 1$, lower proxy bond is the bullet bond, presented in Section 5.1.[48]
- $N = 2$, lower proxy bond is the lower barbell bond, presented in Section 5.2.

[47] In the degenerate case of $N = 1$, the upper proxy bond has no internal nodes, only corner nodes.

[48] $N = 1$, lower proxy bond is a degenerate case with no internal nodes. This case corresponds the classical "barbell" hedge, in which the two hedging zero-coupon bonds have predetermined maturities. Such barbells only match the duration of the original bond. The bright side is that they constitute an upper bound for the true bond price.

- $N = 2$, upper proxy bond is the upper barbell bond, presented in Section 5.3.
- $N = 3$, lower proxy bond is the balanced barbell bond, presented in Section B.1.1.
- $N = 3$, upper proxy bond is the triple barbell bond, presented in Section B.1.2.

Some more complex cases—that is, matching higher moments—are not discussed in detail. But it is remarkable that even for higher order moments (up to $N = 6$), one can obtain closed-form solutions with relatively tractable formulas for the proxy bond payment dates and present values.

- $N = 4$, lower proxy bond is not discussed specifically in this paper. This proxy bond contains on payment at $t = T$. The two other dates are determined as the two roots of the orthogonal polynomial of degree two that stems from measure m_{T-t}. Hence, the formulas are similar to the ones we obtain in the case of the balanced barbell (Section B.1.1).
- $N = 4$, upper proxy bond is similar to the lower case. This proxy bond contains one payment at $t = 0$ instead of $t = T$. The two other dates are determined as the two roots of the orthogonal polynomial of degree two that stems from measure m_t (as opposed to m_{T-t}). Hence, the formulas are similar except that one needs to rely on the standard moments, μ_i, instead of the shifted moments, $\breve{\mu}_i$.
- For $N = 5$ and $N = 6$, one can also obtain proxy bonds in closed form, but the formulas for the proxy bond parameters rely on the Cardano formulas for solving cubic equation and, thus, are slightly more tedious. For $N = 7$ and $N = 8$, formulas for the proxy bond parameters are even more tedious as they rely on the Ferrari's method of solving the quatric equation. Beyond $N = 8$, closed form solutions do not exist.

C Comparison of the bullet method with the exponential duration and the discrete duration methods

In this section, we demonstrate that the bullet approximation is similar to the exponential duration method introduced by Livingston and Zhou (2005).

Definition C.1. We define the exponential Macaulay duration method as follows.

$$\log P(y) \approx \log P(y_0) + \left.\frac{d \log P(y)}{dy}\right|_{y=y_0} \times \Delta y, \quad ^{49}$$

which is equivalent to

$$\log P(y) \approx \log \mu_0 - \mu_{1/0} \cdot \Delta y.$$

Thus, the bond price is approximated as follows:

$$P(y) \approx \mu_0 e^{-\mu_{1/0} \cdot \Delta y}. \quad ^{50} \tag{57}$$

The exponential Macaulay duration method is identical to the method of Livingston and Zhou (2005) except that instead of discretely compounded YtM, Y (and the corresponding duration, D), it relies on the continuously compounded YtM, y (and the corresponding duration, $\mu_{1/0}$).

Comparing equations 3 and 57, allows us to see that the following proposition holds.

Proposition C.1. The bullet approximation is numerically equivalent to the exponential Macaulay duration method.

The discrepancies between our bullet method and the exponential duration method (Livingston and Zhou, 2005) come from the fact that our method relies on continuous compounding. Put differently, Livingston and Zhou (2005) linearize the price–YtM relationship in $(\log P, Y)$ coordinates, whereas we linearize it in log–log coordinates—$(\log P, \log(1 + Y/f))$

In figure 1 we have shown that the Livingston and Zhou's method underperforms compared to our bullet method. The reason is that even for the simplest (i.e., zero-coupon) bonds, Livingston and Zhou's approximation has a slightly smaller convexity relative to that of the original bond due to their functional form.

Bajo et al. (2013) propose another method to address the apparent disadvantage of the Livingston and Zhou's method—the fact that the exponential duration method results in

[49] As opposed to $\log P(Y) \approx P(Y_0) + \left.\frac{d \log P(Y)}{dY}\right|_{Y=Y_0} \times \Delta Y$.

[50] As opposed to $P(Y) \approx \mu_0 e^{-D \cdot \Delta y}$.

a relatively poor approximation of zero-coupon bonds.

Bajo et al. (2013) suggest a formula that is reminiscent of discrete-time discounting,

$$P(Y) \approx \mu_0 \cdot (1 + \Delta Y)^{-D}, \quad ^{51}$$

which the authors motivate by the fact that D and Y are discretely compounded.

Since the authors implicitly assume annual compounding, we will also follow this convention, especially that for zero coupon bonds, this convention is applied in practice. Also, this assumption is correct for bonds with annual coupon payments. We recast the Livingston and Zhou's and Bajo et al.'s approximations in the continuous-time notation to make them comparable with our bullet formula:

Exp. duration (Livingston and Zhou)	$\hat{P}_{\text{exp.dur}}(\Delta y) = \mu_0 \cdot e^{-\mu_{1/0} \cdot \left(e^{\Delta y} - 1\right)}$
Discrete duration (Bajo et al.)	$\hat{P}_{\text{disc.dur}}(\Delta y) = \mu_0 \cdot (1 + e^{y_0 + \Delta y} - e^{y_0})^{-\mu_{1/0}/e^{y_0}}$
Bullet approximation	$\hat{P}_{\text{bullet}}(\Delta y) := \mu_0 \cdot e^{-\mu_{1/0} \cdot \Delta y}$

Now, we consider a zero-coupon bond maturing at time T and compute duration of convexity in continuous time along with that of the three approximations. For zero-coupon bonds, the full valuation function is numerically identical to the bullet approximation as the proxy bond literally coincides with the original bond:

$$\hat{P}(\Delta y) = \hat{P}_{\text{bullet}}(\Delta y) = \mu_0 \cdot e^{-T \cdot \Delta y}.$$

Method	Duration	Convexity
Full valuation	$-\frac{\hat{P}'(0)}{\hat{P}(0)} = T$	$\frac{\hat{P}''(0)}{\hat{P}(0)} = T^2$
Exp. duration (Livingston and Zhou)	$-\frac{\hat{P}'_{\text{exp.dur}}(0))}{\hat{P}_{\text{exp.dur}}(0)} = T$	$\frac{\hat{P}''_{\text{exp.dur}}(0)}{\hat{P}_{\text{exp.dur}}(0)} = T^2 - T$
Discrete duration (Bajo et al.)	$-\frac{\hat{P}'_{\text{disc.dur}}(0)}{\hat{P}_{\text{disc.dur}}(0)} = T$	$\frac{\hat{P}''_{\text{disc.dur}}(0)}{\hat{P}_{\text{disc.dur}}(0)} = T^2 + Y_0 T$
Bullet approximation	$-\frac{\hat{P}'_{\text{bullet}}(0)}{\hat{P}_{\text{bullet}}(0)} = T$	$\frac{\hat{P}''_{\text{bullet}}(0)}{\hat{P}_{\text{bullet}}(0)} = T^2$

As we see from the table above, all thee methods correctly match the duration. As for convexity, the exponential duration method (Livingston and Zhou, 2005) underestimates the convexity of the zero-coupon bond, while the discrete duration (Bajo et al., 2013) method slightly overestimates convexity of the zero-coupon bond.

Generally, the Bajo et al.'s method is more precise than the Livingston and Zhou's. The disadvantage, however, is that Bajo et al.'s method does not always result in a conservative estimate (unlike the other two methods).

To summarize, we stipulate the following proposition:

Proposition C.2. 1. The bullet price approximation is greater than or equal to the exponential duration approximation of the bond price. The equality is attained if and only if the YtM shock is null.

[51] As opposed to $\mu_0 \cdot e^{-D \cdot \Delta Y}$ in Livingston and Zhou (2005).

2. The exponential duration approach is conservative.

3. The discrete duration approximation is above the bullet approximation for $Y_0 > 0$ with $\Delta y \neq 0$.

4. The discrete duration approximation is not conservative for zero-coupon bonds when $Y_0 > 0$ and $\Delta y \neq 0$.

Proof. See Appendix G, Section G.13. □

D Performance of the new methods in comparison to the traditional ones for small changes of YtM

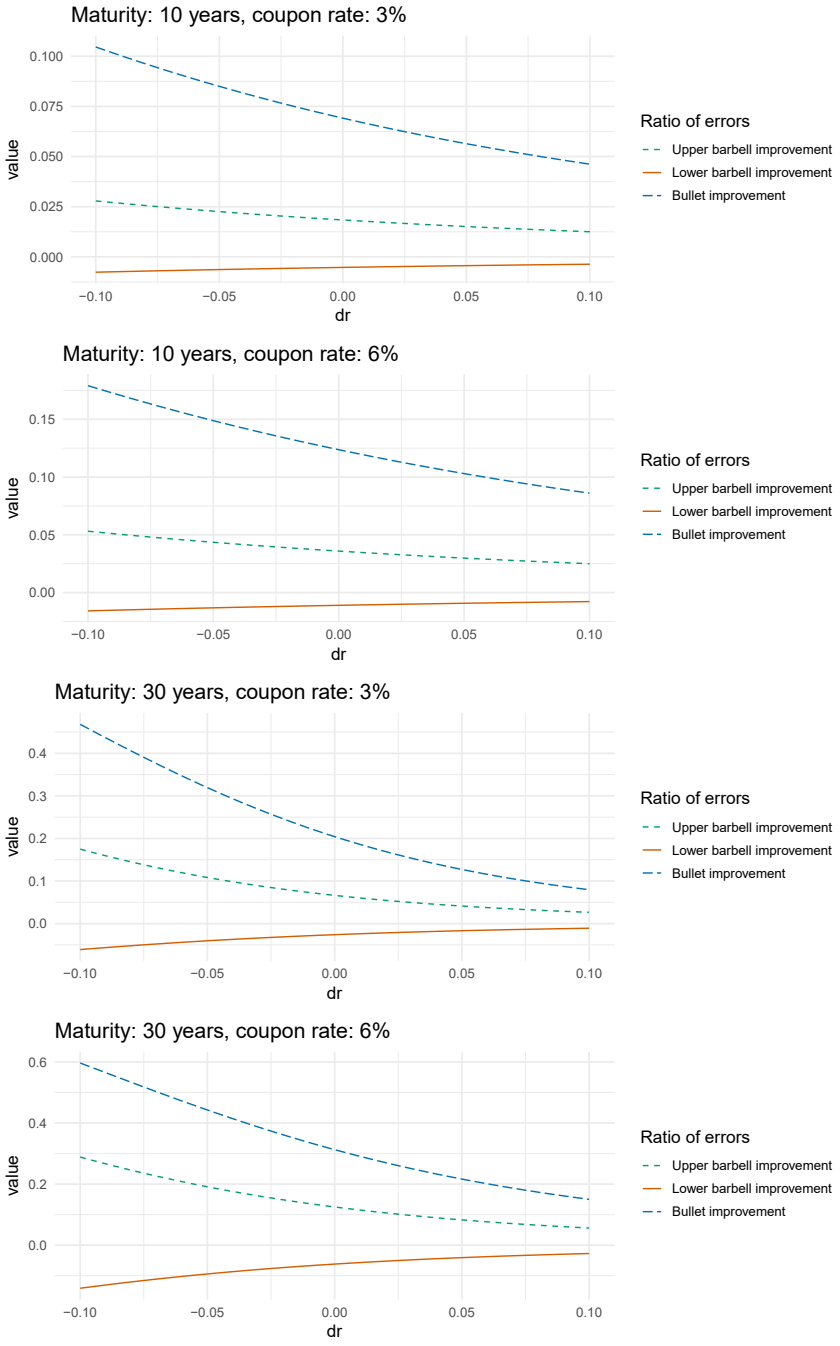


Figure A.3: The ratios of errors: new method errors divided by the traditional method errors. More specifically, we divide the errors of each of the barbell methods by that of the duration–convexity method (all three methods rely on duration *and* convexity). Rather, for the bullet method's error we use the duration method as the benchmark (both methods rely on the duration and ignore convexity). The fact that all ratios are below one in absolute value shows that our new methods are superior to the traditional ones. For small changes of the rates all five methods are very precise. Thus, for small shocks the superiority of our methods becomes visibly noticeable only when the approximation errors are compared in terms of ratios.

