Panel Discussion: Productivity and Technological Progress

Biases in the Estimation of Firm TFP & Markups

Eduardo Morales

Princeton University

July 5, 2024

What is TFP?

- TFP is the linearly additive unobserved term in a production function. E.g.,

$$\underbrace{a_i}_{\text{TFP}_i} = \underbrace{q_i}_{\text{Output}_i} - \underbrace{(\alpha_l l_{it} + \alpha_k k_{it})}_{\text{Inputs}_i},$$

with, e.g., $l_i = \ln(L_i)$ and α_l the causal impact of l_i on q_i.

- We estimate a_i as a residual

$$\hat{a}_i = \hat{q}_i - (\hat{\alpha}_l l_{it} - \hat{\alpha}_k \hat{k}_{it}).$$

- Given $\{\hat{a}_i\}_{i=1}^{N}$, we compute moments of its distribution,

$$\text{e.g.,} \qquad \bar{a}_{2020} - \bar{a}_{1970},$$

or study its determinants

e.g., estimate causal impact of R&D on TFP.

Sources of Bias in TFP Estimates

1. Transmission bias in the absence of instruments.
2. Measurement of output.
3. Measurement of inputs.
4. Multiproduct firms.
5. Dynamics & forward-looking firms.
6. . . .

Sources of Bias in TFP Estimates

1. Transmission bias in the absence of instruments.
2. Measurement of output.
3. Measurement of inputs.
4. Multiproduct firms.
5. Dynamics & forward-looking firms.
6. . . .

OP (1996), LP (2003), and ACF (2015) focus on solving issue number 1

Sources of Bias in TFP Estimates

1. Transmission bias in the absence of instruments.
2. Measurement of output.
3. Measurement of inputs.
4. Multiproduct firms.
5. Dynamics & forward-looking firms.
6. ...

OP (1996), LP (2003), and ACF (2015) focus on solving issue number 1

In this intervention, I want to focus on issue number 2

Measurement of Output

- Rarely do we observe physical output. Often, observe firm's revenue instead.
- In these cases,

$$\hat{q}_i = r_i \neq q_i.$$

- What are the implications of this error in measurement both for the estimated distribution of firm-level productivity and for the study of its determinants?
- The answer to this question depends on the nature of

$$r_i - q_i.$$

- In this intervention, I will consider two settings:
  - Single-market oligopolistic firms: $r_i - q_i = p_i$.
  - Multi-market firms: $r_i - q_i = f(d_{i1}Q_{i1}, \ldots, d_{iM}Q_{iM})$.

SINGLE-MARKET OLIGOPOLISTIC FIRMS

Economic Model

- Index period by t, good by ω, and firm producing each good ω by i.
- Representative consumer with nested CES utility function:

$$\mathcal{U}_t = \left(\int_0^1 Q_t(\omega)^{\frac{\rho-1}{\rho}} d\omega \right)^{\frac{\rho}{\rho-1}} \quad \text{and} \quad Q_t(\omega) = \left(\sum_{i=\{1,2\}} Q_{it}(\omega)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

  with $\sigma \geq \rho > 1$.
- Market structure: duopoly with firms competing à la Cournot.
- Production function (in logs):

$$q_{it}(\omega) = a_{it}(\omega) + \alpha l_{it}(\omega).$$

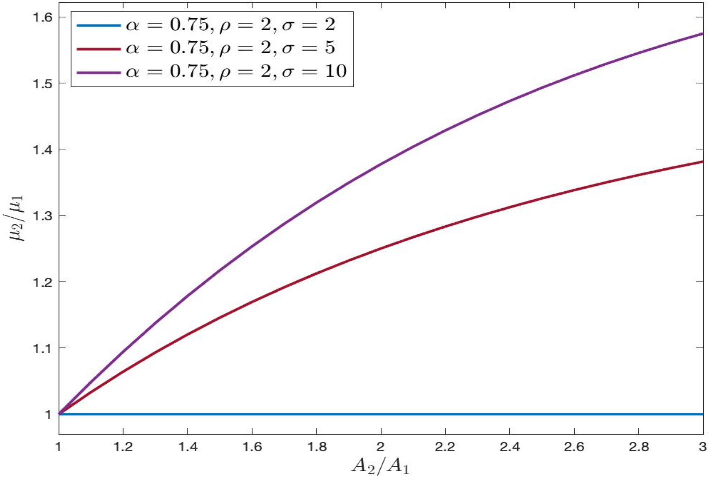
- Evolution of log productivity:

$$a_{it}(\omega) = \gamma a_{it-1}(\omega) + \beta x_{it}(\omega) + \varepsilon_{it}(\omega).$$

- Estimate β, $\{a_{it}(\omega)\}$, and $\{\mu_{it}(\omega)\}$, with $\mu_{it} = p_{it}(\omega)/mc_{it}(\omega)$.

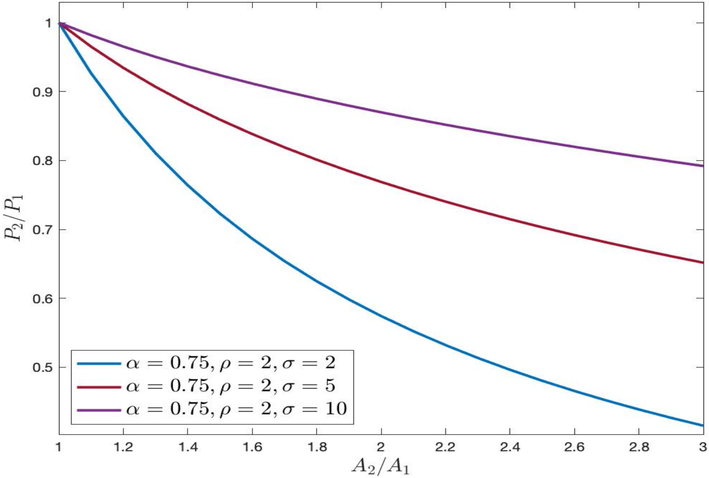
Mechanics of the Model

Markups are (weakly) increasing in productivity. . .



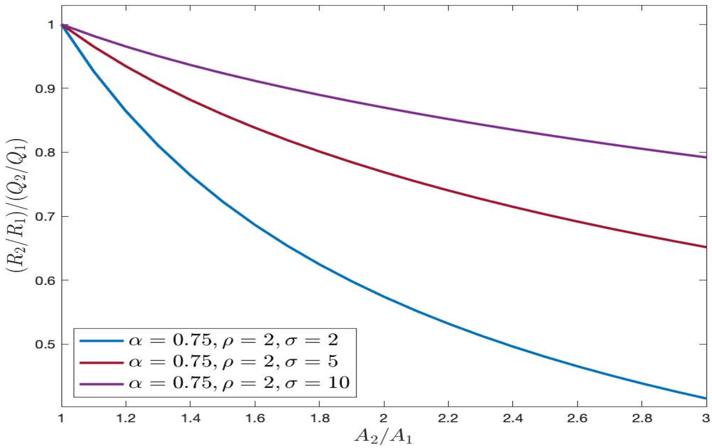
Mechanics of the Model

. . . but prices are decreasing in productivity.



