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## Abstract

This paper investigates analytically and numerically intertemporal equilibrium portfolio policies under time dependent returns. The analysis is performed using a new method for obtaining approximate closed form solutions to the optimal portfolio-consumption problem that does not require the imposition of constraints on the conditional moments of consumption and that allows for autoregressive conditional heteroskedasticity in stock returns. The analytical and numerical results show that the elasticity of intertemporal substitution is irrelevant for the determination of the portfolio policy when returns are persistent and follow GARCH processes. In addition, results show that small departures from the *i.i.d.* assumption produce an important variability in the portfolio holdings that contrasts with the static CAPM constant portfolio policies. However, a conditional version of the static CAPM with the inclusion of a Jensen inequality correction is able to explain the overwhelming majority of the mean and almost all the variability of the equilibrium portfolio weights for a sensible choice of the parameters of the conditional mean and variance processes of asset returns.



KEYWORDS: CAPM, conditional heteroskedasticity, Euler equations, approximation analysis, numerical solutions.

# 1 Introduction

An important part of the literature in Applied Financial Economics in the last few years has documented the existence of significant time-dependency in stock returns. Thus, work by Lo and Mackinley (1988, 1990a, 1990b), Fama and French (1988), Poterba and Summers (1988), and Cutler, Poterba and Summers (1990), among others, has detected the presence of predictable components in stock returns that have the form of negative autocorrelation for individual stocks at high frequencies and for the aggregate market portfolio at low frequencies and positive autocorrelation for aggregate portfolios at high frequencies. On the other hand, papers by Bollerslev (1986), Bollerslev, Engle and Wooldridge (1988), Nelson (1991) and Pagan and Schwert (1989) have provided empirical evidence on the existence of a significant autoregressive structure in the conditional variance of stock returns. Those findings have been extensively used in the asset pricing literature as evidence against the hypothesis of the standard conditional homoskedastic risk neutral random walk model for asset prices, either with the purpose of challenging the efficient markets paradigm or with the purpose of underlining the necessity of assuming more sophisticated risk averse agents to explain asset prices. In the latter tradition, the discrete time versions of the Consumption based Capital Asset Pricing Models (e.g., Hansen and Singleton (1982, 1983), Grossman and Shiller (1984), Breeden, Gibbons and Litzenberger (1989), and its generalizations (e.g., Epstein and Zin (1989), Weil (1990), and Constantinides (1990)) provide an intertemporal framework where efficient pricing is compatible with time dependency in returns.

The direct partial equilibrium analysis of portfolio policies under time-dependent returns in discrete time is less developed than the asset pricing literature. The reason is, to a great extent, the difficulty in obtaining closed form solutions for a sufficiently general problem even under specific distributional assumptions on asset prices. This fact contrasts with the successful developments of portfolio theory in continuous time in the tradition inaugurated with the work of Merton (1971, 1973). In this framework, the mutual fund separation theorems provide an insightful interpretation of closed form solutions of the intertemporal portfolio problem when asset prices follow diffusion processes. However, those results are not directly applicable to empirical analysis with discrete data.

The analysis of portfolio policies in discrete time often relies on CAPM like static, two-period models. Those popular models are, in general, theoretically inappropriate in an intertemporal framework if returns are time-dependent. However, the predictions of those static models are essentially correct in a world where the state variables are independent and identically distributed. This result holds under a variety of preference structures (see Giovanini and Weil (1989) and Kocherlakota (1990)) and constitutes a natural benchmark against which to analyze the relevance of using sophisticated intertemporal models to analyze the optimal portfolio choice.

In this paper, I analyze the role of the intertemporal dependency of returns in the determination of agents' equilibrium portfolio policies. I assume a general specification of preferences, identical to the ones suggested by Epstein and Zin (1989) and Weil (1990), and focus on the analysis of the rational portfolio choice between risky and riskless assets. The specification of preferences allows one to analyze independently the effects of aversion to risk and intertemporal substitution of consumption on the determination of the optimal portfolio. The goal is to characterize the difference between the predictions of the static models with those of the intertemporal models as an interaction between agents' preferences and the distribution of asset returns. In particular, I study how the time-dependency in the distribution of returns affects the mean and the variability of the holdings of the risky asset and the role of risk aversion and the elasticity of substitution in the determination of the optimal portfolio policy.

Given the absence of closed form solutions, the analysis has to rely on linear approximations or numerical solution techniques. In this paper I explore both approaches. I perform an approximation analysis by obtaining a first order Taylor series expansion of the first order conditions of the intertemporal maximization problem expressed in terms of the consumption wealth-ratio. Recently, Campbell (1990) has obtained an approximate asset pricing expression without reference to consumption by combining a linear approximation to the budget constraint with the standard Euler Equation. In order to do that, Campbell assumes that asset returns and consumption are jointly homoskedastic. Unlike Campbell, I do not need to impose exogenous constraints on the conditional moments of consumption, so that the framework is useful for analyzing individuals' equilibrium consumption-portfolio policies. Furthermore, I show that the analysis is robust to the presence of the standard form of autoregressive conditional heteroskedasticity in asset returns. This provides empirical attractiveness to both the portfolio policy and the asset pricing analysis.

Finally, the numerical part of the exercise is useful for two reasons. On one side, it provides the possibility of testing the accuracy of the approximate portfolio expressions. On the other hand, it allows the analysis to be extended to include more general stochastic processes for returns. In particular, I study optimal portfolio policies under time varying persistence and exponential conditional heteroskedasticity in stock returns.

This paper is organized as follows. Section 2 presents the intertemporal decision model. Section 3 obtains an approximate expression for the equilibrium consumption policy under common distributional assumptions on returns and discusses its asset pricing implications. Section 4 uses the results of Section 3 to obtain an approximate expression for the two-asset equilibrium portfolio policy. Section 5 solves numerically the equilibrium consumption-portfolio problem for different distributional assumptions on returns. Section 6 concludes.

## 2 An Intertemporal Consumption-Portfolio Problem

Assume that the agents in the economy behave according to the generalized isoelastic preferences (GIP) introduced by Epstein and Zin (1989) and Weil (1990). The assumption of GIP preferences implies rather sophisticated agents who show independent attitudes toward risk and toward substitution of consumption over time. This allows a relatively complete description of the consumption-portfolio choice in a world with time-dependent uncertainty. In a finite horizon setting, the GIP preferences are defined by a utility function that has the recursive structure

$$
\begin{aligned}
U_t &= V[C_t, E_t U_{t+1}] = \left\{(1-\delta)C_t^{1-\rho} + \delta(E_t U_{t+1})^{\frac{1-\rho}{1-\gamma}}\right\}^{\frac{1-\gamma}{1-\rho}} \qquad t < T && (1) \\
U_T &= \left[(1-\delta)C_T^{1-\rho}\right]^{\frac{1-\gamma}{1-\rho}}. && (2)
\end{aligned}
$$

In this preference structure, $\gamma$ is the coefficient of relative risk aversion for timeless gambles and $1/\rho$ is the elasticity of intertemporal substitution. If $\gamma = \rho$, expression (2) collapses to the time-additive constant relative risk aversion expected utility specification.

Agents can invest in $N+1$ securities whose respective returns are $R_i(t+1)$ $(i = 0, \cdots, N)$. Define $W_t$ as the total wealth of an agent at the beginning of period $t$, and $R_m(t+1)$ as the real gross return of the portfolio of assets held from period $t$ to $t+1$. Then at period $t$, the agents' budget constraint has the form

$$W_{t+1} = (W_t - C_t)R_m(t+1) \tag{3}$$

The first order conditions of an agent who maximizes (2) subject to (3) imply the set of Euler equations

$$E_t\left\{\delta^{\frac{1-\gamma}{1-\rho}}\left(\frac{C_{t+1}}{C_t}\right)^{-\rho\frac{1-\gamma}{1-\rho}} R_m(t+1)^{\frac{1-\gamma}{1-\rho}-1} R_i(t+1)\right\} = 1 \qquad i = 0,\cdots,N, \tag{4}$$

together with the terminal condition ($C_T = W_T$). Now, let's define the consumption-wealth ratio at period t as $a_t = C_t/W_t$. Then, specializing (4) for the return on the equilibrium portfolio we find the following stochastic Euler equation for the equilibrium consumption-wealth ratio

$$\begin{aligned} a_t &= \left\{1 + \delta^{\frac{1}{\rho}}\left[E_t\left(R_m(t+1)^{1-\gamma} a_{t+1}^{-\rho\frac{1-\gamma}{1-\rho}}\right)\right]^{\frac{1-\rho}{\rho(1-\gamma)}}\right\}^{-1} \qquad t < T, &(5)\\ a_T &= 1. &(6)\end{aligned}$$

Then, expressing the system (4) in terms of the consumption-wealth ratio and subtracting the Euler equation corresponding to asset 0 from the remaining N equations we find

$$E_t\left\{a_{t+1}^{-\rho\frac{1-\gamma}{1-\rho}} R_m(t+1)^{-\gamma}\left[R_i(t+1) - R_0(t+1)\right]\right\} = 0 \qquad i = 1\cdots N. \tag{7}$$

Expressions (5) and (7) constitute a set of (N+1) equations that together with a transversality condition completely characterize the optimal portfolio and consumption policies for a given distribution of the asset returns.

