Estimating Nonlinear Heterogeneous Agents Models with Neural Networks

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Introduction

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- Structural estimation of HA models remains highly challenging
    - Estimation of parameters affecting the steady state (DSS) often unfeasible

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  - Methods typically linearize aggregates, abstracting from aggregate risk
- Yet, nonlinear aggregate dynamics are crucial to explain recent macro data
  - ZLB, fiscal limit, deep recessions, sudden inflation rise
- New approach to estimate nonlinear HA models based on neural networks

⇒ Likelihood estimation of a nonlinear HANK model using US data

⇒ Estimation includes parameters affecting the steady state (DSS)

The challenges of estimating HA models

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- This repetion is a major obstacle in estimating nonlinear HANK models.
  - Solving the models is too time-consuming as they feature many states
  - Solving the steady state (DSS) is also a computationally costly
  - Ex-ante unknown how many times you need to take these two steps

Structural estimation with neural networks

Key steps to make estimation of nonlinear HANK models with NNs feasible

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⇒ For likelihood estimation, we introduce the neural-network particle filter

- ⇒ To speed up likelihood evaluation
- ⇒ To mitigate computational inaccuracies of standard particle filters

Main takeaways

- Novel integrated neural-network based estimation procedure
  - Global solution and estimation of models with hundreds of state variables (HA, many countries or sectors) taking into account aggregate risk and nonlinear constraints possible

- Estimation of HANK models with aggregate nonlinearities and risk
  - Idiosyncratic income risk can be a key contributor to aggregate volatility

- Replication of:
  1. Global solution of a RANK model with a ZLB constraint
  2. The SSJ solution of Auclert et al's (2021) one-asset HANK model
  3. Estimation of a quantitative HANK (20+ params) despite filtering hurdles

- Estimation of a HANK model with a ZLB constraint using actual data

Literature

- **Solution methods for HA models**
  Krussel and Smith (1998); Algan et al., (2008); Reiter, (2009); Den Haan et al., (2010); Ahn et al., (2018); Boppart et al., (2018); Bayer et al., (2019); Auclert et al., (2021); and Winberry, (2021); Bhandari et al. (2023) and Bayer et al. (2024)

- **Estimation of HANK models with linearized aggregate dynamics**
  Challe et al., (2017); Bayer et al., (2024); Auclert et al., (2020); Lee, (2020); Auclert et al., (2021); Bilbiie et al., (2023); and Acharya et al., (2023)

- **NNs to solve complex dynamic macroeconomic models globally**
  Scheidegger and Billions (2019); Fernandez-Villaverde et al. (2020), Ebrahimi Kahou et al. (2021), Maliar et al. (2021), Azinovic et al. (2022), Duarte et al. (2024), and Valaitis and Villa (2024)

- **Solve HANK models with global methods**
  NN-based: Maliar and Maliar, (2020); Gorodnichenko et al., (2021); Fernandez-Villaverde et al., (2023); and Han et al., (2021)
  Alternative global methods: Schaab (2020); and Lin and Peruffo (2024)

Model solution

Key step: Pseudo-State Variables

- Solving a model amounts to approximating a set of policy functions:

$$\psi_t = \psi(\mathbb{S}_t | \underbrace{\tilde{\Theta}, \bar{\Theta}}_{\Theta})$$

- Key step: We could treat parameters $\tilde{\Theta}$ as pseudo-state variables

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- A NN can be trained to approximate the "extended" policy functions

$$\psi_t = \psi_{NN}(\mathbb{S}_t, \tilde{\Theta} | \bar{\Theta}; W)$$

What's a NN   Preliminaries

Training of the neural network to solve the model

- Define a loss function as the weighted sum of squared residual errors

$$\Phi^L = \sum_{k=1}^{K} \alpha_k [F_k(\psi(\mathbb{S}_t, \tilde{\Theta}|\bar{\Theta}))]^2.$$

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- The NN is trained by minimizing the average loss over batches of size B

$$\bar{\Phi}^L = \frac{1}{B} \sum_{b=1}^{B} \sum_{k=1}^{K} \alpha_k \left[F_k(\psi(\mathbb{S}_{t,b}, \tilde{\Theta}_b|\bar{\Theta})) \right]^2.$$

- This optimization step is repeated thousands of times
- Stochastic solution domain
  - At the beginning of every step, sample from the approximate ergodic distr.

Why Does This Approach Work?

- Approximate policy functions *without conditioning* on a parameter value
  - ⇒ We only need to train the NNs once, prior to estimation
  - ⇒ No need to re-train the NN multiple times at different parameter values

- NNs are well-suited to dealing with high-dimensional problems (scalability)
  - We expand the dimensionality of the state vector by adding parameters
  - But the increase in computational burden remains manageable

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  - We expand the dimensionality of the state vector by adding parameters
  - But the increase in computational burden remains manageable
- Once the *extended* policy functions are approximated, the model's solution at any parameter values can be obtained in a fraction of seconds
  - ⇒ Repetition of the solution and evaluation steps becomes manageable

Example: Linearized NK model

- Small off-the-shelf linearized three equation NK model with TFP shock
- Features a closed-form analytical solution

$$\begin{aligned}
\hat{X}_t &= E_t \hat{X}_{t+1} - \sigma^{-1} \left(\phi_\Pi \hat{\Pi}_t + \phi_Y \hat{X}_t - E_t \hat{\Pi}_{t+1} - \hat{R}^*_t \right) \\
\hat{\Pi}_t &= \kappa \hat{X}_t + \beta E_t \hat{\Pi}_{t+1} \\
\hat{R}^*_t &= \rho_A \hat{R}^*_{t-1} + \sigma(\rho_A - 1)\omega \sigma_A \epsilon^A_t
\end{aligned}$$

where $\hat{X}_t$ is the output gap, $\hat{\Pi}$ is inflation, R^*_t is the natural rate of interest, and ϵ^A_t is a TFP shock

Example: Solution to Linearized NK Model

- Solution to equation system depends on state variables and parameters

$$\begin{pmatrix} \hat{X}_t \\ \hat{\Pi}_t \end{pmatrix} = \psi \left(\underbrace{\hat{R}^*_t}_{\text{State } S_t}, \underbrace{\beta, \sigma, \eta, \phi, \theta_\Pi, \theta_Y, \rho_A, \sigma_A}_{\text{Parameters } \tilde{\Theta}} \right).$$

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- The analytical solution is given as

$$\hat{X}_t = \frac{1 - \beta\rho_A}{(\sigma(1-\rho_A) + \theta_Y)(1 - \beta\rho_A) + \kappa(\theta_\Pi - \rho_A)} \hat{R}^*_t,$$
$$\hat{\Pi}_t = \frac{\kappa}{(\sigma(1-\rho_A) + \theta_Y)(1 - \beta\rho_A) + \kappa(\theta_\Pi - \rho_A)} \hat{R}^*_t.$$

Solving the NK Model with a Neural Network

1. Approximate the policy function with a deep neural network:
   - Two policy functions:

$$\begin{pmatrix} \hat{X}_t \\ \hat{\Pi}_t \end{pmatrix} \approx \psi_{NN}(\underbrace{\hat{R}_t^*}_{\mathbb{S}_t}, \underbrace{\beta, \sigma, \eta, \phi, \theta_\Pi, \theta_Y, \rho_A, \sigma_A}_{\tilde{\Theta}})$$