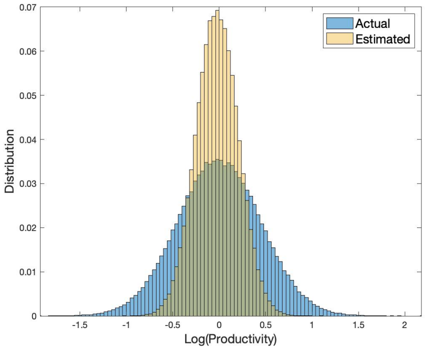
Mechanics of the Model

Revenues increase in productivity less than quantities.

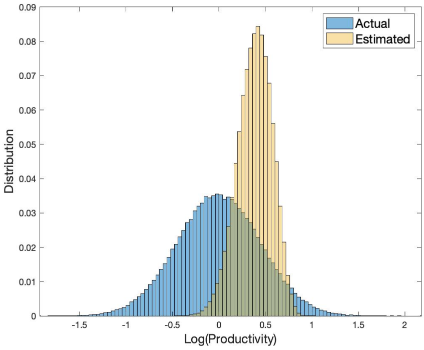


Simulation Results: Distribution of (Log) Productivity

When using revenue (instead of quantities) for estimation, implied distribution of log productivity is too concentrated and often shifted to the right.



(a) $\sigma = 2$

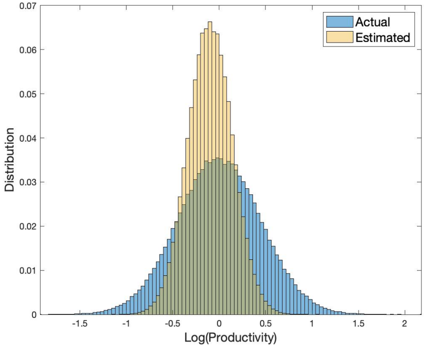


(b) $\sigma = 5$

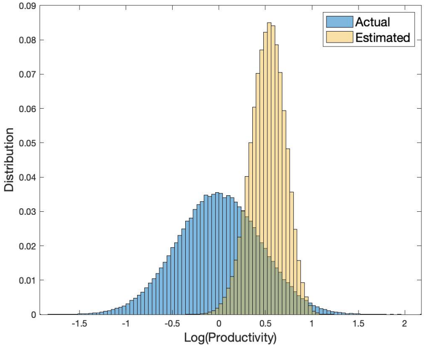
Common parameters: $\alpha = 0.75$ and $\rho = 2$

Simulation Results: Distribution of (Log) Productivity

Patterns are fairly robust, but may vary slightly with the value of α.



(a) $\sigma = 2$

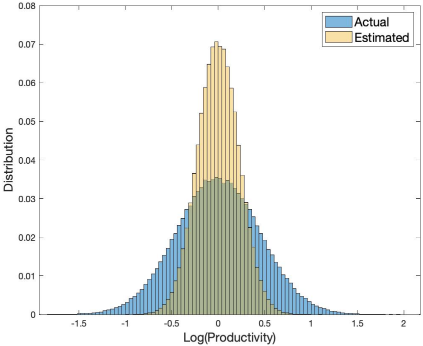


(b) $\sigma = 5$

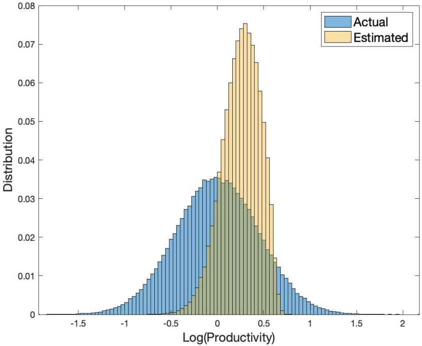
Common parameters: $\alpha = 0.5$ and $\rho = 2$

Simulation Results: Distribution of (Log) Productivity

Patterns are fairly robust, but may vary slightly with the value of α.



(a) $\sigma = 2$

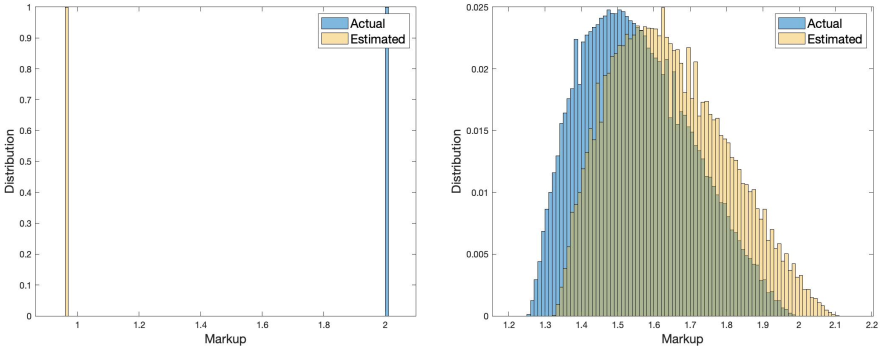


(b) $\sigma = 5$

Common parameters: $\alpha = 1$ and $\rho = 2$

Simulation Results: Distribution of Markups

When using revenue (instead of quantities) for estimation, implied distribution of markups has identical shape to the true one, but shifted to the left or right.



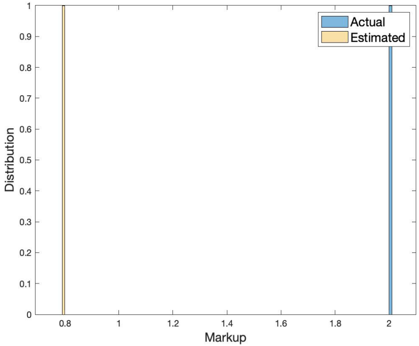
(a) $\sigma = 2$

(b) $\sigma = 5$

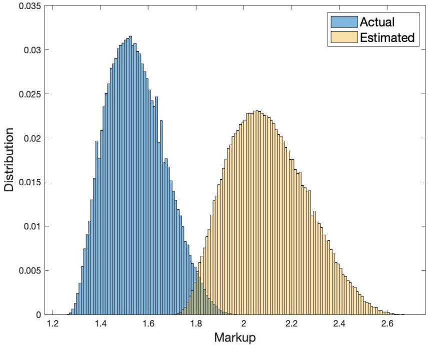
Common parameters: $\alpha = 0.75$ and $\rho = 2$

Simulation Results: Distribution of Markups

Patterns are quite sensitive to the value of α.



