Institutional Asset Pricing, Segmentation, and Household Heterogeneity

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Introduction

- Since 2007-9, extensive analysis of how **frictions within the financial sector**:
  - Explain **asset pricing spreads** (e.g. Koijen-Yogo-19, Vayanos-Vila-21),
  - **Amplify cycles** (e.g. Gertler-Kiyotaki-15, Brunnermeier-Sannikov-14), and
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  . . . *often summarized as a regulatory trade-off between growth and stability.*
- Less analysis of how financial frictions/regulations **affect different households**.
- **Goal**: investigate how the organization of the financial sector impacts inequality

  . . . *and characterize* ***tradeoff between growth, stability, and inequality***.

This Paper

1. Macro-finance model with **heterogeneous households** demanding finance services:
   - ⋆ Asset market participation constraints ⇒ demand for intermediation
   - ⋆ Retire/die at random times ⇒ demand for pension/insurance products.
   - ⋆ Face borrowing/lending constraints ⇒ demand for bank deposits

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3. Counterfactual (today): **re-evaluation of Basel III and IV**
   - ⋆ Stabilizing financial markets typically generates inequality.
   - ⋆ Why? Regulations open up spreads, which wealthy agents are more able to earn.

Literature Review

- Institutional asset pricing
  [Brunnermeier and Sannikov, 2014], [He and Krishnamurthy, 2013], [Koijen and Yogo, 2019], [Koijen and Yogo, 2023], [Vayanos and Vila, 2021], [Payne and Szőke, 2024]
  - *This paper:* introduces household sector.
- Asset pricing and inequality
  [Gomez, 2017], [Gomez and Gouin-Bonenfant, 2024], [Fagereng et al., 2022], [Cioffi, 2021], [Zhang, 2022] [Fernández-Villaverde et al., 2023], [Basak and Chabakauri, 2023], [Fernández-Villaverde and Levintal, 2024]
  - *This paper:* endogenous capital market participation and price volatility.
- Deep learning for macroeconomic models
  [Azinovic et al., 2022], [Han et al., 2021], [Maliar et al., 2021], [Kahou et al., 2021], [Bretscher et al., 2022], [Han et al., 2018], [Huang, 2022], [Duarte et al., 2024], [Gopalakrishna, 2021], [Fernández-Villaverde et al., 2023], [Sauzet, 2021], [Gu et al., 2023], [Barnett et al., 2023], [Payne et al., 2024]
  - *This paper:* non-trivial agent optimization, distribution dynamics, and asset pricing.
- Deep learning and portfolio choice
  [Huang, 2023], [Azinovic and Žemlička, 2023], [Azinovic et al., 2023], [Kubler and Scheidegger, 2018]
  - *This paper:* exploits master equation to enforce market clearing in the neural network.

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Environment: Setting, Production, and Households

- Continuous time $t \in [0, \infty)$. One perishable consumption good, one capital stock.
- Goods production **technology** $y_t = e^{z_t} k_t$, where capital $dk_t = (\phi(\iota_t) - \delta) k_t dt$ and:
  - Aggregate productivity follows: $dz_t = \alpha_z(\bar{z} - z_t)dt + \sigma_z \sqrt{\zeta} dW_{z,t}$, (Ext.: shocks to ζ_t)

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- Continuum of price taking **households** (index by $i \in [0, 1]$)
  - Die at idiosyncratic rate λ_h; replaced by new household with $\underline{a}_h = \varphi_h A$.
  - While alive: households get flow utility $\beta u(c_{i,t}) = \beta c_{i,t}^{1-\gamma}/(1-\gamma)$ from consuming $c_{i,t}$.
  - At death: utility $(1 - \beta)\mathcal{U}(\mathcal{C}_{i,t})$ from consuming $\mathcal{C}_{i,t}$. (Microfoundation: retirement value)

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  - *Friction:* cannot provide death insurance contracts to each other.
  - *Friction:* cannot borrow or lend to each other.
  - *Friction:* penalty on holding assets $\psi_{h,t}(k_{i,t}, a_{i,t})$, $\uparrow$ in capital $k_{i,t}$ and $\downarrow$ in wealth $a_{i,t}$.

$\Rightarrow$ non-degenerate density across household wealth, $g_h(a)$ and need for banks/funds.

Environment: Financial Intermediaries, Government, & Markets

- Two types of financial intermediaries:
  - **Bankers** (b): issue risk-free instantaneous deposits (at r_t^d) and hold assets.
  - **Fund managers** (f): issue (pension/insurance) contracts and hold assets.
  - A contract pays 1 good to the household owning the contract when they die.
  - Financial intermediary $j \in \{b, f\}$ exits at rate λ_j and new one forms with $\underline{a}_j = \varphi_j A$.
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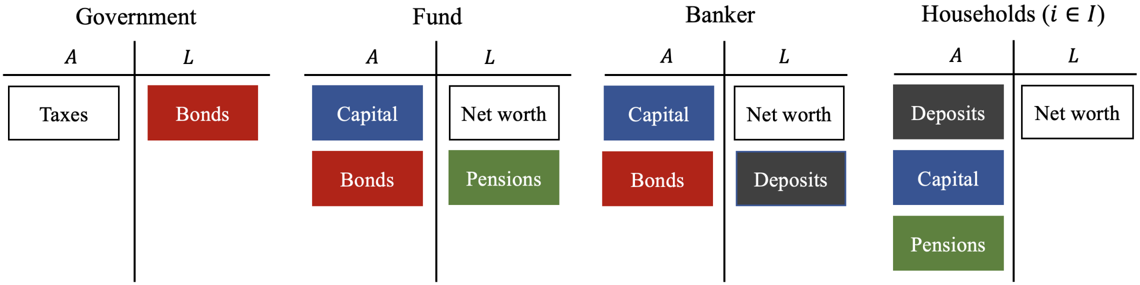
- **Competitive markets**: Asset prices for capital, contracts, bonds, $\boldsymbol{q}_t = (q_t^k, q_t^n, q_t^B)$:

$$\frac{dq_t^j}{q_t^j} = \mu_{q^j,t}dt + \sigma_{q^j,t} \cdot dW_t, \quad dR_t^j := \frac{\text{div}_j}{q_t^j j_t} + \frac{d(q_t^j j_t)}{q_t^j j_t} =: r_t^j dt + \sigma_{q^j,t} \cdot dW_t, \quad j \in (k, n, B)$$

Summary: Balance Sheets



Environment Nests Key Household Macro-Finance Models

- **"Financial accelerator"** banking models with representative households:
  - Remove the fund, set household capital cost to ∞: [Gertler and Kiyotaki, 2010].
  - Remove the fund, remove household capital cost: [Brunnermeier and Sannikov, 2014].
- **"Insurance/pension fund"** models without portfolio choice:
  - Remove flow utility ($\beta = 1$): [Vayanos and Vila, 2021] "preferred-habitat" preferences.
    - Households have demand for long-maturity assets that payoff at death.
  - Remove utility at death ($\beta = 0$): [Blanchard, 1985] "perpetual youth" preferences
    - Households demand annuities (i.e. short the fund contracts).
- Our model: household portfolio choice heterogeneity disciplines how far between.

Optimization and Equilibrium

- Given their belief about the price processes $(\tilde{\mathsf{r}}, \tilde{\mathbf{q}})$, household i with wealth $a_{i,t} := q_t^k k_{i,t} + q_t^n n_{i,t} + d_{i,t}$ solves: (analogous for bank and fund but with no ψ or $\mathcal{U}$)

$$\max_{\substack{\mathbf{c}_i, \boldsymbol{\mathcal{C}}_i, \\ \mathbf{k}_i, \mathbf{n}_i, \mathbf{d}_i, \boldsymbol{\iota}_i}} \left\{ \mathbb{E}_0 \left[\int_0^T e^{-\rho_i t} \left(\beta u(c_{i,t}) - \psi(k_{i,t}, a_{i,t}) \right) dt + e^{-\rho T} (1-\beta) \mathcal{U}(\mathcal{C}_{i,T}) \right] \right\}$$

$$s.t.\ da_{it} = d\tilde{R}_t^k(\iota_{i,t}) k_{i,t} + d\tilde{R}_t^n n_{i,t} + \left(\tilde{r}_t^d d_{i,t} - c_{i,t} - \tau_{i,t} \right) dt =: \mu_{a_i,t} a_{i,t} dt + \sigma_{a,i,t} a_{i,t} dW_t$$

$$\mathcal{C}_{i,T} = d_{i,T} + \tilde{q}_T^k k_{i,T} + n_{i,T}, \quad d_{i,t} \geq 0,$$

- Aggregate processes $(\mathsf{K}, \mathsf{r}, \mathbf{q}, g = (g_h, a_f, a_b))$ & agent decisions $(\mathsf{c}_i, \iota_i, \mathsf{k}_i, \mathsf{n}_i, \mathsf{d}_i, \mathsf{b}_i)_{i \in I}$:
  1. Given beliefs $(\tilde{\mathsf{r}}, \tilde{\mathbf{q}})$, households, bankers, and funds optimize. Banker and fund problems
  2. The price processes $(\mathsf{r}, \mathbf{q})$ satisfies market clearing conditions at each time t.
  3. Agent beliefs are consistent $(\tilde{\mathsf{r}}, \tilde{\mathbf{q}}) = (\mathsf{r}, \mathbf{q})$. Market clearing Recursive characterization

Recursive Characterization: Equilibrium HBJE

- ⋆ Individual state = a_i, Aggregate states = $(z, K, g) = \boldsymbol{S}$.
  (where g is the wealth distribution across households and financial intermediaries)

- ⋆ Given belief about distribution evolution, $(\tilde{\mu}_g(\boldsymbol{S}), \tilde{\sigma}_g(\boldsymbol{S}))$, household i chooses (c_i, ι_i) and asset wealth shares $\theta_i^k := q^k k_i / a_i$, $\theta_i^n := q^n n_i / a_i$, $\theta_i^d := d_i / a_i$ to solve:

$$(\rho + \lambda) V_h(a_i, \boldsymbol{S}) = \max_{c_i, \boldsymbol{\theta}_i, \iota_i} \left\{ u(c_i) - \psi_k(\theta_i^k, a_i) + \lambda \left(\mathcal{U}(\mathcal{C}_i; \theta_i^n, \theta_i^k) - V_h(a_i, \boldsymbol{S}) \right) \right.$$

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$$+ \partial_{a_i} V_h(a_i, \boldsymbol{S}) \mu_{a_i}(a_i, c_i, \boldsymbol{\theta_i}, \boldsymbol{S}) + \partial_z V_h(a_i, \boldsymbol{S}) \mu_z + \partial_K V_h(a_i, \boldsymbol{S}) \mu_K$$

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& + 0.5\left(\partial^2_{a_i^2}V_h(a_i,\boldsymbol{S})\sigma_{a_i}^T\sigma_{a_i}(a_i,\boldsymbol{\theta}_i,\boldsymbol{S}) + \partial^2_{z^2}V_h(a_i,\boldsymbol{S})\sigma_z^2\right) + \partial^2_{a_i z}V_h(a_i,\boldsymbol{S})\sigma_{a_i,z}(a_i,\boldsymbol{\theta}_i,\boldsymbol{S})\sigma_z a
\end{aligned}
$$

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& + \langle \partial_g V_h(a_i, \boldsymbol{S}), \tilde{\mu}_g(\boldsymbol{S}) \rangle + \left\langle \partial^2_{gz} V_h(a_i, \boldsymbol{S}), \tilde{\sigma}_{g,z}(\boldsymbol{S}) \sigma_z \right\rangle + \left\langle \partial^2_{g,g} V(a_i, \boldsymbol{S}), \tilde{\sigma}_g^T \tilde{\sigma}_{g'}(\boldsymbol{S}) \right\rangle \Big\} \; s.t. \; (B.C.)
\end{aligned}
$$

- Equilibrium belief consistency becomes: $(\tilde{\mu}_g(\boldsymbol{S}), \tilde{\sigma}_g(\boldsymbol{S})) = (\mu_g(\boldsymbol{S}), \sigma_g(\boldsymbol{S}))$.
- Banker and fund HJBEs are similar but without ν or ψ_i terms.

Portfolio Choice: Pension/Insurance Contracts

Let $V_j(a_j, \boldsymbol{S})$ denote value function for type $j \in \{h, b, f\}$ and let $\xi_j = \partial_{a_j} V_j(a_j, \boldsymbol{S})$.

Then the FOCs in the pension/insurance contract market:

$$\underbrace{r^n - r^d}_{\substack{\text{Excess}\\ \text{return}}} + \underbrace{\frac{\lambda_h(1-q^n)}{q^n}\frac{\mathcal{U}'(\mathcal{C})}{\xi_i}}_{\substack{\text{"Inelastic demand"}\\ \text{component}}} = - \underbrace{\boldsymbol{\sigma}_{\xi_i} \cdot \boldsymbol{\sigma}_{q^n}}_{\substack{\text{Comovement of}\\ \text{SDF and price}}} \qquad \text{...Household FOC}$$

$$\underbrace{r^n - r^d}_{\substack{\text{Excess}\\ \text{return}}} = - \underbrace{\boldsymbol{\sigma}_{\xi_f} \cdot \boldsymbol{\sigma}_{q^n}}_{\substack{\text{Comovement of}\\ \text{SDF and price}}} \qquad \text{...Fund FOC}$$

Nesting Vayanos-Vila Preferences

Portfolio Choice: Capital

FOCs for Capital market:

$$\underbrace{r^k - r^d}_{\text{Excess return}} = -\boldsymbol{\sigma}_{\xi_i} \cdot \boldsymbol{\sigma}_{q^k} - \underbrace{\partial_k \psi_k(\theta_i^k, a)}_{\text{“Participation” constraint}} \qquad \ldots \text{Household FOC}$$

$$r^k - r^d = -\boldsymbol{\sigma}_{\xi_b} \cdot \boldsymbol{\sigma}_{q^k} \qquad \ldots \text{Bank FOC}$$

$$r^k - r^d = -\boldsymbol{\sigma}_{\xi_f} \cdot \boldsymbol{\sigma}_{q^k} \qquad \ldots \text{Fund FOC}$$

- Bank and fund liabilities have different exposure:
  - Bank short-term deposits are not exposed to TFP shocks,
  - Fund contracts decrease in value when TFP decreases.

…So fund managers are naturally better hedged against TFP.

Equilibrium Characterization Has Three Blocks

1. *Optimization:* Given price processes, agents choose optimally (shown for HHs):

   Euler equations: $\rho_h = r + \mu_{\xi_h}, \quad \xi_h = u'(c_h), \quad \mu_{\xi_h} = \text{Drift of } \xi_h$

   Portfolio FOC: $r^l - r^d = -\sigma_{\xi_h} \cdot \sigma_{q^n} - \text{Wedge}^l, \quad l \in \{k, n\}$

2. *Distribution evolution:* Given prices & choices, household distribution evolution: More

$$dg_{h,t}(a) = \Big[\overbrace{\underbrace{\lambda_h \check{\varphi}_h(a)}_{\text{Birth}} - \underbrace{\lambda_h g_{h,t}(a)}_{\text{Death}} - \underbrace{\partial_a[\mu_a(a, \boldsymbol{s}_t, g_t) a g_{h,t}(a)]}_{\text{Wealth drift}} + \underbrace{\frac{1}{2}\partial_a \left[(\sigma_a^2(a, \boldsymbol{s}_t, g_t)) a^2 g_{h,t}(a) \right]}_{\text{Wealth volatility}}}^{=:\mu_{g_h,t}(a) = \text{distribution ``drift''}} \Big]$$

$$- \underbrace{\partial_a[\sigma_a(a, \boldsymbol{s}_t, g_t) a g_{h,t}(a)]}_{=:\sigma_{g_h,t}(a) = \text{distribution volatility}} \quad dW_{z,t}$$

3. *Market clearing:* Implicit price processes (q^k, q^n, q^B) consistent to Ito's Lemma More

Comparison to Models With Existing Solution Techniques

Models	Non-Trivial Blocks			Method
	1 (Opt.)	2 (Dist.)	3 (Asset q)	
Representative Agent (à la [Lucas, 1978])	simple	NA	simple	Finite difference
Heterogeneous Agents (à la [Krusell and Smith, 1998])	✓	✓	simple	[Gu et al., 2023]
Long-lived assets (à la [Brunnermeier and Sannikov, 2014])	closed-form	low-dim	✓	[Gopalakrishna, 2021]
HA + Long-lived assets	✓	✓	✓	This paper

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High Level Approach ("Projection" onto a Neural Network (NN))

- **1.** Take finite dim. approx. to agent continuum to get finite state space $\hat{\boldsymbol{S}}$: GLMP-24
  - When solving for eqm functions, approximate with finite population $i \leq I$ so state space:

$$\hat{\boldsymbol{S}} := (z, \zeta, K, \eta_1, \ldots, \eta_I, \eta_b, \eta_f), \quad \text{where}$$
$$\eta_j = a_j / A = \text{ wealth share of agent } i$$

  - When simulating, use eqm functions to approximate KFE operator on wealth grid.