Expression (7) is very informative about the importance of the intertemporal aspect of the optimal portfolio decision problem. This is relevant as long as the term in $a_{t+1}$ on the left hand side of (7) is sufficiently related to the distribution of asset returns. For those cases were, according to (5), the equilibrium wealth ratio is independent of the distribution of asset returns (as in the *i.i.d.* case), the intertemporal equilibrium portfolio policy is equivalent to the one corresponding to a sequence of two-period expected utility of wealth maximizing problems. The agents will choose portfolio policies that depart from the simple two-period strategy only if their future consumption decisions depend upon the realization of the asset returns. The existence and the magnitude of that distributional dependence is endogenously determined by the interaction of agent's preferences with the distribution of asset returns. Thus, in order to analyze the importance of intertemporal considerations in deciding portfolio strategies, we have to understand how the information about the distribution of asset returns affects the distribution of the consumption-wealth ratio for given values of the preference parameters. Once we have done that, we will be able to observe how different are the optimal portfolio implications of the intertemporal models from the prescriptions of the static models for empirically sensible specifications of the asset returns.

Unfortunately, outside very specific cases (*i.i.d* returns and/or unit elasticity of intertemporal substitution with either lognormal AR(1) returns or unit coefficient of relative risk aversion[1]), it is not possible to obtain exact closed forms for the equilibrium consumption wealth ratio and the portfolio holdings. Therefore, we have to rely on suitable approximations or numerical solution techniques. I now proceed to exploit both approaches in turn.

# 3 An Approximate Closed Form Solution for the Optimal Consumption Policy Problem

## 3.1 Log-linear Approximation to the First Order Conditions

Campbell and Mankiw (1989) and Campbell (1990) use a log-linear approximation to the budget constraint (3) and variants of the Euler equation (4) in an infinite horizon problem to obtain approximate closed-form solutions for the aggregate consumption-wealth ratio. This approach, however, requires making assumptions on both the form and the moments of the joint conditional distribution of the equilibrium portfolio return and consumption. A more appealing approach from a partial equilibrium individual decision perspective is to make sensible assumptions about the distribution of asset returns and to determine endogenously the moments of the consumption

[1]Giovanini and Weil (1988) doccument these result. The i.i.d. case has been also studied by Kocherlakayota (1990) in discrete time and Svenson (1990) in continuous time.

policy which is optimal under those distributional assumptions. I will accomplish this task by performing a direct linearization of the stochastic difference equation (5). I need however, as in Campbell (1990), to assume that the consumption-wealth ratio and the equilibrium portfolio return have a joint conditional lognormal distribution.

Taking logs on both sides of equation (5) and performing a first order Taylor series expansion of the right hand side around its unconditional mean at $t$ we find

$$\begin{aligned} -\log a_t &= A(T-t) \\ &+ B(T-t)\left\{\frac{1-\rho}{\rho}E_t r_m(t+1) + \frac{(1-\rho)(1-\gamma)}{2\rho}\sigma_{mm}(t) - (1-\gamma)\sigma_{am}(t)\right. \\ &\left. +\frac{\rho(1-\gamma)}{2(1-\rho)}\sigma_{aa}(t) - E_t \log a_{t+1} - D(T-t)\right\} \qquad t < T, \end{aligned} \tag{8}$$

where

$$\begin{aligned} A(T-t) &= \log(1+\delta \exp D(T-t)), \\ B(T-t) &= \frac{\exp D(T-t)}{1+\exp D(T-t)}, \\ D(T-t) &= E\left\{\log \delta^{\frac{1}{\rho}} + \frac{1-\rho}{\rho(1-\gamma)}\log\left[E_t\left(R_m(t+1)^{1-\gamma}a_{t+1}^{-\rho\frac{1-\gamma}{1-\rho}}\right)\right]\right\}, \end{aligned} \tag{9}$$

and $r_m(t) = \log R_m t$ and $\sigma_{mm}(t)$, $\sigma_{aa}(t)$, and $\sigma_{am}(t)$ represent the conditional second order moments of the joint distribution of the (log) equilibrium return and the (log) consumption-wealth ratio. The term $D(T-t)$ and therefore $A(T-t)$ and $B(T-t)$ are deterministic functions of time reflecting the fact that, in this finite-horizon problem, $\log(a_t)$ does not have, in general, a stationary unconditional distribution[2].

As in the Mankiw-Campbell case, the lower the variability of $a_t$ conditional on $t-1$, the higher the accuracy of this approximation. Thus, in the *i.i.d* case it is easy to see that $a_t$ becomes a deterministic sequence. For this case, the approximating expression (8) becomes exact under lognormality of the returns.

Now, using the condition $\log(a_T) = 0$ in equation (8) we find

[2]In the infinite horizon case, if $R_m(t+1)$ has a stationary distribution, then, under some standard regularity conditions, there exists a stationary solution for $a_t$ in (5).

$$
\begin{aligned}
-\log a_t &= P(T-t) \\
&+ E_t \sum_{j=1}^{T-t} Q_j(T-t) \left\{ \frac{1-\rho}{\rho} r_m(t+j) + \frac{(1-\rho)(1-\gamma)}{2\rho} \sigma_{mm}(t+j-1) \right. \\
&- \left. (1-\gamma)\sigma_{am}(t+j-1) + \frac{\rho(1-\gamma)}{2(1-\rho)} \sigma_{aa}(t+j-1) \right\}, \qquad (10)
\end{aligned}
$$

where

$$
P(T-t) = A(T-t) - \sum_{j=0}^{T-1} D(T-t-j+1) Q_j(T-t)
$$

and

$$
Q_j(T-t) = \prod_{i=0}^{j-1} B(T-t-i). \qquad (11)
$$

Expression (10) shows how the consumption-wealth ratio at period $t$ is characterized by the expected changes in the conditional mean of returns and in the second order moment of the joint distribution of the consumption-wealth ratio and the asset returns. If consumption and the portfolio return $r_m(t)$ are assumed to have a conditional homoskedastic distribution (as in Campbell (1991)), then $a_t$ and $r_m(t)$ also have a homoskedastic joint conditional distribution. In this case, expression (10) shows that the (log) consumption-ratio is simply a deterministic term plus a discounted sum of future expected returns. Then, the responsiveness of the consumption-wealth ratio to changes in the opportunity set is essentially independent of the coefficient of relative risk aversion $(\gamma)$[3] and only depends on the elasticity of intertemporal substitution $(1/\rho)$. In the general heteroskedastic case, however, both parameters characterize the effect of expected changes in the moments of the returns on the optimal consumption policy. Not surprisingly, the risk associated with changes in the variability of market returns and the uncertainty about the distribution of future optimal consumption affect the marginal propensity to consume in a way which is not independent of the risk aversion parameter.

The homoskedastic assumption, as formulated in Campbell (1990), is not atractive for two reasons. First, it imposes exogenous conditions on the distribution of both returns and consumption when the former should be sufficient to characterize optimal consumption decisions according

[3]Notice however that the dynamics of $a_t$ will not be completely independent of $\gamma$ since the latter affects the former through the determination of discount factors $(Q_j(T-t))$

to (8). Second, it is inconsistent with the empirical evidence on autoregressive conditional heteroskedasticity in stock returns[4]. (See, e.g., Bollerslev (1986), Engle and Wooldridge (1986) and Pagan and Schwert (1989).) A necessary step forward is, then, to characterize completely the optimal consumption policy from the single assumption of a sufficiently general and empirically attractive conditional heteroskedastic distribution for returns. This is done here by assuming that the equilibrium-portfolio return follows the VAR process [5]

$$r_m(t) = \eta' x(t-1) + m(t) \tag{12}$$

$$x(t) = \Gamma x(t-1) + u(t), \tag{13}$$

where $\eta$ and $\Gamma$ are, respectively, a vector and a matrix of parameters; $m(t)$ and $u(t)$ are non-autocorrelated random variables, and $x(t)$ is a vector of exogenous variables whose first component is $r_m(t)$. Using (13), it can be immediately seen that the expected returns follow a simple first-order autoregressive process

$$E_t r_m(t+1) = \eta' x(t) = H E_{t-1} r_m(t) + v(t) \tag{14}$$

$$\text{where} \quad H = \eta'(\Gamma'\eta\eta'\Gamma)^{-1}\Gamma'\eta$$

$$\text{and} \quad v(t) = H(\Gamma'\eta\eta'\Gamma)^{-1}\Gamma'\eta\eta' u(t). \tag{15}$$

Assume now that, conditional on the information available at period t, $(m_t, v_t)$ is normally distributed with zero mean and second order moments $\sigma_{mm}(t), \sigma_{mv}(t)$ *and* $\sigma_{vv}(t)$ that follow the GARCH processes

$$\sigma_{mm}(t) = \alpha_0^m + \alpha_1^m \sigma_{mm}(t-1) + \alpha_2^m m(t)^2 \tag{16}$$

$$\sigma_{mv}(t) = \alpha_0^{mv} + \alpha_1^{mv} \sigma_{mv}(t-1) + \alpha_2^{mv} m(t) v(t) \tag{17}$$

$$\sigma_{vv}(t) = \alpha_0^v + \alpha_1^v \sigma_{vv}(t-1) + \alpha_2^v v(t)^2. \tag{18}$$

[4] Those two criticisms apply also to the heteroskedastic model proposed in Campbell (1991). In that model a linear combination of the conditional second moments of aggregated returns and consumption is made to depend on expected returns.