2. Construct the loss function $\bar{\Phi}^L$ for optimization
   - Based on minimization of squared residual errors

$$err_{IS} = \hat{X}_t - \left(E_t\hat{X}_{t+1} - \sigma^{-1}\left(\phi_\Pi\hat{\Pi}_t + \phi_Y\hat{X}_t - E_t\hat{\Pi}_{t+1} - \hat{R}_t^*\right)\right)$$
$$err_{PC} = \hat{\Pi}_t - \left(\kappa\hat{X}_t + \beta E_t\hat{\Pi}_{t+1}\right)$$

   - Loss function weighs the errors and averages over batch size B of 500

$$\bar{\Phi}^L = \alpha_1 \frac{1}{B}\sum_{b=1}^{B}({err_{IS}}^b)^2 + \alpha_2 \frac{1}{B}\sum_{b=1}^{B}({err_{PC}}^b)^2$$

Solving the NK Model with a NN (cont'd)

3. Train the deep neural networks using stochastic optimization
   - 500 000 iterations with a batch size of 1000
     1. Draw parameters from a bounded parameter space

Parameters		LB	UB	Parameters		LB	UB
β	Discount factor	0.95	0.99	θ_Π	MP inflation	1.25	2.5
σ	Relative risk aver.	1	3	θ_Y	MP output	0.0	0.5
η	Inverse Frisch	0.25	2	ρ_A	Persistence TFP	0.8	0.95
φ	Price duration	0.5	0.9	σ_A	Std. dev. TFP	0.02	0.1

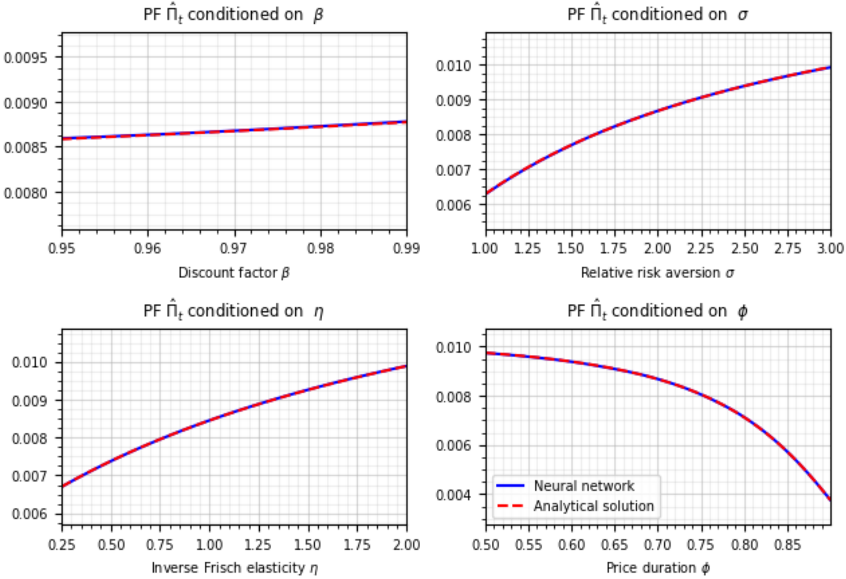
     2. Draw points from the state space via simulation

$$\hat{R}_t^* = \rho_A \hat{R}_{t-1}^* + \sigma(\rho_A - 1)\omega\sigma_A\epsilon_t^A$$

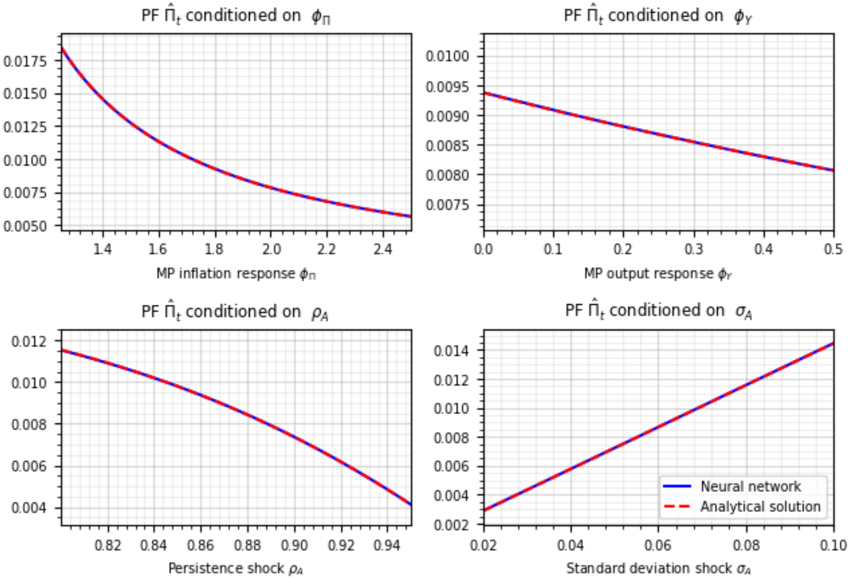
     3. Optimizer (ADAM) to choose the weights of the NN to minimize $\bar{\Phi}^L$

Training

NN: Inflation over the Parameter Space



NN: Inflation over the Parameter Space (cont'd)



Structural estimation

The Neural-Network Particle Filter

- Objective: evaluate the likelihood of the model More
  - For nonlinear models we can evaluate the likelihood using the particle filter
  - This filter requires tracking thousands of particles over multiple periods
  - Calculation is typically noisy and can be time-consuming
  - Some advantages of neural-network based particle filter
    1. Single likelihood evaluation can be done almost instantly
    2. Effectively smooths out noise from the particle filter

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- We run the particle filter to obtain the likelihood value at each draw
  - We employ the trained NNs to approximate the policy functions and the DSS
- The NN is trained to predict the likelihood evaluated at these draws
  - Loss function: the mean squared errors between the likelihood approximated by the NN and computed by the particle filter
  - Minimize the *average* loss function over batches
- This conclude the first optimization step $\Rightarrow \frac{S^{training}}{B}$ steps in one epoch
- Repeat for thousands of epochs by progressively lowering the learning rate

◂ Overfitting

Cutting through the noise of the particle filter

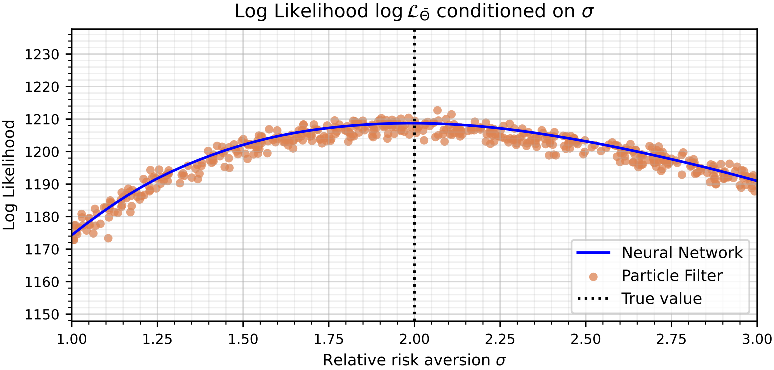

Figure: **Accuracy in likelihood evaluations: NN particle filter vs. standard particle filter.** The logorithm of the likelihood of the model as a function of the risk aversion parameter σ. The value of the fixed parameters are set to the middle of their bounds.