- **2.** Represent equilibrium functions by neural networks with states $\hat{\boldsymbol{S}}$ as inputs.

- **3.** Train NN parameters to minimize loss in eqm conditions on random states.

- *Approach is easy to describe but tricky to implement in practice.*

- *"Art" of deep learning is rewriting the problem to "help" the neural net.*

Some "Practical" Technical Decisions

1. **Q.** Which variables do we represent by neural networks (NN)?
   A. Consumption/wealth, portfolio shares, & price volatilities.
   - ⋆ We fit NNs to the variables that are "easiest" to train (e.g. smooth, bounded).
   - ⋆ Better to represent $\xi = \partial_a V$ than V so we can easily impose V concavity.
   - ⋆ Better to represent $\omega = c/a$, then get $\xi = (\omega \eta q K)^{-\gamma}$ so extreme curvature is analytic.

2. **Q.** Dimension of finite distribution approximation?
   A. Separate dimensionality for training (I = 10-20) and simulation (100 wealth grid).

3. **Q.** How do we take large numbers of derivatives?
   A. Parametrize variables to use vectorized automatic differentiation + Duarte-25

4. **Q.** Which equilibrium conditions go into loss function?
   A. Avoid market clearing. Similar in spirit to [Azinovic and Žemlička, 2023]

Neural Network (NN) Approximation

- ⋆ Define consumption-to-wealth ratios ($\omega := c/a$, $\Omega := \mathcal{C}/a$) and p/f share $\theta_j^l := q^l l/a_j$.
  Represent $(\omega,\ \Omega,\ (\theta_h^l)_{l\in(n,k)}\ (q^l)_{l\in(n,B)},\ \mu_{q^k},\ \boldsymbol{\sigma}_q)$ by NNs with parameters Θ:

$$\hat{\omega}(\hat{\boldsymbol{S}};\Theta),\quad \hat{\Omega}(\hat{\boldsymbol{S}};\Theta),\quad \hat{\theta}_h^l(\hat{\boldsymbol{S}};\Theta)_{l\in(n,k)},\quad \hat{q}^l(\hat{\boldsymbol{S}};\Theta)_{l\in(n,B)},\quad \hat{\mu}_{q^k}(\hat{\boldsymbol{S}};\Theta),\quad \hat{\boldsymbol{\sigma}}_q(\hat{\boldsymbol{S}};\Theta).$$

- ⋆ At state $\boldsymbol{X}$, the error ("loss") in NN representation is: (with $\hat{\Xi} = u'(\hat{\omega}(\boldsymbol{S})\eta\hat{q}^k(\boldsymbol{S})K)$)

$$\mathcal{L}_\omega(\hat{\boldsymbol{S}}) = r^d - \tau_h - \rho_h - \lambda_h + \mu_\Xi(\hat{\boldsymbol{S}}) + D_\theta\boldsymbol{\psi} \qquad \ldots\text{Euler eq.}$$

$$\mathcal{L}_\Omega(\hat{\boldsymbol{S}}) = \hat{\Omega} - \eta A(\hat{\boldsymbol{S}})\mathcal{W}(\theta_h^k(\hat{\boldsymbol{S}}), \theta_h^n(\hat{\boldsymbol{S}})) \qquad \ldots\text{Death cons.}$$

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- ⋆ Define consumption-to-wealth ratios ($\omega := c/a$, $\Omega := \mathcal{C}/a$) and p/f share $\theta^l_j := q^l l/a_j$.
  Represent $(\omega,\ \Omega,\ (\theta^l_h)_{l\in(n,k)}\ (q^l)_{l\in(n,B)},\ \mu_{q^k},\ \boldsymbol{\sigma}_q)$ by NNs with parameters Θ:

$$\hat{\omega}(\hat{\boldsymbol{S}};\Theta),\quad \hat{\Omega}(\hat{\boldsymbol{S}};\Theta),\quad \hat{\theta}^l_h(\hat{\boldsymbol{S}};\Theta)_{l\in(n,k)},\quad \hat{q}^l(\hat{\boldsymbol{S}};\Theta)_{l\in(n,B)},\quad \hat{\mu}_{q^k}(\hat{\boldsymbol{S}};\Theta),\quad \hat{\boldsymbol{\sigma}}_q(\hat{\boldsymbol{S}};\Theta).$$

- ⋆ At state $\boldsymbol{X}$, the error ("loss") in NN representation is: (with $\hat{\Xi} = u'(\hat{\omega}(\boldsymbol{S})\eta\hat{q}^k(\boldsymbol{S})K)$)

$$\mathcal{L}_\omega(\hat{\boldsymbol{S}}) = r^d - \tau_h - \rho_h - \lambda_h + \mu_\Xi(\hat{\boldsymbol{S}}) + D_\theta\boldsymbol{\psi} \qquad \text{...Euler eq.}$$

$$\mathcal{L}_\Omega(\hat{\boldsymbol{S}}) = \hat{\Omega} - \eta A(\hat{\boldsymbol{S}})\mathcal{W}(\theta^k_h(\hat{\boldsymbol{S}}), \theta^n_h(\hat{\boldsymbol{S}})) \qquad \text{...Death cons.}$$

$$\mathcal{L}_{\theta^l_h}(\hat{\boldsymbol{S}}) = r^l - r^d + \boldsymbol{\sigma}_{\xi_i}\cdot\boldsymbol{\sigma}_{q^l} + \text{Wedge}^l \qquad \text{...FOC conditions}$$

Neural Network (NN) Approximation

- ⋆ Define consumption-to-wealth ratios ($\omega := c/a$, $\Omega := \mathcal{C}/a$) and p/f share $\theta^l_j := q^l l/a_j$. Represent (ω, Ω, $(\theta^l_h)_{l\in(n,k)}$ $(q^l)_{l\in(n,B)}$, μ_{q^k}, $\boldsymbol{\sigma}_q$) by NNs with parameters Θ:

$$\hat{\omega}(\hat{\boldsymbol{S}};\Theta), \quad \hat{\Omega}(\hat{\boldsymbol{S}};\Theta), \quad \hat{\theta}^l_h(\hat{\boldsymbol{S}};\Theta)_{l\in(n,k)}, \quad \hat{q}^l(\hat{\boldsymbol{S}};\Theta)_{l\in(n,B)}, \quad \hat{\mu}_{q^k}(\hat{\boldsymbol{S}};\Theta), \quad \hat{\boldsymbol{\sigma}}_q(\hat{\boldsymbol{S}};\Theta).$$

- ⋆ At state $\boldsymbol{X}$, the error ("loss") in NN representation is: (with $\hat{\Xi} = u'(\hat{\omega}(\boldsymbol{S})\eta\hat{q}^k(\boldsymbol{S})K)$)

$$\mathcal{L}_\omega(\hat{\boldsymbol{S}}) = r^d - \tau_h - \rho_h - \lambda_h + \mu_\Xi(\hat{\boldsymbol{S}}) + D_\theta\boldsymbol{\psi} \qquad \text{...Euler eq.}$$

$$\mathcal{L}_\Omega(\hat{\boldsymbol{S}}) = \hat{\Omega} - \eta A(\hat{\boldsymbol{S}})\mathcal{W}(\theta^k_h(\hat{\boldsymbol{S}}), \theta^n_h(\hat{\boldsymbol{S}})) \qquad \text{...Death cons.}$$

$$\mathcal{L}_{\theta^l_h}(\hat{\boldsymbol{S}}) = r^l - r^d + \boldsymbol{\sigma}_{\xi_i}\cdot\boldsymbol{\sigma}_{q^l} + \text{Wedge}^l \qquad \text{...FOC conditions}$$

$$\mathcal{L}_{\mu_{q^k}}(\hat{\boldsymbol{S}}) = (D_{\hat{\boldsymbol{S}}}q^l)^T\boldsymbol{\mu}_{\hat{\boldsymbol{S}}} + 0.5\text{tr}\left\{\boldsymbol{\sigma}^T_{\hat{\boldsymbol{S}}}(\hat{\boldsymbol{S}},\boldsymbol{\theta}_h)\boldsymbol{\sigma}_x(\hat{\boldsymbol{S}},\boldsymbol{\theta}_h)D^2_{\hat{\boldsymbol{S}}}q^l\right\} \qquad \text{...Consistency}$$

$$\mathcal{L}_\sigma(\hat{\boldsymbol{S}}) = \hat{\boldsymbol{\sigma}}_q(\hat{\boldsymbol{S}}) - (\boldsymbol{\sigma}_{\hat{\boldsymbol{S}}})^T(D_{\hat{\boldsymbol{S}}}q^l), \quad \forall l\in(k,n,m) \qquad \text{...Consistency}$$

Neural Network Structure | Algorithm Details | Sampling Approaches | Implementation Details | Test Models

Algorithm (“EMINN” or “Economic Deep Galerkin”)

Initialize neural network objects ($\hat{\omega}_h$, $\hat{\Omega}_h$, $\hat{\boldsymbol{\theta}}_j$, $\hat{q}^l$, $\hat{\mu}_{q^k}$, $\boldsymbol{\sigma}_{q^l}$) with parameters $\boldsymbol{\Theta}$,

Initialize optimizer.

while validation loss > tolerance **do**

  Sample N new training points: $\left(\hat{\boldsymbol{S}}^n = \left(z^n, \zeta^n, K^n, (\eta_i)^n_{i\leq I}, \eta^n_b, \eta^n_f\right)\right)_{n=1}^{N}$.

  Calculate equilibrium at each training point $\hat{\boldsymbol{S}}^n$ (using ($\hat{\omega}_h$, $\hat{\Omega}_h$, $\hat{\boldsymbol{\theta}}_j$, $\hat{q}^l$, $\hat{\mu}_{q^k}$, $\boldsymbol{\sigma}_{q^l}$)):

  Construct loss as following (potentially with different weights):

$$\hat{\mathcal{L}}(\hat{\boldsymbol{S}}^n) = (\mathcal{L}_\omega + \mathcal{L}_\Omega + \mathcal{L}_{\theta^l_h} + \mathcal{L}_{\mu_{q^k}} + \mathcal{L}_\sigma)(\hat{\boldsymbol{S}}^n; \boldsymbol{\Theta})$$

  Update $\boldsymbol{\Theta}$ using a variant stochastic gradient descent.

end while

NN Structure | Algorithm Details | Sampling Approaches | Implementation | Tests | Errors

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Goal: Compare economies under different regulatory regimes

- ⋆ Introduce regulatory constraints: for intermediary $j \in \{b, f\}$ on asset $l \in \{k, B\}$:

$$\Psi_{j,l,t} = \psi_{j,l,t} a_{j,t}, \text{ where } \psi_{j,l,t} = 0.5\psi_j^l \left[\min\{0, \theta_{j,t}^l\}^2 + \max\{0, \theta_{j,t}^l - \bar{\theta}_{j,t}^l\}^2\right]$$

- ⋆ Compare three regulatory regimes:

  1. Baseline: calibrated to 2010Q1-2024Q4 under Basel III & Dodd-Frank.
     - ⋆ Regulatory constraints and recapitalization rates target bank and fund leverage.
     - ⋆ Household capital constraint targets median household portfolio share.
     - ⋆ Household, bank, and fund risk aversion is set to target spreads.
  2. Bank restrictions: imposes tighter constraint on bank leverage (25% decrease in $\bar{\theta}_b^k$).
     (Admati and Hellwig (2013), Acharya, Engle and Pierret (2014), Basel IV)
  3. Fund restrictions: regulate funds similarly to banks (set $\bar{\theta}_f^k$ similar to $\bar{\theta}_b^k$).
     (Koijen and Yogo (2022), Begenau, Liang and Siriwardane (2024))

Baseline Calibration | Untargetted moments | Regulatory Parameters

Ergodic Mean Outcomes Across Regulatory Regimes

	Baseline	Counterfactual: Bank-regulated	Counterfactual: Fund-regulated
Growth-Stability-Inequality Tradeoff			
Output Growth (%)	2.446	↓ 2.245	↓ 2.303
Wealth share. Risk (hh.)	0.281	↓ 0.222	↓ 0.242
Gini Coefficient	0.093	↑ 0.101	↑ 0.097
Price and spreads			
Sharpe Ratio $(r^k - r^d)/\sigma_{q^k}$	0.463	0.567	0.738
Govt. bond price q^B	0.135	0.241	0.221
Pension spread $r^n - r^d$ (%)	-2.833	-3.492	-3.433
Sector level results			
Fund Risky A/E	1.686	1.907	1.655
Bank Risky A/E	1.076	0.869	1.084

Sector Level: Funds Are "Natural Backstops" For Absorbing Risk

- In a recession, long-term asset prices drop (goods are scarce): $\downarrow q_t^k, q_t^n, q_t^B$
- Banks and funds issue liabilities with different exposure to aggregate shocks:
  - Bank short-term risk-free deposits are not exposed to TFP or volatility shocks,
  - Fund pension/insurance contracts decrease in market value: $\downarrow q_t^n n_t$

$\Rightarrow$ All else equal, fund has more "balance sheet space" to absorb business cycle risk.

Intuition 1: Funds absorb TFP risk by purchasing capital in recessions. IRFs

Intuition 2: Restricting bank leverage leads to higher fund leverage (substitutes) but …restricting fund leverage leads to lower bank leverage (complements)

Within Household Sector: Evolution of Household Wealth Inequality

- ⋆ Difference between the drift of the wealth share of any two households i and j is:

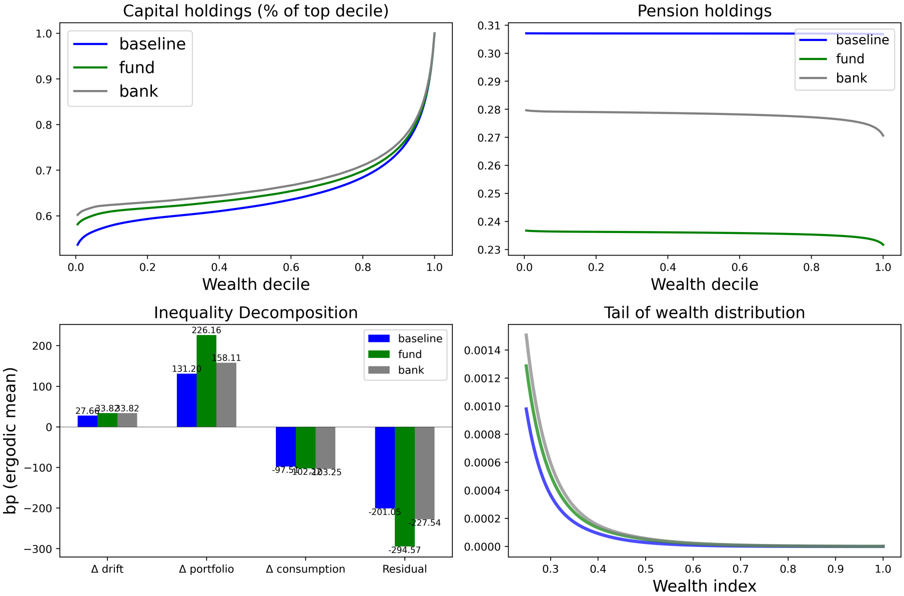
$$\mu_{\eta_j,t} - \mu_{\eta_i,t} = (\theta^k_{j,t} - \theta^k_{i,t})(r^k_t - r^d_t - \sigma^k_{q,t} \cdot \sigma^k_{q,t}) + (\theta^n_{j,t} - \theta^n_{i,t})(r^n_t - r^d_t - \sigma^k_{q,t} \cdot \sigma^n_{q,t})$$
$$- (\omega_j - \omega_i) + \varphi_h \lambda(\eta^{-1}_{j,t} - \eta^{-1}_{i,t}), \quad \theta^k_i, \theta^n_i = \text{capital \& pension portfolio}$$

1. Participation constraint: low wealth agents hold less capital and earn less risk premium.
2. Pension needs: low wealth agents save through pension/insurance contracts.
3. Consumption: low wealth agents consume less to escape participation constraint.
4. Redistribution: through death (and wealth taxes).

Intuition 3: Wealth distribution shape $\sim (r^k - r^d_t, r^n - r^d_t, VAR(\theta^k_j), VAR(\theta^n_j))$

Intuition 4: ↑ Leverage penalty ⇒ ↓ volatility but ↑ spreads, helping wealthy agents

Household Sector: Inequality Dynamics



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Conclusion

- Built a macroeconomic model with:
  - Intermediary asset pricing
  - Heterogeneous agents facing asset market participation constraints,
  - Aggregate TFP shocks, a financial sector, and endogenous asset price volatility
- Developed new "deep learning" global solution technique class of models.
- Complicated tradeoffs between growth, stability, and inequality.
  - Leverage restrictions dampen volatility but opens up return spreads.
  - Rich households able to circumvent financial intermediaries and directly earn spreads.
  - So, leverage restrictions also increase inequality.
- Opens up the possibility of studying new areas of macro-finance.