[5] Ideally one would like to make distributional assumptions on the returns of the elementary securities. Unfortunately the problem becomes immediately analytically intractable. On the other hand, the numerical simulations presented in Section 4 show that the results of this section hold in a two asset problem when the distributional assumptions are made on the return of the risky asset.

Once we have specified the distribution of asset returns, we can solve (10) backwards using the terminal condition $\log(a_T) = 0$. Notice that $\log(a_{T-1})$ depends only on the mean and variance of $r_m(T)$ conditional on the information available at $T-1$. Continuing the recursion we are able to express the (log) consumption-wealth ratio in terms of the expected values of the future conditional moments of $m(t)$ and $v(t)$ without reference to the moments of the consumption-wealth ratio. Furthermore, the VAR-GARCH process specified above has two interesting properties that help to simplify expression (10). First, the expected values of next period's conditional second order moments are linear in the current period's conditional second order moments. Second, the conditional moments of $m(t)$ and $v(t)$ of odd order are zero, and the even order moments are functions of the second order moments. These properties permit the (log) of the consumption–wealth ratio at period $t$ to be expressed in terms of future expected returns and the second order moments of the joint conditional distribution of $r_m(t+1)$ and $E_{t+1}r_m(t+2)$ at period $t$. This result is formalized in the following proposition.

**PROPOSITION 1:**

If $r_m(t)$ and $E_t r_m(t+1)$ follow the VAR-GARCH process (12) to (18), then equation (8) for the (log) consumption-wealth ratio can be written as

$$\begin{aligned} -\log(a_t) &= K(T-t) + E_t \sum_{j=1}^{T-t} Q_j(T-t)\left(\frac{1-\rho}{\rho}\right) r_m(t+j) \\ &+ \sum_{p=0}^{2^{T-t-1}} \sum_{q=0}^{2^{T-t-1}} \sum_{r=0}^{2^{T-t-1}} s(p,q,r,T-t)\sigma_{mm}(t)^p \sigma_{mv}(t)^q \sigma_{vv}(t)^r, \end{aligned} \tag{19}$$

where $K(T-t)$ and $s(p,q,r,T-t)$ are deterministic terms.

**Proof:**

Appendix.

Thus, the (log) consumption-wealth ratio at period $t$ is a function of the revisions in expectations about future returns and the conditional second-order moments of the joint distribution of $r_m(t+1)$ and $E_{t+1}r_m(t+2)$. This expression does not make any reference to the conditional second-order moments of consumption or consumption-wealth ratios as in expression (10). By recursive evaluations of the stochastic equation (10) we have been able to determine those moments endogenously using the distribution of asset returns. The terms $K(T-t)$ and $s(p,q,r,T-t)$ are extremely complex functions of the preference parameters ($\gamma$ and $\rho$), the parameters of the GARCH processes and the linearization terms ($P(T-t)$ and $B_j(T-t)$). Their exact expression is uninteresting in the analysis that follows.

Expression (19) has powerful implications for the analysis of optimal portfolio policies in this time-dependent return environment. I will deal with this issue in the next section. From an asset pricing prospective, it allows us to obtain an approximate intertemporal asset pricing relation without reference to the conditional moments of aggregate consumption when asset returns follow autoregressive conditional heteroskedastic processes. This issue is briefly analyzed below.

### 3.2 Asset Pricing without Consumption

For simplicity I focus on the pricing of the aggregate market portfolio in the presence of a riskless asset[6]. Let us interpret $R_m$ in equation (7) as the market return of risky assets and $a_t$ as the aggregate consumption-wealth ratio. Assuming that the asset 0 is riskless and taking a lognormal approximation to the specialization of equation (7) for the aggregate market return and the riskless return we find

$$E_t\left[r_m(t+1) - r_0(t+1)\right] = -\frac{1}{2}\sigma_{mm}(t) + \gamma\sigma_{mm}(t) + \rho\frac{1-\gamma}{1-\rho}\sigma_{am}(t). \tag{20}$$

Thus, the risk premium is explained by the variance of the risky return and its covariance with the consumption-wealth ratio. The intertemporal aspect of the asset pricing equation (20) is summarized in the term in $\sigma_{am}(t)$. If $\gamma = 1$ or $\sigma_{am}(t) = 0$, equation (20) is indistinguishable from a discrete time version of the continuous time static CAPM. Notice that it differs from the Merton (1970) continuous-time version of the static CAPM by the Jensen Inequality term $-1/2\sigma_{mm}(t)$[7].

Using the VAR-GARCH specification for the return on the market portfolio and its conditional expected value we can prove the following proposition

[6] A disaggregated analysis would follow from the addition of the assumption of normality in the joint distribution of the rates of returns of the elementary securities.

[7] In discrete time, by the Jensen Inequality, the expected value of the continuously compounded gross return between $t$ and $t+1$ is not one plus the expected value of the net rate of return of investing one dollar during that period. The difference is precisely $-1/2\sigma_{mm}(t)$ for the equilibrium portfolio return.

**PROPOSITION 2:**

If $r_m(t)$ and $E_t r_m(t+1)$ follow the VAR-GARCH process (12) to (18) and the consumption--wealth ratio follows expression (19), then the conditional covariance of the (log) consumption--wealth ratio ($\log(a_t)$) with the equilibrium portfolio return ($r_m(t)$) is

$$\sigma_{am}(t) = -\frac{1-\rho}{\rho}\sigma_{mf}(t), \tag{21}$$

where

$$\begin{aligned} \sigma_{mf}(t) &= Cov_t\left[r_m(t+1), E_{t+1}\sum_{j=1}^{T-t} Q_j(T-t) r_m(t+j+1)\right] \\ &= S(T-t) Cov_t(r_m(t+1), r_m(t+2)), \end{aligned} \tag{22}$$

and

$$S(T-t) = \sum_{j=1}^{T_t} Q_j(T-t) H^{j-1}. \tag{23}$$

**Proof**

Appendix.

Therefore, according to expression (21) the intertemporal component of the risk premium is only composed of the discounted future expected revisions on the conditional mean of returns. This expression is the result of combining the normality assumption with the GARCH processes in the distribution of $m$ and $v$. Those assumptions guarantee that the last term in (19) can be expressed as a weighted sum of even-order moments of the conditional joint distribution of $m$ and $v$. Naturally, the conditional covariance of those terms with $r_m$ is zero by conditional normality and, therefore, future conditional variances cannot be predicted from the current realization of returns. Thus, since returns do not convey any information about future changes in conditional variances, this element is absent in the asset pricing expression. Finally, the autoregressive process (14) for the conditional mean of returns allows one to express this term at time $t$ as a function of the covariability of next period's return with the return one period after.

From proposition 2 we immediately obtain the following asset pricing expression

$$E_t(r_m(t+1) - r_0(t+1)) = -\frac{1}{2}\sigma_{mm}(t) + \gamma\sigma_{mm}(t) - (1-\gamma)\sigma_{mf}(t). \tag{24}$$

This result is essentially the Campbell (1991) approximate asset pricing formula. In this expression, the conditional mean of the risk premium is explained by a Jensen inequality term (see footnote 7), the variance of the market return and the predictability of next period's return. Naturally, if returns are not predictable (as in the *i.i.d.* case), expression (24) is just a discrete time approximation to the continuous time CAPM. However, the *i.i.d.* assumption is not necessary for the CAPM to hold under an arbitrary level of risk aversion. Thus, if returns are not persistent but follow an autoregressive conditional variance process, the static version of the CAPM is approximately valid. This result implies that the popular static models for pricing assets are sensible in a world were returns are hardly predictable but the conditional variances are persistent.