(a) $\sigma = 2$

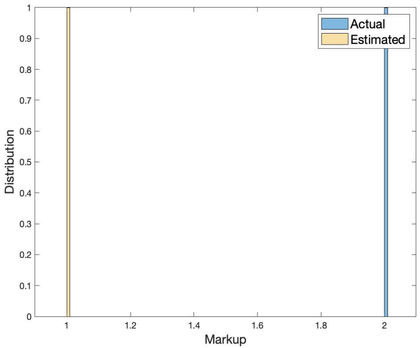


(b) $\sigma = 5$

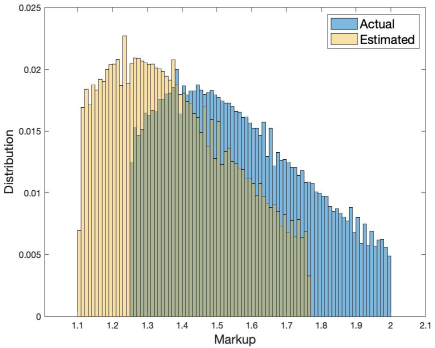
Common parameters: $\alpha = 0.5$ and $\rho = 2$

Simulation Results: Distribution of Markups

Patterns are quite sensitive to the value of α.



(a) $\sigma = 2$



(b) $\sigma = 5$

Common parameters: $\alpha = 1$ and $\rho = 2$

Simulation Results

Estimated return to productivity-enhancing investments is downward biased.

σ	α	Bias in...		
		β	$sd(a_i)$	$\bar{\mu}_i$
2	0.50	-46%	-56%	-60%
	0.75	-49%	-55%	-52%
	1	-50%	-55%	-50%
5	0.50	-62%	-40%	36%
	0.75	-64%	-48%	5%
	1	-63%	-50%	-11%
10	0.50	-60%	-32%	46%
	0.75	-59%	-39%	10%
	1	-56%	-39%	-7%

Note: Bias in θ equals $(\hat{\theta} - \theta_0)/\theta_0$, with $\hat{\theta}$ the estimator of θ and θ_0 its true value.

Common parameter: $\rho = 2$.

Simulation Results

Estimated dispersion of productivity is downward biased.

σ	α	Bias in...		
		β	$sd(a_i)$	$\bar{\mu}_i$
2	0.50	-46%	-56%	-60%
	0.75	-49%	-55%	-52%
	1	-50%	-55%	-50%
5	0.50	-62%	-40%	36%
	0.75	-64%	-48%	5%
	1	-63%	-50%	-11%
10	0.50	-60%	-32%	46%
	0.75	-59%	-39%	10%
	1	-56%	-39%	-7%

Note: Bias in θ equals $(\hat{\theta} - \theta_0)/\theta_0$, with $\hat{\theta}$ the estimator of θ and θ_0 its true value.

Common parameter: $\rho = 2$.

Simulation Results

Estimated average markups may be upward or downward biased.

σ	α	Bias in...		
		β	$sd(a_i)$	$\bar{\mu}_i$
2	0.50	-46%	-56%	-60%
	0.75	-49%	-55%	-52%
	1	-50%	-55%	-50%
5	0.50	-62%	-40%	36%
	0.75	-64%	-48%	5%
	1	-63%	-50%	-11%
10	0.50	-60%	-32%	46%
	0.75	-59%	-39%	10%
	1	-56%	-39%	-7%

Note: Bias in θ equals $(\hat{\theta} - \theta_0)/\theta_0$, with $\hat{\theta}$ the estimator of θ and θ_0 its true value.

Common parameter: $\rho = 2$.

MULTI-MARKET FIRMS

Multi-market Firms

- Firms often sell in multiple markets; e.g., export markets.
- Firms facing a downward sloping demand curve in a market will have revenue functions that are concave in the quantity they sell in that market.
- Given a total quantity sold by a firm i, Q_i, its total revenue, R_i, will increase in the number of markets this firm sells to.
- Holding constant a firm's TFP, the residual of the revenue function increases in the number of markets the firm sells to.
- Using revenue as the outcome variable leads to upward biased estimates of the TFP of the firms that sell in multiple markets.

Example: Firms Operating in Two Markets

- Firm i's production function (in logs) is:

$$q_i = a_i + \alpha l_i.$$

- All firms sell at Home, and may sell or not in Foreign.
- Firm i faces the following demand functions in home and foreign markets:

$$Q_{iH} = P_{iH}^{-\sigma} K_H \qquad \text{and} \qquad Q_{iF} = P_{iF}^{-\sigma} K_F.$$

From these, we can write the potential revenue in each of the two markets as

$$R_{iH} = Q_{iH}^{\frac{\sigma-1}{\sigma}} K_H^{\frac{1}{\sigma}} \qquad \text{and} \qquad R_{iF} = Q_{iF}^{\frac{\sigma-1}{\sigma}} K_F^{\frac{1}{\sigma}},$$

and total revenue (in logs) as

$$r_i = \underbrace{\ln(\kappa) + \ln\left[\frac{(1 + d_{Fi}\lambda_i^{\frac{\sigma-1}{\sigma}})}{(1 + d_{iF}\lambda_i)^{\frac{\sigma-1}{\sigma}}}\right] + \frac{\sigma-1}{\sigma} a_i}_{a_i^R} + \frac{\sigma-1}{\sigma}\alpha l_i,$$

with $\lambda_i \equiv Q_{iF}/Q_{iH}$.

Example: Firms Operating in Two Markets

- Firm i's production function (in logs) is:

$$q_i = a_i + \alpha l_i.$$

- All firms sell at Home, and may sell or not in Foreign.
- Firm i faces the following demand functions in home and foreign markets:

$$Q_{iH} = P_{iH}^{-\sigma} K_H \qquad \text{and} \qquad Q_{iF} = P_{iF}^{-\sigma} K_F.$$

From these, we can write the potential revenue in each of the two markets as

$$R_{iH} = Q_{iH}^{\frac{\sigma-1}{\sigma}} K_H^{\frac{1}{\sigma}} \qquad \text{and} \qquad R_{iF} = Q_{iF}^{\frac{\sigma-1}{\sigma}} K_F^{\frac{1}{\sigma}},$$

and total revenue (in logs) as

$$r_i = \underbrace{\ln(\kappa) + \ln\left[\frac{(1 + d_{Fi}\lambda_i^{\frac{\sigma-1}{\sigma}})}{(1 + d_{iF}\lambda_i)^{\frac{\sigma-1}{\sigma}}}\right] + \frac{\sigma - 1}{\sigma} a_i}_{a_i^R} + \frac{\sigma - 1}{\sigma} \alpha l_i,$$

with $\lambda_i \equiv Q_{iF}/Q_{iH}$.

Example: Firms Operating in Two Markets

- Firm i's production function (in logs) is:

$$q_i = a_i + \alpha l_i.$$

- All firms sell at Home, and may sell or not in Foreign.
- Firm i faces the following demand functions in home and foreign markets:

$$Q_{iH} = P_{iH}^{-\sigma} K_H \qquad \text{and} \qquad Q_{iF} = P_{iF}^{-\sigma} K_F.$$

From these, we can write the potential revenue in each of the two markets as

$$R_{iH} = Q_{iH}^{\frac{\sigma-1}{\sigma}} K_H^{\frac{1}{\sigma}} \qquad \text{and} \qquad R_{iF} = Q_{iF}^{\frac{\sigma-1}{\sigma}} K_F^{\frac{1}{\sigma}},$$

and total revenue (in logs) as

$$r_i = \underbrace{\ln(\kappa) + \ln\left[\frac{(1 + d_{Fi}\lambda_i^{\frac{\sigma-1}{\sigma}})}{(1 + d_{iF}\lambda_i)^{\frac{\sigma-1}{\sigma}}}\right] + \frac{\sigma-1}{\sigma} a_i}_{a_i^R} + \frac{\sigma-1}{\sigma}\alpha l_i,$$

with $\lambda_i \equiv Q_{iF}/Q_{iH}$.