E Relation to other literature

Our methods, especially when applied to scenarios of non-parallel shifts, resemble the methodology for hedging in the portfolio immunization literature (Falkenstein and Hanweck, 1996, 1997; Barber and Copper, 1996). The common theme is that both methodologies substitute (or hedge) a given bond or fixed-income portfolio with a more parsimonious fixed-income portfolio designed to exhibit a similar sensitivity to a specified class of yield curve shocks. Nevertheless, there are four major distinctions:

- Our goal is to simplify computations within the stress testing/risk management framework. Our methodology also enables risk managers/regulators to make computations with incomplete information. Meanwhile, the immunization literature targets practically implementable portfolio hedging strategies.

- Our methods are approximately immune to parallel shift and to addition of small linear functions $a(t) = a_1 t$, while the immunization literature targets immunizing portfolio with respect to yield curve changes most often statistically observed (PCA factors).

- Immunization methods rely on existing bonds to create mimicking portfolios and calibrate weights, while our methods also calibrate the timings of the proxy bond cash flows and choose them so as to achieve a good approximation irrespective of whether or not zero-coupon bonds of this maturity exist in the market.

- Our methods rely exclusively on the characteristics of the original bond (or bond portfolio) and ignore the past history of yield curve. Thus, they are much easier to apply in practice.

To sum up, from the computational point of view our methodology is similar to that of the immunization literature in that the proxy bond (or portfolio of bonds) reacts similarly to a linear space of yield curve shocks. The difference is that our space of shocks—spanned by the factors $\Delta y(t) = 1$ and $\Delta y(t) = t$—is simpler than the space of shocks in the immunization literature—the space spanned by the first two principal components (factors).[52] A bond portfolio market value depends on an infinitely dimensional object—namely, the yield curve. Our average barbell method reduces the number of dimensions to three.[53] This reduction in dimensionality can be exploited in order to compute a Value-at-Risk or an expected shortfall.[54]

[52] In that sense, our approach is similar to the Gauss integration quadrature—a method of numerical integration that replaces the true integral with a weighted sum of the function values evaluated at a given set of nodes. The nodes and the weights are calibrated so that the approximation error is null as long as the integrand is a polynomial whose degree does not exceed a given number, as explained in many texts, for example, Gautschi (2004).

[53] Recall that one of the four cash flows of the average barbell bond is at time $t = 0$.

[54] Also known as conditional Value-at-Risk.

F Proxy bonds' reaction to non-parallel shocks: comparison with the quadratic approximation

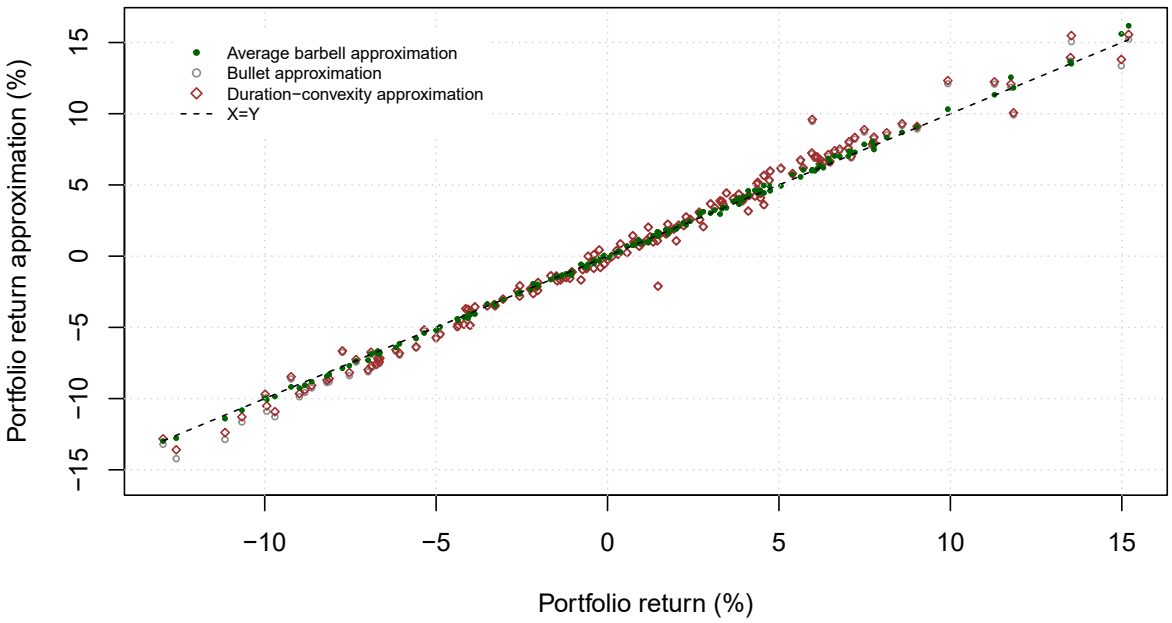


Figure A.4: Approximation of portfolio returns using proxy bonds for each bond in the portfolio: comparison with the duration–convexity approach

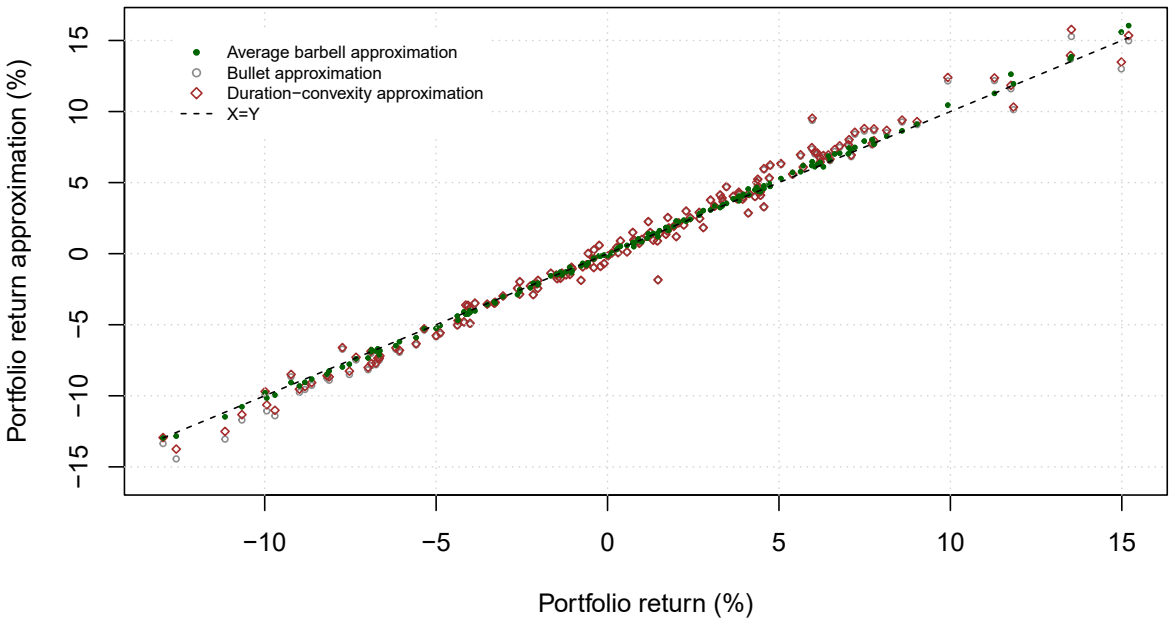


Figure A.5: Approximation of portfolio returns using proxy bonds for subportfolios: comparison with the duration–convexity approach