Expression (24) has two interesting features. First, the risk premium is explained in an intertemporal framework without reference to consumption. Thus, it has a great deal of empirical attractiveness. Second, asset prices are essentially independent of the elasticity of intertemporal substitution[8]. However, Campbell derives this expression assuming that the conditional second order moments of the joint distribution of consumption and market returns are constant. This implies that the excess returns are *i.i.d.* according to the standard CCAPM. In this section, I have shown that the above results hold with time-dependence in excess returns, without imposing conditions on the conditional moments of aggregate consumption and allowing for a standard form of heteroskedasticity in stock returns.

## 4 Approximate Optimal Portfolio Policies

Once we have obtained an expression for the optimal consumption policy, we can concentrate on obtaining an approximate expression for the optimal portfolio weights in this intertemporal framework. Let's assume for simplicity that there are only two assets in the economy: a risky asset whose return is $R_1(t+1)$, and a riskless asset whose return is $R_0(t+1)$. Specializing

[8]One should consider though, that the linearization term $Q_j(T_t)$ depends on the mean of the consumption-wealth ratio and this is not independent of the elasticity of intertemporal substitution.

expression (7) for those two returns and taking a lognormal approximation we find

$$0 = E_t\left[R_1(t+1) - R_0(t+1)\right] + \frac{1}{2}\sigma_{11}(t) - \gamma\sigma_{1m}(t) - \rho\frac{1-\gamma}{1-\rho}\sigma_{a1}(t). \tag{25}$$

Now, denote by $\omega(t)$ the portfolio weight of the risky asset chosen at period $t$. Then

$$r_m(t+1) = \omega(t)r_1(t+1) + (1-\omega(t))r_0(t+1), \tag{26}$$

and, therefore, $\sigma_{1m}(t) = \omega(t)\sigma_{11}(t)$.

Thus, the equilibrium portfolio weight for the risky asset has to satisfy

$$\omega(t) = \frac{E_t r_1(t+1) - r_0(t+1)}{\gamma\sigma_{11}(t)} + \frac{1}{2\gamma} - \frac{\rho(1-\gamma)}{\gamma(1-\rho)}\frac{\sigma_{a1}(t)}{\sigma_{11}(t)}. \tag{27}$$

The first term in (27) ($MV$) is simply the mean-variance ratio of excess returns characteristic of the portfolio expression in the static CAPM. The second term ($JI$) is a consequence of the Jensen Inequality. The last term ($INT$) summarizes the intertemporal considerations of people deciding on their optimal portfolio policies. This term is characterized by the conditional correlation of the risky return and the consumption-wealth ratio. Therefore, the $INT$ component involves the effect on next period's consumption of changes in the opportunity set. Under *i.i.d.* returns the equilibrium consumption-wealth ratio is a deterministic term, and the *intertemporal* component of the equilibrium portfolio policy is zero. In this case, the risky asset portfolio weight expression is just the discrete version of the instantaneous portfolio weight which is optimal when the price of the risky asset follows a Geometric Brownian Motion.

Now, using proposition 2 we can write the *intertemporal* component of $\omega(t)$ ($INT(t)$) as

$$\begin{aligned} INT(t) &= -\frac{\rho(1-\gamma)}{\gamma(1-\rho)}\frac{\sigma_{a1}(t)}{\sigma_{11}(t)} = \frac{(1-\gamma)}{\gamma\omega(t)\sigma_{11}(t)}\sigma_{af}(t) \\ &= \frac{(1-\gamma)}{\gamma}S(T-t)\mu_m(t)\omega(t), \end{aligned} \tag{28}$$

where

$$\mu_m(t) = \frac{Cov_t(r_m(t+1), r_m(t+2))}{\sigma_{mm}(t)}. \tag{29}$$

The intertemporal component of the portfolio weight of the risky asset is a proportion of the total portfolio holding. This proportion depends on the coefficient of relative risk aversion, the conditional autocorrelation coefficient of the equilibrium portfolio returns and the deterministic term $H(T-t)$.

Notice that the equilibrium portfolio policy is independent of the elasticity of intertemporal substitution. On one hand, from expression (27), an increase in $\rho$ reduces the demand for the risky asset by increasing the absolute value of $INT(t)$ for a given covariance of the risky return with the consumption-wealth ratio. On the other hand, a higher $\rho$ implies a higher aversion to time-variations in the consumption-wealth ratio. Thus, the covariance between returns and the consumption-wealth ratio becomes smaller. According to expression (29) those effects cancel each other.

The sign of $INT(t)$ depends on the magnitude of the coefficient of relative risk aversion ($\gamma$) and the sign of the conditional autocorrelation. The reason is that a positive autocorrelation in the portfolio return, for example, causes two effects. On one side, it provides incentives to invest in the risky asset since it implies a larger expected opportunity set for next period's consumption. On the other side, it provides desincentives to risky investments by increasing the agent's exposure to changes in the opportunity set for a given position in the risky asset. Positive conditional correlation in returns only increases (decreases) the intertemporal component of the demand for the risky asset with respect to the static CAPM if $\gamma$ is below (above) one. For $\gamma = 1$, the effects cancel each other and the demand for the risky asset is equivalent to the one predicted by the static CAPM.

Expression (25) establishes a bound on the proportion of the portfolio weights explained by the intertemporal component. It can be shown that $S(T-t) < 1$ for $t < T$. Therefore, if $\gamma > 1/2$ the proportion of $\omega(t)$ explained by the intertemporal term is bounded by the conditional autocorrelation of the equilibrium portfolio returns. For non-autocorrelated returns that proportion is zero. Notice also that, if the equilibrium returns are not *i.i.d.* but the only source of time-dependency is conditional heteroskedasticity of the GARCH form, the intertemporal component in (27) is zero and the portfolio implications of the static CAPM with the Jensen Inequality correction are approximately valid. Thus, as in the asset pricing model of the previous section, the solution of an unrelated sequence of two-period portfolio choice problems produces correct intertemporal portfolio policies in a non-*i.i.d.* environment characterized by returns that follow GARCH processes but are not autocorrelated.

The analysis above has been able to relate the magnitude of the intertemporal component of the portfolio policy with the predictability of the equilibrium portfolio returns. This naturally involves the predictability of the performance of the elementary securities and the next period's portfolio policy. In order to analyze directly the effect of the predictability of the risky return on the determination of the intertemporal component of the optimal portfolio policy, it is useful to solve explicitly the problem at period (T-2).

First, we know that at period $T-1$ the portfolio policy is trivially

$$\omega(T-1) = \frac{E_{T-1}r_1(T) - r_0(T) + 1/2\sigma_{11}(T-1)}{\gamma\sigma_{11}(T-1)}. \tag{30}$$

Therefore, according to (28), the intertemporal component of the proportion of wealth allocated to the risky asset is

$$\begin{aligned} INT(T-2) &= \frac{(1-\gamma)}{\gamma}S(2)\mu_m(T-2)\omega(T-2) \\ &= \frac{(1-\gamma)}{\gamma}S(2)\frac{Cov_t\left[(r_1(T-1)-r_0(T-1)), r_0(T)+\omega(T-1)r_m(T))\right]}{\sigma_{11}(t)} \\ &= \frac{1-\gamma}{\gamma}S(2)\left(\mu_{10}(T-2)+\frac{1}{2\gamma}\mu_{11}(T-2)+\frac{1}{\gamma}\mu_{1R^2}(T-2)\right), \end{aligned} \tag{31}$$

where

$$\begin{aligned} \mu_{10}(t) &= \frac{Cov_t(\tilde{r}(t+1), r_0(t+2))}{\sigma_{11}(t)}, \\ \mu_{11}(t) &= \frac{Cov_t(\tilde{r}(t+1), \tilde{r}(t+2))}{\sigma_{11}(t)}, \\ \mu_{1R^2}(t) &= \frac{Cov_t(\tilde{r}(t+1), \tilde{R}^2(t+1)}{\sigma_{11}(t)}, \end{aligned}$$

and $\tilde{R}^2(t) = (E_t\tilde{r}(t+1))^2/\sigma_{11}(t)$ and $\tilde{r}(t) = r_1(t) - r_0(t)$.

Thus $INT(T-2)$ is composed of three terms related to the forecasting power of a particular realization of the excess returns $(\tilde{r}(t))$. The first term $(\mu_{10})$ measures the conditional correlation between next period's excess return and the riskless rate two periods ahead. Similarly, $(\mu_{11})$ is the conditional autocorrelation coefficient on excess returns and, finally, $(\mu_{1R^2})$ measures the conditional correlation between next period's excess return and the predictability of the two period ahead excess returns. Notice that the riskless asset is typically a very smooth variable and therefore hardly predictable by the excess returns. Second, by definition, $\mu_{1R^2}$ is of second order importance in relation to the autocorrelation of the excess returns. Therefore, in a world where the predictability of the excess returns and the riskless rate are not very volatile, the first and third term are small relative to the autocorrelation of excess returns. Thus, the intertemporal component of the equilibrium portfolio, expressed in equation (28) in terms of the predictability of the equilibrium portfolio return, is mainly characterized by the conditional autocorrelation of excess returns.