Thank you

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Roadmap

Optimization

- *Banker problem:* Given their belief about price processes, $(\tilde{\mathsf{r}}, \tilde{\mathsf{q}})$, and initial wealth, $a_{b,0}$, a banker chooses processes $(\mathsf{c}_b, \boldsymbol{\theta}_b, \iota_b)$ to solve the Problem (1) below:

$$\begin{aligned} &\max_{\mathsf{c}_b, \boldsymbol{\theta}_b, \iota_b} \left\{ \int_0^\infty e^{-\rho_b t} u(c_{b,t}) dt \right\} \quad s.t. \\ &\frac{da_{b,t}}{a_{b,t}} = \theta^k_{b,t} d\tilde{R}^k_t + \theta^m_{b,t} d\tilde{R}^m_t + \Big((1 - \theta^k_{b,t} - \theta^m_{b,t}) \tilde{r}^d_t - c_{b,t}/a_{b,t} - \tau_{b,t} \Big) dt \end{aligned} \tag{1}$$

- *Fund problem:* Given their belief about price processes, $(\tilde{\mathsf{r}}, \tilde{\mathsf{q}})$, and initial wealth, $a_{f,0}$, a fund manager chooses processes $(\mathsf{c}_f, \boldsymbol{\theta}_f, \iota_f)$ to solve the Problem (2) below:

$$\begin{aligned} &\max_{\mathsf{c}_f, \boldsymbol{\theta}_f, \iota_f} \left\{ \int_0^\infty e^{-\rho_f t} u(c_{f,t}) dt \right\} \quad s.t. \\ &\frac{da_{f,t}}{a_{f,t}} = \theta^k_{f,t} d\tilde{R}^k_t + \theta^m_{f,t} d\tilde{R}^m_t + (1 - \theta^k_{f,t} - \theta^m_{f,t}) d\tilde{R}^n_t + (-c_{f,t}/a_{f,t} - \tau_{f,t}) \, dt \end{aligned} \tag{2}$$

Back

Equilibrium

An equilibrium is a collection of aggregate processes $(\mathbf{K}, \mathbf{r}, \mathbf{q}, g)$ and agent decision processes $(\mathbf{c}_i, \boldsymbol{\iota}_i, \mathbf{k}_i, \mathbf{n}_i, \mathbf{d}_i)$:

1. Given beliefs $(\tilde{\mathbf{r}}, \tilde{\mathbf{q}})$, households, bankers, and fund managers optimize.

2. The price processes $(\mathbf{r}, \mathbf{q})$ satisfies market clearing conditions at each time t: (Where capital letters refer to sector aggregates.)

   (i) Goods market: $C_{h,t} + C_{b,t} + C_{f,t} + \lambda \mathcal{C}_{h,t} = e^{z_t} K_t - \iota_t K_t$,

   (ii) Pension share and deposit: $\sum_i N_{i,t} = \sum_i D_{i,t} = 0$,

   (iii) Capital market: $\sum_i K_{i,t} = K_t$,

   (iv) Bond market clears: $B_{b,t} + B_{f,t} = B$

3. Agent beliefs are consistent with equilibrium $(\tilde{\mathbf{r}}, \tilde{\mathbf{q}}) = (\mathbf{r}, \mathbf{q})$.

Back

Recursive Characterization: Master Equation

- ⋆ Individual state = a_i, Aggregate states = $(z, \zeta, K, g) = (\cdot)$.
- ⋆ Given belief about evolution of other agents, $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot))$, household i chooses (c_i, ι_i) and asset wealth shares $\theta_i^k := q^k k_i / a_i$, $\theta_i^n := q^n n_i / a_i$, $\theta_i^d := d_i / a_i$ to solve:

$$(\rho + \lambda) V(a_i, \cdot) = \max_{c_i, \boldsymbol{\theta}_i, \iota_i} \left\{ u(c_i) - \psi(\theta_i^k, a_i) + \lambda \left(\mathcal{U}(\mathcal{C}_i; \theta_i^n, \theta_i^k) - V(a_i, \cdot) \right) \right.$$

Recursive Characterization: Master Equation

- ⋆ Individual state $= a_i$, Aggregate states $= (z, \zeta, K, g) = (\cdot)$.
- ⋆ Given belief about evolution of other agents, $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot))$, household i chooses (c_i, ι_i) and asset wealth shares $\theta_i^k := q^k k_i / a_i$, $\theta_i^n := q^n n_i / a_i$, $\theta_i^d := d_i / a_i$ to solve:

$$(\rho + \lambda) V(a_i, \cdot) = \max_{c_i, \boldsymbol{\theta}_i, \iota_i} \left\{ u(c_i) - \psi(\theta_i^k, a_i) + \lambda \left(\mathcal{U}(\mathcal{C}_i; \theta_i^n, \theta_i^k) - V(a_i, \cdot) \right) \right.$$

$$+ \partial_{a_i} V(a_i, \cdot) \mu_{a_i}(a_i, c_i, \boldsymbol{\theta_i}, \cdot) + \partial_z V(a_i, \cdot) \mu_z + \partial_\zeta V(a_i, \cdot) \mu_\zeta + \partial_K V(a_i, \cdot) \mu_K$$

Recursive Characterization: Master Equation

- ⋆ Individual state $= a_i$, Aggregate states $= (z, \zeta, K, g) = (\cdot)$.
- ⋆ Given belief about evolution of other agents, $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot))$, household i chooses (c_i, ι_i) and asset wealth shares $\theta_i^k := q^k k_i / a_i$, $\theta_i^n := q^n n_i / a_i$, $\theta_i^d := d_i / a_i$ to solve:

$$
\begin{aligned}
(\rho + \lambda) V(a_i, \cdot) = \max_{c_i, \boldsymbol{\theta}_i, \iota_i} \Big\{ & u(c_i) - \psi(\theta_i^k, a_i) + \lambda \left(\mathcal{U}(\mathcal{C}_i; \theta_i^n, \theta_i^k) - V(a_i, \cdot) \right) \\
& + \partial_{a_i} V(a_i, \cdot) \mu_{a_i}(a_i, c_i, \boldsymbol{\theta_i}, \cdot) + \partial_z V(a_i, \cdot) \mu_z + \partial_\zeta V(a_i, \cdot) \mu_\zeta + \partial_K V(a_i, \cdot) \mu_K \\
& + 0.5 \left(\partial^2_{a_i^2} V(a_i, \cdot) \boldsymbol{\sigma}^T_{a_i} \boldsymbol{\sigma}_{a_i}(a_i, \boldsymbol{\theta}_i, \cdot) + \partial^2_{z^2} V(a_i, \cdot) \sigma_z^2 + \partial^2_{\zeta^2} V_i(a_i, \cdot) \sigma_\zeta^2 \right)
\end{aligned}
$$

Recursive Characterization: Master Equation

- $\star$ Individual state $= a_i$, Aggregate states $= (z, \zeta, K, g) = (\cdot)$.
- $\star$ Given belief about evolution of other agents, $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot))$, household i chooses (c_i, ι_i) and asset wealth shares $\theta_i^k := q^k k_i / a_i$, $\theta_i^n := q^n n_i / a_i$, $\theta_i^d := d_i / a_i$ to solve:

$$
\begin{aligned}
(\rho + \lambda) V(a_i, \cdot) = \max_{c_i, \boldsymbol{\theta}_i, \iota_i} \Big\{ & u(c_i) - \psi(\theta_i^k, a_i) + \lambda \left(\mathcal{U}(\mathcal{C}_i; \theta_i^n, \theta_i^k) - V(a_i, \cdot) \right) \\
& + \partial_{a_i} V(a_i, \cdot) \mu_{a_i}(a_i, c_i, \boldsymbol{\theta_i}, \cdot) + \partial_z V(a_i, \cdot) \mu_z + \partial_\zeta V(a_i, \cdot) \mu_\zeta + \partial_K V(a_i, \cdot) \mu_K \\
& + 0.5 \left(\partial^2_{a_i^2} V(a_i, \cdot) \boldsymbol{\sigma}^T_{a_i} \boldsymbol{\sigma}_{a_i}(a_i, \boldsymbol{\theta}_i, \cdot) + \partial^2_{z^2} V(a_i, \cdot) \sigma_z^2 + \partial^2_{\zeta^2} V_i(a_i, \cdot) \sigma_\zeta^2 \right) \\
& + \partial^2_{a_i z} V(a_i, \cdot) \sigma_{a_i, z}(a_i, \boldsymbol{\theta}_i, \cdot) \sigma_z + \partial^2_{a_i \zeta} V(a_i, \cdot) \sigma_{a_i, \zeta}(a_i, \boldsymbol{\theta}_i, \cdot) \sigma_\zeta + \partial^2_{z\zeta} V(a_i, \cdot) \sigma_z \sigma_\zeta + \sigma_\lambda \sigma_V
\end{aligned}
$$

Recursive Characterization: Master Equation

- ⋆ Individual state $= a_i$, Aggregate states $= (z, \zeta, K, g) = (\cdot)$.
- ⋆ Given belief about evolution of other agents, $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot))$, household i chooses (c_i, ι_i) and asset wealth shares $\theta_i^k := q^k k_i/a_i$, $\theta_i^n := q^n n_i/a_i$, $\theta_i^d := d_i/a_i$ to solve:

$$
\begin{aligned}
(\rho+\lambda)V(a_i,\cdot) = \max_{c_i,\boldsymbol{\theta}_i,\iota_i} \Big\{ & u(c_i) - \psi(\theta_i^k, a_i) + \lambda \left(\mathcal{U}(\mathcal{C}_i; \theta_i^n, \theta_i^k) - V(a_i,\cdot)\right) \\
& + \partial_{a_i} V(a_i,\cdot)\mu_{a_i}(a_i, c_i, \boldsymbol{\theta_i}, \cdot) + \partial_z V(a_i,\cdot)\mu_z + \partial_\zeta V(a_i,\cdot)\mu_\zeta + \partial_K V(a_i,\cdot)\mu_K \\
& + 0.5\left(\partial^2_{a_i^2} V(a_i,\cdot)\boldsymbol{\sigma}_{a_i}^T \boldsymbol{\sigma}_{a_i}(a_i,\boldsymbol{\theta}_i,\cdot) + \partial^2_{z^2} V(a_i,\cdot)\sigma_z^2 + \partial^2_{\zeta^2} V_i(a_i,\cdot)\sigma_\zeta^2\right) \\
& + \partial^2_{a_i z} V(a_i,\cdot)\sigma_{a_i,z}(a_i,\boldsymbol{\theta}_i,\cdot)\sigma_z + \partial^2_{a_i\zeta} V(a_i,\cdot)\sigma_{a_i,\zeta}(a_i,\boldsymbol{\theta}_i,\cdot)\sigma_\zeta + \partial^2_{z\zeta} V(a_i,\cdot)\sigma_z\sigma_\zeta + \sigma_\lambda\sigma_V \\
& + \int_{\mathcal{A}} \partial_g V(a_i,\cdot)\tilde{\mu}_{a_j}(\cdot) + \int_{\mathcal{A}} \partial^2_{gz} V(a_i,\cdot)\tilde{\sigma}_{a_j,z}(\cdot)\sigma_z + \int_{\mathcal{A}} \partial^2_{g\zeta} V(a_i,\cdot)\tilde{\sigma}_{a_j,\zeta}(\cdot)\sigma_\zeta \\
& \quad + 0.5 \int_{\mathcal{A}}\int_{\mathcal{A}} \partial^2_{g,g} V_i \tilde{\boldsymbol{\sigma}}_g^T \tilde{\boldsymbol{\sigma}}_{g'}(\cdot) \Big\}
\end{aligned}
$$

- ⋆ Banker and fund HJBEs are similar but without ν or ψ_i terms.
- ⋆ Equilibrium belief consistency becomes: $(\hat{\mu}_{a_j}(\cdot), \hat{\sigma}_{a_j}(\cdot)) = (\mu_{a_j}(\cdot), \sigma_{a_j}(\cdot))$.

Back

Recursive Characterization: Finite Agent Approximation

- Individual state $= a_i$, Aggregate states $= (z, \zeta, K, \{a_j\}_{j \neq i}) = (\cdot)$.

- Given belief about evolution of other agents, $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot))$, household i chooses (c_i, ι_i) and asset wealth shares $\theta_i^k := q^k k_i / a_i$, $\theta_i^n := q^n n_i / a_i$, $\theta_i^d := d_i / a_i$ to solve:

$$\rho_h V(a_i, \cdot) = \max_{c_i, \boldsymbol{\theta}_i, \iota_i} \Big\{ u(c_i) - \psi(\theta_i^k, a_i) + \lambda \Big(\mathcal{U}(\mathcal{C}_i; \theta_i^n, \theta_i^k) - V(a_i, \cdot) \Big)$$

$$+ (\mathcal{L}_h V)(a_i, \cdot, c_i, \boldsymbol{\theta}_i)$$

$$+ \textstyle\sum_{j \neq i} \partial_{a_j} V(a_i, \cdot) \tilde{\mu}_{a_j}(\cdot) + \displaystyle\sum_{j \neq i} \partial^2_{a_j z} V(a_i, \cdot) \tilde{\sigma}_{a_j, z}(\cdot) \sigma_z$$

$$+ \sum_{j \neq i} \partial^2_{a_j \zeta} V(a_i, \cdot) \tilde{\sigma}_{a_j, z}(\cdot) \sigma_z + 0.5 \sum_{j \neq i, j' \neq i} \partial^2_{a_j, a_{j'}} V_i \tilde{\boldsymbol{\sigma}}^T_{a_j} \tilde{\boldsymbol{\sigma}}_{a'_j}(\cdot) \Big\}$$

- Banker and fund HJBEs are similar but without ψ terms and with log utility and closed form solution.

- Equilibrium belief consistency becomes: $(\tilde{\mu}_{a_j}(\cdot), \tilde{\sigma}_{a_j}(\cdot)) = (\mu_{a_j}(\cdot), \sigma_{a_j}(\cdot))$.

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Market for Pension Shares (Vayanos-Vila Preferences)

Individual state $= a_i$, Aggregate states $= (z, K, \{a_j\}_{j\neq i}) = (\cdot)$

Recursive characterization

Let $V_j(a_j, \cdot)$ denote value function for type $j \in \{h, b, f\}$ and let $\xi_j = \partial_{a_j} V_j(a_j, \cdot)$.

Then the FOCs in the pension share market:

$$\underbrace{r^n - r^d}_{\text{Excess return}} + \underbrace{\frac{\lambda}{q^n} \exp\left(-\alpha \frac{q^k}{q^n} \theta_i^n \eta_i\right)}_{\text{"Preferred habitat" component}} = -\underbrace{\boldsymbol{\sigma}_{\xi_i} \cdot \boldsymbol{\sigma}_{q^n}}_{\text{"shifter"}} \qquad \ldots \text{Household FOC}$$

$$\underbrace{r^n - r^d}_{\text{Excess return}} = -\underbrace{\boldsymbol{\sigma}_{\xi_f} \cdot \boldsymbol{\sigma}_{q^n}}_{\text{Comovement of SDF and price}} \qquad \ldots \text{Fund FOC}$$

Back

Block 1: Optimization

- Given price processes $(r, r_k, q, \mu_q, \sigma_q)$, household optimization implies:

Euler equation:

$$\rho_h \xi_h = r\xi_h + \frac{\partial \xi_h}{\partial z}\mu_z + \frac{\partial \xi_i}{\partial K}\mu_K + \frac{1}{2}\frac{\partial^2 \xi_h}{\partial z^2}\sigma_z^2 + \sum_j \frac{\partial \xi_h}{\partial \eta_j}\eta_j \mu_{\eta_j,t} + \sum_j \frac{\partial^2 \xi_h}{\partial z \partial \eta_j}\eta_j \sigma_{\eta_j,t}\sigma_z + \frac{1}{2}\sum_{j,j'} \frac{\partial^2 \xi_h^2}{\partial \eta_j \partial \eta_{j'}}\eta_j \eta_{j'} \sigma_{\eta_j,t}\sigma_{\eta_{j'},t}$$

Consumption FOC:

$$\xi_h = u'(c_h)$$

Portfolio FOC:

$$\xi_h(r_k - r) = -\left(\frac{\partial \xi_h}{\partial z}\sigma_z + \sum_j \frac{\partial \xi_h}{\partial \eta_j}\sigma_{j,\eta}\right)\sigma_q - \frac{\partial \Psi_h}{\partial k_i}$$

- Expert optimization is similar but adjusted for Epstein-Zin More

Block 2: Distribution Evolution

At the sector level, the wealth share for financial intermediary $j \in \{b, f\}$ evolves by:

$$\frac{d\eta_{j,t}}{\eta_{j,t}} = \left(\mu_{A_j,t} - \mu_{A,t} + (\sigma_{A,t} - \sigma_{A_j,t})\sigma_{A,t}\right) dt + (\sigma_{A_j,t} - \sigma_{A,t}) dW_t \tag{3}$$

where:

$$\mu_{A_b,t} = r^d + \theta_h^k (r^k - r^d) - (\rho_b + \lambda_b) - \tau_b + \lambda_b \left(\phi_b \eta_{b,t}^{-1} - 1\right) \tag{4}$$

$$\mu_{A_f,t} = r^b + \theta_f^k (r^k - r_f^b) + \theta_f^m (r^m - r_f^n) - (\rho_f + \lambda_f) - \tau_f + \lambda_f \left(\phi_f \eta_{f,t}^{-1} - 1\right) \tag{5}$$

$$\mu_{A,t} = \vartheta_t (\mu_{q^k} + \Phi(\iota) - \delta) + (1 - \vartheta_t)\mu_{q^m} \tag{6}$$

and where aggregate wealth is given by $A_t = q_t^k K_t + q_t^m M$ and the aggregate wealth in capital is $\vartheta_t := q_t^k K_t / (q^k K_t + q^m M)$.