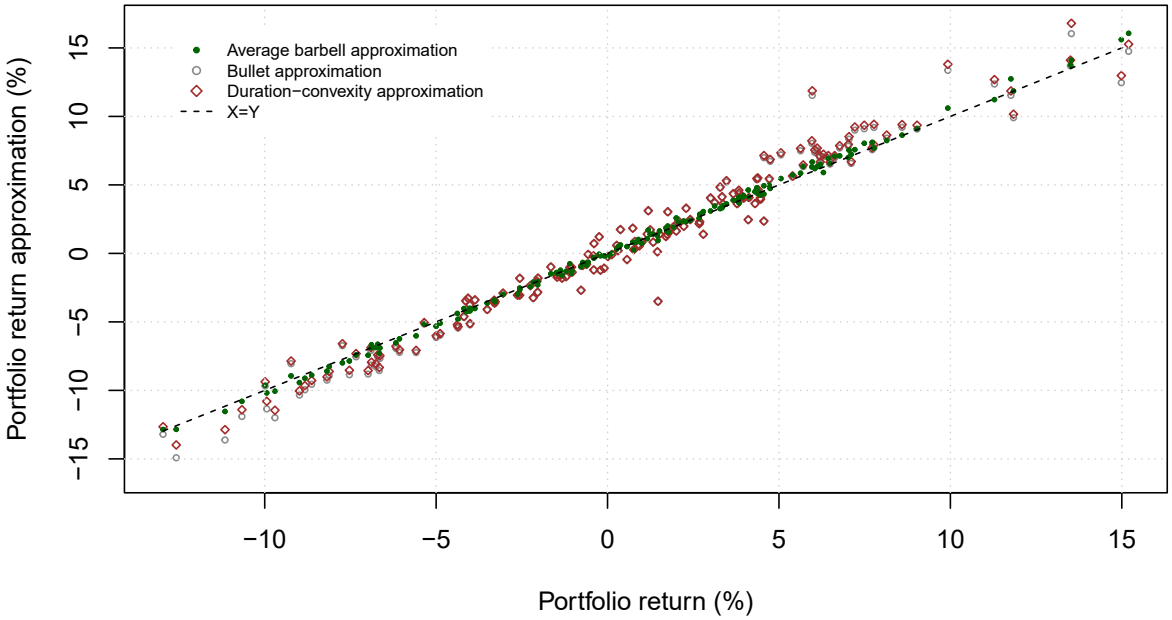
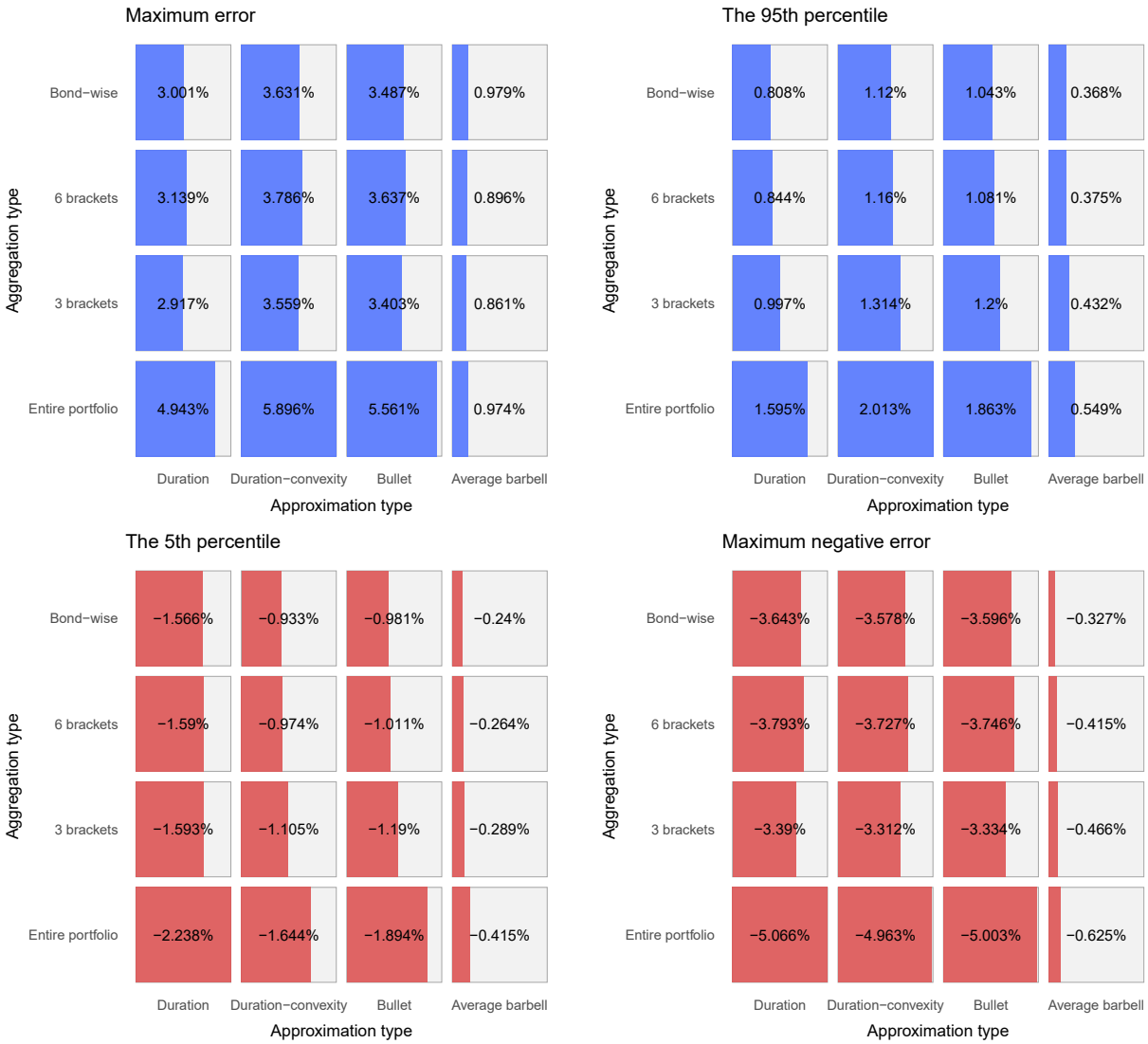


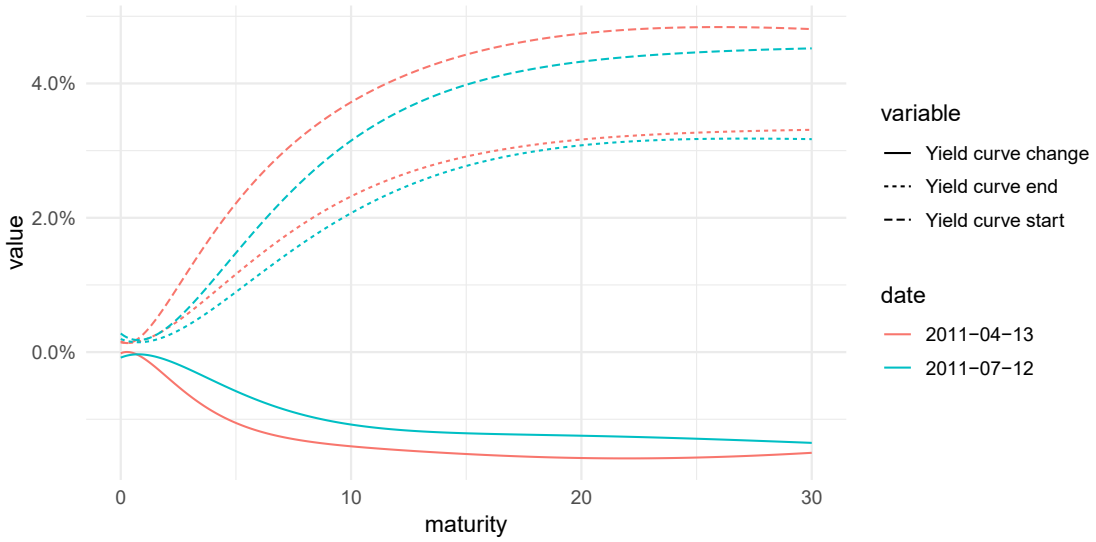
Figure A.6: Approximation of portfolio returns using proxy bonds for the entire portfolio: comparison with the duration–convexity approach

Figure A.7: Extreme error values by method and the aggregation type



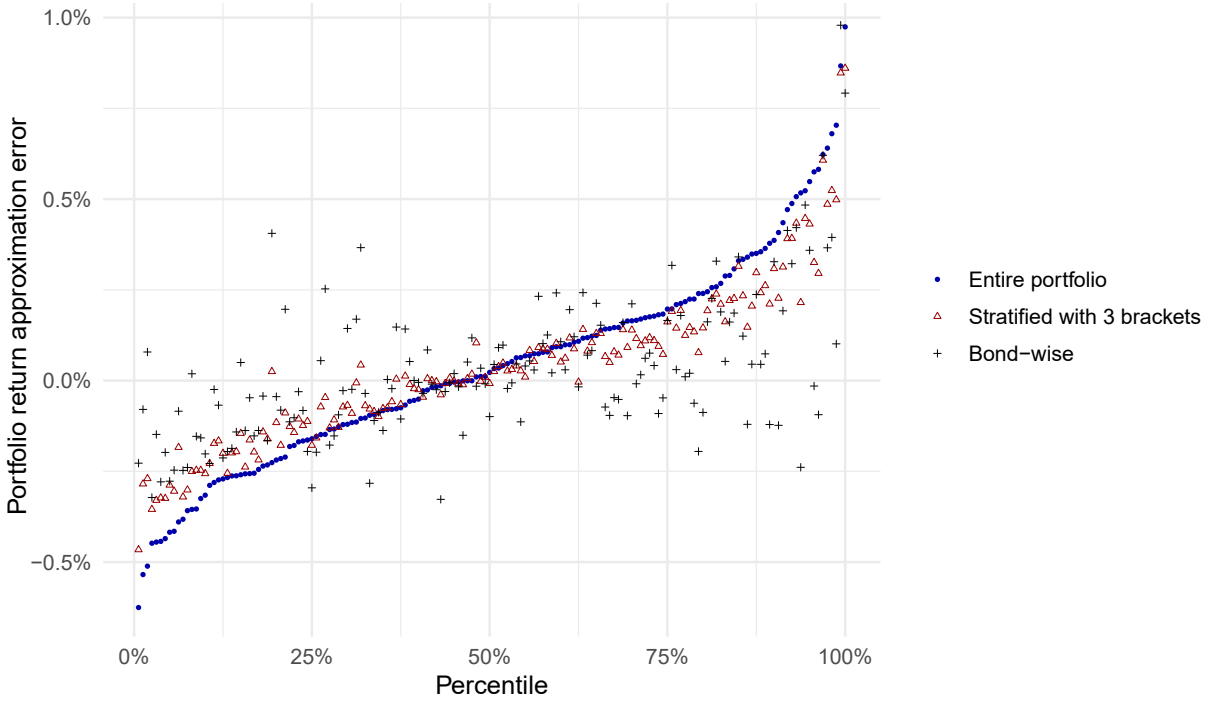
Note: The figure reports the approximation methods' errors—the deviation of the approximated portfolio value in a stress scenario from the actual portfolio value—for scenarios where the errors reach extreme sizes. For each method and level of portfolio aggregation in the respective cell, there reported the size of the error that corresponds to the maximum, 95th percentile, 5th percentile or minimum value when all scenarios are ranked by the error size for this method and aggregation level. The total area of a cell corresponds to the largest error size for each percentile. Hence, the colored area of other cells represent the relative size of the errors of the respective method compared to the worst error.
The key observation of this analysis, is that, across all scenarios drawn from the historical dynamic of the US Treasury yield curve, the error of the average barbell method does not exceed 1% of the actual portfolio value, while traditional methods' error can reach 5%.

Figure A.8: Yield curve changes corresponding to the biggest errors in the average barbell method.