Notice finally that the static CAPM part of the equilibrium portfolio policy (27) is characterized by the mean–variance ratio of excess returns. The intertemporal component $INT(t)$, however, is composed of terms that have the form of conditional correlation coefficients whose admissible range of variation can be bounded for mathematical and economic reasons. Thus, the CAPM component is expected not only to have a higher magnitude if returns are only slightly predictable but also to be much more volatile than the intertemporal term under any level of predictability. Therefore, according to this analysis, most of the variability of the portfolio holdings is attributable to its static mean-variance component.

In summary, the approximation analysis of the equilibrium portfolio choice between risky and riskless assets suggests that the part of this policy which is not explained by the traditional mean-variance trade-off characteristic of two-period portfolio strategies is typically small and smoothly behaved in an environment with time-dependent returns that show small persistence and autoregressive conditional variance.

## 5 Numerical Solution of the Optimal Portfolio Problem

In the previous sections we performed a linear approximation to the first order conditions of the individual's consumption-portfolio problem. That analysis has offered an approximate closed form solution of the optimal portfolio problem that involves a static mean-variance term, a Jensen inequality term and an intertemporal term. The approximate solution has shown that the elasticity of intertemporal substitution is essentially irrelevant to explaining portfolio choice, and that the intertemporal term in the portfolio expression is likely to play a small role in the explanation of the portfolio holdings for a standard GARCH-VAR specification of the stochastic process of asset returns.

In this section, I apply numerical solution techniques to the two-asset optimal portfolio problem. By doing so, I will be able to estimate the impact of the departures from the *i.i.d.* uncertainty assumption on the mean and variability of the portfolio holdings of the risky asset. In addition, the analysis will yield a test of the accuracy of the linear approximation to the first order conditions and the robustness of the results to different specifications of the conditional mean and variance of the returns.

Accomplishing those exercises requires us to solve numerically expressions (5) and (7) for the consumption-two asset portfolio problem. In order to make the analysis independent of the time-horizon, I focus on obtaining stationary solutions to the consumption-portfolio problem for the infinite horizon case.

### 5.1 Method

Let's assume for simplicity that the return on the riskless asset is a constant and that the only source of uncertainty is the realization of the risky asset return[9]. Now, according to expressions (5) and (7) and assuming the risky return follows a K-state homogenous Markov Process with transition probabilities ($\pi_{ij},\ i,j = 1,\cdots,K$), any stationary consumption- portfolio policy must satisfy the equations:

$$a(i) = \left\{1 + \delta^{\frac{1}{\rho}} \sum_{j=1}^{K} \pi_{ij} \left\{ [\omega(i)R_1(j) + (1-\omega(i))R_0)]^{1-\gamma} a(j)^{-\rho\frac{1-\gamma}{1-\rho}} \right\}^{\frac{1-\rho}{\rho(1-\gamma)}} \right\}^{-1} \tag{32}$$

[9]The assumption of a time invariant riskfree rate can be relaxed. However, a more realistic description of its stochastic process complicates the numerical computations without altering substantially the results given the relative smoothness of this series in relation to the risky returns.

and

$$\sum_{j=0}^{m} \pi_{ij} a(j)^{-\rho\frac{1-\gamma}{1-\rho}} \left[\omega(i)R_1(j) + (1-\omega(i))R_0\right]^{-\gamma} \left[R_1(j) - R_0\right] = 0 \qquad i = 1, \cdots, K. \tag{33}$$

Therefore, the optimal consumption-portfolio problem consists of obtaining the pairs $\{a(i), \omega(i), i = 1, \cdots, K\}$ that simultaneously solve the system of nonlinear equations (32) and (33).

In order to obtain the discrete Markov distribution for $R_1(t)$, I initially specified a continuous Markov process with conditional heteroskedastic errors for the risky rate of return ($r_1(t) = \log R_1(t)$). Then, using the quadrature method suggested by Tauchen and Hussey (1991)[10], I obtained a discrete set of values and their associated transition probabilities that approximate the underlying continuous distribution. The procedure was applied to different specifications of the conditional mean and variance of the risky return. In all the cases I used combinations of the parameters of the stochastic process that are compatible with the unconditional mean and variance of the quarterly real value weighted returns of the New York Stock Exchange for the period 1947.2-1988.4. The riskfree rate was set at the unconditional mean of the return on 3-month Treasury Bills during that period[11].

## 5.2 Model with Constant Persistence in Returns

As a first exercise, I obtained numerical solutions for the optimal equilibrium portfolio when the risky return follows an autoregressive process of the form:

$$r_1(t+1) = c_0 + \eta r_1(t) + u(t+1) \tag{34}$$

$$\sigma_{11}(t) = \beta_0 + \beta_1[r_1(t) - Er_1(t)]^2. \tag{35}$$

The first order autoregressive process is a simple way of parametrizing the level of autocorrelation in returns. According to the approximation analysis, this element is the core of the intertemporal component of the portfolio policy. The process for the conditional variance differs from the

[10] Unlike the usual approximation by rectangles, this method endogenously determines the optimal grid that approximates the continuous space state as the roots of a series of Hermite polynomials.

[11] The unconditional mean of the quarterly VWR of the NYSE deflated by CPI is 1.722 %. Its variance is .648 %. The mean of the real return on three-month T.Bills is .136%.

standard ARCH specification in that it includes the squared deviation from the unconditional mean of returns instead of the deviation from the conditional mean. However, for the realistic case of small values of $\eta$, the difference is immaterial and allows us to reduce the dimensionality of the space-state[12]. With this assumption, the state space is simply characterized by $K$ different values of the risky return. $K$ is set to 10 in the usual compromise between accuracy of the approximation and time consumption of the numerical routine[13].

Table 1 presents the mean and variance of the optimal portfolio weight ($\omega(t)$) for different values of the persistence parameter ($\eta$) and the slope parameter ($\beta_1$) in the conditional variance specification. In all the cases $c_0$ and $\beta_0$ are set so that the unconditional mean and variance remain unaltered[14]. The range of values for the persistence parameter ($\eta \in [0, .15]$) is chosen to match the existing empirical evidence on small, positive, short term autocorrelation in aggregate stock returns[15].

The results show that a coefficient of relative risk aversion higher than 3 is required to obtain portfolio policies that do not imply borrowing at the riskless rate. Similarly, $\gamma$ has to be higher than 5 to imply a portfolio weight for the risky asset below 50%. Friend and Blume (1975) find that the proportion of risky assets to total wealth in the U.S. economy was between .5 and .74 depending on income brackets. According to table 1, that proportion can only be obtained for values of the coefficient of relative risk aversion between 4 and 5.

As expected, the mean of the holdings of the risky asset decreases with an increase in the level of persistence in returns ($\eta$) and increases with the slope of the conditional variance equation ($\beta_1$). The effect of the increased persistence is produced by the reduction in the ability to hedge against changes in the opportunity set for a given position in the risky asset. This effect is larger than the one produced by the increased profit opportunities for the empirically relevant values of $\gamma$. The effect of an increase in $\beta_1$ is a consequence of the reduction in the conditional variance

[12]If we consider the standard ARCH formulation and the risky return is assumed to take $k$ values randomly, the state space should be composed of the $k^2$ possible pairs of those values. On the other hand, work by French, Schwert and Stambaugh (1987) has shown that the specification of the conditional mean is not highly relevant for the estimation of the conditional variance of stock returns.

[13]As it has been reported by Tauchen and Hussey (1991) using Monte Carlo experiments, a higher number of states only increases marginally the accuracy of the discrete approximation to stationary AR(1) processes.

[14]Thus $c_0 = (1-\eta)Er_1(t)$ and $\beta_0 = (1-\eta^2-\beta_1)Var(r_1(t))$

[15]Cutler, Poterba and Summers (1990) find for the period 1926-1988 with monthly data an autocorrelation coefficient of .106 with a standard error of .06. Similar results are found by Campbell (1990) for the same period. For quarterly returns in the period 1947-1988 the autocorrelation coefficient is .101 with standard error of .07

of next period's risky return when the most likely values (close to the unconditional mean) are realized.

Overall, the response of the mean of the risky portfolio holdings moves only moderately with unconditional-moment preserving changes in the conditional mean and variance of the risky return. By contrast, the volatility of the proportion of wealth allocated to the risky asset is highly sensitive to movements in the parameters of the stochastic process of returns and, in particular, to changes in the persistence parameter. Thus while in the *i.i.d.* case the portfolio weights are constant, an autocorrelation coefficient in returns of only .05 makes the standard deviation of the equilibrium portfolio weight reach values between 20% and 30% of its mean. Those percentages becomes 50% and 66% if $\eta$ is equal to .15. The sensitivity in the portfolio holdings is essentially due to the high volatility that the time-dependency of returns causes on the mean-variance ratio. Thus, even a small level of time dependency in returns obliges people to radically change their allocation of wealth across assets according to the realization of the risky return.