◂ Back

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- Real-time estimation and forecasting
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  - ⇒ Treat future data as pseudo-state variables when training the NN filter
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- Extensions to other non-likelihood methods (GMM or IRFs matching)

Four proofs of concept

Proofs of Concept

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3. Solution of nonlinear one-asset HANK model (Auclert et al. 2021)
   - Comparison with SSJ for a model with hundreds of state variables

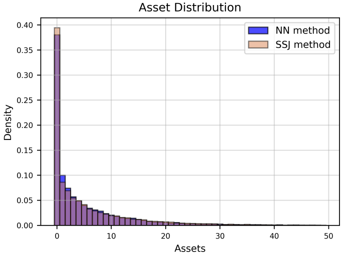
   ⇒ Model nearly linear ⇒ we replicate asset distrib., policy functions, and IRFs

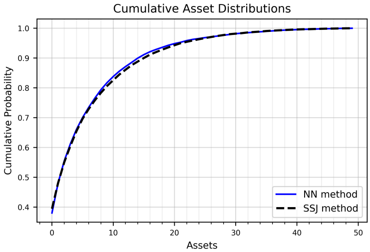
4. Estimation of a quantitative HANK model (Auclert et al. 2021)
   - More than 20 parameters estimated, also those affecting the DSS

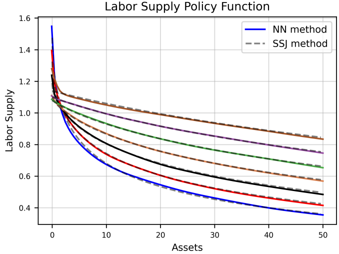
   ⇒ Estimates closely match the data-generating process — the NN-PF alleviates the bottleneck of the standard PF applied to highly dimensional models

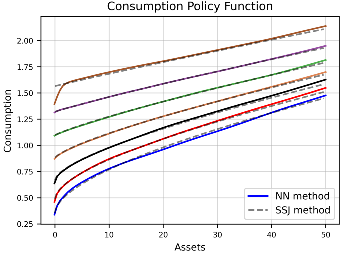
Proof of concept 3: Auclert et al.'s (2021) one-asset HANK model

Asset distribution and individual policy functions



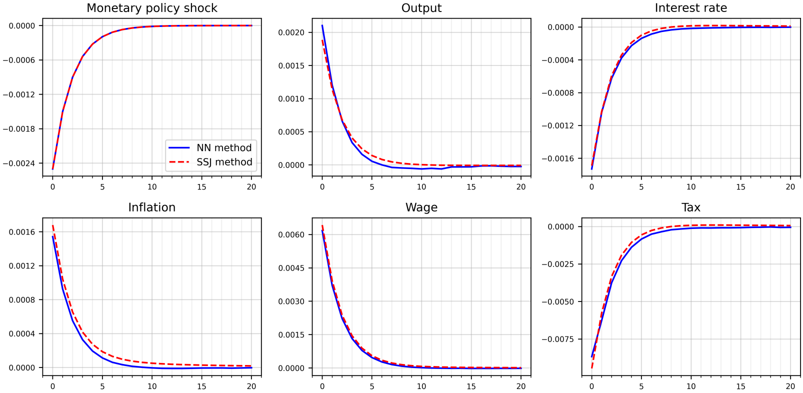






Small justifiable differences; Poor approx of policy functions at very unlikely states

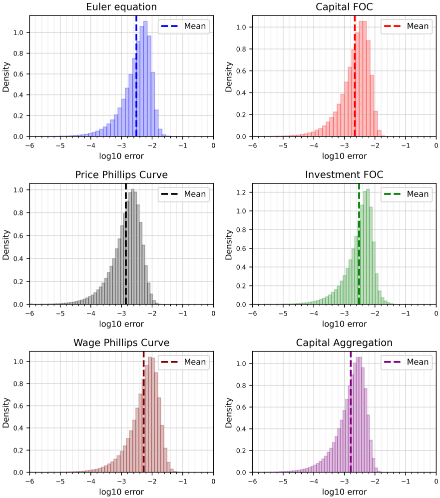
Impulse response functions



The two methods deliver similar propagation of the aggregate shock, as the model is quite linear and features only one aggregate shock

Proof of concept 4: estimation of a quantitative HANK model

Solution accuracy of quantitative HANK model



◂ Compare redPost-training residual errors are bell-shaped with contained left tails

Estimation of quantitative HANK model

- Simulate seven observables used in prior predictive checks as in Auclert et al. (2020) and Bayer et al. (2024)
    - GDP growth, consumption growth, investment growth, real wage growth, hours worked growth, inflation, and the nominal interest rate
- Similar model as in Auclert et al. (2020) and Bayer et al. (2024)
- Sample length: 262 quarters
- Model estimated using the NN Particle Filter

Parameters		True	Prior					Posterior	
		Value	Type	Mean	Std	Lower Bound	Upper Bound	Mean	Std
Persistence idiosyncratic risk	ρ_s	0.90	Trc.N	0.90	0.01	0.85	0.95	0.882	0.007
Std. dev. idiosyncratic risk	σ_s	0.20	Trc.N	0.20	0.01	0.15	0.25	0.206	0.008
MP inflation response	ϕ_π	2.0	Trc.N	2.00	0.5	1.50	3.00	2.317	0.084
MP output response	ϕ_y	0.25	Trc.N	0.25	0.25	0.00	0.50	0.397	0.032
Inflation indexation	ι_p	0.25	Trc.N	0.25	0.25	0.00	0.50	0.289	0.080
Wage indexation	ι_w	0.25	Trc.N	0.25	0.03	0.00	0.50	0.247	0.041
Rotemberg costs price PC	φ^p	100	Trc.N	100	2.50	75	125	103.9	2.75
Rotemberg costs wage PC	φ^w	100	Trc.N	100	2.50	75	125	98.5	2.93
Persistence risk-premium	ρ_ζ	0.90	Trc.N	0.90	0.05	0.75	0.95	0.882	0.022
Persistence wage mark-up	ρ_{μ^W}	0.90	Trc.N	0.90	0.05	0.75	0.95	0.884	0.022
Persistence price mark-up	ρ_{μ^P}	0.90	Trc.N	0.90	0.01	0.75	0.95	0.920	0.010
Persistence MP shock	ρ_m	0.90	Trc.N	0.90	0.01	0.75	0.95	0.907	0.010
Persistence investment eff.	ρ_ν	0.90	Trc.N	0.90	0.01	0.75	0.95	0.897	0.011
Persistence government	ρ_g	0.90	Trc.N	0.90	0.005	0.75	0.95	0.900	0.005
Std. dev. risk-premium	$100\sigma_\zeta$	0.2	Trc.N	0.2	0.5	0.1	0.4	0.240	0.021
Std. dev. growth rate	$100\sigma_z$	0.2	Trc.N	0.2	0.5	0.1	0.4	0.187	0.018
Std. dev. wage mark-up	$100\sigma_{\mu^W}$	5.0	Trc.N	5.0	5.0	2.0	8.0	5.40	0.258
Std. dev. price mark-up	$100\sigma_{\mu^P}$	5.0	Trc.N	5.0	5.0	2.0	8.0	5.26	0.222
Std. dev. MP shock	$100\sigma_m$	0.1	Trc.N	0.1	0.01	0.05	0.2	0.093	0.011
Std. dev. government	$100\sigma_g$	0.5	Trc.N	0.5	0.25	0.1	1.0	0.419	0.022
Std. dev. investment eff.	$100\sigma_\nu$	1.0	Trc.N	1.0	0.5	0.5	1.5	0.834	0.055

Estimation of a nonlinear HANK model with the ZLB using actual data

Estimation

- Estimation of a nonlinear, one-asset HANK model with the ZLB constraint Model
- US time-series data from 1990:Q1 to 2019:Q4
    - GDP growth per capita, GDP deflator, and shadow interest rate
- Measurement equations

$$\begin{bmatrix} \text{Output Growth} \\ \text{Inflation} \\ \text{Interest Rate} \end{bmatrix} = C + \begin{bmatrix} 400\left(\frac{Y_t}{Y_{t-1}/g_t} - 1\right) \\ 400\left(\Pi_t - 1\right) \\ 400\left(R_t - 1\right) \end{bmatrix} + u_t,$$