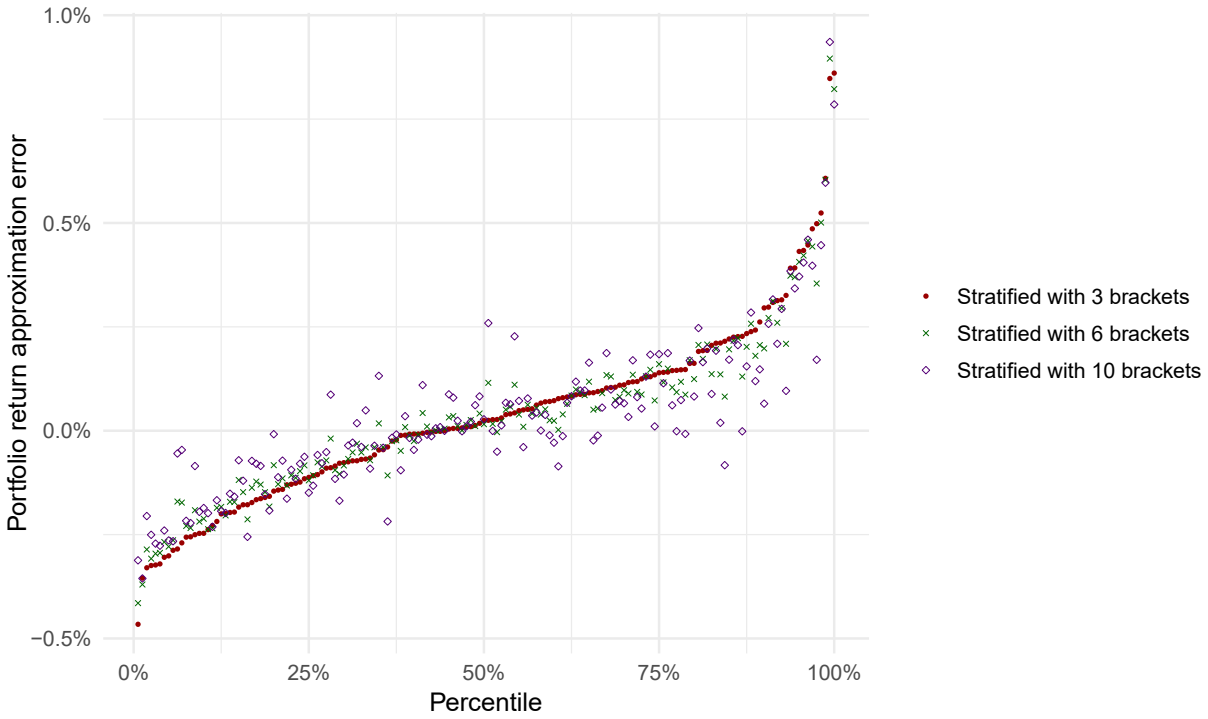
Note: The figure depicts the yield curve changes for two scenarios that generate the largest errors of the average barbell approximation method applied to three subportfolios of bond. We can observe that the cases of the biggest approximation errors feature non-linear shape of the yield curve shock. Note that in Appendix H, we show theoretically that the barbell approximations should be roughly immune to affine shocks to the yield curve: parallel shifts and changes in the slope without changes in the curvature. In the present figure, both cases exhibit substantial changes in curvature.

Figure A.9: Comparisons of different types of aggregation



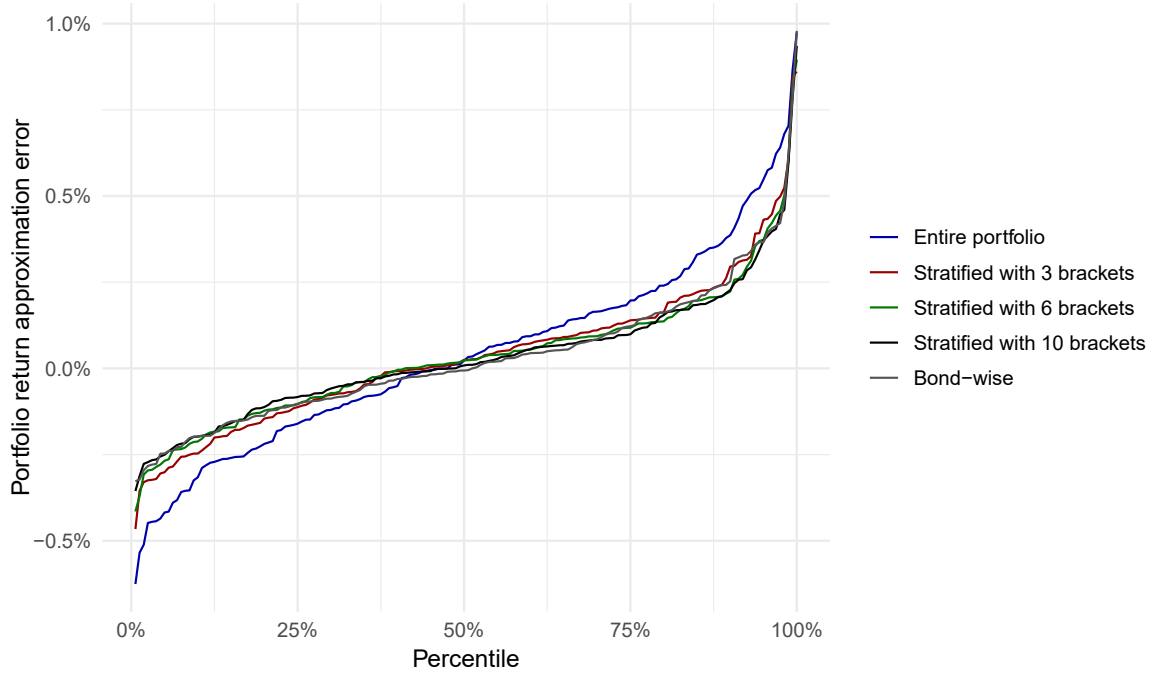
Note: The figure plots the errors of the average barbell approximation method applied to different levels of bond portfolio aggregation: the entire portfolio, three subportfolios of bonds split by maturity brackets, and bond-wise. The observations (scenarios) are ordered by the size of the errors of the average barbell method applied to the entire portfolio in ascending order.

Figure A.10: Comparisons of different types of aggregation: stratification



Note: The figure plots the errors of the average barbell approximation method applied to different levels of bond portfolio aggregation: three, six, and ten subportfolios of bonds split by maturity brackets. The observations (scenarios) are ordered by the size of the errors of the average barbell method applied to three subportfolios in ascending order.

Figure A.11: Comparison of different types of aggregation: distribution of approximation errors



Note: The figure plots the distribution of the errors of the average barbell approximation method applied to different levels of bond portfolio aggregation: the entire portfolio, three, six, and ten subportfolios of bonds split by maturity brackets, and bond-wise.

G Proofs

In this section, we provide the proofs that have been excluded from the main text for the sake of conciseness.

G.1 Proof of the Summary statement (Section 4)

First, let us consider the case $y - y_0 > 0$.

- $P_l(y) < P_b(y)$ follows from the fact that for at $y = y_0$, the plots of the functions P_l and P_b are tangent[55] to each other and the fact that the plot of P_l is a straight line, while the plot of P_b is convex.
- $P_b(y) < P_{lb}(y)$ holds because the bullet proxy bond can be also viewed as bullet bond for the lower barbell bond. So, from Theorem 5.1, its price constitutes a lower bound for the price of the lower barbell bond.
- $P_{lb}(y) < P(y)$ follows from Theorem 5.10.
- $P(y) < P_{ub}(y)$ follows from Theorem 5.6.
- Finally, $P_{ub}(y) < P_q(y)$ is the least trivial. This follows in three steps, in each of those steps we apply the Newton-Leibnitz formula. We start from comparing the third derivatives and end up obtaining the needed inequality for the approximating functions themselves.
  1. $P'''_{ub}(y) < P'''_q(y) \equiv 0$ for any $y > y_0$ together with $P''_{ub}(y_0) = P''_q(y_0)$ implies that $P''_{ub}(y) < P''_q(y)$ for any $y > y_0$.
  2. In turn, $P''_{ub}(y) < P''_q(y)$ for any $y > y_0$ together with $P'_{ub}(y_0) = P'_q(y_0)$ implies that $P'_{ub}(y) < P'_q(y)$ for any $y > y_0$.
  3. Finally, $P'_{ub}(y) < P'_q(y)$ for any $y > y_0$ together with $P_{ub}(y_0) = P_q(y_0)$ implies that $P_{ub}(y) < P_q(y)$ for any $y > y_0$.

Now, let us consider the case $y - y_0 < 0$.

- $P_l(y) < P_q(y)$ and $P_l(y) < P_b(y)$ because there is a tangency at $y = y_0$ between P_l and each of the two other function. At the same time, both P_b, and P_q are convex.
- $P_q(y) < P_{ub}(y)$ follows in three steps, in each of those steps we apply the Newton-Leibnitz formula. We start from comparing the third derivatives and end up obtaining the needed inequality of the approximations themselves.

[55] The first derivatives of all the functions in inequality 1 have the same derivative at $y = y_0$ by construction.

1. $P'''_{ub}(y) < P'''_{q}(y) \equiv 0$ for any $y < y_0$ together with $P''_{ub}(y_0) = P''_{q}(y_0)$ implies that $P''_{ub}(y) > P''_{q}(y)$ for any $y < y_0$.
2. In turn, $P''_{ub}(y) > P''_{q}(y)$ for any $y < y_0$ together with $P'_{ub}(y_0) = P'_{q}(y_0)$ implies that $P'_{ub}(y) < P'_{q}(y)$ for any $y < y_0$.
3. Finally, $P'_{ub}(y) < P'_{q}(y)$ for any $y < y_0$ together with $P_{ub}(y_0) = P_{q}(y_0)$ implies that $P_{ub}(y) > P_{q}(y)$ for any $y < y_0$.

- $P_b(y) < P_{ub}(y)$ follows from the fact that the bullet proxy bond is also a proxy bond for the upper barbell bond.
- $P_{ub}(y) < P(y)$ follows from Theorem 5.6.
- $P(y) < P_{lb}(y)$ follows from Theorem 5.10.

This concludes the proof.

Remark G.1. Note that the comparison of the polynomial approximations (linear and quadratic) against the proxy bond approximations are preserved even if the polynomial approximations are in (Y, P) domain. The proofs of those inequalities remain the same except that they rely on Y as the argument for the approximating functions involved.

G.2 Proof of Theorem 5.1

The after-shock price of the proxy bond is as follows:

$$P_{\text{bullet}}(y) = C_b e^{-yT_b} = \mu_0 e^{-(y-y_0)\mu_{1/0}} = \mu_0 e^{-\Delta y \cdot \mu_{1/0}}.$$

Meanwhile, the after-shock price of the original bond is as follows:

$$P(y) = \int e^{-\Delta y \cdot t} d\text{PV}(t).$$

We need to show that

$$P(y) \geq P_{\text{bullet}}(y).$$

The following chain of inequalities boils down our problem to the Jensen inequality:

$$\int e^{-\Delta y \cdot t} d\text{PV}(t) \geq e^{-\Delta y \cdot \mu_{1/0}} \mu_0.$$

Let $F(t) = e^{-\Delta y \cdot t}$.

$$\int F(t) d\text{PV}(t) \geq F(\mu_{1/0}) \mu_0.$$

$$\frac{\int F(t) d\text{PV}(t)}{\int d\text{PV}(t)} \geq F\left(\frac{\int t d\text{PV}(t)}{\int d\text{PV}(t)}\right).$$

$$\frac{\int F(t)d\mathrm{PV}(t)}{\int d\mathrm{PV}(t)} \geq F\left(\frac{\int td\mathrm{PV}(t)}{\int d\mathrm{PV}(t)}\right).$$
$$\int F(t)\frac{d\mathrm{PV}(t)}{\mu_0} \geq F\left(\int td\frac{\mathrm{PV}(t)}{\mu_0}\right),$$

which is Jensen's inequality.