The most striking feature presented in Table 1 is the complete irrelevance of the elasticity of intertemporal substitution $(1/\rho)$ in the determination of the portfolio policy. Thus, the result obtained in section 3 using a linear approximation to the first order conditions holds with surprising precision in the numerical solutions of the equilibrium portfolio policy problem. The numerical computations confirm that the direct influence of changes in the elasticity of substitution on the portfolio policy is completely offset by the associated movements in the consumption-wealth ratio.

Tables 2 and 3 analyze the accuracy of the lognormal approximation and the contribution of each component of the portfolio policy to the mean and variability of the proportion of wealth allocated to the risky asset. The mean-variance term $(MV)$ and the Jensen inequality term $(JI)$ are calculated using the conditional moments of the excess returns and the preference parameters. The intertemporal element $(INT)$ is calculated using the covariance of the risky returns with the consumption-wealth ratio obtained in the numerical procedure. In those tables, I present the mean and variance of the portfolio weights obtained by both the numerical procedure and the lognormal approximation (27). In addition, I present the percentage ratio of the unconditional mean and variance of the components $MV$, $JI$, and $INT$ relative to the unconditional mean and variance of the portfolio weights obtained from expression (27). The unconditional mean and variance were obtained using the unconditional probabilities associated with the transition probability matrix. The tables present results only for $\gamma = 4$ since for other values they do not qualitatively differ from the ones displayed. The numerical computations show first that the log-linear approximation to the portfolio weight matches with reasonable accuracy the mean and variance of the numerical solution and displays a perfect correlation with this series. Second, as expected, the intertemporal component has a negative sign and a participation in the equilibrium portfolio weight of a magnitude noticeably below the participation of the Jensen Inequality term and the autocorrelation coefficient of the return on the risky asset. This participation decreases slightly with an increase in the slope parameter of the conditional variance equation. Third, the participation of the intertemporal term in the total variability of the portfolio holdings of the risky asset is almost negligible. Therefore, in this model with lognormal returns which exhibit constant conditional autocorrelation and autoregressive conditional heteroskedasticity, the insights obtained with the linear approximation analysis are confirmed by the numerical solution to the portfolio problem. If returns are conditionally heteroskedastic but only slightly persistent, the optimal portfolio policy is remarkably well approximated by the conditional version of the static-CAPM portfolio expression with the inclusion of the Jensen inequality term. Furthermore, the possible discrepancy in the static CAPM approximation from the optimal portfolio weights is essentially unvarying across states.

### 5.3 Time Varying Persistence

One obvious criticism of the above exercise is that it relies on constant conditional autocorrelation of returns. It is natural to think that given the usual statistical imprecision with which the autocorrelation of returns is estimated, the agents only perceive the persistence of returns with some measurement error. In order to test the robustness of the results to the presence of randomness in the conditional autocorrelation coefficients, I solved numerically the portfolio problem for the following return process:

$$\begin{aligned} r_1(t+1) &= c_0 + \eta(t) r_1(t) + u(t), \\ \eta(t) &= \eta + \epsilon(t), \end{aligned}$$

where $\epsilon(t)$ has a $N(0,\sigma_\epsilon)$ univariate distribution and is independent of $u(t)$. The conditional variance for $r_1(t)$ follows specification (35).

In order to solve the model, I approximated the distribution of the persistence term by a two-state Markov Process. The global state space is then composed of the 20 possible pairs of values for the risky return and the persistence parameter. Again, the transition probabilities were obtained using the Tauchen-Hussey method.

Tables 4 through 6 present the results obtained for the time-varying persistence process. The numerical routine was applied assuming mean values of the persistence term equivalent to those taken for $\eta$ in the constant-persistence model. The standard deviation is in all cases $\sigma_\eta = .1$. This number implies movements in the persistence parameters across states that clearly exceed the usual sample estimation errors. Results do not differ substantially from the constant autocorrelation case. The elasticity of substitution remains irrelevant to the determination of the equilibrium portfolio policy and the lognormal approximation is still acceptable in determining the moments and the variation across states of the portfolio weights. On the other hand, the new source of uncertainty slightly decreases the mean proportion of the risky asset and moderately increases its variability. However, the proportion of the mean explained by the intertemporal term increases only marginally while its contribution to the total variability of the holdings of the risky asset is similar to the ones obtained for the constant persistence case. This result is not surprising since a non-zero variance of the autocorrelation coefficient not only increases the variability of the intertemporal term but it also affects the variability of the conditional mean of excess returns in the Mean-Variance component. According to the numerical results, these effects approximately cancel each other. Therefore, the main features of the optimal portfolio policies obtained in the approximation analysis and the numerical computations for the constant persistence process hold with time-varying conditional autocorrelation.

## 5.4 Exponential Autoregressive Conditional Variance

The approximation analysis made in Section 3 and the numerical exercises of this section have used a specification of the conditional variances that implied a lack of predictability of future conditional variances from the current realization of returns. Therefore, it can be argued that dropping this feature might affect the accuracy of the approximation. In addition, if the conditional variances were correlated with the returns, the terms of the mean-variance tradeoff and the expected movements in the opportunity set would differ with respect to the conditionally normal case with standard autoregressive conditional variance. In particular, if the conditional variance of the risky asset increased more with negative shocks than with positive shocks, one would expect two effects. On one hand, the investment in the risky asset would become more attractive because the expected risk-premium per unit of variance would be higher in mean. On the other hand, the risky investment would become less attractive because the asymmetry enlarges the exposure of agents to changes in the opportunity set for a given position in the risky asset.

In order to analyze the effect on the conditional variances that can be predicted by returns, I solved numerically the model assuming a stochastic process for the risky return composed of the conditional mean equation (34) and the conditional variance process:

$$\log \sigma_{11}(t) = \beta_0 + \beta_1[r_1(t) - Er_1(t)] + \beta_2|r_1(t) - Er_1(t)|. \tag{36}$$

This process is an adaptation of the Nelson (1991) EGARCH process[16]. The main feature of this process is that it allows for asymmetric responses of the conditional variance processes to positive and negative deviations of the returns from their expected values. This asymmetry is driven by the parameter $\beta_1$. If $\beta_1$ is negative (positive), conditional variances are higher (lower) when the shocks are negative than when they are positive. This type of effect is the one found by Nelson (1991) and Pagan and Schwert (1989) in their EGARCH estimations of the conditional variance process of asset returns in different samples.

In the numerical routine I chose $\beta_1 = -1.638$ and $\beta_2 = 2.497$. Those parameters resulted from a Maximum Likelihood estimation of the process (36) with U.S. quarterly data over the period 1947-1988. Those estimates imply a strong asymmetry in the variance process in its response to positive or negative shocks[17].

The numerical results of the estimation with the exponential conditional variance process are

[16] Nelson, however, considers shocks to the conditional rather than unconditional expectation and normalizes each error term by the conditional standard deviation.

[17] The implied asymmetry is noticeably higher than the one found by Nelson (1991) and Pagan and Schwert (1989).

presented in tables 7 through 9. Table 7 shows that the departure from the conditions of the lognormal approximation of sections 3 and 4 is not sufficient to provide a role for the elasticity of intertemporal substitution in the determination of the equilibrium portfolio policy. Thus, the results are identical for $\rho$ equal to 2 and 10 . Table 8 shows that the exponential process only modifies slightly the results obtained for the symmetric conditional variance process. As before, the lognormal approximation explains almost perfectly the variation of the portfolio holdings across states and is only marginally less accurate in fitting the mean and the variance of the portfolio weights.

The relative magnitude of the intertemporal term is higher than the one obtained for the regular variance process. This result is a consequence of the stronger sensitivity of the opportunity set to the realization of the risky asset return and agents' desire to hedge against movements in their next period's consumption possibilities. As an example, unlike in the symmetric conditional variance case, when returns are not autocorrelated ($\eta = 0$) there is a non-zero participation of the intertemporal component in the portfolio weights. In this case, the mean of $INT$ is, in absolute terms, .5% of the mean of the portfolio weight allocated to the risky asset. However, with the exception of the $\eta = 0$ case, the proportion of the portfolio weights due to the intertemporal term is below the magnitude of the autocorrelation coefficient and, in all cases, it is smaller than the share of the Jensen inequality term. Finally, there is no modification in the distribution of the variance of the portfolio holdings of the risky assets among its components. The portfolio variability is almost completely explained by the mean-variance component of the equilibrium portfolio expression.

In summary, the presence of predictable conditional variances in the distribution of returns increases the importance of the intertemporal component of the portfolio policy as long as it enlarges the information conveyed by returns about the future movements in the opportunity set. However, it is still true that, for reasonable conditional mean and variance parameters, the overwhelming majority of the mean value of the risky holdings and practically all its variability are explained by the static valuation of the traditional mean-variance tradeoff between the returns of the risky and the riskless asset.