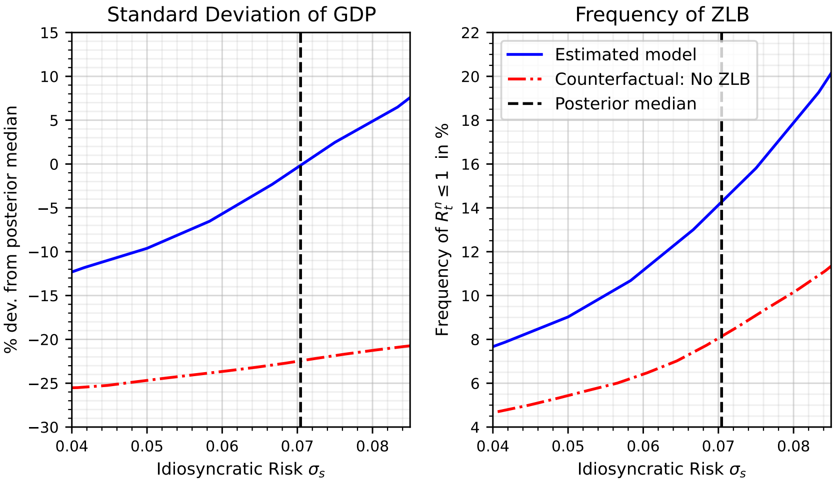
- Measurement error u_t follows a Gaussian distribution $\mathcal{N}(0, \Sigma_u)$

Prior and Posterior Moments

Estimation									
Parameters		Prior					NN		
		Type	Mean	Std	Lower Bound	Upper Bound	Posterior Median	5%	95%
Parameters affecting the DSS									
Std. dev. idiosyncratic risk	$100\sigma_s$	Trc.N	5.00	1.000	2.50	10.0	7.04	5.67	8.10
Borrowing limit	$\underline{B}$	Trc.N	−0.50	0.010	−0.65	−0.35	−0.50	−0.54	−0.46
Other parameters									
Rotemberg costs NKPC	φ	Trc.N	100	5.00	70	120	101	94	107
MP inflation response	θ_Π	Trc.N	2.25	0.125	1.75	2.75	2.43	2.20	2.67
MP output response	θ_Y	Trc.N	1.00	0.025	0.75	1.25	0.96	0.92	1.00
Persistence growth rate	ρ_z	Trc.N	0.40	0.025	0.2	0.6	0.43	0.39	0.47
Persistence MP shock	ρ_m	Trc.N	0.90	0.005	0.85	0.95	0.91	0.90	0.91
Std. dev. preference	$100\sigma_\zeta$	Trc.N	1.50	0.100	1.00	2.00	1.22	1.10	1.33
Std. dev. growth rate	$100\sigma_z$	Trc.N	0.40	0.100	0.30	0.60	0.47	0.43	0.53
Std. dev. MP shock	$100\sigma_m$	Trc.N	0.06	0.010	0.05	0.20	0.15	0.14	0.16

Calibrated Parameters

Interactions between nonlinearities and heterogeneity



⇒ The estimated idiosyncratic risk affected by the volatility of the observables

More results

Conclusions

- Novel integrated neural-network based estimation procedure
    - Global solution and estimation of models with hundreds of state variables (HA, many countries or sectors) and nonlinear constraints possible

- Estimation of HANK models with aggregate nonlinearities and risk
    - Interactions between nonlinearities, aggregate uncertainty, and heterogeneity

- New step-by-step learning guide – available for some of the applications

- New techniques to solve and estimate the economic models of the future

Example Codes

Code for the analytical example!

`https://github.com/tseep/estimating-hank-nn`

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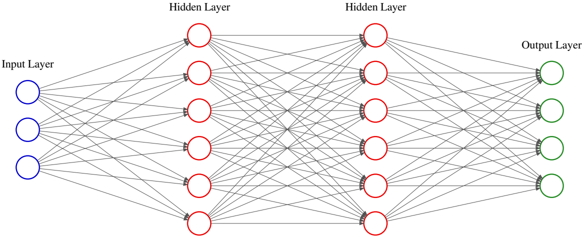
`https://github.com/tseep/estimating-hank-nn`

Or run the code directly in the cloud with Google Colab

`https://colab.research.google.com/github/tseep/estimating-hank-nn/blob/main/examples/colab_analytical.ipynb`

Appendix

What is a neural network?



- A NN is a combination of mathematical functions performed at every neuron
- Consider a function $Y = \psi(X)$ to be approximated by a NN defined as

$$Y = \psi_{NN}(X|W),$$

What is a neural network? (cont'd)

- A single neuron assigns its inputs $x_1, x_2, \ldots x_S$ some weights $w_1, w_2, \ldots, w_S$ and computes their sum (adjusted by a bias w_0) to return a single output $\tilde{y}$

$$\tilde{y} = h(w_0 + \sum_{i=1}^{S} w_i x_i).$$

- The activation function $h(\cdot)$ helps the NN to capture nonlinear dynamics
- The vector W with all weights is optimized (trained) to minimize a loss fct
- All the optimizing weights across individual neurons (W) define a NN
- The neural network training usually exploits graphics processing units (GPUs) as they can process multiple computations simultaneously
  - e.g. PyTorch, Google Jax and TensorFlow

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Model solution with neural networks

- Objective: Solving a nonlinear DSGE model
  - State variables $\mathbb{S}_t$, shocks ν_t and structural parameters Θ

- Model's dynamics summarized by a set of (nonlinear) transition equations

$$\mathbb{S}_t = f\left(\mathbb{S}_{t-1}, \nu_t; \Theta\right),$$

where function f is generally unknown and needs to be obtained numerically

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where function f is generally unknown and needs to be obtained numerically

- **Heterogeneity**: Approximate distributions with a finite number of agents
  - The state variables and the vector of shocks

$$\mathbb{S}_t = \left\{ \left\{ \mathbb{S}_t^i \right\}_{i=1}^L, \mathbb{S}_t^A \right\} \quad \text{and} \quad \nu_t = \left\{ \left\{ \nu_t^i \right\}_{i=1}^L, \nu_t^A \right\}.$$

  - Individual and aggregate policy functions

$$\psi_t^i = \psi^I(\mathbb{S}_t^i, \mathbb{S}_t|\Theta) \quad \text{and} \quad \psi_t^A = \psi^A(\mathbb{S}_t|\Theta).$$

Back

Incorporation of Heterogeneity

- Heterogeneity usually assumes the existence of a continuum of agents

  → Distribution of states and shocks is infinite

$$\int \mathbb{S}_t^i d\Omega \quad \text{and} \quad \int \nu_t^i d\Omega,$$

- We approximate the distribution with a finite number agents L

$$\left\{\mathbb{S}_t^i\right\}_{i=1}^L \quad \text{and} \quad \left\{\nu_t^i\right\}_{i=1}^L.$$