Example: Firms Operating in Two Markets

- Firm i's production function (in logs) is:

$$q_i = a_i + \alpha l_i.$$

- All firms sell at Home, and may sell or not in Foreign.
- Firm i faces the following demand functions in home and foreign markets:

$$Q_{iH} = P_{iH}^{-\sigma} K_H \qquad \text{and} \qquad Q_{iF} = P_{iF}^{-\sigma} K_F.$$

From these, we can write the potential revenue in each of the two markets as

$$R_{iH} = Q_{iH}^{\frac{\sigma-1}{\sigma}} K_H^{\frac{1}{\sigma}} \qquad \text{and} \qquad R_{iF} = Q_{iF}^{\frac{\sigma-1}{\sigma}} K_F^{\frac{1}{\sigma}},$$

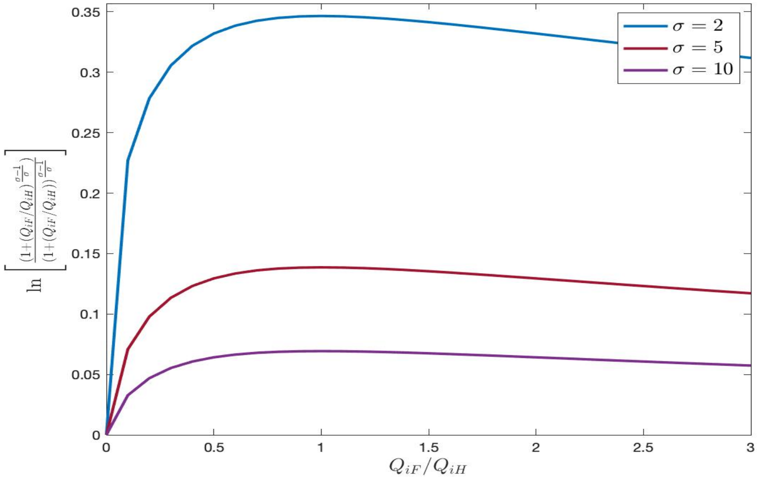
and total revenue (in logs) as

$$r_i = \underbrace{\ln(\kappa) + \ln\left[\frac{(1 + d_{Fi}\lambda_i^{\frac{\sigma-1}{\sigma}})}{(1 + d_{iF}\lambda_i)^{\frac{\sigma-1}{\sigma}}}\right] + \frac{\sigma-1}{\sigma} a_i}_{a_i^R} + \frac{\sigma-1}{\sigma}\alpha l_i,$$

with $\lambda_i \equiv Q_{iF}/Q_{iH}$.

Example: Firms Operating in Two Markets

Productivity estimates of exporting firms may be upward biased in up to 35%



CONCLUSION

Conclusion

- In the absence of output quantity data or data on firm sales by market (e.g., customs data), estimates of firm-level productivity and markups should be taken with a grain of salt.

EXTRA SLIDES

I. Deriving Demand Function

- Given utility function $\mathcal{U}_t$, representative consumer solves

$$\max_{\{Q_{it}(\omega)\}} \left(\int_0^1 \left(\sum_{i=\{1,2\}} Q_{it}(\omega)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma(\rho-1)}{(\sigma-1)\rho}} d\omega \right)^{\frac{\rho}{\rho-1}}$$

subject to

$$\int_0^1 \sum_{i=\{1,2\}} P_{it}(\omega) Q_{it}(\omega) d\omega = I_t.$$

- Given Lagrange multiplier λ, first-order condition with respect to $Q_{it}(\omega)$ is:

$$\left(\int_0^1 \left(\sum_{i=\{1,2\}} Q_{it}(\omega)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma(\rho-1)}{(\sigma-1)\rho}} d\omega \right)^{\frac{1}{\rho-1}} \left(\sum_{i=\{1,2\}} Q_{it}(\omega)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\rho-\sigma}{(\sigma-1)\rho}} Q_{it}(\omega)^{-\frac{1}{\sigma}} = \lambda P_{it}(\omega).$$

I. Deriving Demand Function

- Therefore,

$$\frac{Q_{1t}(\omega)}{Q_{2t}(\omega)} = \left(\frac{P_{1t}(\omega)}{P_{2t}(\omega)}\right)^{-\sigma} \quad \Rightarrow \quad Q_{2t}(\omega) = \left(\frac{P_{2t}(\omega)}{P_{1t}(\omega)}\right)^{-\sigma} Q_{1t}(\omega),$$

and

$$\begin{aligned}
\frac{Q_{1t}(\omega)}{Q_{1t}(\omega')} &= \left(\frac{P_{1t}(\omega)}{P_{1t}(\omega')}\right)^{-\sigma} \frac{\left(\sum_{i=\{1,2\}} Q_{it}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma(\rho-\sigma)}{(\sigma-1)\rho}}}{\left(\sum_{i=\{1,2\}} q_{it}(\omega')^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma(\rho-\sigma)}{(\sigma-1)\rho}}} \\
&= \left(\frac{P_{1t}(\omega)}{P_{1t}(\omega')}\right)^{-\sigma} \frac{\left(Q_{1t}(\omega)^{\frac{\sigma-1}{\sigma}} + Q_{1t}(\omega)^{\frac{\sigma-1}{\sigma}}\left(\frac{P_{2t}(\omega)}{P_{1t}(\omega)}\right)^{1-\sigma}\right)^{\frac{\sigma(\rho-\sigma)}{(\sigma-1)\rho}}}{\left(Q_{1t}(\omega')^{\frac{\sigma-1}{\sigma}} + Q_{1t}(\omega')^{\frac{\sigma-1}{\sigma}}\left(\frac{P_{2t}(\omega')}{P_{1t}(\omega')}\right)^{1-\sigma}\right)^{\frac{\sigma(\rho-\sigma)}{(\sigma-1)\rho}}} \\
&= \left(\frac{Q_{1t}(\omega)}{Q_{1t}(\omega')}\right)^{\frac{\rho-\sigma}{\rho}} \left(\frac{P_{1t}(\omega)}{P_{1t}(\omega')}\right)^{-\frac{\sigma^2}{\rho}} \frac{\left(P_{1t}(\omega)^{1-\sigma} + P_{2t}(\omega)^{1-\sigma}\right)^{\frac{\sigma(\rho-\sigma)}{(\sigma-1)\rho}}}{\left(P_{1t}(\omega')^{1-\sigma} + P_{2t}(\omega')^{1-\sigma}\right)^{\frac{\sigma(\rho-\sigma)}{(\sigma-1)\rho}}}.
\end{aligned}$$

I. Deriving Demand Function

- (cont.)