# 6 Conclusions

This paper has discussed the implications of time-dependence in stock returns for the allocation between riskless and risky assets in the equilibrium portfolio. In a world with *i.i.d.* uncertainty, the portfolio policies associated with the static CAPM are approximately optimal for a wide range of preference specifications. The static CAPM predicts constant portfolio weights that are determined by a proportion of the mean-variance ratio of excess returns. Since this strategy constitutes a natural benchmark against which to analyze the effect of time-dependency on returns, I have focused on analyzing how agents' intertemporal preferences interact with the characteristics of the distribution of asset returns to form optimal portfolio policies that depart from the predictions of the static CAPM.

The analysis has employed a very general model of preferences and empirically attractive specifications of the stochastic process for asset returns. On one hand, I have assumed Generalized Isoelastic Preferences in order to distinguish between the effects of attitudes toward risk and attitudes toward intertemporal substitution of consumption in the determination of the portfolio policy. On the other hand, returns have been assumed to follow lognormal processes that include different autoregressive specifications of the conditional first and second order moments.

Since closed form solutions of the optimal portfolio problem are not available, the analysis has relied on linear approximations and numerical solution techniques. Thus, I have first proposed an approach that allows approximate explicit solutions for the consumption-portfolio problem to be obtained using a standard GARCH specification in the conditional variance of stock returns and without making assumptions on the conditional moments of the distribution of consumption. This analysis has yielded a decomposition of the optimal portfolio weight into three components corresponding to a static CAPM-like conditional Mean-Variance term, a Jensen inequality term and an intertemporal term that involves the conditional autocorrelation of returns. Second, I have employed recent developments in discrete approximations to continuous distributions to obtain numerical solutions for the equilibrium consumption-portfolio policy from a useful representation of the Euler Equations expressed in terms of the consumption-wealth ratio. The numerical analysis has provided strong support for the loglinear approximation to the portfolio weights under the assumed distributions of asset returns.

The approximation and the numerical analysis have provided similar answers to the question of determining the main components of equilibrium portfolio policy under credible specifications of the asset returns distribution. First, even when the elasticity of intertemporal substitution is an important determinant of the equilibrium consumption-wealth ratio, it is irrelevant for the portfolio policy. Thus, the GIP predicts exactly the same portfolio policies as the standard, constant relative risk aversion expected utility representations. Second, small departures from the *i.i.d.* assumption produce an important variability in the portfolio holdings that contrasts with the CAPM constant portfolio policies. However, a conditional version of the static CAPM with the inclusion of a Jensen inequality correction is able to explain the overwhelming majority of the mean, and almost all the variability, of the equilibrium portfolio weights for a sensible choice of the parameters of the stochastic processes. Thus, the proportion of the portfolio holdings which is not explained by the corrected conditional mean-variance ratio has typically a magnitude below the assumed persistence in returns, is practically invariant across states and is independent of the parameters of the conditional variance process. Those results are robust to the assumption of time-varying persistence and an empirically reasonable degree of asymetry in the response of the conditional variance to positive and negative movements in asset returns.

| | | | $\eta = 0$ | $\eta = .05$ | $\eta = .1$ | $\eta = .15$ |
|---|---|---|---|---|---|---|
| $\gamma = 3$ | $\rho = 2$ | $\beta_1 = 0$ | .982 | .952 | .927 | .907 |
| | | | (0) | (.20) | (.42) | (.63) |
| | | $\beta_1 = .2$ | 1.024 | 1.000 | .981 | .968 |
| | | | (.16) | (.23) | (.37) | (.54) |
| | | $\beta_1 = .5$ | 1.277 | 1.257 | 1.246 | 1.246 |
| | | | (.40) | (.43) | (.52) | (.64) |
| | $\rho = 10$ | $\beta_1 = 0$ | .982 | .952 | .927 | .907 |
| | | | (0) | (.20) | (.41) | (.63) |
| | | $\beta_1 = .2$ | 1.024 | 1.000 | .981 | .968 |
| | | | (.16) | (.23) | (.37) | (.54) |
| | | $\beta_1 = .5$ | 1.277 | 1.257 | 1.246 | 1.246 |
| | | | (.40) | (.43) | (.52) | (.64) |
| | | | $\eta = 0$ | $\eta = .05$ | $\eta = .1$ | $\eta = .15$ |
| $\gamma = 4$ | $\rho = 2$ | $\beta_1 = 0$ | .737 | .712 | .691 | .674 |
| | | | (0) | (.16) | (.31) | (.47) |
| | | $\beta_1 = .2$ | .768 | .748 | .732 | .720 |
| | | | (.12) | (.17) | (.28) | (.40) |
| | | $\beta_1 = .5$ | .958 | .940 | .931 | .929 |
| | | | (.30) | (.33) | (.39) | (.48) |
| | $\rho = 10$ | $\beta_1 = 0$ | .737 | .712 | .691 | .674 |
| | | | (0) | (.16) | (.31) | (.47) |
| | | $\beta_1 = .2$ | .768 | .748 | .732 | .720 |
| | | | (.12) | (.17) | (.28) | (.40) |
| | | $\beta_1 = .5$ | .958 | .940 | .931 | .929 |
| | | | (.30) | (.33) | (.39) | (.48) |
| | | | $\eta = 0$ | $\eta = .05$ | $\eta = .1$ | $\eta = .15$ |
| $\gamma = 5$ | $\rho = 2$ | $\beta_1 = 0$ | .589 | .567 | .550 | .536 |
| | | | (0) | (.12) | (.25) | (.38) |
| | | $\beta_1 = .2$ | .614 | .597 | .583 | .573 |
| | | | (.10) | (.14) | (.23) | (.32) |
| | | $\beta_1 = .5$ | .766 | .751 | .742 | .740 |
| | | | (.24) | (.28) | (.31) | (.38) |
| | $\rho = 10$ | $\beta_1 = 0$ | .589 | .567 | .550 | .536 |
| | | | (0) | (.12) | (.25) | (.38) |
| | | $\beta_1 = .2$ | .614 | .597 | .583 | .573 |
| | | | (.10) | (.14) | (.23) | (.32) |
| | | $\beta_1 = .5$ | .766 | .751 | .742 | .740 |
| | | | (.24) | (.28) | (.31) | (.38) |

Table 1: Means and standard deviations (in parenthesis) of optimal portfolio weight for the risky asset. The conditional mean and variance processes for the risky asset are $E_t r_1(t+1) = c_0 + \eta r_1(t)$ and $\sigma_{11}(t) = \beta_0 + \beta_1[r_1(t) - Er_1(t)]^2$. In all the cases $\delta = .99$.

| | $\eta$ | $E(\omega)$ | $E(\omega_{log})$ | CORR | MV(%) | JI(%) | INT(%) |
|---|---|---|---|---|---|---|---|
| $\beta_1 = .2$ | 0 | .768 | .771 | 1 | 83.78 | 16.22 | 0 |
| | .05 | .748 | .750 | 1 | 86.39 | 16.68 | -3.06 |
| | .1 | .732 | .731 | 1 | 89.30 | 17.09 | -6.39 |
| | .15 | .720 | .716 | 1 | 92.53 | 17.46 | -9.98 |
| $\beta_1 = .5$ | 0 | .958 | .965 | 1 | 87.05 | 12.95 | 0 |
| | .05 | .940 | .946 | 1 | 89.17 | 13.21 | -2.38 |
| | .1 | .931 | .934 | 1 | 91.53 | 13.38 | -4.91 |
| | .15 | .929 | .930 | 1 | 94.14 | 13.44 | -7.59 |

Table 2: Contribution of each component to the mean of the equilibrium portfolio weight. $E(\omega)$ is the mean of the equilibrium portfolio weight obtained in the numerical procedure. $E(\omega_{log})$ is the mean portfolio weight obtained using the lognormal approximation. CORR is the correlation coefficient between $\omega$ and $\omega_{log}$. MV(%), JI(%) and INT(%) are, respectively, the percentage ratio of the unconditional mean of the mean–variance term, the Jensen Inequality term, and the Intertemporal term to $E(\omega_{log})$. Preference parameters are $\gamma = 4$ and $\rho = 10$. Conditional mean and variance of the risky return are as in Table 1.

| | $\eta$ | $\sigma^2_\omega$ | $\sigma^2_{\omega log}$ | MV(%) | INT(%) |
|---|---|---|---|---|---|
| $\beta_1 = .2$ | 0 | .0150 | .0151 | 100.0 | 0 |
| | .05 | .0308 | .0309 | 100.8 | .006 |
| | .1 | .0796 | .0793 | 101.0 | .011 |
| | .15 | .1635 | .1613 | 102.0 | .018 |
| $\beta_1 = .5$ | 0 | .0930 | .0980 | 100.0 | 0 |
| | .05 | .1063 | .1109 | 102.3 | .02 |
| | .1 | .1517 | .1557 | 103.1 | .061 |
| | .15 | .2321 | .2345 | 103.4 | .096 |