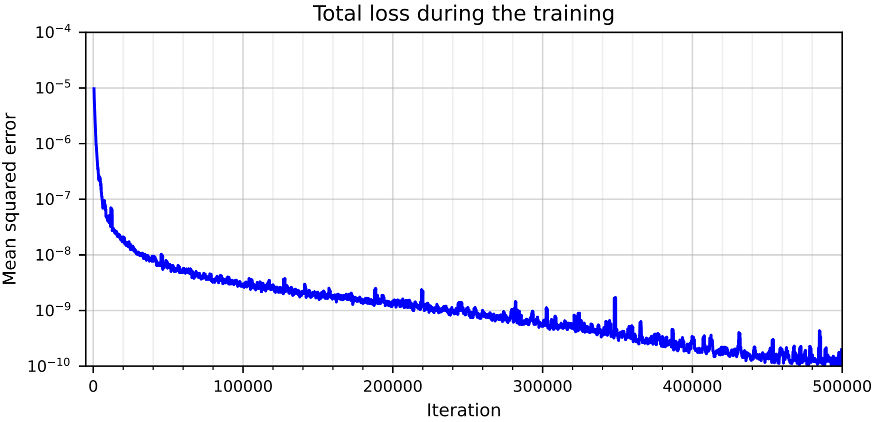
- The state variables and the vector of shocks

$$\mathbb{S}_t = \left\{\left\{\mathbb{S}_t^i\right\}_{i=1}^L, \mathbb{S}_t^A\right\} \quad \text{and} \quad \nu_t = \left\{\left\{\nu_t^i\right\}_{i=1}^L, \nu_t^A\right\},$$

- Individual and aggregate policy functions

$$\psi_t^i = \psi^I(\mathbb{S}_t^i, \mathbb{S}_t | \bar{\Theta}) \quad \text{and} \quad \psi_t^A = \psi^A(\mathbb{S}_t | \bar{\Theta}).$$

Training of Neural Network and Loss Function



Back

Particle Filter

- Observation equation connects the state variables with the observables $\mathbb{Y}_t$:

  $$\mathbb{Y}_t = g(\mathbb{S}_t|\tilde{\Theta}) + u_t,$$

  where g is a function and u_t is a measurement error

- Particle filter determines the likelihood

  $$\mathcal{L}\left(\mathbb{Y}_{1:T}; \tilde{\Theta}\right) = \Omega^{PF}\left(\mathbb{Y}_{1:T}; \tilde{\Theta}\right)$$

- Particle filter can be noisy and very time consuming for complex models

- Using a filter to calculate the likelihood is still a bottleneck Back

Nonlinear RANK Model with ZLB

- Off-the-shelf New Keynesian model
  - Shocks to households' preference to consumption
  - Price rigidities a la Rotemberg
  - Zero lower bound constraint on the nominal interest rate

$$R_t = \max\left[1, R\left(\frac{\Pi_t}{\Pi}\right)^{\theta_\Pi}\left(\frac{Y_t}{Y}\right)^{\theta_Y}\right]$$

- We are interested in solving and estimating it in its nonlinear specification

◂ Back

Neural Network and Estimation

- Training NN over 100000 iterations and batch size of 200 economies
- We train the NN by drawing from the bounded parameter space
- Stochastic solution from simulating the model after each draw

- Observation equation with a sample size of 1000 periods

$$\begin{bmatrix} \text{Output Growth} \\ \text{Inflation} \\ \text{Interest Rate} \end{bmatrix} = \begin{bmatrix} 400\left(\frac{Y_t}{Y_{t-1}-1}\right) \\ 400\left(\Pi_t - 1\right) \\ 400\left(R_t - 1\right) \end{bmatrix} + u_t$$

- Estimation includes five structural parameters
- Priors are truncated normal densities
- 15000 data points to train neural network based particle filter Back

◂ Back

Estimation Results

Estimation							
Par.	Cal.	Neural Network			Conventional Approach		
	True Value	Posterior Median	5%	95%	Posterior Median	5%	95%
θ_Π	2.0	2.02	1.87	2.17	2.06	1.94	2.20
θ_Y	0.25	0.251	0.238	0.263	0.248	0.237	0.258
φ	1000	988.6	935.1	1036.7	973.7	911.2	1037.2
ρ_ζ	0.8	0.686	0.669	0.701	0.691	0.670	0.710
σ^ζ	0.02	0.020	0.020	0.021	0.020	0.019	0.020

- Neural network based estimation works very well
  - Posterior median is very close to the true value
- The bounds of neural network and conventional method are very similar
- Neural network method is much faster and much more scalable!

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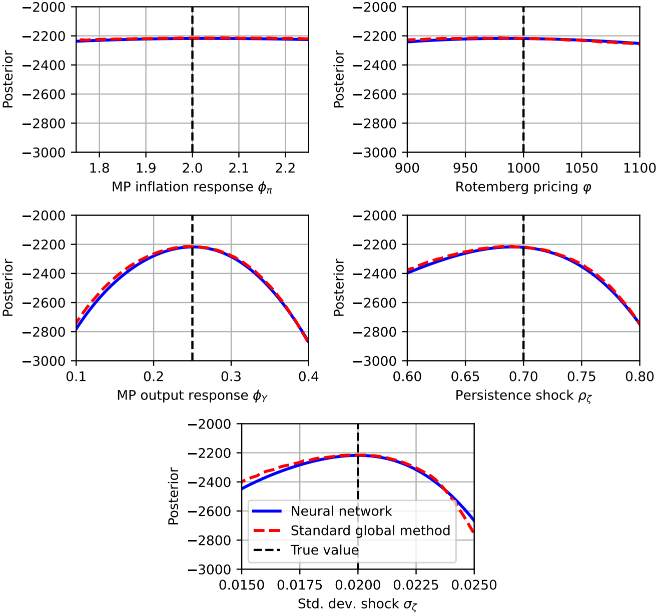
Bayesian Estimation with NN: Posterior



Calibration HANK Model

Calibrated Parameters

Parameters		Value	Parameters		Value
β	Discount factor	0.9975	γ^{τ}	Tax progressivity	0.18
σ	Relative risk aversion	1	D	Government debt	1
η	Inverse Frisch elasticity	0.72	Π	Inflation target	1.00625
ϵ	Price elasticity demand	11	ρ_s	Persistence labor prod	0.9
χ	Disutility labor	0.74	ρ_{ζ}	Persistence pref. shock	0.7
g	Average growth rate	1.0033			

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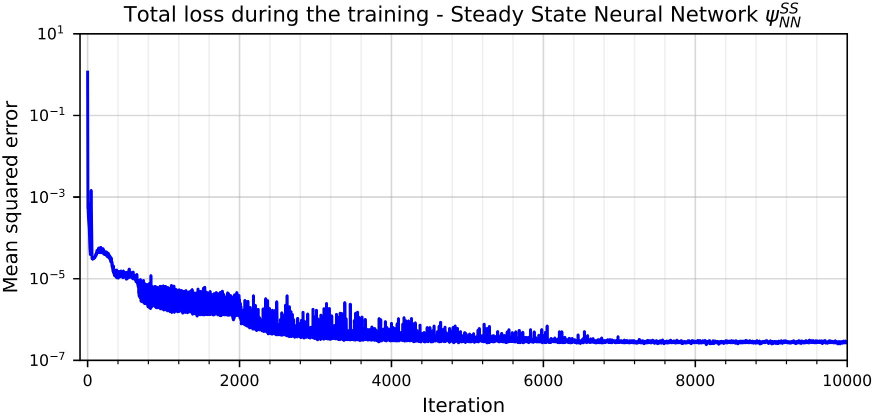
Estimation - Step 1a: Model Solution and Neural Networks

- NN training of model without aggregate risk (Aiyagari version)