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Block 2: Distribution Evolution

Within the household sector, the density of household wealth shares evolves according to:

$$dg_{h,t}(\eta) = \Big(\lambda_h \phi(\eta) - \lambda_h g_{h,t}(\eta) - \partial_\eta[\mu_\eta(\eta, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(\eta)] \tag{7}$$

$$+ \frac{1}{2}\partial_\eta \left[(\sigma^2_{\eta,z}(\eta, \boldsymbol{s}_t, g_{h,t}) + \sigma^2_{\eta,\zeta}(\eta, \boldsymbol{s}_t, g_{h,t})) g_{h,t}(\eta) \right] \Big) dt \tag{8}$$

$$- \partial_\eta[\sigma_{\eta,z}(\eta, \boldsymbol{s}_t, g_{h,t}) g_{h,t}(\eta)] dW_{z,t} \tag{9}$$

where:

$$\mu_{\eta_{i,t}} = \mu_{a_i,t} - \mu_{A,t} + (\sigma_{A,t} - \sigma_{a_i,t})\sigma_{A,t} \tag{10}$$

$$\sigma_{\eta_i,t} = \sigma_{a_i,t} - \sigma_{A,t} \tag{11}$$

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Block 3: Equilibrium Consistency

$$\mu_{q^l} q^l(\boldsymbol{S}) = (D_x q^l(\boldsymbol{S}))^T \boldsymbol{\mu}_s + \frac{1}{2}\text{tr}\left\{(\boldsymbol{\sigma}_s(\boldsymbol{S}, \boldsymbol{\theta}_h) \odot \boldsymbol{S})^T (\boldsymbol{\sigma}_s(\boldsymbol{S}, \boldsymbol{\theta}_h) \odot \boldsymbol{s}) D_s^2 q^l(\boldsymbol{S})\right\} + \mathcal{L}_g q^l(\boldsymbol{S})$$

$$\sigma_{q^l} q^l(\boldsymbol{S}) = (\boldsymbol{\sigma}_s \odot \boldsymbol{S})^T (D_s q^l(\boldsymbol{S}))$$

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Block 3: Equilibrium Consistency

- Clearing conditions pin down the prices:

$$\sum_i c_{i,t} + \Phi(\iota_t) K_t = y_t \qquad \sum_i (1-\theta_{i,t}) a_{i,t} = 0 \qquad \sum_i \theta_{i,t} a_{i,t} = q_t K_t$$

- But q process is implicit so we must impose consistency conditions on q to close the model:

$$\begin{aligned} q\mu_{q,t} &= \sum_j \frac{\partial q}{\partial \eta_j} \eta_j \mu_{\eta_j,t} + \frac{\partial q}{\partial z}\mu_{z,t} + \frac{\partial q}{\partial K}\mu_{K,t} + \sum_j \frac{\partial^2 \xi_i}{\partial z \partial \eta_j} \eta_j \sigma_{\eta_j,t}\sigma_z \\ &\quad + \frac{1}{2}\sum_{j,j'} \frac{\partial^2 q}{\partial \eta_j \partial \eta_{j'}} \eta_j \eta_{j'} \sigma_{\eta_j,t}\sigma_{\eta_{j'},t} + \frac{1}{2}\frac{\partial^2 q}{\partial z^2}\sigma_z^2 \\ q\sigma_{q,t} &= \sum_j \frac{\partial q}{\partial \eta_j} \eta_j \sigma_{\eta_j,t} + \frac{\partial q}{\partial z}\sigma_{z,t} \end{aligned}$$

Roadmap

Equilibrium Concepts and Algorithms

Back

- **Rational expectations equilibrium** (our approach):
  - Agents have a "perceived law of motion" for the economic variables that is the true law of motion if agents follow their perceived law of motion (on and off the equilibrium path)
- **Self-confirming belief "indoctrination" equilibrium** (reinforcement learning):
  - "Perceived law of motion" is the best statistical fit to data generated by an economy where agents follow that perceived law of motion.
  - So, agents would never reject a misspecification test but may have off-equilibrium beliefs that differ from rational expectations.
- **Misspecified belief equilibrium** (Krusell-Smith):
  - Agents have a "perceived law of motion" for the equilibrium variables in the economy that *satisfies a particular parametric form*.
  - In equilibrium, parameters in their perceived law of motion are the statistical best fit to the data generated by an economy where all agents follow that perceived law of motion.
  - However, if agents were to perform a misspecification test on the parametric form for their perceived law of motion, they would typically reject it.

Ingredient 1: Finite Dimensional "Distribution" Approximation

	Finite Population	Discrete State	Projection
Params $\hat{\varphi}$	Agent states $\hat{\varphi}_t = \{(a^i_t)\}_{i \leq N}$	Masses on grid $\hat{\varphi}_{i,t}, \ \forall (a^i)_{i \leq N}$	Basis coefficients $\hat{\varphi}_{i,t}, \ \forall b_i(a)\|_{i \leq N}$
Dist. approx.	$\frac{1}{N}\sum_{i=1}^N \delta_{(a^i_t)}$	$\sum_{i=1}^N \hat{\varphi}_{i,t}\delta_{(a^i)}$	$\sum_{i=0}^N \hat{\varphi}_{i,t} b_i(a)$
KFE approx. ($\mu^{\hat{\varphi}}$)	Evolution of other agents' states	Evolution of mass between grid points (e.g. finite diff.)	Evolution of projection coefficients (least squares)
Dimension (N)	≈ 50	≈ 200	≈ 5

Projections more easily capture shape but have complicated KFE approximation Back

Sampling Approaches

- ⋆ Sampling (a, z, ζ, K): draw from uniform distribution, then add draws where error high.

- ⋆ Sampling the parameters in the distribution approximation $(\hat{\varphi}^i)_{i \leq N}$:
  - ⋆ *Moment sampling:*
    1. Draw samples for selected moments of the distribution (that are important for $\hat{Q}(z, \hat{\varphi})$).
    2. Sample $\hat{\varphi}$ from a distribution that satisfies the moments drawn in the first step.
  - ⋆ *Mixed steady state sampling:*
    1. Solve for the steady state for a collection of fixed aggregate states z.
    2. Draw random, perturbed mixtures of this collection of steady state distributions.
  - ⋆ *Ergodic sampling:*
    1. Simulate economy using current value function approximation.
    2. Use simulated distributions as training points.

Need to choose economically relevant subspace on which to sample. Back

Finite Population: Why Are 10-20 Agents Sufficient?

Back

Simulation-based finite-agent methods in discrete time often need a large number of agents to be accurate. Why not here?

1. We sample individual states (x) and distribution states ($\hat{\eta}$) separately when generating training sets
   - ⋆ Without separation: need large N to have sufficiently many x in high-curvature regions (e.g., close to borrowing constraint) to learn value function well.
   - ⋆ With separation: can independently control where we learn V in the x-dimension and how we represent g.
   - ⋆ Ultimately, 10-20 agents is sufficient for approximating key moments of distribution.
2. We can eliminate/reduce finite agent noise
   - ⋆ During training, we can eliminate the finite agent noise analytically in the "KFE".
   - ⋆ During simulation: approximate KFE operator instead of simulating finite population economy.

Finite Population KFE With Averaged Idiosyncratic Noise

- Applying Ito's Lemma to $\frac{1}{N}\sum_{i=1}^{N}\phi_t(a_{i,t})$ and rearranging gives:

$$\frac{1}{N}\sum_{i=1}^{N}\phi_t(a_t^i) - \frac{1}{N}\sum_{i=1}^{N}\phi_0(a_0^i) = \frac{1}{N}\sum_{i=1}^{N}\int_0^t \Big(\partial_s\phi_s(a_{i,s}) + \mu_a a_{i,s}\partial_a\phi_s(a_{s,i}) + \frac{1}{2}\sigma_{a,s}^2 a_{i,s}^2\partial_{aa}\phi_s(a_{i,s})\Big)ds$$

$$+ \frac{1}{N}\sum_{i=1}^{N}\int_0^t(\phi_s(\varphi_h A_s) - \phi_s(a_{i,s}))dN_s^i + \frac{1}{N}\sum_{i=1}^{N}\int_0^t \sigma_{a,s}a_{i,s}\partial_x\phi_s(a_{i,s})dW_{z,s}$$

- Take the limit as $N\to\infty$ selectively (so only the idiosyncratic noise averages out)

$$\frac{1}{N}\sum_{i=1}^{N}\phi_t(a_t^i) - \frac{1}{N}\sum_{i=1}^{N}\phi_0(a_0^i) = \frac{1}{N}\sum_{i=1}^{N}\int_0^t \Big(\partial_s\phi_s(a_{i,s}) + \mu_a a_{i,s}\partial_a\phi_s(a_{s,i}) + \frac{1}{2}\sigma_{a,s}^2 a_{i,s}^2\partial_{aa}\phi_s(a_{i,s})\Big)d$$

$$+ \int_{\mathcal{A}}\int_0^t(\phi_s(\varphi_h A) - \phi_s(a))\lambda_h g_s(a)dsda + \frac{1}{N}\sum_{i=1}^{N}\int_0^t \sigma_{a,s}a_{i,s}\partial_x\phi_s(a_{i,s})dW_{z,}$$

Finite Population: Simulation

Back

Start from an initial g_0 equal to the ergodic distribution.
for $n = 0, \ldots, N_T - 1$ **do**
    Sample ΔB_t^0 from normal distribution $N(0, \Delta t)$, construct TFP shock path by: $z_{t+\Delta t} = z_t + \eta(\bar{z} - z_t) + \sigma \Delta B_t^0$.
    for $k = 1, \ldots, N_{sim}$ **do**
        Draw states for N agents $\{\eta_i^k : i = 1, \ldots, N\}$ from density g_t at $t = n\Delta t$.
        At state (z_t, η_t^k), compute equilibrium variables $v_t^k = \hat{v}(z_t, \eta_t^k)$
    end for
    Calculate the average variables $\bar{v}_t = \frac{1}{N_{sim}} \sum_{k=1}^{N_{sim}} v_t^k$
    At each grid point $a_m \in \underline{\boldsymbol{a}}$, calculate consumption $\hat{\mu}_a((a_m), z_t, \bar{v}_t)$.
    Construct the transition matrix $\mathcal{A}_t$ using finite difference on the grid $\underline{\boldsymbol{a}}$
    Update g_t by implicit method: $g_{t+\Delta t} = (I - \bar{\mathcal{A}}_t^\top \Delta t)^{-1} g_t$
end for

Approximate ω by Neural Network (Feed Forward, Fully Connected)

- Let $\boldsymbol{X} = (z, K, \{\eta_i\}_{i \leq I})$. We approximate surplus $\omega(\boldsymbol{X})$ by neural network with form:

$$
\begin{aligned}
\boldsymbol{h}^{(1)} &= \phi^{(1)}(W^{(1)}\boldsymbol{X} + \boldsymbol{b}^{(1)}) && \text{...Hidden layer 1} \\
\boldsymbol{h}^{(2)} &= \phi^{(2)}(W^{(2)}\boldsymbol{h}^{(1)} + \boldsymbol{b}^{(2)}) && \text{...Hidden layer 2} \\
&\vdots \\
\boldsymbol{h}^{(H)} &= \phi^{(H)}(W^{(H)}\boldsymbol{h}^{(H-1)} + \boldsymbol{b}^{(H)}) && \text{...Hidden layer H} \\
\omega &= \sigma(\boldsymbol{h}^{(H)}) && \text{...Variable}
\end{aligned}
$$

- Terminology (our parameter choices are in blue):
  - H: is the number of *hidden layers*, $(H = 4)$
  - Length of vector $\boldsymbol{h}^{(i)}$: number of *neurons* in hidden layer i, $(Length = 32)$
  - $\phi^{(i)}$: is the *activation function* for hidden layer i, $(\phi^i = tanh)$
  - σ: is the *activation function* for the final layer, $(\sigma = tanh)$
  - $\boldsymbol{\Theta} = (W^1, \ldots W^{(H)}, b^{(1)}, \ldots, b^{(H)})$ are the *parameters*,

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Algorithm ("EMINN" or "Economic Deep Galerkin")

Initialize neural networks $\{\hat{\omega}, \hat{\Omega}, \hat{q}^n, \hat{\boldsymbol{\sigma}}\}$ with parameters $(\Theta_\omega, \Theta_\Omega, \Theta_{\boldsymbol{\sigma}_q}, \Theta_q)$.

while Loss > tolerance **do**

  Sample M new training points: $(\boldsymbol{X}^m = (z^m, \zeta^m, K^m, (\eta_i^m)_{i \leq I}))_{m=1}^N$.

  Calculate equilibrium at each training point $\boldsymbol{X}^m$ given current $\{\hat{\omega}, \hat{\Omega}, \hat{q}^n, \hat{\boldsymbol{\sigma}}\}$:

  (a) Compute $(\hat{\omega}_i^m)_{i \leq I}$ using current approximation $\hat{\omega}$ evaluated at $\boldsymbol{X}^m$.

  (b) Compute $(q^k)^m$ and $(\Xi_i^m)_{i \leq I}$ using $(\hat{\omega}_i^m)_{i \leq I}$.

  (c) Solve for $(\boldsymbol{\theta}^m, \boldsymbol{\sigma}_{\boldsymbol{\eta}}^m, s^m)$ given the current approximations for $\{\hat{\omega}, \hat{\Omega}, \hat{q}^n, \hat{\boldsymbol{\sigma}}\}$.

  (d) Compute $\boldsymbol{\mu}_\eta, \boldsymbol{\mu}_q, \boldsymbol{r}$.

  Loss: $\hat{\mathcal{L}}(\boldsymbol{X}) = \frac{1}{M} \sum_m \left(\kappa_\omega |\hat{\mathcal{L}}_{\omega_h}(\boldsymbol{X}^m)| + \kappa_\Omega |\hat{\mathcal{L}}_{\omega_h}(\boldsymbol{X}^m)| + \kappa_\sigma |\hat{\mathcal{L}}_{\boldsymbol{\sigma}}(\boldsymbol{X}^m)| + \kappa_q |\hat{\mathcal{L}}_q(\boldsymbol{X}^m)| \right)$

  Update $(\Theta_\omega, \Theta_\Omega, \Theta_{\boldsymbol{\sigma}_q}, \Theta_q)$ using stochastic gradient descent to decrease $\hat{\mathcal{L}}(\boldsymbol{X})$. Back

end while

Imposing Market Clearing

- Deep learning difficulty—adding market clearing loss functions creates instability.
  $\Rightarrow$ We want to impose market clearing in the sampling.

- Key point 1: Wealth share space makes market clearing sampling easier:
  - If we sample in the "a" space, then imposing market clearing means restricting a to a hyperplane that depends upon equilibrium prices. E.g. $\sum_i a_i = \hat{q}(\boldsymbol{X})K$
  - Instead, we solve for the equilibrium ω as a function of $(z, \zeta, K, (\eta_i)_{i \leq I})$, which means capital market clearing is satisfied by $\sum_i \eta_i = 1$.

- Key point 2: Solving for the equilibrium Ξ avoids nesting neural networks:
  - If we worked with $\xi(a, \boldsymbol{X})$, then we need to compute $\xi_i = \hat{\xi}(\eta_i \hat{q}(\boldsymbol{X})K, \boldsymbol{X})$.
  - But we work directly with $\Xi(\boldsymbol{X})$, which does not require nested neural networks.

- Better to move market clearing conditions out of the loss function.