For $\Delta y \neq 0$, the function $F(t) = e^{-\Delta yt}$ is strictly convex. Therefore, by the properties of Jensen's inequality, the equality holds if and only if the normalized present-value measure is concentrated at a single date, which corresponds to a zero-coupon bond. If $\Delta y = 0$, the function is constant and the equality holds trivially. This concludes the proof.

G.3 Proof of Proposition 5.2

Let $\int f(t)\, d\mathrm{PV}$ be a Lebesgue integral with respect to the measure associated with the present value of payments promised by a bond. Let T be the remaining time to maturity of the original bond.

All-in-all, we have six inequalities in the proposition.

The second inequality can be proven as follows. For any of the payments promised by the original bond, $t \leq T$. Therefore, we have the following.

$$\mu_2 = \int t^2 d\mathrm{PV} \leq \int tTd\mathrm{PV} = T\mu_1.$$

Thus, we obtain that

$$T_{ub} = \mu_2/\mu_1 \leq T.$$

The first and the third inequalities follow from the fact that $\mu_1^2 \leq \mu_2\mu_0$, which can be shown using Cauchy-Bunyakovsky-Schwartz inequality. Note that $\mu_i = \int t^i d\mathrm{PV}$. So, we need to show that

$$\left(\int 1 \cdot t\, d\mathrm{PV}\right)^2 \leq \int t \cdot t\, d\mathrm{PV} \cdot \int 1 \cdot 1\, d\mathrm{PV}.$$

This can be rewritten as follows

$$\langle 1, t\rangle^2 \leq \langle t, t\rangle \cdot \langle 1, 1\rangle, \tag{58}$$

where is an inner product defined as follows:

$$\langle a, b\rangle := \int a \cdot b\, d\mathrm{PV}.$$

Noting that Eq. 58 is the Cauchy-Bunyakovsky-Schwartz inequality concludes the proof of the first and third inequalities.

The fourth, fifth, and sixth inequalities are obvious. This concludes the proof.

G.4 Proof of Lemma 5.3

The first stage is to show that the third derivatives of the original and proxy bonds are equal to the negatives of the third moments of their respective cash flow streams. To show this, we start with the price of the bond (regardless of which bond we consider: the original or the proxy one):

$$P_{\text{Bond X}}(y) = \int e^{-t(y-y_0)} d\text{PV}_{\text{Bond X}}.$$

Thus, the third derivative is as follows:

$$P'''_{\text{Bond X}}(y) = \int (-t)^3 e^{-t(y-y_0)} d\text{PV}_{\text{Bond X}}.$$

Next, we evaluate the third derivative at $y = y_0$:

$$P'''_{\text{Bond X}}(y_0) = -\int t^3 d\text{PV}_{\text{Bond X}}.$$

So, the third derivatives are indeed equal to the negatives of the third moments. It remains to compare the third moments of the original and the proxy bonds. The third moment of the proxy bond is as follows:

$$\mu_3^{ub} = cT^3 = cT^2 \cdot T = \mu_2 T = \mu_2^2/\mu_1 \leq \mu_3.$$

The last inequality follow from $\mu_2^2 \leq \mu_1\mu_3$, which is a partial case of Cauchy-Bunyakovsky-Schwartz inequality:

$$\mu_2^2 = \left\langle t^{0.5}, t^{1.5} \right\rangle^2 \leq \left\langle t^{0.5}, t^{0.5} \right\rangle \cdot \left\langle t^{1.5}, t^{1.5} \right\rangle = \mu_1\mu_3,$$

in which the equality attained if and only if the vector $(t_1^{0.5}, ..., t_I^{0.5})$ is collinear with $(t_1^{1.5}, ..., t_I^{1.5})$—that is, when there exist a real number k such that

$$(t_1^{1.5}, ..., t_I^{1.5}) = k \cdot (t_1^{0.5}, ..., t_I^{0.5}).$$

In turn, this means that for all non-zero dates $t_i = k$. In essence, this means that all non-zero dates of payments must coincide.

G.5 Proof of Theorem 5.4

For the non-trivial case, it suffices to use the third-order Taylor expansion of $P_{ub}(y) - P(y)$ around $y = y_0$. The first three terms of the Tailor expansion—the intercept, the linear,

and the quadratic term—are null. The forth term—that is, the cubic term—has positive coefficient, as we know from Lemma 5.3. Thus, the third-order Taylor expansion is positive for $y > y_0$ and is negative for $y < y_0$. Finally, in some neighborhood, the error of the Taylor expansion has to be negligible compared to the third-order Taylor expansion. Thus, the signs must coincide.

G.6 Proof of Lemma 5.5 (Generalized Descartes' rule of signs)

Our argument proceeds in three stages.

The first stage is to show that $V(f) - Z(f) \equiv 0 \,(\text{mod } 2)$. Obviously,

$$\text{sign}(a_1 a_2) \cdot ... \cdot \text{sign}(a_{N-1} a_N) = \text{sign}(a_1 a_2 \cdot a_2 a_3 \cdot ... \cdot a_{N-1} a_N) = \text{sign}(a_1 a_N).$$

So, $V(f) \equiv 0 \,(\text{mod } 2)$ when $a_1 a_N > 0$, and $V(f) \equiv 1 \,(\text{mod } 2)$ when $a_1 a_N < 0$.

Next, we notice that a polynomial with $a_1 a_N > 0$ has an even number of zeros (counting their order), and a polynomial with $a_1 a_N > 0$ has an odd number of zeros (counting their order). To show that, we normalize the polynomial by dividing it by $a_1 x^{\alpha_1}$. The higher order coefficient, a_N/a_1 has the same sign as $a_1 a_N$. Note that $f(x) \to 1$ for $x \to +0$ because the intercept term dominates, and $f(x) \to +\infty \times a_n/a_1$ for $x \to +\infty$ because the senior term—$\frac{a_n}{a_1} x^{\alpha_N - \alpha_1}$—dominates. So, function f changes sign even (odd) number of times between $x = 0$ and $x = +\infty$ if $a_1 a_n > 0$ ($a_1 a_n < 0$). Each sign change of function f corresponds to an odd-order zero. If the function's plot touches the horizontal axis without changing the sign, this corresponds to an even-order zero.

All-in-all, if $a_1 a_N > 0$, we have an even number of odd-order zeros and any number of even-order zeros. Counting the zeros with their orders gives an even number. If $a_1 a_N < 0$, we have an odd number of odd-order zeros and any number of even-order zeros. Counting the zeros with their orders gives an odd number.[56] This concludes the first stage.

The second stage is to analyze how the following two operations with polynomials affect the number of sign changes and the number of zeros:

- Normalizing the junior term to 1: $f \mapsto n(f)$.
- Taking the derivative of a normalized polynomial: $n(f) \mapsto (n(f))'$.

[56] Infinite order is impossible for because f is an analytical function—that is, its Taylor expansion should coincide with it within a neighborhood—which would imply that the function be identically equal to zero. Infinite number of zeros is also impossible too. If there were infinitely many zeros, one could have found a sequence $x_1, x_2, ...$ such that $\lim x_i = x_\infty$ be finite (all "zeros" are separated from 0 and $+\infty$ because for small/large values of x, $f(x)$ is dominated by its junior/senior term, which guarantees the existence of x_∞ distinct from 0 and $+\infty$). By continuity, x_∞ would be also a zero of f. The Taylor expansion of f around $x = x_\infty$ must be identically equal to zero because in any small neighborhood of x_∞ there are zeros of f. For a non-zero Taylor expansion, this would have been impossible because first non-zero term would have dominated all the senior terms in some neighborhood of x_∞.

Obviously, the first operation results in no-change in Z and V:

$$Z(n(f)) = Z(f),$$

$$V(n(f)) = V(f).$$

The result of the second operation is as follows:

$$Z((n(f))') \geq Z(n(f)) - 1$$

This follows from Rolle's theorem: between any two positive zeros of function f there is a zero of its derivative. First order zeros disappear, while the zeros of higher order lower their order by one. So, if the $n(f)$ has zeros $x_1, ..., x_M$ with orders $\omega_1, ..., \omega_M$. Then $Z(n(f)) = \sum_{i=1}^{N} \omega_i$. The minimum number of zeros of the derivative counting their order is as follows:

$$\underbrace{\sum_{i=1}^{M}(\omega_i - 1)}_{\text{Zeros of } (n(f))' \text{ coinciding with those of } n(f)} + \underbrace{M - 1}_{\text{Zeros of } (n(f))' \text{ between those of } n(f) \text{ (at least)}} = Z\left(n(f)\right) - 1.$$

So, we have that

$$Z\left(n(f)'\right) \geq Z\left(n(f)\right) - 1.$$

The result regarding $V(n(f)')$ depends on the sign of the second coefficient of $n(f)$, namely, a_2/a_1.

If $a_1 a_2 > 0$, then $V(n(f)') = V(n(f))$, otherwise $V(n(f)') = V(n(f)) - 1$.

In any case, we have that

$$V(n(f)) - 1 \leq V(n(f)') \leq V(n(f)).$$

This third stage is to show that $Z(f) \leq V(f)$ by induction. The induction is by the number of terms in the polynomial, N. For $N = 1$ and $N = 2$, the statement is obvious. Now let us proceed to the induction step. So, suppose the statement is true for the polynomials with less than N terms and prove that it also holds for N terms.

Let us prove the induction step by contradiction. Thus, let us assume that the number of terms in f equals N and that $Z(f) > V(f)$, and bring it to an inconsistency. From the first stage of the proof, we know that $Z(f)$ and $V(f)$ are of the same parity, so in effect, our assumption allows us to conclude that $Z(f) \geq V(f) + 2$—which is the same as $Z(f) - V(f) \geq 2$.

From the second stage of the proof, we have that

$$Z(n(f)') \geq Z(f) - 1. \tag{59}$$

On the other hand, we have that

$$V(n(f)') \leq V(f). \tag{60}$$

Put simply, when we apply the normalization and then the first derivative, V does not increase, while Z can drop by 1 at most. Therefore, the difference $Z - V$ cannot change by more then 1:

$$Z(n(f)') - V(n(f)') \geq Z(f) - V(f) - 1. \tag{61}$$

But we know that $Z-V$ applied to f is at least 2. Thus, we have $Z(n(f)')-V(n(f)') \geq 1$, which contradicts that assumption of the induction step and concludes the proof.

G.7 Proof of Lemma 5.8

We know from the proof of Lemma 5.3 that the third derivative of a bond price equals to the negative of the third moment of the present values of its cash flows.

Hence, it is enough to show the inequality for the third moments $\mu_3^{lb} \geq \mu_3$. Moreover, the inequality for the third moments around $t = 0$ is equivalent to that for the third moments around $t = T$.