$$\frac{Q_{1t}(\omega)}{Q_{1t}(\omega')} = \left(\frac{P_{1t}(\omega)}{P_{1t}(\omega')}\right)^{-\sigma} \frac{\left(P_{1t}(\omega)^{1-\sigma} + P_{2t}(\omega)^{1-\sigma}\right)^{\frac{(\sigma-\rho)}{(1-\sigma)}}}{\left(P_{1t}(\omega')^{1-\sigma} + P_{2t}(\omega')^{1-\sigma}\right)^{\frac{(\sigma-\rho)}{(1-\sigma)}}}.$$

- Defining

$$P_t(\omega) = \left(P_{1t}(\omega)^{1-\sigma} + P_{2t}(\omega)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}, \tag{1}$$

we can further rewrite

$$Q_{1t}(\omega) = \left(\frac{P_{1t}(\omega)}{P_{1t}(\omega')}\right)^{-\sigma} \left(\frac{P_t(\omega)}{P_t(\omega')}\right)^{\sigma-\rho} Q_{1t}(\omega'),$$

and

$$Q_{2t}(\omega) = \left(\frac{P_{2t}(\omega)}{P_{1t}(\omega)}\right)^{-\sigma} \left(\frac{P_{1t}(\omega)}{P_{1t}(\omega')}\right)^{-\sigma} \left(\frac{P_t(\omega)}{P_t(\omega')}\right)^{\sigma-\rho} Q_{1t}(\omega').$$

- Using the budget constraint, we can therefore write

$$\int_0^1 (P_{1t}(\omega)Q_{1t}(\omega) + P_{2t}(\omega)Q_{2t}(\omega))d\omega = I_t$$

$$\int_0^1 Q_{1t}(\omega)P_{1t}(\omega)^{\sigma}(P_{1t}(\omega)^{1-\sigma} + P_{2t}(\omega)^{1-\sigma})d\omega = I_t$$

$$\int_0^1 Q_{1t}(\omega)P_{1t}(\omega)^{\sigma}P_t(\omega)^{1-\sigma}d\omega = I_t$$

$$\int_0^1 \left(\frac{P_{1t}(\omega)}{P_{1t}(\omega')}\right)^{-\sigma}\left(\frac{P_t(\omega)}{P_t(\omega')}\right)^{\sigma-\rho} Q_{1t}(\omega')P_{1t}(\omega)^{\sigma}P_t(\omega)^{1-\sigma}d\omega = I_t$$

$$P_{1t}(\omega')^{\sigma}Q_{1t}(\omega')P_t(\omega')^{\rho-\sigma}\int_0^1 P_t(\omega)^{1-\rho}d\omega = I_t$$

and, therefore,

$$Q_{1t}(\omega') = \left(\frac{P_{1t}(\omega')}{P_t(\omega')}\right)^{-\sigma} P_t(\omega')^{-\rho}\Big(\int_0^1 P_t(\omega)^{1-\rho}d\omega\Big)^{-1} I_t.$$

I. Deriving Demand Function

- Defining

$$P_t = \left(\int_0^1 P_t(\omega)^{1-\rho} d\omega \right)^{\frac{1}{1-\rho}},$$

note that

$$\left(\int_0^1 P_t(\omega)^{1-\rho} d\omega \right)^{-1} = P_t^{\rho-1}$$

and, therefore,

$$Q_{1t}(\omega') = \left(\frac{P_{1t}(\omega')}{P_t(\omega')} \right)^{-\sigma} \left(\frac{P_t(\omega')}{P_t} \right)^{-\rho} Q_t, \quad \text{with} \quad Q_t = \frac{I_t}{P_t}.$$

- Generally, for any $\omega \in [0,1]$ and $i \in \{1,2\}$, the demand function is

$$Q_{it}(\omega) = \left(\frac{P_{it}(\omega)}{P_t(\omega)} \right)^{-\sigma} \left(\frac{P_t(\omega)}{P_t} \right)^{-\rho} Q_t = P_{it}(\omega)^{-\sigma} P_t(\omega)^{\sigma-\rho} P_t^{\rho} Q_t. \qquad (2)$$

II. Deriving Marginal Cost Function

- Given production function $Q_{it}(\omega) = A_{it}(\omega) L_{it}(\omega)^{\alpha}$ and a cost w_t per unit of $L_{it}(\omega)$, the total cost of firm i of producing $Q_{it}(\omega)$ units of good ω is:

$$c_{it}(Q_{it}(\omega);\omega) = w_t L_{it}(Q_{it}(\omega);\omega) = w_t \left(\frac{Q_{it}(\omega)}{A_{it}(\omega)} \right)^{\frac{1}{\alpha}}.$$

- The marginal cost to firm i of producing $Q_{it}(\omega)$ units of good ω thus is:

$$\frac{\partial c_{it}(Q_{it}(\omega);\omega)}{\partial Q_{it}(\omega)} = w_t \frac{1}{\alpha} \left(\frac{1}{A_{it}(\omega)} \right)^{\frac{1}{\alpha}} Q_{it}(\omega)^{\frac{1-\alpha}{\alpha}}$$

- Given the demand function in eq. (2), we can write these marginal costs as:

$$\frac{\partial c_{it}(Q_{it}(\omega);\omega)}{\partial Q_{it}(\omega)} = w_t \frac{1}{\alpha} \left(\frac{1}{A_{it}(\omega)} \right)^{\frac{1}{\alpha}} \left(P_{it}(\omega)^{-\sigma} P_t(\omega)^{\sigma-\rho} P_t^{\rho} Q_t \right)^{\frac{1-\alpha}{\alpha}}, \qquad (3)$$

with $P_t(\omega)$ as defined in eq. (1).

III. Deriving Pricing Equation: Bertrand Competition

- Firm i producing good ω solves the following problem

$$\max_{P_{it}(\omega)} \{P_{it}(\omega) Q_{it}(P_{it}(\omega);\omega) - c_{it}(Q_{it}(P_{it}(\omega);\omega);\omega)\}$$

where $Q_{it}(\omega) = Q_{it}(P_{it}(\omega);\omega)$ is the quantity demanded of good ω produced by firm i when it sets the price $P_{it}(\omega)$, and $c_{it}(Q_{it}(\omega);\omega)$ is the total cost of producing quantity $Q_{it}(\omega)$ for firm i producing good ω.