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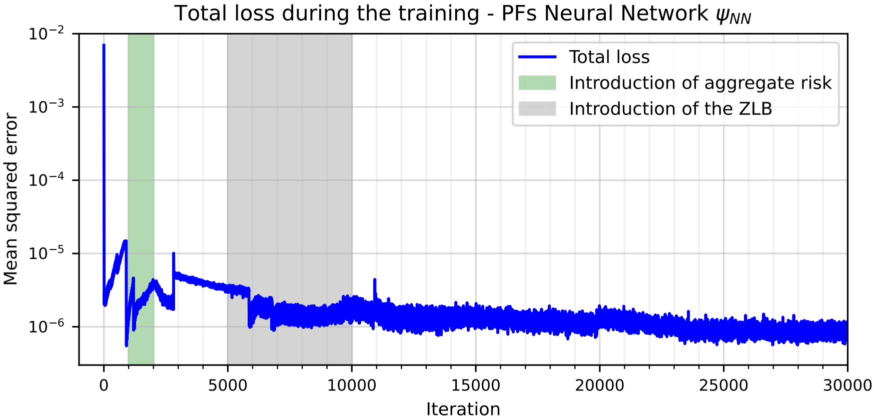
Estimation - Step 1b: Nonlinear Model Solution

- NN training of the full nonlinear model version
  - Stepwise introduction of aggregate risk and the ZLB



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Overfitting

- Overfitting occurs when NNs learn too much from the training sample
  - e.g. the noise generated by computational inaccuracies of the particle filter

⇒ Obtain a validation sample of randomly drawn parameters and likelihoods

  - We do not optimize the weights of the NN
  - We compute the loss function implied by the NN in the validation sample
  - We compare the loss in the validation sample to that in the training sample
  - Losses should be similar to dispel concerns of overfitting

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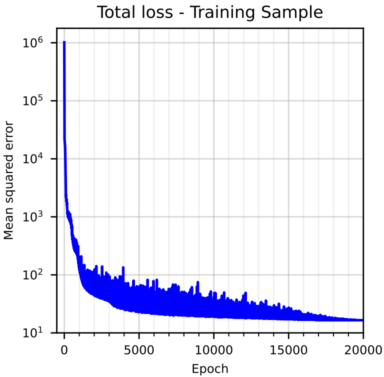
Overfitting



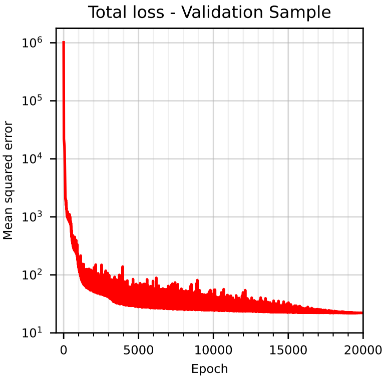


Figure: **Training and validation convergence.** The figure shows the total loss over the the training sample (left) and over the validation sample (right). An epoch is completed when all the points in the training or validation sample are utilized. The vertical axis is expressed in a logarithmic scale.

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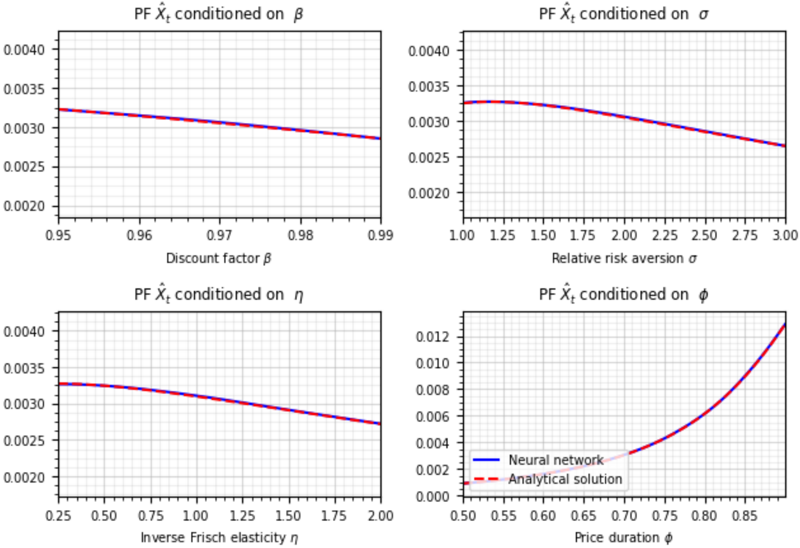
PoF 1: NN-approximation of policy functions



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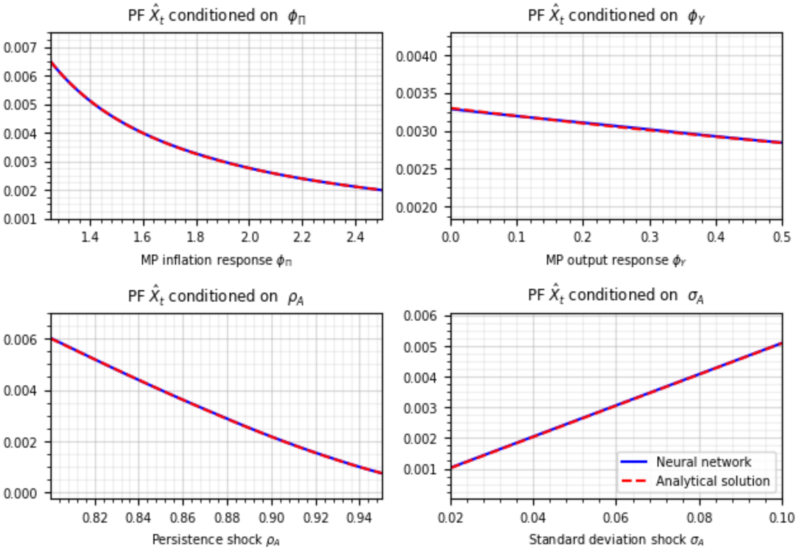
PoF 1: NN-approximation of policy functions (cont'd)



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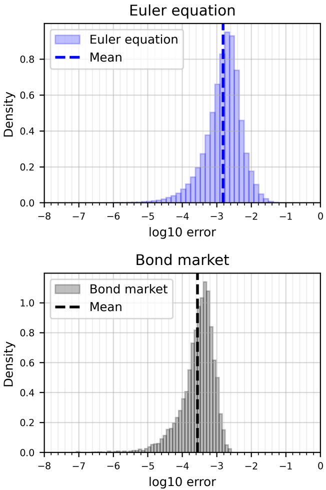
Solution accuracy: One-asset model



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State and pseudo state variables and policy functions

- Discretization of the number of agents: $L = 100$ agents
- 2 individual state variables:

$$\left\{\tilde{B}^i_{t-1}\right\}_{i=1}^{L} \quad \text{and} \quad \left\{s^i_t\right\}_{i=1}^{L}$$

- 4 aggregate state variables:

$$R^N_{t-1}, \quad \zeta_t, \quad z_t, \quad \text{and} \quad mp_t$$

- One idiosyncratic shock and three aggregate shocks

$$\left\{ \left\{\epsilon^{s,i}_t\right\}_{i=1}^{L}, \epsilon^{\zeta}_t, \epsilon^{z}_t, \epsilon^{m}_t \right\}$$

- 10 pseudo state variables

$$\tilde{\Theta} = \{\sigma_s, \underline{B}, \varphi, \theta_\Pi, \theta_Y, \rho_z, \rho_m, \sigma_\zeta, \sigma_z, \sigma_m\}$$

- 10 calibrated parameters

$$\bar{\Theta} = \{\beta, \eta, \sigma, \bar{a}, \chi, \gamma^\tau, \Pi, D, \rho_s, \rho_\zeta\}$$

Back

Loss functions to minimize for the training of NNs

The error associated with the Fisher-Burmeister function – smooth way to represent the Kuhn-Tucker conditions: $\mu_t^i \geq 0$, $\left(\tilde{B}_t^i - \underline{B}\right) \geq 0$, and $\mu_t^i \times \left(\tilde{B}_t^i - \underline{B}\right) = 0$ – so as to enforce the borrowing limit at the individual household level:

$$\left\{ L^{1,i} = \left(\Psi^{FB} \left(1 - \bar{\lambda}_t^i, \tilde{B}_t^i - \underline{B} \right) \right)^2 \right\}_{i=1}^{L}, \tag{1}$$

where $\bar{\lambda}_t^i$ are the multipliers associated with the Euler equation of each household i and $L^{1,i}$ is the squared error of agent i.