Let us now introduce third moments around $t = T$:

$$\breve{\mu}_3 := \int (t-T)^3 d\mathrm{PV}_{\text{original}}$$

and

$$\breve{\mu}_3^{lb} := \int (t-T)^3 d\mathrm{PV}_{\text{lower barbell}}.$$

It is easy to see that

$$\breve{\mu}_3 = \mu_3 - 3T\mu_2 + 3T^2\mu_1 - T^3\mu_0$$

and

$$\breve{\mu}_3^{lb} = \mu_3^{lb} - 3T\mu_2^{lb} + 3T^2\mu_1^{lb} - T^3\mu_0^{lb}.$$

Since, the 0-th, the first and the second moments match,[57] it is enough to show that $\breve{\mu}_3^* \geq \breve{\mu}_3$:

$$\breve{\mu}_3^* = c_{lb}(t_{lb} - T)^3 = c_{lb}(t_{lb} - T)^2 \cdot (t_{lb} - T) = \breve{\mu}_2 \cdot (t_{lb} - T) = \breve{\mu}_2^2/\breve{\mu}_1 \geq \breve{\mu}_3,$$

where the last inequality follows from $\breve{\mu}_2^2 \leq \breve{\mu}_1\breve{\mu}_3$, which is a partial case of Cauchy-Bunyakovsky-Schwartz inequality:

$$\breve{\mu}_2^2 = \left\langle (T-t)^{0.5}, (T-t)^{1.5} \right\rangle^2 \leq \left\langle (T-t)^{0.5}, (T-t)^{0.5} \right\rangle \cdot \left\langle (T-t)^{1.5}, (T-t)^{1.5} \right\rangle = \breve{\mu}_1\breve{\mu}_3.$$

[57] Mathematically, $\mu_0 = \mu_0^{lb}$, $\mu_1 = \mu_1^{lb}$, and $\mu_2 = \mu_2^*$.

Note that the equality is attained if and only if the vector $((T-t_1)^{0.5},...,(T-t_I)^{0.5})$ is collinear with $((T-t_1)^{1.5},...,(T-t_I)^{1.5})$—that is, when there exist a real number k such that

$$((T-t_1)^{1.5},...,(T-t_I)^{1.5}) = k\cdot((T-t_1)^{0.5},...,(T-t_I)^{0.5}).$$

In turn, this means that the third derivatives match if and only if for all payment dates of the original bond different from T, we have $t_i = T-k$ for the same value k. So, the third derivatives match if and only if the original bond has no more than one payment before maturity (in which case the lower barbell approximation becomes tautological).

G.8 Proof of Proposition 6.1

$$\{\hat{P}^x(\Delta y)\}_{x\in\mathbf{X}} \subseteq \left[\min(\hat{P}^X_{ub}(\Delta y),\hat{P}^X_{lb}(\Delta y)),\max(\hat{P}^X_{ub}(\Delta y),\hat{P}^X_{lb}(\Delta y))\right]$$

follows from the fact that the upper and the lower barbell approximations constitute bounds for the true bond price (Theorems 5.6 and 5.10.)

On the other hand, the two barbell bonds themselves lie in the bond class $\mathbf{X}$, so the ends of the interval

$$\left[\min(\hat{P}^X_{ub}(\Delta y),\hat{P}^X_{lb}(\Delta y)),\max(\hat{P}^X_{ub}(\Delta y),\hat{P}^X_{lb}(\Delta y))\right]$$

belong to $\{\hat{P}^x(\Delta y)\}_{x\in\mathbf{X}}$.[58] Furthermore, all convex combinations of the two barbells (long only portfolios composed of the two barbells) are also in the X. Thus, all the points between $\hat{P}^X_{ub}(\Delta y)$ and $\hat{P}^X_{lb}(\Delta y)$ are also in $\{\hat{P}^x(\Delta y)\}_{x\in\mathbf{X}}$, which concludes the proof.

G.9 Proof of Corollary 6.1.1

Lemma G.1. One can always accomplish a non-trivial—that is, having two positive cash flows—upper barbell bond with an extra payment at an arbitrary date T after T_{ub} so that the price, duration and convexity do not change. Moreover, for a arbitrary shock Δy and arbitrary discrepancy margin, δ, it is possible to make sure that this bond's price is close to the price of the upper barbell bond:

$$\left|\hat{P}^*_{ub}(\Delta y) - \hat{P}_{ub}(\Delta y)\right| < \delta.$$

Proof. Recall that

$$T_{ub} = \mu_2/\mu_1 > 0,$$

$$c_{ub} = \mu_1^2/\mu_2 > 0,$$

[58] Note that the maturity of the upper barbell bond is always below that of that of bond X, and thus below $\overline{T}_X$. The maturity of the lower barbell bond is $\overline{T}_X$ by construction.

and

$$c_0 = \mu_0 - \mu_1^2/\mu_2 > 0.$$

Suppose, we wish to construct a modified bond with payments c'_0, c'_{ub}, and ϵ at dates 0, T'_{ub}, and T, respectively. Then, for the modified bond the moment matching conditions are as follows:

$$\mu'_0 := \mu_0 - \epsilon = c'_0 + c'_{ub},$$

$$\mu'_1 := \mu_1 - \epsilon T = c'_{ub} T'_{ub},$$

and

$$\mu'_2 := \mu_2 - \epsilon T^2 = c'_{ub} \left(T'_{ub}\right)^2.$$

Solving for unknown parameters T'_{ub}, c'_{ub}, and c'_0, we obtain the following:

$$T'_{ub} = \mu'_2/\mu'_1,$$

$$c'_{ub} = (\mu'_1)^2/\mu'_2,$$

and

$$c'_0 = \mu'_0 - (\mu'_1)^2/\mu'_2.$$

Note that for $\epsilon \to 0$, $T'_{ub} \to T_{ub}$, we have that $c'_{ub} \to c_{ub}$, and $c'_0 \to c_0$. Thus, there is a positive ϵ so that both payments, c'_{ub} and c'_0 are strictly positive, and the date T'_{ub} is strictly between 0 and T.

The newly constructed bond has the time to maturity T, and the moments (0-th, 1-st, and 2-nd) identical to that of the original bond. Moreover $\epsilon \to 0$ implies

$$\hat{P}^*_{ub}(\Delta y) \to \hat{P}_{ub}(\Delta y).$$

So, for small enough $\epsilon > 0$, the discrepancy is below δ, which concludes the proof. □

The case in which the upper barbel bond is a zero-coupon bond is trivial as in this case the two approximations coincide with the true bond. Clearly, one cannot improve two bonds that coincide.

In the rest of the proof, we are dealing with the non-trivial case. By construction, the maturity of the original bond, T (known by the *Risk manager*) is longer than that of the upper barbell bond, as follows from Proposition 5.2.

Suppose $\mathbf{X}$ is the set of bonds satisfying the three moment conditions and the maturity restriction. Then the set $\{\hat{P}^x(\Delta y)\}_{x\in\mathbf{X}}$ still contains the lower barbell approximation (in this case based on the known maturity rather than a maturity cap) because the *Risk manager* cannot rule out the possibility that she actually deals with a bond/portfolio whose cash flows exactly coincide with the lower barbel bond.

Even though the upper barbell itself does not satisfy the maturity restriction. Never-

theless, thanks to Lemma G.1, we know that $\mathbf{X}$ contains bonds arbitrarily similar to the upper barbell bond (in the sense that their after-shock price is arbitrarily close to that of the upper barbell bond).

So a tighter bound than the upper barbell bound cannot exist by contradiction: if there were a tighter bound (suppose it is tighter by ϵ), then this bound would be violated by a bond similar to the upper barbell bond but with a small payment at the required maturity. By Lemma G.1, we can always synthesize a bond less than $\epsilon/2$ away from the upper barbell bond.

G.10 Proof of Theorem B.1

The second point follows from the observation that for the proxy-bond measure, the following identities must hold (easy to directly verify):

$$e_1\mu_{i+1}^{\text{Balanced barbell}} = \mu_{i+2}^{\text{Balanced barbell}} + e_2\mu_i^{\text{Balanced barbell}} \text{ for any non-negative integer } i, \quad (62)$$

where $e_1 := t_1 + t_2$ and $e_2 := t_1 t_2$. The moment matching conditions allow to trivially recover e_1 and e_2 from the moments of the original bond (or portfolio). By Vieta's theorem, $t_{1,2}$ are the roots of the quadratic equation of $q(t) := t^2 - e_1 t + e_2$. Note that $q(t)$ is the orthogonal polynomial, and thus must have two real roots strictly between $t = 0$ and $t = T$, as explained in Gautschi (2004).[59,60] This concludes the proof of point 2 of Theorem B.1.

To show that the balanced barbell approximation constitutes the lower bound for the true bond price (portfolio value), we again rely on our generalized version of the Descartes' rule of signs (Lemma 5.5). We know that the discrepancy function, $\hat{P}^{\text{Balanced barbell}}(\Delta y) - \hat{P}(\Delta y)$, is null for $\Delta y = 0$ with three first derivatives being null as well. This has a root of order four. Therefore, by Lemma 5.5, the net cash flow stream from the long-short portfolio composed of one unit of the proxy bond and minus one unit of the original bond has to switch the direction at least four times (except for the degenerate case of a two-cash-flow original bond leading to an identical proxy bond). This means that both cash flows of the proxy bond are positive, which concludes the proof of point 2 of Theorem B.1.

Finally, given that the proxy bond has only two payments, the net cash flow stream from the long-short portfolio cannot switch the direction more than four times. Thus, the quadruple root of the discrepancy function exhausts the maximum of number of possible

[59] By definition, an orthogonal polynomial of degree two is a polynomial orthogonal to 1 and t. It is easy to see that the orthogonality conditions hold (from equation 62 and the moment matching conditions, $\mu_i^{\text{Balanced barbell}} = \mu_i$ for $i = 0, 1, 2, 4$): $\int 1 \cdot q(t) d\text{PV}_{\text{True}} \equiv \mu_2 - e_1\mu_1 + e_2\mu_0 = 0$, $\int t \cdot q(t) d\text{PV}_{\text{True}} \equiv \mu_3 - e_1\mu_2 + e_2\mu_1 = 0$.

[60] By contradiction, an orthogonal polynomial of degree two cannot have zero roots or one root between 0 and T. If it had zero roots (or one double root), then it would not be orthogonal to 1. Similarly, if it had only one root t^* in the interval $(0, T)$, then it could not be orthogonal to $t = t^*$. Thus, the polynomial has at least two distinct roots. Since it is of degree two, it has exactly two roots.

roots and is unique. For $\Delta y \to +\infty$, the original bond dominates the proxy bond as its first cash flow (senior term in the exponential polynomial) arrives earlier. In the absence of non-trivial roots, this implies that the original bond dominates everywhere except for $\Delta y = 0$. This concludes the proof.