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Additional Implementation Details

	Description
Neural Network	
(i) Structure	Feed-forward with 4 hidden layers and 80 neurons each layer
(ii) Initialization	Random
Sampling	
(i) (z, ζ, K)	Uniform sampling
(ii) $(\eta_i)_{i \leq N}$	Moment sampling: sample moments of the distribution and then population distributions satisfying moments
Loss Function	
(i) Learning rate	0.0005

- ⋆ Average training error $\approx 10^{-5}$
- ⋆ Test on [Lucas, 1978], [Basak and Cuoco, 1998], [Brunnermeier and Sannikov, 2014] More

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Training Loss

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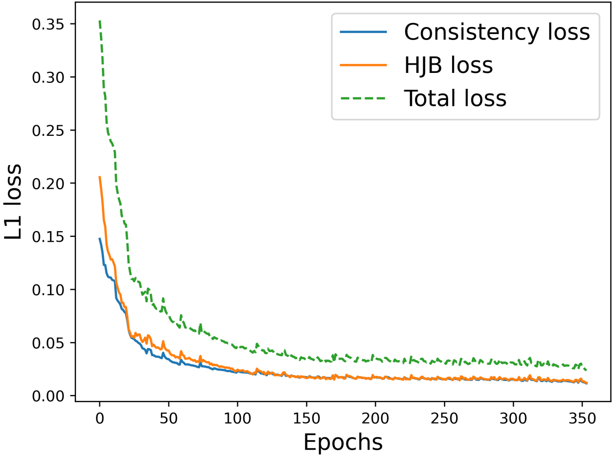

Algorithm Success

	Euler Equation	Other Equations
MSE	5.0×10^{-5}	0.9×10^{-4}

- ⋆ Training (10-20 agents) takes around 20-40 minutes on a laptop.
- ⋆ Simulating (100 grid points) takes around 15-30 minutes on a laptop.

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Roadmap

Three Testable Models

Model	Layers	Neurons	Learning Rate	Error
"As-if" Complete Model	4	64	0.001	1.0×10^{-5}
[Basak and Cuoco, 1998]	5	64	0.001	4.9×10^{-4}
[Brunnermeier and Sannikov, 2014]	5	32	0.001	7.0×10^{-5}

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Representative Agent Model ([Lucas, 1978])

Suppose there are no financial frictions: $\Psi(a_{i,t}, b_{i,t}) = 0$ and no investment.

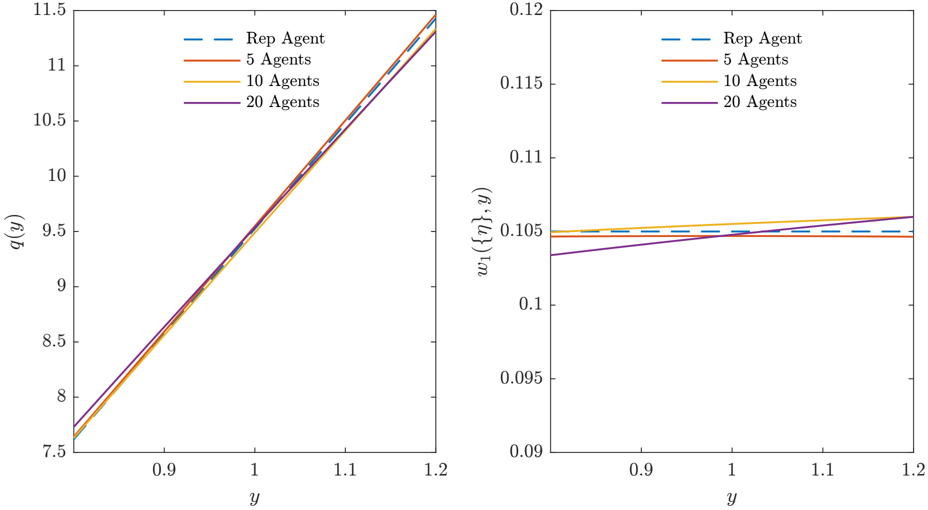
In this case, households are identical so there is a "representative agent".

The model has closed form solution:

$$q(y) = \frac{y}{\rho + (\gamma - 1)\mu - \frac{1}{2}\gamma(\gamma - 1)\sigma^2}$$

$$\omega(y) = \left[\rho + (\gamma - 1)\mu - \frac{1}{2}\gamma(\gamma - 1)\sigma^2\right]$$

[Lucas, 1978] Model solution. MSE: $< 10^{-4}$



As-if Complete Market Model, $\gamma = 5$, $\mu = 2\%$, $\sigma = 5\%$, $\rho = 5\%$.

Limited Participation Model ([Basak and Cuoco, 1998])

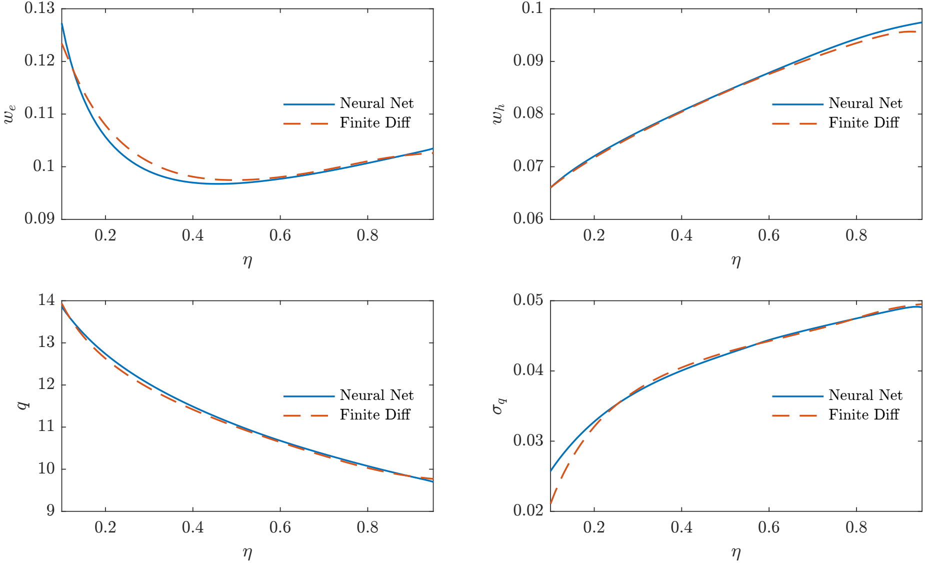
Two types of agents: experts (e) and households (h).

Expert sector can hold stocks and bonds.

Household sector can only hold bonds: $\Psi_h(a_{h,t}, b_{h,t}) = a_{h,t} - b_{h,t} = 0$.

State space is (y, η), where η is expert's wealth share.

[Basak and Cuoco, 1998] Model solution. L2 Loss: $< 10^{-5}$



2 Agents Limited Participation Model, $\gamma = 5, \rho_e = \rho_h = 5\%,\ \mu = 2\%,\ \sigma = 5\%.$

Productivity Gap Model ([Brunnermeier and Sannikov, 2014])

Two types of agents: experts (e) and households (h).

We allow households to hold capital but in a less productive way. The productivity of experts and households is z_h, z_e $(z_h < z_e)$ respectively. Their relative risk-aversion are both γ.

Output grows at exogenous drift $\mu_y = y\mu$, volatilty $y\sigma$, and experts cannot issse outside equities.

State space is (y, η), where η is expert's wealth share.

[Brunnermeier and Sannikov, 2014] Model solution. L2 Loss: $< 10^{-5}$

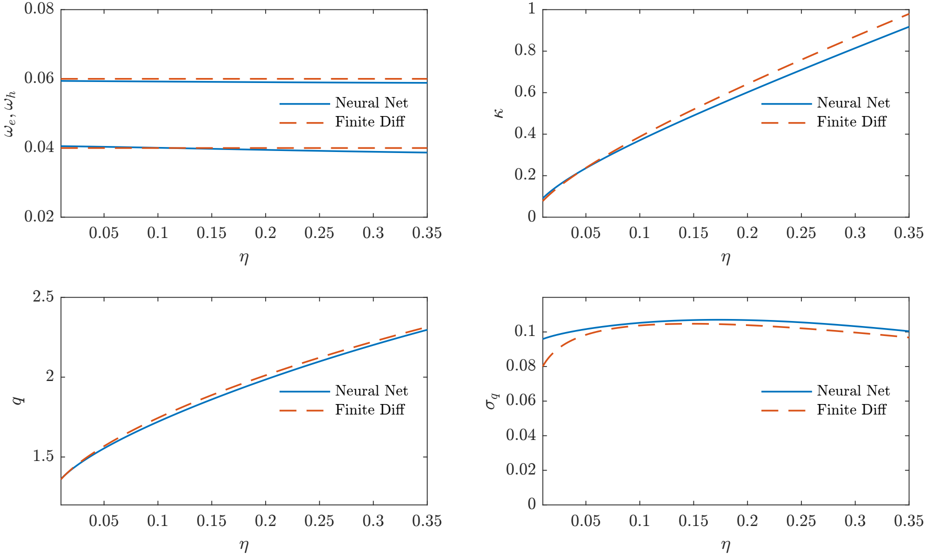
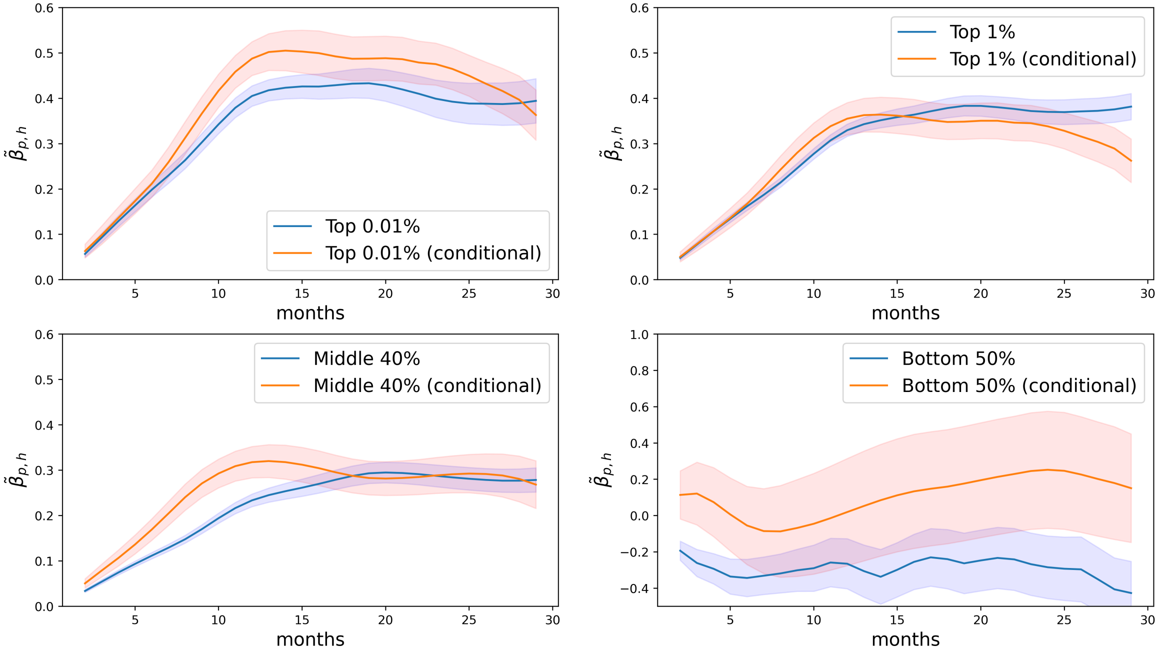


Figure: Solution to the model with productivity gap.



Top 0.01%
Top 0.01% (conditional)
Top 1%
Top 1% (conditional)
Middle 40%
Middle 40% (conditional)
Bottom 50%
Bottom 50% (conditional)
months
$\tilde{\beta}_{p,h}$

Table of Contents

Untargeted Moments

- Approximately match the risk premium, term premium, and investment rate. More
- Approximately match pension/insurance risk exposure from [Koijen and Yogo, 2022]:
  - Match exposure to stock market returns,
  - Directly match exposure to long-term bond returns. More
- Approximately match household portfolio choice and risk exposure:
  - Portfolio shares in risky, pension, and safe assets,
  - Directionally match local projections of household percentile wealth shares on shocks to financial intermediary net worth. More

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Key Spreads (Untargeted)

Untargeted moments	Data	Model
Risk premium - shadow (%)	8.5	7.3
Term premium (%)	0.7	0.8
Investment/capital rate (%)	14.0	11.8

Table: All values are annualized. The time period is from 2010 Q1 to 2025Q2.

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Fund Sector Risk Exposure (Untargeted)

Factor	Data (1999-2017)	Data (2010-2017)	Model
Stock market return	1.36	1.11	1.84
	(0.19)	(0.08)	(0.00)
Long term bond return	-0.01	-1.28	-1.67
	(0.32)	(0.43)	(0.00)
Observations	228	96	1499

Table: Risk exposure of fund sector. The table reports betas from a factor regression of fund equity returns on stock returns and long-term bond returns. Data values are taken from [Koijen and Yogo, 2022], and corresponds to the period 2010-2017. Baseline refers to the economy where fund is allowed to hold capital. Regulated refers to the economy where the fund is shut out of the capital market, and holds only government bonds. Heteroskedasticity adjusted standard errors are given in parenthesis.

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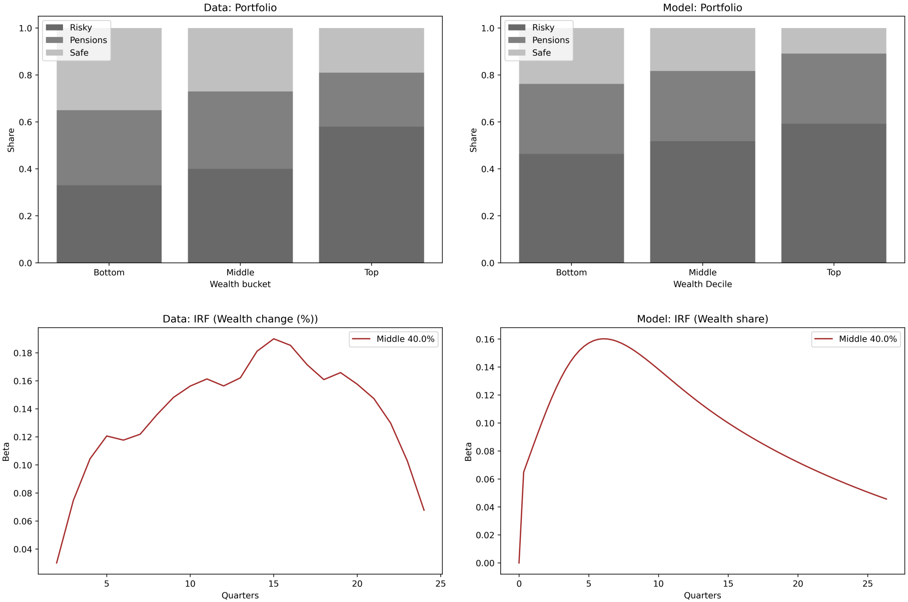
Portfolios And Local Projections (Untargeted)



Regulatory Parameters

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	Baseline	Counterfactual: Fund-regulated	Counterfactual: Bank-regulated
Bank regulation parameters			
ψ_b^k	0.1	0.1	0.15
ψ_b^m	2.0	2.0	2.0
$\bar{\theta}_b^k$	1.0	1.0	1.0
$\bar{\theta}_b^m$	5.0	5.0	5.0
Fund regulation parameters			
ψ_f^k	0.1	0.1	0.1
ψ_f^m	2.0	2.0	2.0
$\bar{\theta}_f^k$	10.0	1.0	10.0
$\bar{\theta}_f^m$	5.0	5.0	5.0

Table: Regulatory parameters across different regulatory regimes.