Loss functions to minimize for the training of NNs (cont'd)

$$L^2 = \left(\left[\varphi \left(\frac{\Pi_t}{\Pi} - 1 \right) \frac{\Pi_t}{\Pi} \right] - (1-\epsilon) - \epsilon MC_t \right.$$
$$\left. - \beta \varphi \frac{1}{M} \sum_{m=1}^{M} \left[\left(\frac{\exp(\zeta_{t+1}^m)}{\exp(\zeta_t)} \right) \left(\frac{\tilde{C}_{t+1}^m}{\tilde{C}_t} \right)^{-\sigma} \left(\frac{\Pi_{t+1}^m}{\Pi} - 1 \right) \frac{\Pi_{t+1}^m}{\Pi} \frac{\tilde{Y}_{t+1}^m}{\tilde{Y}_t} \right] \right)^2, \tag{2}$$

$$L^3 = \left(D - \frac{1}{L} \sum_{i=1}^{L} B_t^i \right)^2, \tag{3}$$

$$L^4 = \frac{1}{M} \sum_{m=1}^{M} \left(D - \frac{1}{L} \sum_{i=1}^{L} B_{t+1}^{i,m} \right)^2, \tag{4}$$

$$L^5 = \left(\tilde{Y}_t - \tilde{C}_t \right)^2, \tag{5}$$

$$L^6 = \frac{1}{M} \sum_{m=1}^{M} \left(\tilde{Y}_{t+1}^m - \tilde{C}_{t+1}^m \right)^2. \tag{6}$$

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PoF 3: Estimation of nonlinear HANK with simulated data

Estimation									
Par.	True	Prior					NN		
	Value	Type	Mean	Std	Lower Bound	Upper Bound	Posterior		
							Median	5%	95%
Parameters affecting the DSS									
$100\sigma_s$	5.00	Trc.N	5.00	1.000	2.50	10.0	4.28	3.17	5.31
$\underline{B}$	-0.50	Trc.N	-0.50	0.010	-0.65	-0.35	-0.50	-0.54	-0.46
Other parameters									
φ	100	Trc.N	100	5.000	70	120	100	92	108
θ_Π	2.25	Trc.N	2.25	0.125	1.75	2.75	2.40	2.25	2.55
θ_Y	1.00	Trc.N	1.00	0.025	0.75	1.25	1.01	0.97	1.05
ρ_z	0.40	Trc.N	0.40	0.025	0.20	0.60	0.40	0.37	0.45
ρ_m	0.90	Trc.N	0.90	0.005	0.85	0.95	0.90	0.89	0.91
$100\sigma_\zeta$	1.50	Trc.N	1.50	0.100	1.00	2.00	1.45	1.34	1.57
$100\sigma_z$	0.40	Trc.N	0.40	0.100	0.30	0.60	0.36	0.32	0.40
$100\sigma_m$	0.06	Trc.N	0.06	0.010	0.05	0.20	0.06	0.05	0.07

◂ Back

Estimation - Step 1: Nonlinear Model Solution

- Model features 214 state and pseudo-state variables
  - 200 individual, 4 aggregate and 10 pseudo-states (parameters to estimate)
  - Approximation of continuum with 100 agents
- The policy functions to approximate numerically with NNs
  - Aggregate PFs: Inflation and wage
  - Individual PFs: Labor choice
- Training of neural networks to approximate policy functions
  - Minimization of the squared residual error of 303 equations
  - 5 hidden layers with 128 neurons in each
  - CELU and PReLU activation functions

◂ Appendix Back

Estimation - Step 1 (cont'd)

- Training of neural networks is a two-step procedure in this case
  1. Deterministic steady state (DSS) - Aiyagari version
     - We need nominal rate and output as input for the Taylor rule
     - We train a neural network over the parameter range Training

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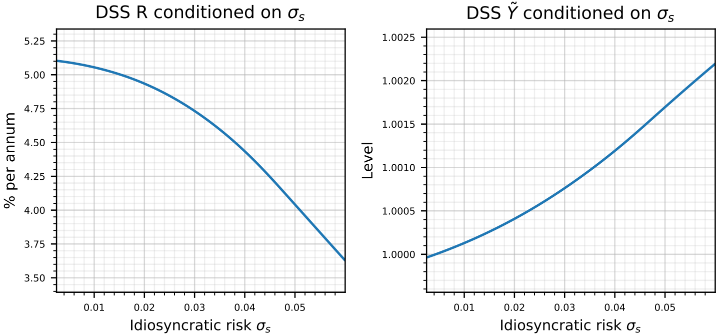
Estimation - Step 1a: Deterministic Steady State

- Impact of idiosyncratic risk on DSS over the parameter space
  - Increase in idiosyncratic risk lowers market clearing rate



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Estimation of Nonlinear HANK with Neural Networks

- HANK with individual and aggregate nonlinearities
  - Households face idiosyncratic income risk s_t^i and a borrowing limit $\underline{B}$

$$E_0 \sum_{t=0}^{\infty} \beta^t \exp(\zeta_t) \left[\left(\frac{1}{1-\sigma} \right) \left(\frac{C_t}{A_t} \right)^{1-\sigma} - \chi \left(\frac{1}{1+\eta} \right) (H_t^i)^{1+\eta} \right]$$

$$\text{s.t.} \;\; C_t^i + B_t^i = \tau_t \left(\frac{W_t}{A_t} \exp(s_t^i) H_t^i \right)^{1-\gamma_\tau} + \frac{R_{t-1}}{\Pi_t} B_{t-1}^i + Div_t \exp(s_t^i)$$

$$B_t^i \geq \underline{B}$$

  where idiosyncratic risk follows an AR(1) process: $s_t^i = \rho_s s_{t-1}^i + \sigma_s \epsilon_t^{s,i}$

  - Aggregate shocks: preference ζ, growth rate z_t, and monetary policy mp_t
  - Monopolistically competitive firms and Rotemberg pricing
  - Monetary policy is constrained by the zero lower bound

$$R_t = \max \left[1, \; R \left(\frac{\Pi_t}{\Pi} \right)^{\theta_\Pi} \left(\frac{Y_t}{A_t Y} \right)^{\theta_Y} \exp(mp_t) \right]$$

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Estimation - Step 1 (cont'd)

1. Training of neural networks is a two-step procedure in this case
   a. Deterministic steady state (DSS) - Aiyagari version
      - Need nominal rate and output as input for the Taylor rule
      - Train 2 NNs for the individual and aggregate *extended* policy functions under the assumption that aggregate shocks are shut down Training

   b. Nonlinear model solution with aggregate and idiosyncratic risk
      - Train 2 NNs for the individual and aggregate *extended* policy functions Training
      - Introduce aggregate risk and the ZLB slowly over the training process