G.11 Proof of Theorem B.2

The moment matching conditions can be written as follows:

$$\begin{pmatrix} 1 & 1 & 1 & \mu_0 \\ 0 & \tau & T & \mu_1 \\ 0 & \tau^2 & T^2 & \mu_2 \\ 0 & \tau^3 & T^3 & \mu_3 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_\tau \\ \gamma_T \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{63}$$

Thus, τ has to be chosen so as to render the 4×4 matrix singular. Thus, the matrix determinant has to be null.

$$\begin{vmatrix} \tau & T & \mu_1 \\ \tau^2 & T^2 & \mu_2 \\ \tau^3 & T^3 & \mu_3 \end{vmatrix} = \tau T \begin{vmatrix} 1 & 1 & \mu_1 \\ \tau & T & \mu_2 \\ \tau^2 & T^2 & \mu_3 \end{vmatrix} = \tau T \begin{vmatrix} 0 & 1 & \mu_1 \\ \tau - T & T & \mu_2 \\ \tau^2 - T^2 & T^2 & \mu_3 \end{vmatrix} =$$

$$\tau T(\tau - T) \begin{vmatrix} 0 & 1 & \mu_1 \\ 1 & T & \mu_2 \\ \tau + T & T^2 & \mu_3 \end{vmatrix} = \tau T(T - \tau) \begin{vmatrix} 1 & 0 & \mu_1 \\ T & 1 & \mu_2 \\ T^2 & \tau + T & \mu_3 \end{vmatrix} =$$

$$\tau T(T - \tau) \begin{vmatrix} 1 & 0 & 0 \\ T & 1 & \mu_2 - \mu_1 T \\ T^2 & \tau + T & \mu_3 - \mu_1 T^2 \end{vmatrix} = \tau T(T - \tau) \begin{vmatrix} 1 & \mu_2 - \mu_1 T \\ \tau + T & \mu_3 - \mu_1 T^2 \end{vmatrix} =$$

$$\tau T(T - \tau)[\mu_3 - \mu_2 T - (\mu_2 - \mu_1 T)\tau] = 0,$$

which yields the formula for τ. Once τ is known, the rest follows trivially: we find (γ_τ, γ_T) from moment matching conditions for the first and the second moments. Finally, we find γ_0 from the zeroth moment matching condition.

The positivity of all three payments and the fact that the triple barbell approximation constitutes the upper bound, follow from Lemma 5.5. Indeed, the discrepancy function has a root of the fourth degree, which implies that the long-short portfolio's cash flow stream has to switch back at least four times (unless the proxy bond precisely coincides with the true bond/portfolio). This only can happen if τ is in $(0, T)$ and when all payments of the proxy bond are positive.

Finally, given that the first and the last payments of the proxy bond are before and after all the payments of the original bond, the cash flow stream cannot switch back more than four times. This implies that the quadruple root at $\Delta y = 0$ of discrepancy function

exhausts the cap for the number of roots. Hence, the discrepancy turns null only for $\Delta y = 0$ and has no other roots.

G.12 Proof of Theorem B.3

The structure of the proof corresponds to the structure of the statement of the theorem: it is subdivided into the same cases and subcases.

1. Even N.

   (a) By construction the moments $\mu_0^{m_t}, ..., \mu_{N-1}^{m_t}$ are equal to $\mu_1, ..., \mu_N$, respectively. So the quadrature without the supplementary node matches all the required moments of the original bond except for μ_0, perhaps. Clearly, adding the extra node allows to correct this potential mismatch without interfering with moments $\mu_1, ..., \mu_N$.

   The inequality follows from the fact that the proxy bond matches $N + 1$ derivatives of the original bond price—from the 0-th derivative to the N-th derivative. Thus, the discrepancy function has a root of degree $N + 1$ at $\Delta y = 0$. The proxy bond has $N/2 + 1$ distinct payments (at this point, we are not claiming that they are positive), the first one being at $t = 0$. Thus, the cash flow stream associated with the long-short portfolio flips the direction $N + 1$ times at most. Hence, from the generalized Descartes' rule of signs (Lemma 5.5), the discrepancy function cannot have other roots in addition to the root $\Delta y = 0$ of order $N + 1$. For $\Delta y \to +\infty$, the proxy bond dominates because its first payment precedes all the payments of the true bond. Hence, for large enough positive values of Δy, the upper proxy bond will be upper bound for the true bond/portfolio. Given the absence of positive roots of the discrepancy function, the upper proxy bond produced an upper bound of for the true bond/portfolio for any positive Δy. Given that the root at $\Delta y = 0$ has an odd order $(N + 1)$, the discrepancy changes sign at this point, and constitutes the lower bound for negative values of Δy.

   (b) Analogous to the previous point. The first step allows us to match all the required shifted moment starting from the first one. The second step (adding a node at $t = T$) does not change the shifted moments of positive order while allowing us to match the zeroth moment. Hence, the procedure provided in the theorem ensures matching all the shifted moments (about $t = T$) of the original bond/portfolio of orders from 0 to N. This matching is equivalent to matching the standard (non-shifted) moments.

2. Odd N.

(a) The first stage (building the quadrature) allows to match the moments $\mu_i^{m_{(T-t)t}} \equiv T\mu_{i+1} - \mu_{i+2}$ for $i = 0, 1, ..., N-2$. The second stage (adding the two corner nodes) allows to match μ_1 and $-\breve{\mu}_1 \equiv \mu_0 T - \mu_1$. Thus, the procedure provided in the theorem ensures matching the following vectors of moments:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ T & -1 & 0 & \cdots & 0 & 0 \\ 0 & T & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T & -1 \end{pmatrix} \begin{pmatrix} \mu_0^{\text{proxy}} \\ \mu_1^{\text{proxy}} \\ \mu_2^{\text{proxy}} \\ \vdots \\ \mu_N^{\text{proxy}} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ T & -1 & 0 & \cdots & 0 & 0 \\ 0 & T & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & T & -1 \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix}$$

It is easy to see that the matrix pre-multiplying the vectors of standard moments is invertible. Thus, the standard moments are also matched.

The inequality follows from the generalized Descartes' rule of signs. Since we've matched $N+1$ moments (and thus $N+1$ derivatives), the discrepancy function has a root of order $N+1$ at $\Delta y = 0$. The proxy bond has $(N-1)/2$ internal nodes and 2 corner nodes. Thus the long-short portfolio's cash flow stream switches signs at most $N+1$ times. This implies that the discrepancy function, does not have roots different from $\Delta y = 0$. At the same time, for large enough positive Δy, the proxy bond dominates since its first cash flow precedes that of the original bond/portfolio. This implies that the upper proxy bond is an upper bound for any $\Delta y \neq 0$ (recall that the root $\Delta 0$ has an even order).

(b) The moment matching is trivial in the absence of corner nodes. The proof of inequality relies on the generalizes Descartes rule of signs: the moment matching implies matching of $N+1$ derivatives. Thus, the long-short portfolio's cash flow stream switches signs $N+1$ times, which means that the first payment of the true bond precedes that of the proxy bond. Thus, the original bond dominates for large enough positive Δy. The Descartes' rule also implies the absence of roots of the discrepancy function different from $\Delta y = 0$. Given that the root at $\Delta y = 0$ has an even order, the inequality is extrapolated to all $\Delta y \neq 0$.

G.13 Proof of Proposition C.2

The first statement is shown as follows:

$$\underbrace{\mu_0 e^{-\mu_{1/0}\cdot\left(e^{\Delta y}-1\right)}}_{\text{Exponential duration method}} < \underbrace{\mu_0 e^{-\mu_{1/0}\cdot\Delta y}}_{\text{Bullet method}}$$

because $e^{\Delta y} - 1 > \Delta y$ (for $\Delta y \neq 0$).

The second statement follows automatically from Theorem 5.1 and the fact that the exponential duration approximation is below the bullet approximation.

The third statement is proven as follows:

$$\underbrace{\mu_0 \cdot (1 + \Delta Y)^{-D}}_{\text{Discrete duration method}} > \underbrace{\mu_0 \cdot \left(\frac{1 + Y_0 + \Delta Y}{1 + Y_0} \right)^{-\mu_{1/0}}}_{\text{Bullet method}}$$

because

$$(1 + \Delta Y) < \left(\frac{1 + Y_0 + \Delta Y}{1 + Y_0} \right)^{1+Y_0},$$

With notation $\delta = \frac{\Delta Y}{1+Y_0}$ and $k = 1 + Y_0$, the last statement is recast. as

$$1 + k\delta < (1 + \delta)^k,$$

which in turn follows form the following three statements:

1. For $\delta = 0$, the left-hand side coincides with the right-hand side.

2. For positive δ, the derivative of the left-hand side is below that of the right-hand side:
$$k < k(1 + \delta)^{Y_0}.$$

3. For negative δ, the derivative of the left-hand side is above that of the right-hand side:
$$k > k(1 + \delta)^{Y_0}.$$

Finally, the fourth statement follows from the two facts:

- The bullet approximation of a zero-coupon bond price coincides with the true bond price.

- The discrete duration approximation exceeds the bullet approximation (the third statement of the proposition).

H Yield curve changes: level and slope factors

The approximations based on proxy bonds are built to capture the dependency between yield to maturity and price without knowing the bond coupon structure. Nevertheless, use cases of the proxy bonds go beyond the approximation of the univariate dependency: the proxy bonds also can be used as a substitute for the original bond when we discount cash flows using a yield curve.

In this section, we analyze a special case: the change of the level or the slope of the yield curve and show theoretically that, in some sense, the proxy bonds designed in Sections 5.2 and 5.3—that is, the upper barbell and the lower barbell—are valid substitutes for such term structure transformations.

We have two bonds:

1. the original bond and
2. a proxy bond.

The following moment-matching conditions are satisfied:

$$\int d\mathrm{PV}_{\mathrm{orig}} = \int d\mathrm{PV}_{\mathrm{proxy}},$$

$$\int t d\mathrm{PV}_{\mathrm{orig}} = \int t d\mathrm{PV}_{\mathrm{proxy}},$$

and

$$\int t^2 d\mathrm{PV}_{\mathrm{orig}} = \int t^2 d\mathrm{PV}_{\mathrm{proxy}},$$

where the present values of cash flows are computed using the current yield to maturity of the original bond, y_0.