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Sector Level: Response to Negative TFP Shocks

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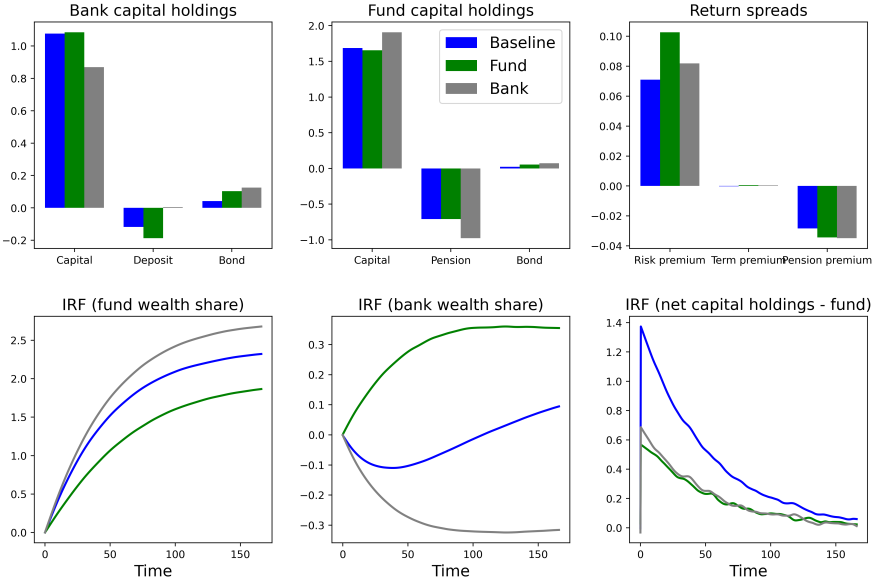
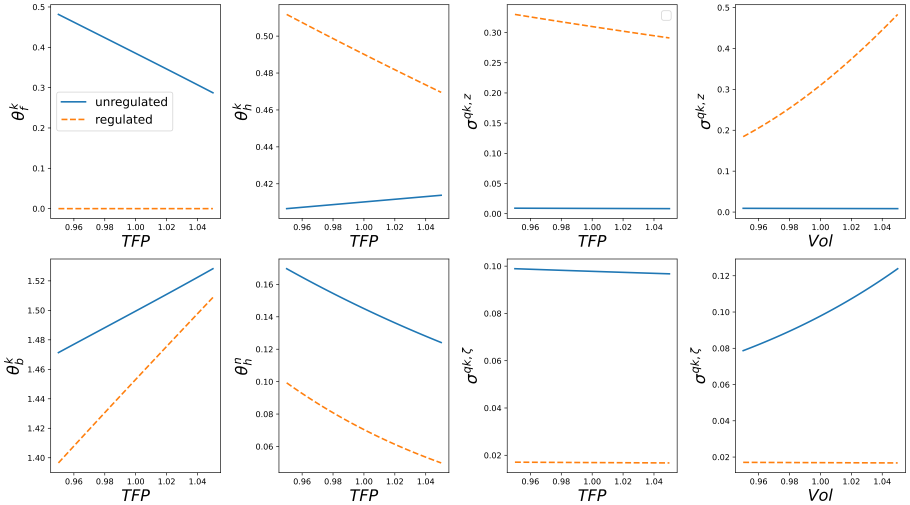


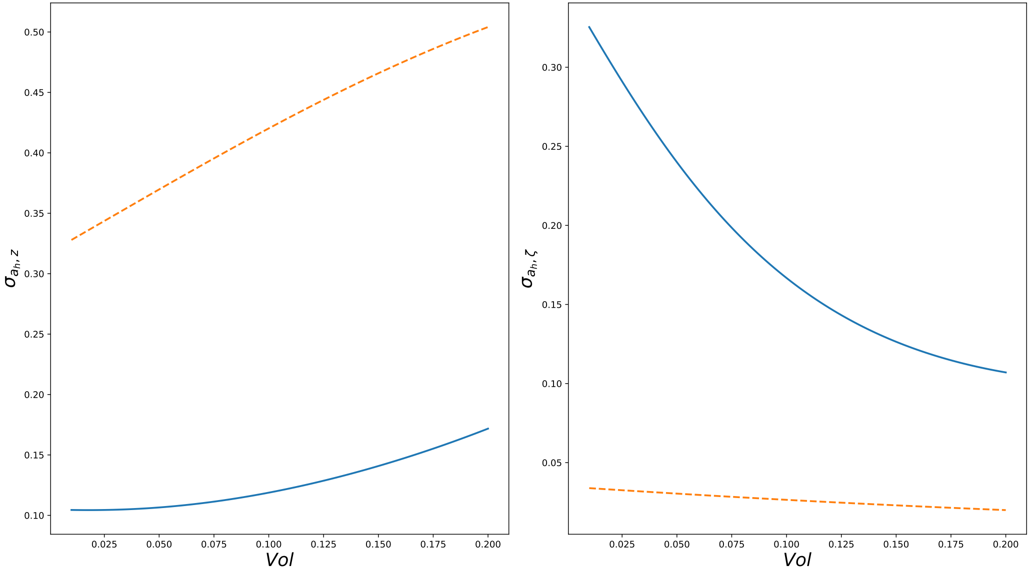
Fund Regulation Determines Where Endogenous Volatility Appears



Back Volatilities Asset Prices Spreads Welfare

Funds Insure Household TFP-risk But Not Volatility-risk

Prices Spreads



Baseline Calibration to 2010Q1-2024Q4 Back

Targeted moments	Parameter	Value	Target data moment	Target
TFP mean reversion	β_z	0.30	Literature	-
TFP volatility	σ_z	0.10	Output growth vol = 0.034	
Depreciation	δ	0	Output growth =0.025	0.0263
Investment friction	κ	95	Investment vol =0.076	0.086
Household risk aversion	γ, Γ	1	Literature	-
Bank risk aversion	γ_b	0.85	Capital Sharpe Ratio = 1.44	1.64
Fund risk aversion	γ_f	0.10	Annuity return = 0.035	0.033
Discount rate (hh.)	ρ_h	0.05	Literature	-
Discount rate (fund)	ρ_e	0.05	Literature	-
Discount rate (bank)	ρ_b	0.05	Literature	-
Capital constraint	$\bar{\psi}_k$	1e-5	Median hh cap. share =0.40	0.42
Death shock intensity (hh.)	λ_h	0.10	Average life =35	10yrs
Exit shock intensity (fund)	λ_f	0.20	Equity recap.	5yrs
Exit shock intensity (bank)	λ_b	0.20	Equity recap.	5yrs
Transfer weight (hh.)	ϕ_h	0.03	Median hh. annuity share = 0.37	0.40
Transfer weight (fund)	ϕ_f	0.16	Fund A/E =1.40	1.45
Transfer weight (bank)	ϕ_b	0.10	Bank A/E =1.50	1.53
Bond maturity	λ_m	0.033	LT bond maturity	30 yrs

Equilibrium Prices

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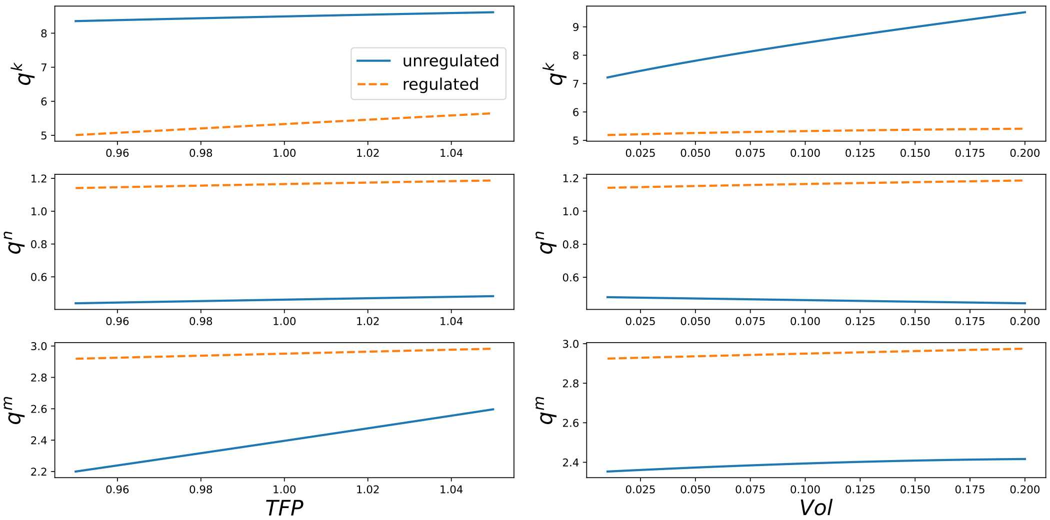
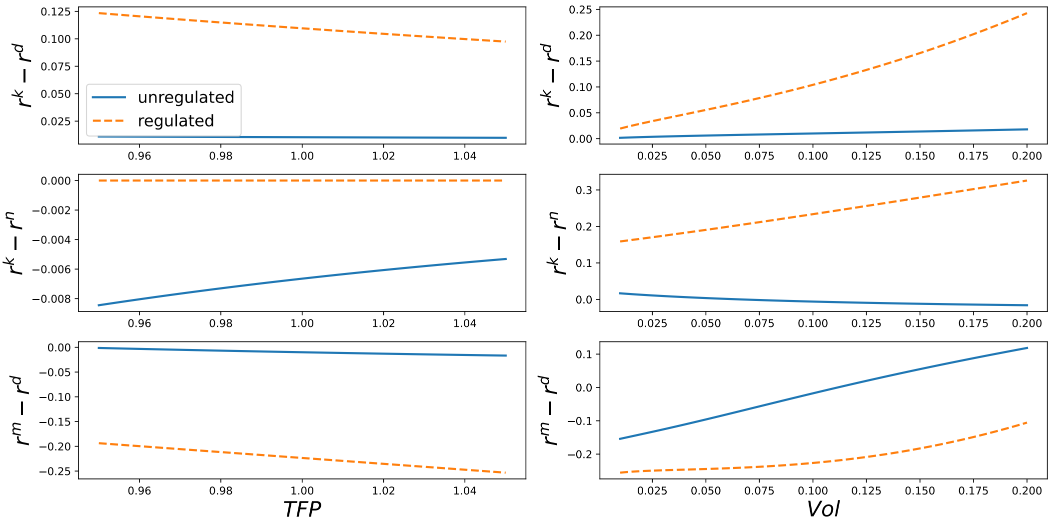


Equilibrium Spreads

Back



Portfolio Choices For Different Wealth Share

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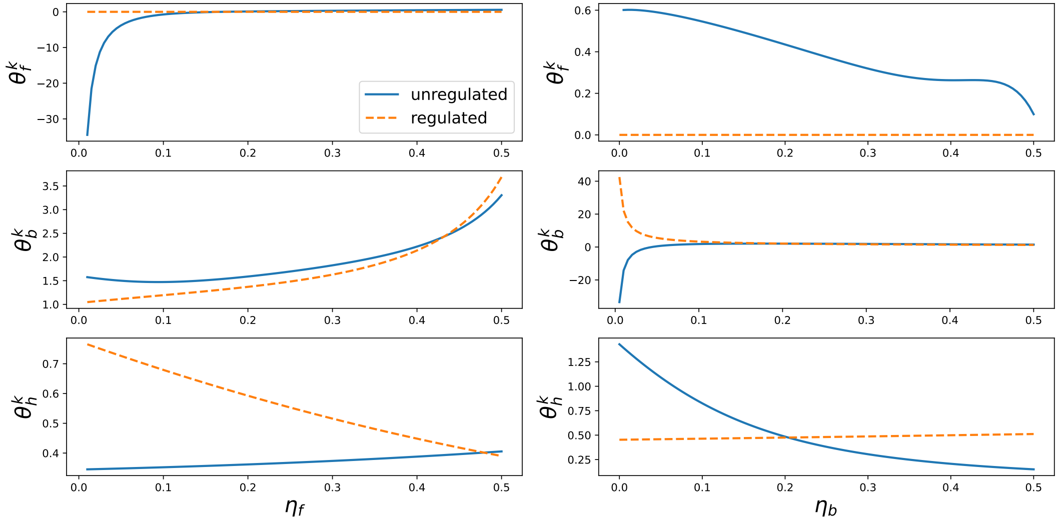

Spreads For Different Wealth Shares

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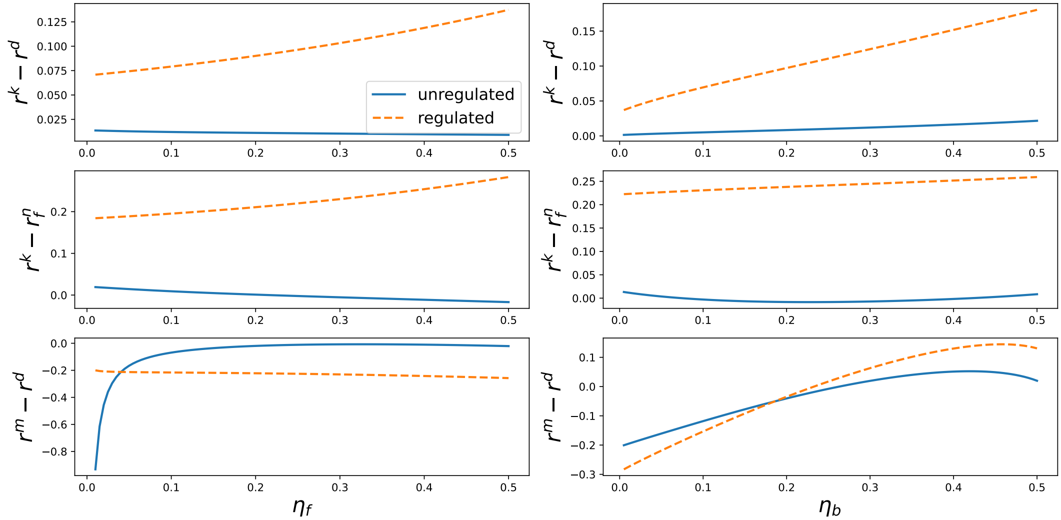

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Q. How Does Asset Pricing Impact Inequality? Within Households

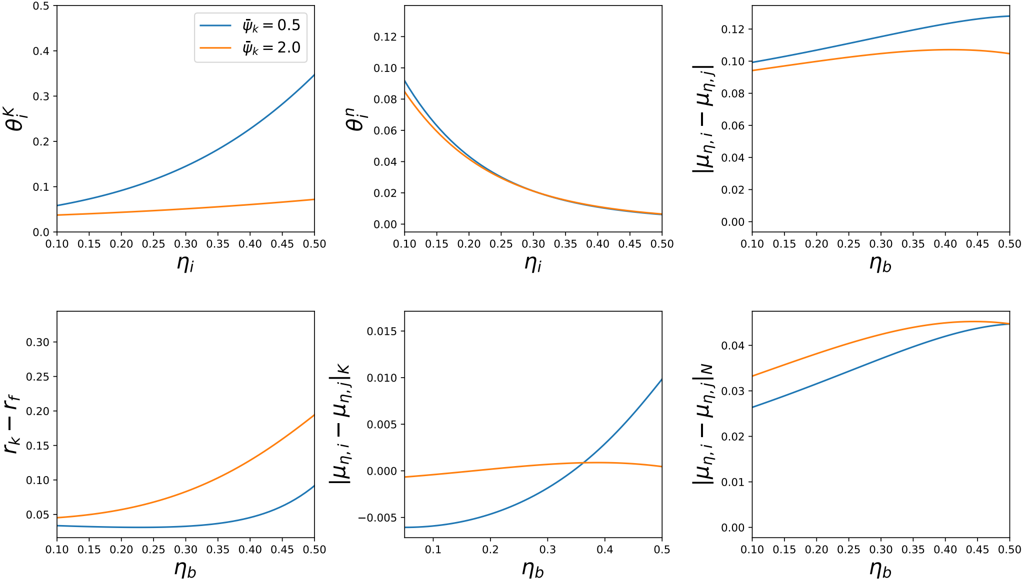
⋆ Difference between the drift of the wealth share of any two households i and j is:

$$\mu_{\eta_j,t} - \mu_{\eta_i,t} = \underbrace{(\theta^k_{j,t} - \theta^k_{i,t})(r^k_t - r^l_t - \boldsymbol{\sigma}^k_{q,t} \cdot \boldsymbol{\sigma}^k_{q,t})}_{=:(\mu_{\eta_j,t} - \mu_{\eta_i,t})^K} + \underbrace{(\theta^n_{j,t} - \theta^n_{i,t})(r^n_t - r^l_t - \boldsymbol{\sigma}^k_{q,t} \cdot \boldsymbol{\sigma}^n_{q,t})}_{=:(\mu_{\eta_j,t} - \mu_{\eta_i,t})^N}$$
$$- (\omega_j - \omega_i) + \varphi_h \lambda \left(\eta^{-1}_{j,t} - \eta^{-1}_{i,t}\right)$$

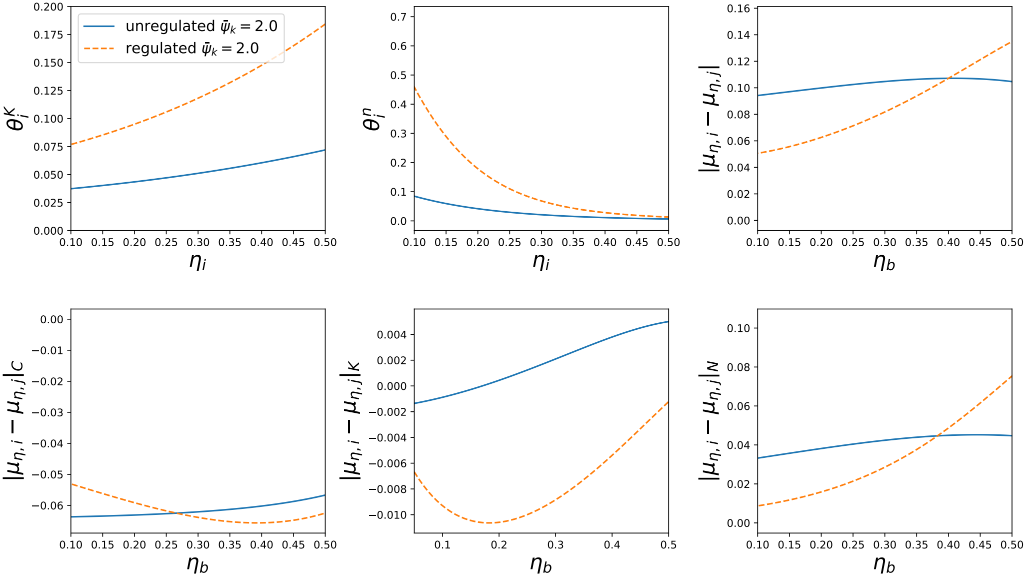
1. Participation constraint: low wealth agents hold less capital and earn less risk premium. (θ^k_i is agent i's wealth share in capital)
2. Pension needs: low wealth agents save through low return pensions. (θ^n_i is agent i's wealth share in pensions)
3. Consumption: low wealth agents consume less to escape participation constraint.
4. Redistribution: through death (and wealth taxes).