- Given eqs. (1) and (2), it holds

$$Q_{it}(P_{it}(\omega);\omega) = P_{it}(\omega)^{-\sigma}\left(P_{1t}(\omega)^{1-\sigma} + P_{2t}(\omega)^{1-\sigma}\right)^{\frac{\sigma-\rho}{1-\sigma}} P_t^{\rho} Q_t$$

and

$$\frac{\partial Q_{it}(P_{it}(\omega);\omega)}{\partial P_{it}(\omega)} = -\sigma P_{it}(\omega)^{-\sigma-1}\left(P_{1t}(\omega)^{1-\sigma} + P_{2t}(\omega)^{1-\sigma}\right)^{\frac{\sigma-\rho}{1-\sigma}} P_t^{\rho} Q_t \qquad (4)$$
$$+ P_{it}(\omega)^{-2\sigma}(\sigma-\rho)\left(P_{1t}(\omega)^{1-\sigma} + P_{2t}(\omega)^{1-\sigma}\right)^{\frac{\sigma-\rho}{1-\sigma}-1} P_t^{\rho} Q_t.$$

III. Deriving Pricing Equation: Bertrand Competition

- Therefore, the first-order condition with respect to $P_{it}(\omega)$ is

$$Q_{it}(P_{it}(\omega);\omega) + P_{it}(\omega)\frac{\partial Q_{it}(P_{it}(\omega);\omega)}{\partial P_{it}(\omega)} - \frac{\partial c_{it}(Q_{it}(\omega))}{\partial Q_{it}(\omega)}\frac{\partial Q_{it}(P_{it}(\omega);\omega)}{\partial P_{it}(\omega)} = 0;$$

or, equivalently,

$$Q_{it}(P_{it}(\omega);\omega)(1 - \varepsilon_{it}(P_{it}(\omega);\omega)) + \frac{\partial c_{it}(Q_{it}(\omega))}{\partial Q_{it}(\omega)}\varepsilon_{it}(P_{it}(\omega);\omega)\frac{Q_{it}(P_{it}(\omega);\omega)}{P_{it}(\omega)} = 0,$$

with

$$\varepsilon_{it}(P_{it}(\omega);\omega) = -\frac{\partial Q_{it}(P_{it}(\omega);\omega)}{\partial P_{it}(\omega)}\frac{P_{it}(\omega)}{Q_{it}(P_{it}(\omega);\omega)}.$$

- We can thus rewrite the first-order condition as

$$P_{it}(\omega)(1 - \varepsilon_{it}(P_{it}(\omega);\omega)) + \frac{\partial c_{it}(Q_{it}(\omega))}{\partial Q_{it}(\omega)}\varepsilon_{it}(P_{it}(\omega);\omega) = 0.$$

III. Deriving Pricing Equation: Bertrand Competition

- Re-arranging terms, we obtain the expression

$$P_{it}(\omega)(1-\varepsilon_{it}(P_{it}(\omega);\omega)) + \frac{\partial c_{it}(Q_{it}(\omega))}{\partial Q_{it}(\omega)}\varepsilon_{it}(P_{it}(\omega);\omega) = 0,$$

or, equivalently,

$$P_{it}(\omega) = \frac{\varepsilon_{it}(P_{it}(\omega);\omega)}{\varepsilon_{it}(P_{it}(\omega);\omega)-1}\frac{\partial c_{it}(Q_{it}(\omega))}{\partial Q_{it}(\omega)}.$$

- Finally, using eq. (4), note that

$$\begin{aligned}\varepsilon_{it}(P_{it}(\omega);\omega) &= \sigma - P_{it}(\omega)^{1-\sigma}(\sigma-\rho)\big(P_{1t}(\omega)^{1-\sigma} + P_{2t}(\omega)^{1-\sigma}\big)^{-1} \\ &= \sigma - (\sigma-\rho)\left(\frac{P_{it}(\omega)}{P_t(\omega)}\right)^{1-\sigma},\end{aligned}$$

with $P_t(\omega)$ as defined in eq. (1).

III. Deriving Equilibrium: Bertrand Competition

- For any good ω, the equilibrium prices $(P_{1t}(\omega), P_{2t}(\omega))$ when firms compete à la Bertrand are given by the following system of equations.

$$P_{1t}(\omega) = \frac{\sigma P_t(\omega)^{1-\sigma} - (\sigma-\rho) P_{1t}(\omega)^{1-\sigma}}{(\sigma-1) P_t(\omega)^{1-\sigma} - (\sigma-\rho) P_{1t}(\omega)^{1-\sigma}} \frac{\alpha^{-1}\kappa_t}{A_{1t}(\omega)^{\frac{1}{\alpha}}} \left(P_{1t}(\omega)^{-\sigma} P_t(\omega)^{\sigma-\rho}\right)^{\frac{1-\alpha}{\alpha}}$$

$$P_{2t}(\omega) = \frac{\sigma P_t(\omega)^{1-\sigma} - (\sigma-\rho) P_{2t}(\omega)^{1-\sigma}}{(\sigma-1) P_t(\omega)^{1-\sigma} - (\sigma-\rho) P_{2t}(\omega)^{1-\sigma}} \frac{\alpha^{-1}\kappa_t}{A_{2t}(\omega)^{\frac{1}{\alpha}}} \left(P_{2t}(\omega)^{-\sigma} P_t(\omega)^{\sigma-\rho}\right)^{\frac{1-\alpha}{\alpha}}$$

with $\kappa_t = w_t (P_t^{\rho} Q_t)^{\frac{1-\alpha}{\alpha}}$ and $P_t(\omega)$ as defined in eq. (1).

- When $\sigma = \rho$ and $\alpha = 1$, these pricing equations become

$$P_{1t}(\omega) = \frac{\sigma}{\sigma-1} \frac{w_t}{A_{1t}(\omega)},$$

$$P_{2t}(\omega) = \frac{\sigma}{\sigma-1} \frac{w_t}{A_{2t}(\omega)}.$$

IV. Deriving Pricing Equation: Cournot Competition

- Firm i producing good ω solves the following problem

$$\max_{Q_{it}(\omega)} \{P_{it}(Q_{it}(\omega);\omega)Q_{it}(\omega) - c_{it}(Q_{it}(\omega);\omega)\} \tag{5}$$

with $P_{it}(\omega) = P_{it}(Q_{it}(\omega);\omega)$ the price of firm i's variety of ω if it produces $Q_{it}(\omega)$ units, and $c_{it}(Q_{it}(\omega);\omega)$ the cost of producing $Q_{it}(\omega)$ for this firm.

- Given eqs. (1) and (2), it holds

$$P_{it}(\omega) = Q_{it}(\omega)^{-\frac{1}{\sigma}} P_t(\omega)^{\frac{\sigma-\rho}{\sigma}} (P_t^{\rho} Q_t)^{\frac{1}{\sigma}}$$

with

$$\begin{aligned} P_t(\omega) &= (P_{1t}(\omega)^{1-\sigma} + P_{2t}(\omega)^{1-\sigma})^{\frac{1}{1-\sigma}} \\ &= \left(Q_{1t}(\omega)^{\frac{\sigma-1}{\sigma}} + Q_{2t}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{1-\sigma}} P_t(\omega)^{\frac{\sigma-\rho}{\sigma}} (P_t^{\rho} Q_t)^{\frac{1}{\sigma}}. \end{aligned}$$