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Estimation - Step 1b: Convergence

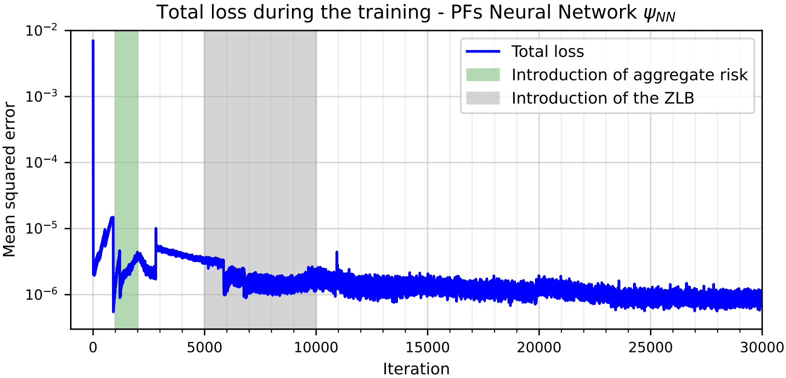

Figure: **Convergence of the NN solution for the HANK model.** The figure shows the dynamics of the mean squared error during the training of the extended individual and aggregate policy functions. The shaded areas indicate the periods in which we introduce aggregate risk and the ZLB. The vertical axis has a logarithmic scale.

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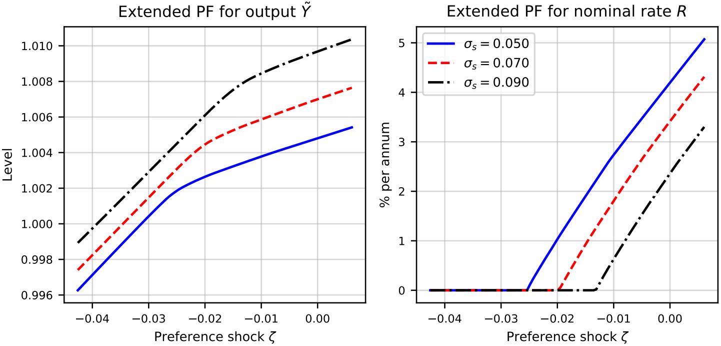
Estimation - Step 1b: Aggregate Policy Functions

- Policy functions for output and inflation for varying preference shock
  - Zero lower bound creates nonlinearity
  - Degree of nonlinearity depends on degree of idiosyncratic risk



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Estimation - Step 2 and 3

2. Neural network particle filter
    - We create 15,000 likelihood evaluations with the particle filter
    - Direct mapping from parameters to likelihood value via neural network
3. Random Walk Metropolis-Hastings Algorithm
    - 1 million draws (after burn-in)
    - Very fast as costs are frontloaded

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Estimation - Step 2 and 3

2. Neural network particle filter
    - We create 15,000 likelihood evaluations with the particle filter
    - Direct mapping from parameters to likelihood value via neural network
3. Random Walk Metropolis-Hastings Algorithm
    - 1 million draws (after burn-in)
    - Very fast as costs are frontloaded

- Estimation (Steps 1 - 3) is conducted in a few days
    - Requires a decent GPU

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How good is what we got?

Standard deviations

	Model	Data
GDP growth	0.6947	0.5831
Inflation	1.1511	0.9045
Federal funds rate	2.561	2.7537

Autocorrelations

	Model	Data
GDP growth	0.1355	0.4050
Inflation	0.8146	0.5456
Federal funds rate	0.7219	0.9707

Avg. Gini coefficient

	Model	Data
Wealth distrib.	0.8793	0.8410

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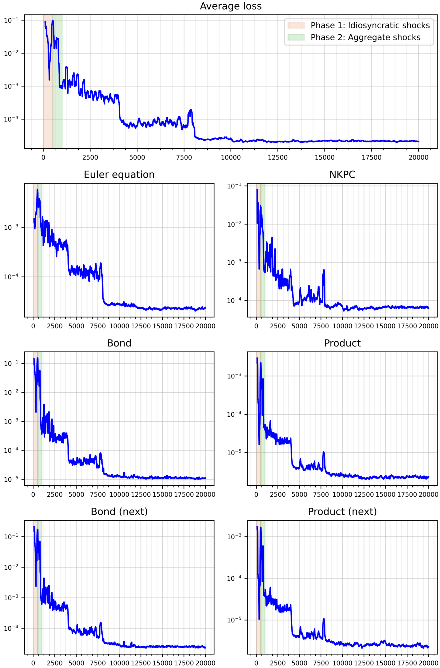
Average loss
Phase 1: Idiosyncratic shocks
Phase 2: Aggregate shocks
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
2500
5000
7500
10000
12500
15000
17500
20000
Euler equation
10^{-3}
10^{-4}
NKPC
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Bond
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
Product
10^{-3}
10^{-4}
10^{-5}
Bond (next)
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Product (next)
10^{-3}
10^{-4}
10^{-5}
0 2500 5000 7500 10000 12500 15000 17500 20000

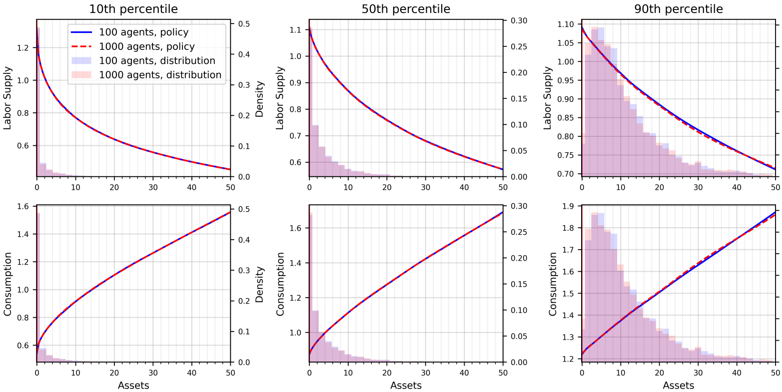

Figure: **Policy functions for variations in number of agents L.** Policy functions (labor supply - upper panels - and consumption - lower panels) and asset distribution conditional on individual productivity (10th percentile, 50th percentile, 90th percentile) for $L = 100$ and $L = 1,000$.

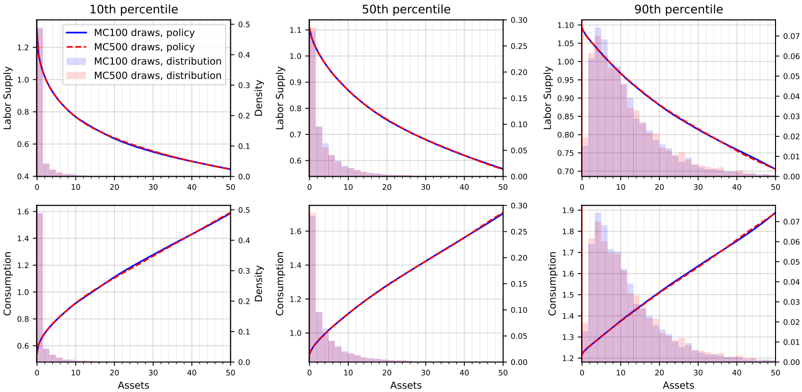

Figure: **Policy functions for variations in number of Monte Carlo draws MC.** Policy functions (labor supply - upper panels - and consumption - lower panels) and asset distribution conditional on individual productivity (10th percentile, 50th percentile, 90th percentile) for $MC = 100$ and $MC = 500$.