Inequality Impact on Asset Pricing

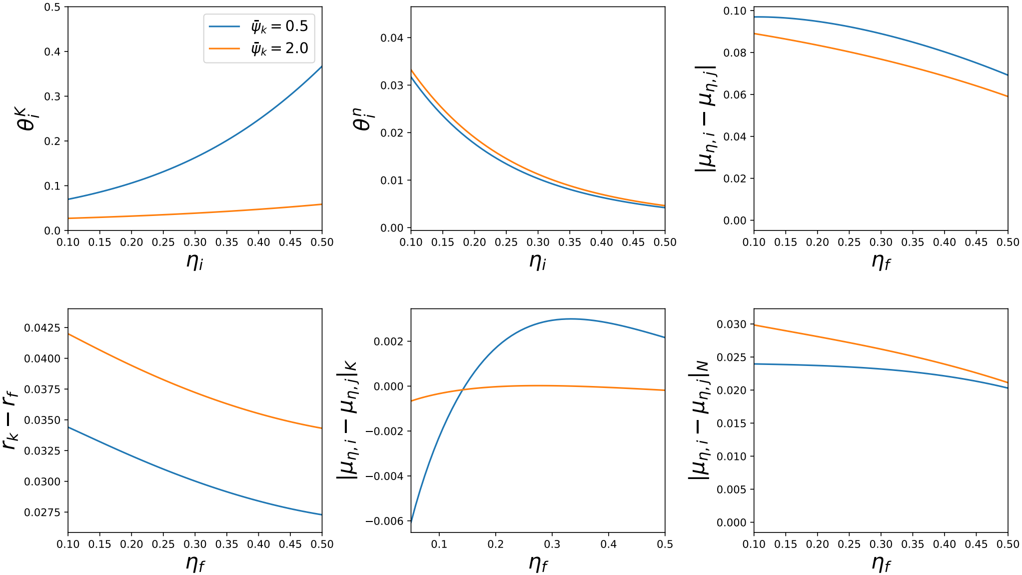
Inequality Decomposition: Wealthy Household Earn Higher Returns



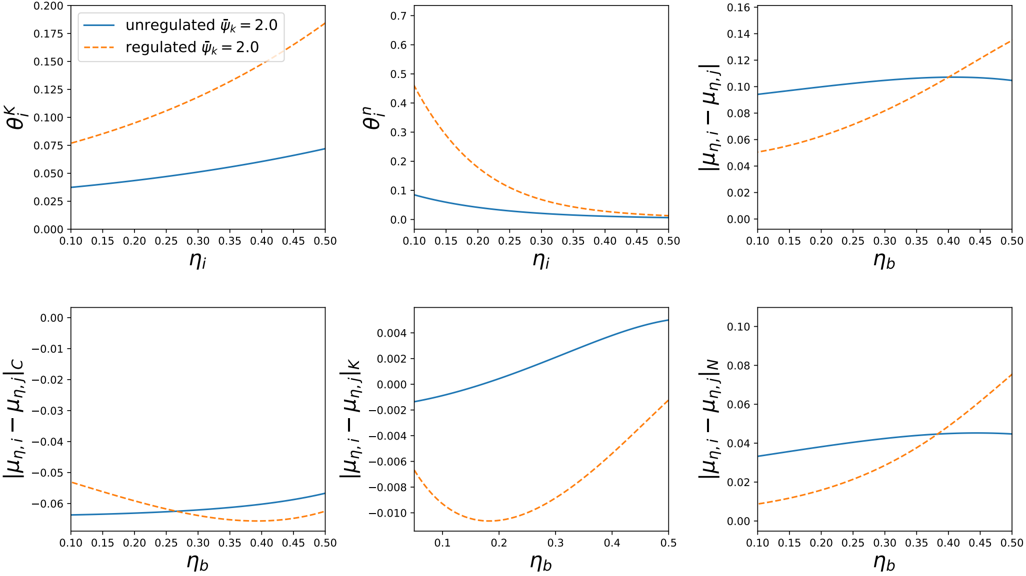
Inequality Decomposition: Segmentation Has Ambiguous Impact



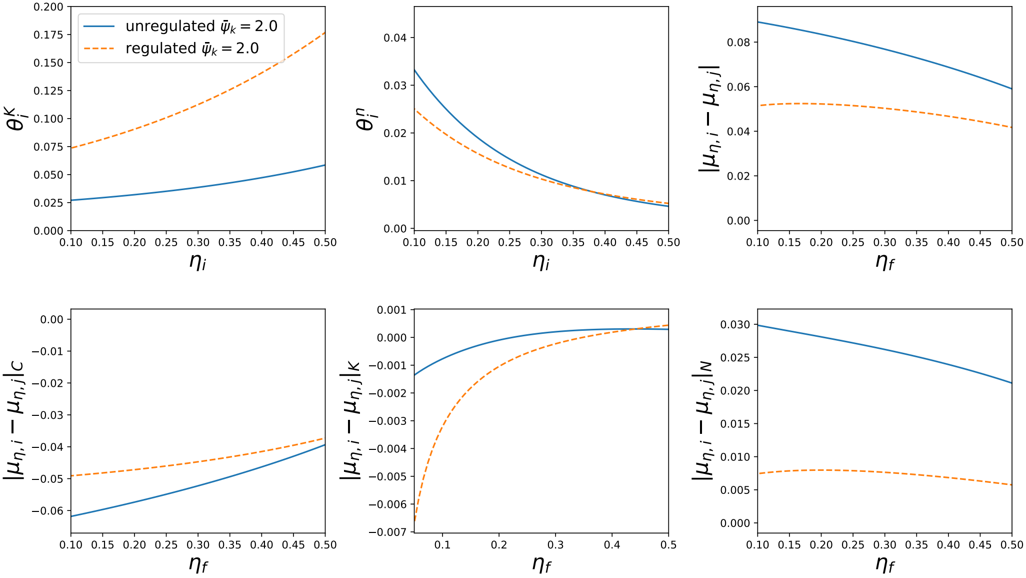
Decomposition of Inequality Evolution



Decomposition of Inequality Evolution: Regulation



Decomposition of Inequality Evolution: Regulation



Q. How Does Asset Pricing Impact Inequality? Within Households

- Difference between the drift of the wealth share of any two households i and j is:

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$$- (\omega_j - \omega_i) + \frac{\tau\lambda}{I-1}\left(\frac{1}{\eta_{j,t}} - \frac{1}{\eta_{i,t}}\right)$$

1. Participation constraint: low wealth agents hold less capital and earn less risk premium. (θ^k_i is agent i's wealth share in capital)
2. Pension needs: low wealth agents save through low return pensions. (θ^n_i is agent i's wealth share in pensions)
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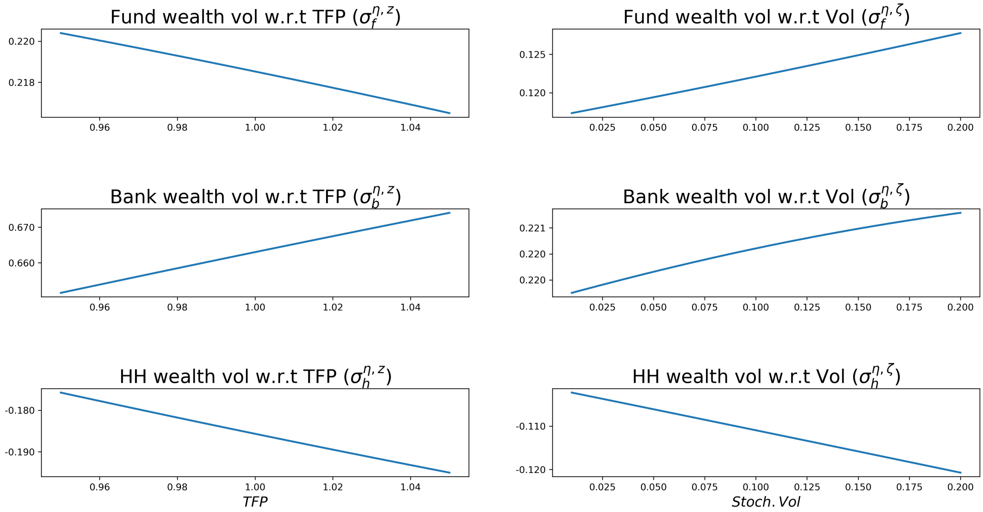
Three Mechanisms

Q. How do changes to TFP and volatility change risk exposure between households, bankers, and fund managers?

Q. How does asset pricing impact inequality amongst households?

Q. How does inequality impact asset pricing amongst households?

Q. How does risk exposure change with TFP and volatility?



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Inequality Evolution (Different Participation Constraints, Just Banks)

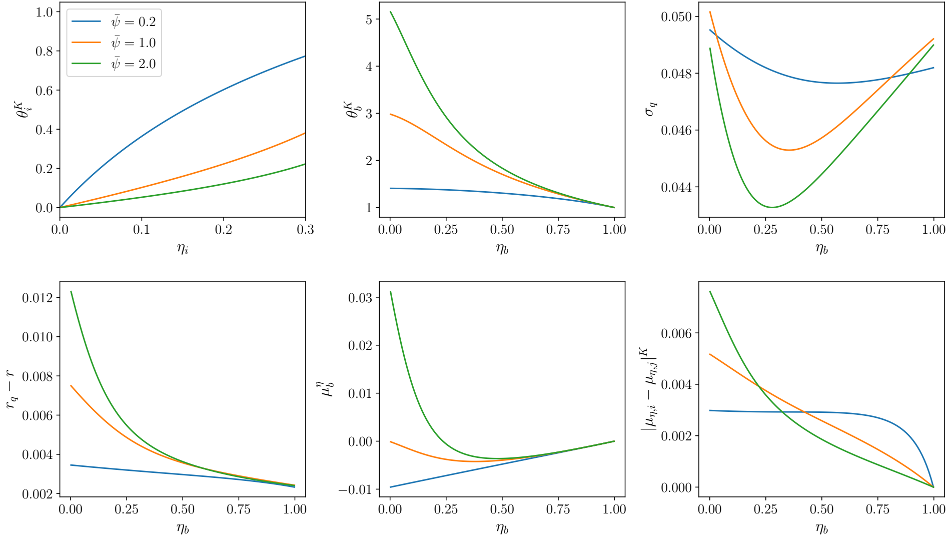

Figure: Equal household wealth distribution. $\rho_b = 0.04, \rho_h = 0.03, \mu = 0.02, \sigma = 0.05$.

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Inequality Evolution (Different Participation Constraints, With Funds)

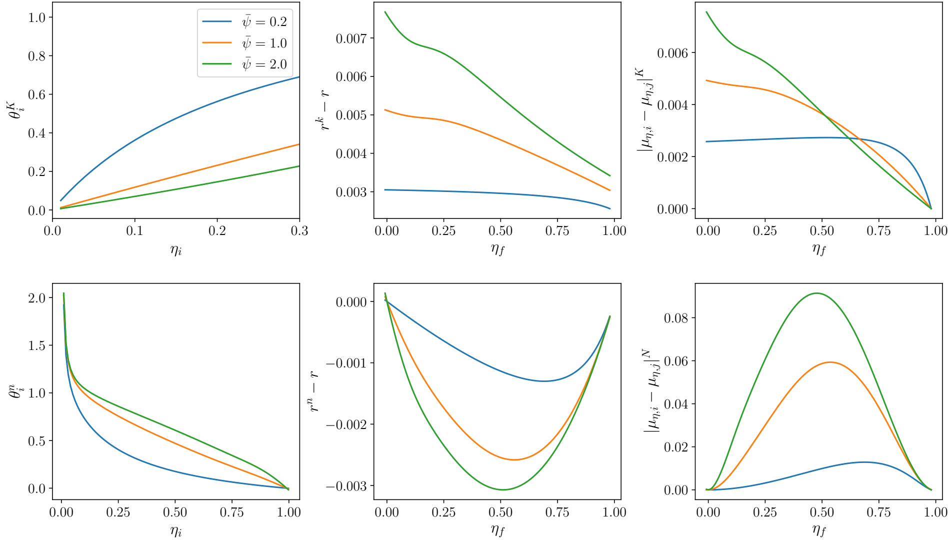

Figure: Equal household wealth distribution. $\rho_b = \rho_f = 0.04, \rho_h = 0.03, \mu = 0.02, \sigma = 0.05.$

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Q. How Does Inequality Impact Asset Pricing?

- Aggregate capital demand is:

$$\sum_{i=1}^{I-1} \theta_{i,t}\eta_{i,t}A_t + \theta_{b,t}\eta_{b,t}A_t + \theta_{f,t}\eta_{f,t}A_t$$

- The capital market participation breaks aggregation in household sector.

- More unequal distribution

  ⇒ households purchase more capital when expert wealth drops.

  ⇒ wealth distribution influences whether household acts as a "buffer" in recessions.

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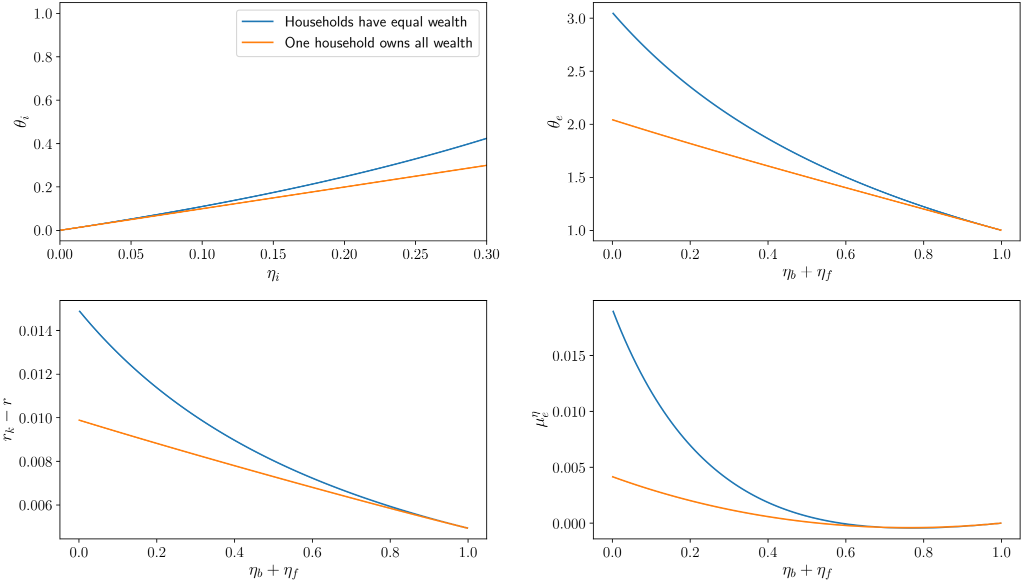
Equilibrium at Different Wealth Distributions