IV. Deriving Pricing Equation: Cournot Competition

- Therefore,

$$P_t(\omega)^{\frac{\rho}{\sigma}} = \left(Q_{1t}(\omega)^{\frac{\sigma-1}{\sigma}} + Q_{2t}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{1-\sigma}} (P_t^\rho Q_t)^{\frac{1}{\sigma}},$$

and, consequently,

$$P_t(\omega) = \left(Q_{1t}(\omega)^{\frac{\sigma-1}{\sigma}} + Q_{2t}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{(1-\sigma)\rho}} (P_t^\rho Q_t)^{\frac{1}{\rho}},$$

and, finally,

$$P_t(\omega)^{\frac{\sigma-\rho}{\sigma}} = \left(Q_{1t}(\omega)^{\frac{\sigma-1}{\sigma}} + Q_{2t}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma-\rho}{(1-\sigma)\rho}} (P_t^\rho Q_t)^{\frac{\sigma-\rho}{\rho\sigma}},$$

- Consequently,

$$P_{it}(Q_{it}(\omega);\omega) = Q_{it}(\omega)^{-\frac{1}{\sigma}} \left(Q_{1t}(\omega)^{\frac{\sigma-1}{\sigma}} + Q_{2t}(\omega)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma-\rho}{(1-\sigma)\rho}} (P_t^\rho Q_t)^{\frac{1}{\rho}}.$$

IV. Deriving Pricing Equation: Cournot Competition

- Therefore,

$$
\begin{aligned}
\frac{\partial P_{it}(Q_{it}(\omega);\omega)}{\partial Q_{it}(\omega)} &= -\frac{1}{\sigma}\frac{P_{it}(\omega)}{Q_{it}(\omega)}(1+\frac{\sigma-\rho}{\rho}\frac{Q_{it}(\omega)^{\frac{\sigma-1}{\sigma}}}{Q_{1t}(\omega)^{\frac{\sigma-1}{\sigma}}+Q_{2t}(\omega)^{\frac{\sigma-1}{\sigma}}}) \\
&= -\frac{1}{\sigma}\frac{P_{it}(\omega)}{Q_{it}(\omega)}(1+\frac{\sigma-\rho}{\rho}\frac{P_{it}(\omega)^{1-\sigma}}{(P_{1t}(\omega)^{1-\sigma}+P_{2t}(\omega)^{1-\sigma})}) \\
&= -\frac{1}{\sigma}\frac{P_{it}(\omega)}{Q_{it}(\omega)}(\frac{1}{\sigma}+\frac{\sigma-\rho}{\sigma\rho}\frac{P_{it}(\omega)^{1-\sigma}}{(P_{1t}(\omega)^{1-\sigma}+P_{2t}(\omega)^{1-\sigma})}) \\
&= -\frac{P_{it}(\omega)}{Q_{it}(\omega)}(\frac{1}{\sigma}+\left(\frac{1}{\rho}-\frac{1}{\sigma}\right)\frac{P_{it}(\omega)^{1-\sigma}}{(P_{1t}(\omega)^{1-\sigma}+P_{2t}(\omega)^{1-\sigma})}).
\end{aligned}
$$

- Therefore,

$$
-\frac{\partial P_{it}(Q_{it}(\omega);\omega)}{\partial Q_{it}(\omega)}\frac{Q_{it}(\omega)}{P_{it}(Q_{it}(\omega);\omega)} = \frac{1}{\sigma}-\left(\frac{1}{\sigma}-\frac{1}{\rho}\right)\left(\frac{P_{it}(\omega)}{P_t(\omega)}\right)^{1-\sigma}.
$$

IV. Deriving Pricing Equation: Cournot Competition

- Given eq. (5), the first-order condition with respect to $Q_{it}(\omega)$ is

$$\frac{\partial P_{it}(Q_{it}(\omega);\omega)}{\partial Q_{it}(\omega)} Q_{it}(\omega) + P_{it}(Q_{it}(\omega);\omega) - \frac{\partial c_{it}(Q_{it}(\omega);\omega)}{\partial Q_{it}(\omega)} = 0;$$

or, equivalently,

$$P_{it}(Q_{it}(\omega);\omega)(1 - \tilde{\varepsilon}_{it}(P_{it}(\omega);\omega)) - \frac{\partial c_{it}(Q_{it}(\omega);\omega_{it})}{\partial Q_{it}(\omega)} = 0,$$

with

$$\tilde{\varepsilon}_{it}(P_{it}(\omega);\omega) = -\frac{\partial P_{it}(Q_{it}(\omega);\omega)}{\partial Q_{it}(\omega)} \frac{Q_{it}(\omega)}{P_{it}(Q_{it}(\omega);\omega)}.$$

- Given $\check{\varepsilon}_{it}(P_{it}(\omega);\omega) = 1/\tilde{\varepsilon}_{it}(P_{it}(\omega);\omega)$, we can write first-order condition as

$$P_{it}(Q_{it}(\omega);\omega)(\frac{\check{\varepsilon}_{it}(P_{it}(\omega);\omega) - 1}{\check{\varepsilon}_{it}(P_{it}(\omega);\omega)}) - \frac{\partial c_{it}(Q_{it}(\omega);\omega_{it})}{\partial Q_{it}(\omega)} = 0,$$

IV. Deriving Pricing Equation: Cournot Competition

- Re-arranging terms, we obtain the expression

$$P_{it}(Q_{it}(\omega);\omega) = \frac{\check{\varepsilon}_{it}(P_{it}(\omega);\omega)}{\check{\varepsilon}_{it}(P_{it}(\omega);\omega) - 1} \frac{\partial c_{it}(Q_{it}(\omega);\omega_{it})}{\partial Q_{it}(\omega)}.$$

- More conveniently, we can rewrite this expression as

$$P_{it}(Q_{it}(\omega);\omega) = \frac{1}{1 - \tilde{\varepsilon}_{it}(P_{it}(\omega);\omega)} \frac{\partial c_{it}(Q_{it}(\omega);\omega_{it})}{\partial Q_{it}(\omega)}.$$

and, given eq. (3), we can rewrite further as

$$P_{it}(Q_{it}(\omega);\omega) =$$

$$\left(1 - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\rho}\right)\left(\frac{P_{it}(\omega)}{P_t(\omega)}\right)^{1-\sigma}\right)^{-1} \frac{\alpha^{-1}\kappa_t}{A_{it}(\omega)^{\frac{1}{\alpha}}} (P_{it}(\omega)^{-\sigma} P_t(\omega)^{\sigma-\rho})^{\frac{1-\alpha}{\alpha}},$$

with $\kappa_t = w_t(P_t^{\rho} Q_t)^{\frac{1-\alpha}{\alpha}}$ and $P_t(\omega)$ as defined in eq. (1).

V. Production Function Estimation Procedure

- For a random sample of sectors $\omega = 1, \ldots, \Omega$ and a period t, we observe

$$\{(\hat{q}_{it}(\omega), \hat{q}_{it-1}(\omega), l_{it}(\omega), l_{it-1}(\omega), x_{it}(\omega))\}_{i=\{1,2\}},$$

where $\hat{q}_{it}(\omega)$ is a proxy of $q_{it}(\omega)$.