This translates into approximate moment-matching conditions[61] for the present values as computed using the yield curves.

$$\int d\mathrm{PV}^*_{\mathrm{orig}} \approx \int d\mathrm{PV}^*_{\mathrm{proxy}},$$

$$\int t d\mathrm{PV}^*_{\mathrm{orig}} \approx \int t d\mathrm{PV}^*_{\mathrm{proxy}},$$

and

$$\int t^2 d\mathrm{PV}^*_{\mathrm{orig}} \approx \int t^2 d\mathrm{PV}^*_{\mathrm{proxy}},$$

[61] We do not control for discrepancies between the YTM-implied moments and the yield-curve-implied moments. The calculations presented in this section are intended to provide additional mathematical motivation for the moment-matching approach, rather than to constitute a rigorous proof.

which can be rewritten as

$$\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \frac{C_i^{\text{orig}}}{\exp\left(\text{yc}_0(t_i^{\text{orig}}) \cdot t_i^{\text{orig}}\right)} \approx \sum_{i=1}^{N_{\text{cash flows}}^{\text{proxy}}} \frac{C_i^{\text{proxy}}}{\exp\left(\text{yc}_0(t_i^{\text{proxy}}) \cdot t_i^{\text{proxy}}\right)},$$

$$\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \frac{t_i^{\text{orig}} C_i^{\text{orig}}}{\exp\left(\text{yc}_0(t_i^{\text{orig}}) \cdot t_i^{\text{orig}}\right)} \approx \sum_{i=1}^{N_{\text{cash flows}}^{\text{proxy}}} \frac{t_i^{\text{orig}} C_i^{\text{proxy}}}{\exp\left(\text{yc}_0(t_i^{\text{proxy}}) \cdot t_i^{\text{proxy}}\right)},$$

and

$$\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \frac{\left(t_i^{\text{orig}}\right)^2 C_i^{\text{orig}}}{\exp\left(\text{yc}_0(t_i^{\text{orig}}) \cdot t_i^{\text{orig}}\right)} \approx \sum_{i=1}^{N_{\text{cash flows}}^{\text{proxy}}} \frac{\left(t_i^{\text{orig}}\right)^2 C_i^{\text{proxy}}}{\exp\left(\text{yc}_0(t_i^{\text{proxy}}) \cdot t_i^{\text{proxy}}\right)}.$$

Also, this can be rewritten in terms of yield-curve discounted present values.

$$\mu_0 = \mu_0^{\text{orig, yc}} := \sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}} \approx \sum_{i=1}^{N_{\text{cash flows}}^{\text{proxy}}} \text{PVYC}_i^{\text{proxy}} =: \mu_0^{\text{proxy, yc}},$$

$$\mu_1^{\text{orig, yc}} := \sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} t_i^{\text{orig}} \text{PVYC}_i^{\text{orig}} \approx \sum_{i=1}^{N_{\text{cash flows}}^{\text{proxy}}} t_i^{\text{proxy}} \text{PVYC}_i^{\text{proxy}} =: \mu_1^{\text{proxy, yc}},$$

and

$$\mu_2^{\text{orig, yc}} := \sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \left(t_i^{\text{orig}}\right)^2 \text{PVYC}_i^{\text{orig}} \approx \sum_{i=1}^{N_{\text{cash flows}}^{\text{proxy}}} (t_i^{\text{proxy}})^2 \text{PVYC}_i^{\text{proxy}} =: \mu_2^{\text{proxy, yc}},$$

where the quantities "PVYC" designate the present values of the cash flows of the two bonds discounted to time 0 using the current state of the yield curve, $\text{yc}_0(\cdot)$.

After the shock to the yield curve, the respective prices of the original bond and the proxy bond are as follows:

$$\hat{P}_{\text{orig}}(\Delta\text{yc}(\cdot)) = \sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}} \exp\left(-\Delta\text{yc}(t_i^{\text{orig}}) \cdot t_i^{\text{orig}}\right)$$

$$\hat{P}_{\text{proxy}}(\Delta\text{yc}(\cdot)) = \sum_{i=1}^{N_{\text{cash flows}}^{\text{proxy}}} \text{PVYC}_i^{\text{proxy}} \exp\left(-\Delta\text{yc}(t_i^{\text{proxy}}) \cdot t_i^{\text{proxy}}\right),$$

Suppose that the shock to the yield curve is small and has an affine structure:

$$\Delta\text{yc}(t) = \Delta r + \Delta s \times t,$$

where Δr is a small shock to the short rate, $\text{yc}(0)$—also known as the overnight rate, and Δs is a small shock to the slope of the yield curve.

Then, the relative changes to each of the two bonds—the original bond and the proxy bond—can be approximately computed as follows:

$$\frac{\hat{P}_{\text{orig}}(\Delta\text{yc}(\cdot)) - \hat{P}_{\text{orig}}(0)}{P_{\text{orig}}(0)} = \frac{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}} \left[\exp\left(-\Delta\text{yc}(t_i^{\text{orig}}) \cdot t_i^{\text{orig}}\right) - 1\right]}{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}}} \approx$$

$$\approx -\frac{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}} \Delta\text{yc}(t_i^{\text{orig}}) \cdot t_i^{\text{orig}}}{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}}} \approx$$

$$\approx -\frac{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}} (\Delta r + \Delta s \times t_i^{\text{orig}}) \cdot t_i^{\text{orig}}}{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}}} =$$

$$= -\Delta r \frac{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}} t_i^{\text{orig}}}{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}}} - \Delta s \frac{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}} \left(t_i^{\text{orig}}\right)^2}{\sum_{i=1}^{N_{\text{cash flows}}^{\text{orig}}} \text{PVYC}_i^{\text{orig}}} = -\mu_{1/0}^{\text{orig, yc}} \Delta r - \mu_{2/0}^{\text{orig, yc}} \Delta s.$$

Similarly, we obtain an analogous result for the relative change of the proxy bond price:

$$\frac{\hat{P}_{\text{proxy}}(\Delta\text{yc}(\cdot)) - \hat{P}_{\text{proxy}}(0)}{P_{\text{proxy}}(0)} = -\mu_{1/0}^{\text{proxy, yc}} \Delta r - \mu_{2/0}^{\text{proxy, yc}} \Delta s.$$

Finally, since the duration and convexity of the two bonds match, their duration and convexity computed using the current state of the yield curve match approximately:

$$\mu_{1/0}^{\text{orig, yc}} \approx \mu_{1/0}^{\text{proxy, yc}}$$

and

$$\mu_{2/0}^{\text{orig, yc}} \approx \mu_{2/0}^{\text{proxy, yc}}.$$

Therefore, the two bonds react approximately similarly to small affine changes of the yield curve.

I The simulated bond portfolio

The sample bond portfolio is generated using the R programming language, as shown in the following code.

- We generate bond parameters at random and create a portfolio of bonds based on those parameters.
- Each bond has a face value of one dollar.
- All parameters are generated independently according to the following probability distributions:
  - remaining maturities: uniformly from 0 to 30 years;[62]
  - coupon rates: uniformly from 0 to 15%;
  - coupon frequencies: 1, 2, 4, and 12 coupons per year with equal probabilities.
- The number of lots in each security is generated as a real number uniformly distributed between 0 and 10.

```
set.seed(1)
#generating a list of bonds
n.bonds <- 50
bond.params <- data.frame(issuance.date=runif(n.bonds,min=-1, max=0),
                          coupon.rate=runif(n.bonds,min = 0,max = 0.15),
                          maturity=sample(x = 1:30, size = n.bonds,
                          replace = TRUE),
                          coupons.p.y=
                          sample(x = c(1, 2, 4, 12),
                          size = n.bonds, replace = TRUE))
quantities <- runif(n.bonds,0,10)
```

Table 1 lists the fifty bonds in our simulated portfolio. For the simulation in Section 6, the vector of the assumed initial dirty prices is also calculated randomly. For each bond, we randomly generate the pre-shock YtM as the coupon rate multiplied by a random factor uniformly distributed between 0.8 and 1.2. The dirty price is then computed as the present value of future cash flows. With this choice of YtM, approximately one half of the bonds in the portfolio are traded at a premium (that is, their clean prices are above par), and the other half are traded at a discount (that is, their clean prices are below par).

[62] We generate bonds with initial maturities equal to integer number of years, and independently generate the time passed since issuance as uniformly distributed quantities ranging from 0 to 1.

For the simulation in Section 7, the initial prices of the bonds are computed given the empirically observed yield curves (as opposed to random YtMs). Thus, the column "Dirty price" in Table 1 is irrelevant for Section 7.

```
prices <- numeric(n.bonds)
for(i in 1:n.bonds) {
    prices[i] <-
        discounted.value(b.list[[paste0("bond_",i)]],
            rate=bond.params$coupon.rate[i]*runif(1,0.8,1.2))
}
```

Issuance date (years)	Coupon rate	Maturity (years)	Coupon frequency	Remaining maturity	Dirty price	# of lots in the portfolio
-0.73	7.16	28	2	27.27	1.18	5.19
-0.63	12.92	12	2	11.37	1.12	5.63
-0.43	6.57	16	2	15.57	1.04	1.29
-0.09	3.67	1	12	0.91	1.00	2.56
-0.80	1.06	13	12	12.20	0.98	7.18
-0.10	1.49	21	1	20.90	1.00	9.61
-0.06	4.74	6	1	5.94	1.02	1.00
-0.34	7.78	17	1	16.66	1.09	7.63
-0.37	9.93	9	2	8.63	0.99	9.48
-0.94	6.10	7	1	6.06	1.03	8.19
-0.79	13.69	23	4	22.21	1.19	3.08
-0.82	4.40	19	2	18.18	1.12	6.50
-0.31	6.89	22	4	21.69	0.95	9.53
-0.62	4.99	18	4	17.38	0.99	9.54
-0.23	9.76	26	2	25.77	1.18	3.40
-0.50	3.87	16	4	15.50	1.09	2.62
-0.28	7.18	11	4	10.72	1.10	1.65
-0.01	11.49	30	12	29.99	1.10	3.22
-0.62	1.26	10	1	9.38	1.02	5.10
-0.22	13.13	7	1	6.78	1.08	9.24
-0.07	5.09	28	12	27.93	1.12	5.11
-0.79	12.59	19	2	18.21	1.15	2.58
-0.35	5.20	29	12	28.65	1.12	0.46
-0.87	5.01	2	2	1.13	1.02	4.18
-0.73	7.15	10	1	9.27	1.00	8.54
-0.61	13.38	1	2	0.39	1.03	3.47
-0.99	12.97	11	4	10.01	1.06	1.31
-0.62	5.85	27	4	26.38	0.92	3.74
-0.13	11.66	26	4	25.87	1.10	6.31
-0.66	14.41	15	4	14.34	1.24	3.90
-0.52	6.52	26	4	25.48	1.15	6.90
-0.40	10.69	29	12	28.60	1.12	6.89
-0.51	6.00	24	1	23.49	1.14	5.55
-0.81	4.88	10	12	9.19	1.06	4.30
-0.17	11.36	16	4	15.83	1.15	4.53
-0.33	3.04	21	1	20.67	1.06	3.06
-0.21	10.67	12	12	11.79	1.15	5.78
-0.89	1.83	7	4	6.11	0.98	9.10
-0.28	3.68	24	12	23.72	1.05	1.43
-0.59	2.15	21	2	20.41	1.00	4.15
-0.18	3.59	28	12	27.82	0.97	2.11
-0.35	0.88	22	1	21.65	1.03	4.29
-0.22	9.63	8	12	7.78	1.11	1.33
-0.45	13.14	1	4	0.55	1.04	4.60
-0.47	11.68	19	1	18.53	0.92	9.43
-0.21	11.96	26	12	25.79	0.99	7.62
-0.98	6.83	3	2	2.02	1.05	9.33
-0.52	6.15	11	1	10.48	1.03	4.71
-0.27	12.16	1	2	0.73	1.04	6.04
-0.31	9.07	29	2	28.69	1.10	4.85

Table 1: Portfolio composition

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