Impulse Response: Inequality is Procylical

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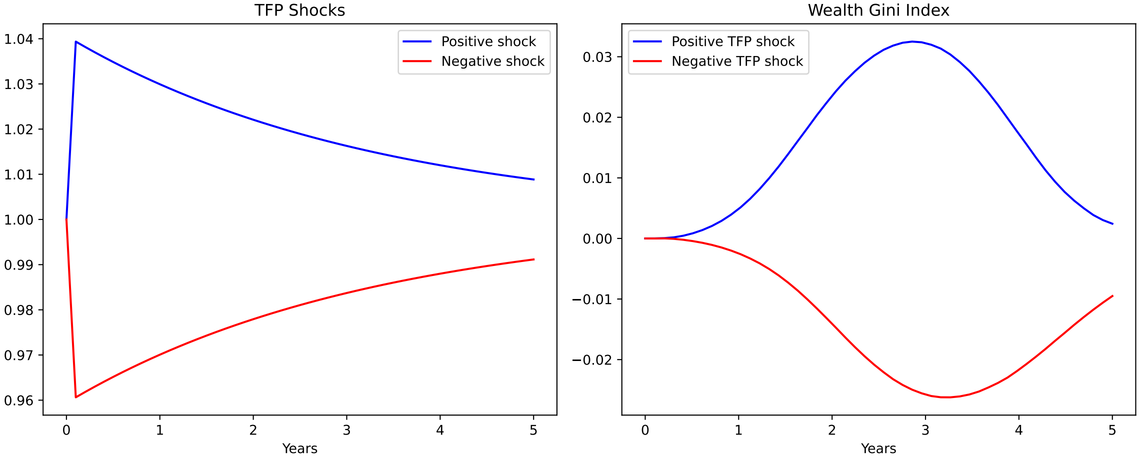

Impulse Response: Prices and Returns

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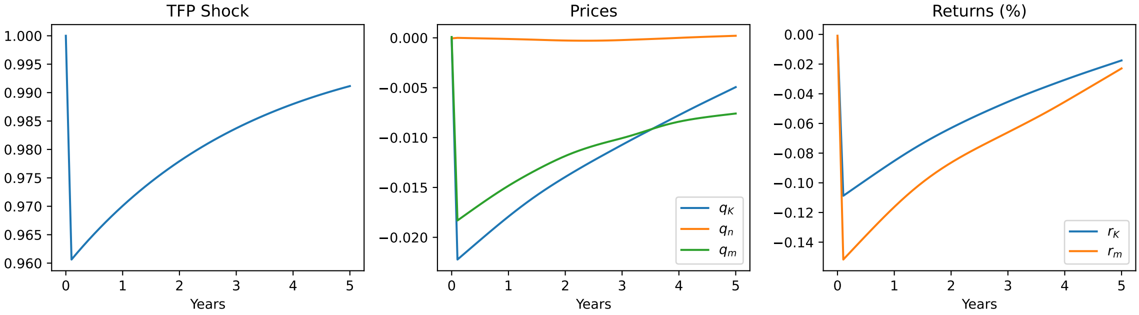

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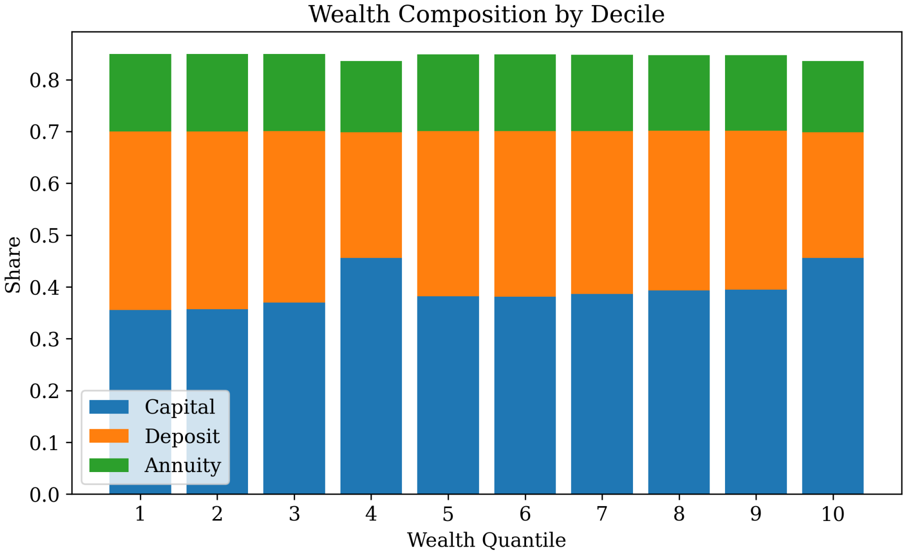
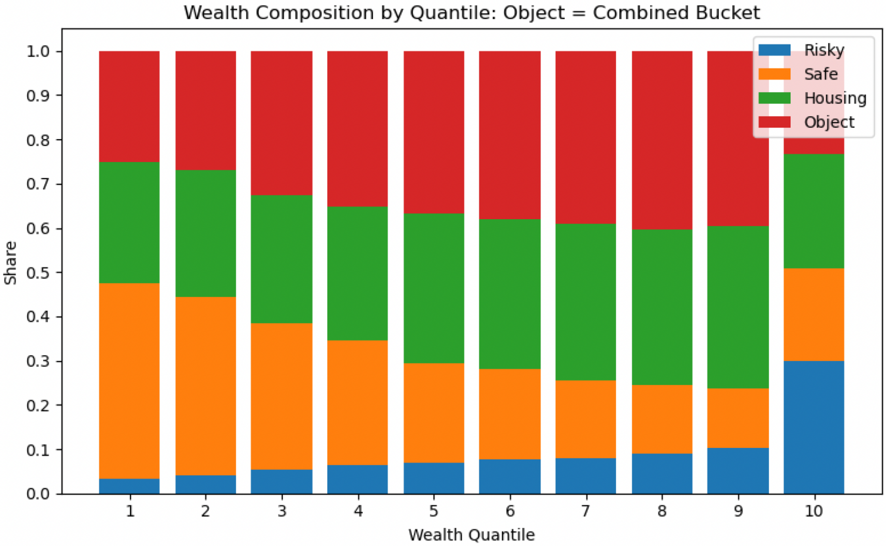
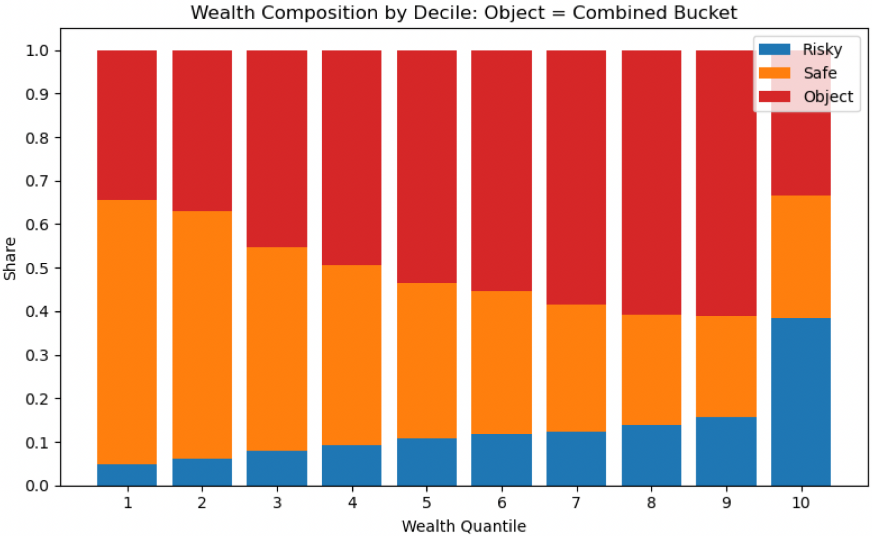
Portfolio Shares (Model)



Portfolio Shares (Data)



Portfolio Shares (Data)



Parameter Values

Parameter	Symbol	Value	Target	Data	Model
Risk aversion	γ	7.0	Capital risk premium	3.4%	3.3%
Households' Discount rate	ρ_h	0.03	Literature	-	-
Intermediaries' Discount rate	ρ_e	0.03	Literature		
TFP Reversion rate	β_z	0.02	Data - TFP persistence		
TFP Volatility	σ_z	0.05	Data - fundamental volatility	-	-
ζ Reversion rate	β_ζ	0.02	Data		
ζ Volatility	σ_ζ	0.30	Data	-	-
Portfolio constraint	$\bar{\psi}$	87.5	Bottom 50 pctl. wealth share	3.3%	2.2%
Number of agents	N	30			

Table: Source: Data for risk premium is computed from [Welch and Goyal, 2008]. The wealth share data is compouted from [Saez and Zucman, 2016]. The time period is from 1976 till 2020.

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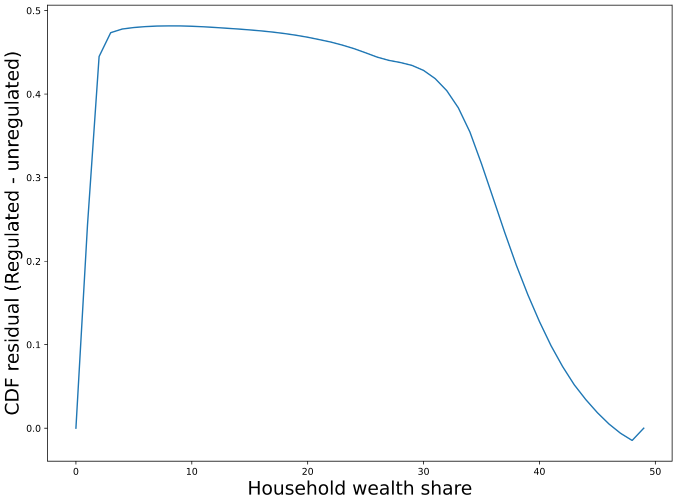
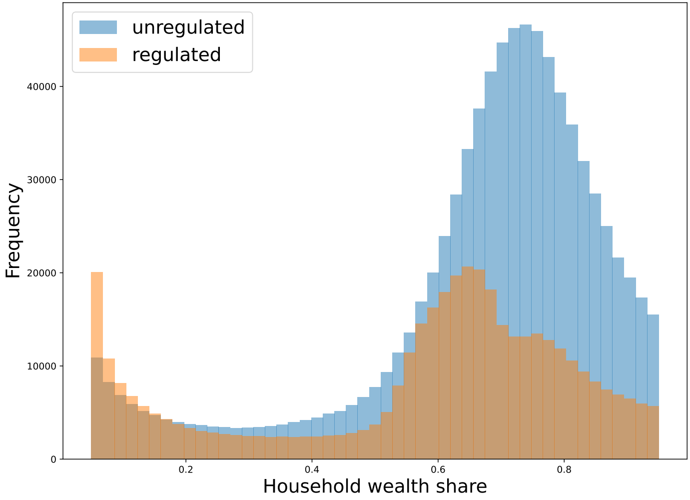
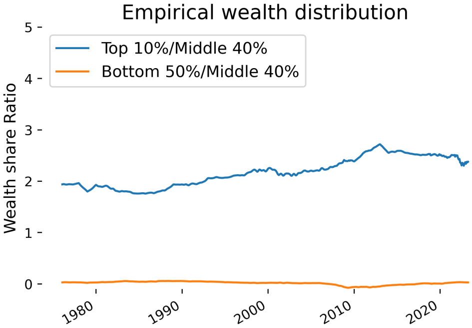
Stationary Model: Ergodic Wealth Distribution Back

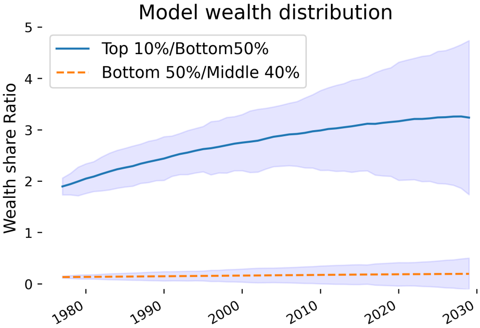


Stationary Model: Ergodic Wealth Distribution



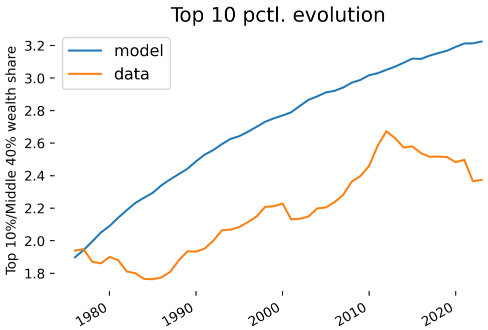
Non-Stationary Model: Can Explain Significant Inequality Change

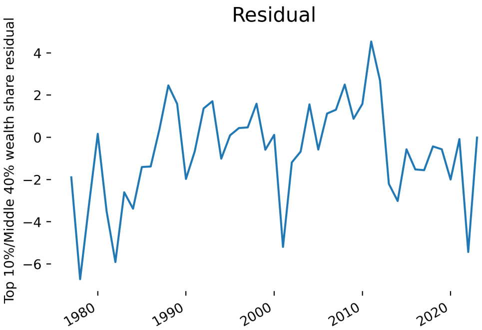




Is Inequality Stationary? Residuals Back

Residuals Back





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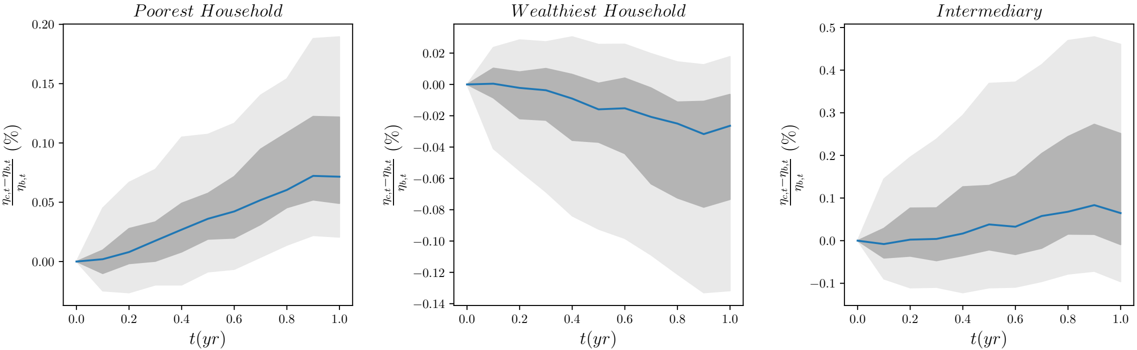
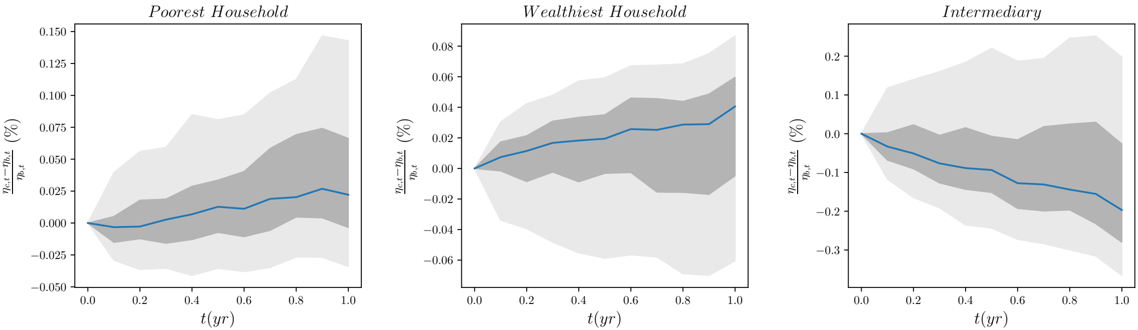
Distributional Impact of Macroprudential Policy

- ⋆ We introduce an exogenous leverage constraint on financial expert: $\theta_e \leq \bar{\ell}$.
- ⋆ Then simulate a collection of recession paths and track the distribution evolution.

Looser Leverage Constraint ($\theta \leq 2.0$)



Tighter Leverage Constraint ($\theta \leq 1.5$)



Approximate Solutions

- Consider HJB equation for the Merton problem (consumption and portfolio choice):

$$\rho V(a) = \max_{c,\theta} u(c) + V'(a)((r + (\bar{R} - r)\theta)a - c) + \frac{1}{2}\sigma^2\theta^2 a^2 V''(a) \tag{12}$$

- Suppose V_0 is the exact solution of Merton's problem, we plug in a scaled solution $k^{-\gamma}V_0$:

$$\rho k^{-\gamma} V_0 = \frac{c^{1-\gamma}k^{1-\gamma}}{1-\gamma} + k^{-\gamma}V_0'((r + (\bar{R} - r)\theta)a - kc) + \frac{1}{2}\sigma^2\theta^2 a^2 k^{-\gamma} V_0''(a) \tag{13}$$

  Which implies that the loss function (with no loss of generality, we use L1 loss here) will be:

$$Loss = \left|\left(k^{1-\gamma} - k^{-\gamma}\right)\right| \cdot \underbrace{\left|-cV_0' + \frac{c^{1-\gamma}}{1-\gamma}\right|}_{\text{Finite value}}$$

  - $\gamma < 1$, no problem because $k^{1-\gamma}$ will explode while $k^{-\gamma}$ vanishes as $k \to \infty$.
  - $\gamma > 1$, a very large k can be problematic because both $k^{1-\gamma}$ and $k^{-\gamma}$ vanish as $k \to \infty$.

- Hence, in the economically relevant case $\gamma > 1$, computer is very good at finding "cheat solution" by simply push value function to be very close to zero.

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