- Given these data, we obtain estimates of (α, γ, β) and $\{a_{it}(\omega)\}_{\omega,i}$ following the procedure described in slide 42.
- We present results for two different scenarios:
  1. Firm output is observed:

$$\hat{q}_{it}(\omega) = q_{it}(\omega).$$

  2. Only firm revenue & sectoral price indices are observed:

$$\hat{q}_{it}(\omega) = r_{it}(\omega) - p_t(\omega), \text{ with } p_t(\omega) = \ln(P_t(\omega)).$$

V. Production Function Estimation Procedure

1. Rewrite productivity as a function of measured output, measured inputs, and production function parameters:

$$\hat{a}_{it}(\omega; \alpha) = \hat{q}_{it}(\omega) - \alpha L_{it}(\omega). \tag{6}$$

2. Given the assumption $a_{it}(\omega) = \gamma a_{it-1}(\omega) + \beta x_{it}(\omega) + \varepsilon_{it}(\omega)$, we can write

$$\hat{q}_{it}(\omega) - \alpha l_{it}(\omega) = \gamma(\hat{q}_{it-1}(\omega) - \alpha l_{it-1}(\omega)) + \beta x_{it}(\omega) + \varepsilon_{it}(\omega),$$

or, equivalently,

$$\hat{q}_{it}(\omega) = \alpha l_{it}(\omega) + \gamma(\hat{q}_{it-1}(\omega) - \alpha l_{it-1}(\omega)) + \beta x_{it}(\omega) + \varepsilon_{it}(\omega).$$

3. We can use this expression to estimate (α, γ, β) and, given $\hat{\alpha}$, we compute $\{\hat{a}_{it}(\omega; \hat{\alpha})\}_{\omega,i}$ using the expression in eq. (6).

VI. Monopolistically Competitive Firms in Two Markets

- Each firm i faces CES demand functions with elasticity of substitution σ both in the home and in the foreign markets. Assuming monopolistic competition, the firm sets the following prices in both markets:

$$P_{iH} = \frac{\sigma}{\sigma - 1}\frac{\tau_{iH} W}{A_i}, \qquad \text{and} \qquad P_{iF} = \frac{\sigma}{\sigma - 1}\frac{\tau_{iF} W}{A_i}.$$

Note: we assume $\sigma > 1$ all throughout the analysis.

- Given the following demand functions in both home and foreign markets:

$$Q_{iH} = \xi_{iH} P_{iH}^{-\sigma} P_H^{\sigma} Q_H, \qquad \text{and} \qquad Q_{iF} = \xi_{iF} P_{iF}^{-\sigma} P_F^{\sigma} Q_F,$$

where ξ_{iH} and ξ_{iF} are demand shifters.

- Combining the expressions for prices and quantities above, we can write the firm's revenue in each market as:

$$R_{iH} = P_{iH} Q_{iH} = \kappa_H \xi_{iH} \tau_{iH}^{1-\sigma} A_i^{\sigma-1} \quad \text{and} \quad R_{iF} = P_{iF} Q_{iF} = \kappa_F \xi_{iF} \tau_{iF}^{1-\sigma} A_i^{\sigma-1},$$

where κ_H and κ_F are constants that do not vary by firm.

VI. Firms Operating in Two Markets

- Firm i's production function (in logs) is:

$$q_i = a_i + \alpha l_i. \tag{7}$$

- Firm i faces the following demand functions in home and foreign market:

$$Q_{iH} = P_{iH}^{-\sigma} P_H^{\sigma} Q_H \qquad \text{and} \qquad Q_{iF} = P_{iF}^{-\sigma} P_F^{\sigma} Q_F.$$

From these, we can write the inverse demand functions as

$$P_{iH} = Q_{iH}^{-\frac{1}{\sigma}} P_H Q_H^{\frac{1}{\sigma}} \qquad \text{and} \qquad P_{iF} = Q_{iF}^{-\frac{1}{\sigma}} P_F Q_F^{\frac{1}{\sigma}},$$

and the potential revenue in each of the two markets as

$$R_{iH} = P_{iH} Q_{iH} = Q_{iH}^{\frac{\sigma-1}{\sigma}} P_H Q_H^{\frac{1}{\sigma}} \quad \text{and} \quad R_{iF} = P_{iF} Q_{iF} = Q_{iF}^{\frac{\sigma-1}{\sigma}} P_F Q_F^{\frac{1}{\sigma}}.$$

VI. Firms Operating in Two Markets

- We can therefore write

$$R_i = R_{iH} + d_{iF} R_{iF} = Q_{iH}^{\frac{\sigma-1}{\sigma}} P_H Q_H^{\frac{1}{\sigma}} + d_{iF} Q_{iF}^{\frac{\sigma-1}{\sigma}} P_F Q_F^{\frac{1}{\sigma}},$$

where d_{iF} is a dummy variable that equals one if firm i sells in market F.

- If $\kappa = P_H Q_H^{\frac{1}{\sigma}} = P_F Q_F^{\frac{1}{\sigma}}$, we can then write

$$R_i = (Q_{iH}^{\frac{\sigma-1}{\sigma}} + d_{iF} Q_{iF}^{\frac{\sigma-1}{\sigma}})\kappa$$

- Without loss of generality, we can write $Q_{iF} = \lambda_i Q_{iH}$ for $\lambda_i > 0$ and, thus,

$$\begin{aligned} R_i &= (Q_{iH}^{\frac{\sigma-1}{\sigma}} + d_{Fi} \lambda_i^{\frac{\sigma-1}{\sigma}} Q_{iH}^{\frac{\sigma-1}{\sigma}})\kappa \\ &= Q_{iH}^{\frac{\sigma-1}{\sigma}} \kappa (1 + d_{Fi} \lambda_i^{\frac{\sigma-1}{\sigma}}). \end{aligned}$$

Similarly, note that

$$Q_i = Q_{iH} + d_{iF} Q_{iF} = Q_{iH} + d_{iF} \lambda_i Q_{iH} = Q_{iH}(1 + d_{iF}\lambda_i) \Rightarrow Q_{iH} = \frac{Q_i}{1 + d_{iF}\lambda_i}.$$

VI. Firms Operating in Two Markets

- We can therefore write

$$R_i = Q_i^{\frac{\sigma-1}{\sigma}} \kappa \frac{(1 + d_{Fi}\lambda_i^{\frac{\sigma-1}{\sigma}})}{(1 + d_{iF}\lambda_i)^{\frac{\sigma-1}{\sigma}}},$$

or, in logs,

$$r_i = \frac{\sigma-1}{\sigma} q_i + \ln(\kappa) + \ln\left[\frac{(1 + d_{Fi}\lambda_i^{\frac{\sigma-1}{\sigma}})}{(1 + d_{iF}\lambda_i)^{\frac{\sigma-1}{\sigma}}}\right].$$

- Given eq. (7), we can further write

$$r_i = \underbrace{\ln(\kappa) + \ln\left[\frac{(1 + d_{Fi}\lambda_i^{\frac{\sigma-1}{\sigma}})}{(1 + d_{iF}\lambda_i)^{\frac{\sigma-1}{\sigma}}}\right] + \frac{\sigma-1}{\sigma} a_i}_{a_i^R} + \frac{\sigma-1}{\sigma} \alpha l_i.$$

VI. Firms Operating in Two Markets