Table 3: Contribution of each component to the variance of the equilibrium portfolio weight. $\sigma^2_\omega$ is the unconditional variance of the portfolio weights obtained in the numerical procedure. $\sigma^2_{\omega log}$ is the unconditional variance of the portfolio weights obtained using the lognormal approximation. MV(%) and INT(%) are, respectively, the percentage ratio of the variance of the mean-variance term and the intertemporal term to the variance of $\omega$. Preference parameters are $\gamma = 4$ and $\rho = 10$.

| | | $E(\eta) = .0$ $\sigma_\eta = .1$ | $E(\eta) = .05$ $\sigma_\eta = .1$ | $E(\eta) = .1$ $\sigma_\eta = .1$ | $E(\eta) = .15$ $\sigma_\eta = .1$ |
|---|---|---|---|---|---|
| $\beta_1 = .2$ | $\rho = 2$ | .766 (.29) | .745 (.32) | .730 (.39) | .718 (.48) |
| | $\rho = 10$ | .766 (.29) | .745 (.32) | .730 (.39) | .718 (.48) |
| $\beta_1 = .5$ | $\rho = 2$ | .954 (.40) | .936 (.42) | .926 (.47) | .925 (.55) |
| | $\rho = 10$ | .954 (.40) | .936 (.42) | .926 (.47) | .925 (.55) |

Table 4: Means and standard deviations (in parenthesis) of optimal portfolio weights for the risky asset ($\gamma = 4$). The conditional mean and variance processes for the risky asset are $E_t r_1(t+1) = c_0 + \eta(t) r_1(t)$, $\eta(t) = \eta + \epsilon(t)$, and $\log \sigma_{11}(t) = \beta_0 + \beta_1 [r_1(t) - E r_1(t)]^2$.

| | $\mathbf{E}(\eta)(\sigma_\eta)$ | $\mathbf{E}(\omega)$ | $\mathbf{E}(\omega_{\log})$ | CORR | MV(%) | JI(%) | INT(%) |
|---|---|---|---|---|---|---|---|
| $\beta_1 = .2$ | 0(.1) | .765 | .766 | 1 | 84.19 | 16.32 | -.51 |
| | .05(.1) | .745 | .745 | 1 | 86.82 | 16.78 | -3.60 |
| | .1(.1) | .730 | .726 | 1 | 89.74 | 17.20 | -6.94 |
| | .15(.1) | .718 | .711 | 1 | 92.98 | 17.57 | -10.55 |
| $\beta_1 = .5$ | 0(.1) | .954 | .959 | 1 | 87.36 | 13.03 | -.39 |
| | .05(.1) | .936 | .941 | 1 | 89.49 | 13.29 | -2.78 |
| | .1(.1) | .926 | .928 | 1 | 91.87 | 13.46 | -5.33 |
| | .15(.1) | .925 | .924 | 1 | 94.48 | 13.52 | -8.00 |

Table 5: Contribution of each component to the mean of the equilibrium portfolio weight. Parameters and conditional mean and variance of the risky return are as in Table 4.

| | $\mathbf{E}(\eta)(\sigma_\eta)$ | $\sigma^2_\omega$ | $\sigma^2_{\omega\log}$ | MV(%) | INT(%) |
|---|---|---|---|---|---|
| $\beta_1 = .2$ | 0 (.1) | .0840 | .0839 | 100.2 | 0 |
| | .05(.1) | .0991 | .0997 | 100.7 | 0 |
| | .1 (.1) | .1492 | .1483 | 101.5 | .01 |
| | .15(.1) | .2340 | .2305 | 102.7 | .02 |
| $\beta_1 = .5$ | 0 (.1) | .1602 | .1654 | 99.6 | 00 |
| | .05(.1) | .1737 | .1785 | 101.2 | .02 |
| | .1 (.1) | .2199 | .2239 | 102.2 | .05 |
| | .15(.1) | .3019 | .3040 | 102.9 | .08 |

Table 6: Contribution of each component to the variance of the equilibrium portfolio weight. Parameters and conditional mean and variance of the risky return are as in Table 4.

|  | $\eta = 0$ | $\eta = .05$ | $\eta = .1$ | $\eta = .15$ |
|---|---|---|---|---|
| $\rho = 2$ | .638 | .636 | .634 | .632 |
|  | (.08) | (.19) | (.32) | (.44) |
| $\rho = 10$ | .638 | .636 | .634 | .632 |
|  | (.08) | (.19) | (.32) | (.44) |

Table 7: Means and standard deviations (in parenthesis) of optimal portfolio weights for the risky asset. The conditional mean and variance processes for the risky asset are $E_t r_1(t+1) = c_0 + \eta r_1(t)$ and $\log \sigma_{11}(t) = \beta_0 + \beta_1[r_1(t) - Er_1(t)] + \beta_2|r_1(t) - Er_1(t)|$. Parameters are: $\gamma = 4$, $\beta_1 = -1.638$, $\beta_2 = 2.497$.

| $\eta$ | E($\omega$) | E($\omega_{\log}$) | CORR | MV(%) | JI(%) | INT(%) |
|---|---|---|---|---|---|---|
| 0 | .638 | .639 | 1 | 81.09 | 19.57 | -.66 |
| .05 | .636 | .634 | 1 | 84.42 | 19.69 | -4.11 |
| .1 | .634 | .630 | 1 | 88.03 | 19.85 | -7.88 |
| .15 | .632 | .624 | 1 | 91.96 | 20.04 | -12.00 |

Table 8: Contribution of each component to the mean of the equilibrium portfolio weight. Parameters and conditional mean and variance of the risky return are as in Table 7.

| $\eta$ | $\sigma^2_\omega$ | $\sigma^2_{\omega\log}$ | MV(%) | INT(%) |
|---|---|---|---|---|
| 0 | .0071 | .0071 | 100.4 | 0 |
| .05 | .0374 | .0372 | 100.7 | 0 |
| .1 | .1009 | .9994 | 101.6 | .01 |
| .15 | .1980 | .1923 | 103.3 | .03 |

Table 9: Contribution of each component to the variance of the equilibrium portfolio weight. Parameters and conditional mean and variance of the risky return are as in Table 7.

## APPENDIX

In order to prove propositions 1 and 2 we require the following Lemmas that are general properties of conditionally normal random variables that follow GARCH processes.

**LEMMA1:** Assume $m$ and $v$ are random variables with a joint normal conditional distribution. Assume the conditional second order moments of that distribution at period $t$ ($\sigma_{mm}(t)$, $\sigma_{mv}(t)$, and $\sigma_{vv}(t)$) follow the GARCH processes (16) to (18). Then

$$\begin{aligned} & E_{t-1}\left[\sigma_{mm}(t)^c\sigma_{mv}(t)^d\sigma_{vv}(t)^e\right] \\ = & \sum_{i=0}^{c}\sum_{j=0}^{d}\sum_{k=0}^{e}\zeta_{c,d,e}(i,j,k)\sigma_{mm}(t-1)^i\sigma_{mv}(t-1)^j\sigma_{vv}(t-1)^k, \end{aligned} \tag{37}$$

for some family of constant terms ($\{\zeta_{c,d,e}(i,j,k)\}$).

**Proof:**

$$\begin{aligned} & E_{t-1}\left[\sigma_{mm}(t)^c\sigma_{mv}(t)^d\sigma_{vv}(t)^e\right] \\ & = E_{t-1}\left\{\left[\alpha_0^m+\alpha_1^m\sigma_{mm}(t-1)+\alpha_2^m m(t)^2\right]^c\left[\alpha_0^{mv}+\alpha_1^{mv}\sigma_{mv}(t-1)+\alpha_2^{mv}m(t)v(t)\right]^d\right. \\ & \left.\left[\alpha_0^v+\alpha_1^v\sigma_{vv}(t-1)+\alpha_2^v v(t)^2\right]^e\right\}. \end{aligned}$$

Now, notice that

$$\sigma_{mm}(t)^c = \left[\alpha_0^m+\alpha_1^m\sigma_{mm}(t-1)+\alpha_2^m m(t)^2\right]^c = \sum_{i=0}^{c}h_{m,i}(t-1)m^{2i}(t),$$

where

$$\begin{aligned} h_{m,i}(t-1) &= \binom{c}{i}\left(\alpha_0^m+\alpha_1^m\sigma_{mm}(t-1)\right)^{c-i}\left(\alpha_2^m\right)^i \\ &= \left(\alpha_2^m\right)^i\binom{c}{i}\sum_{s=0}^{c-i}\left[\left(\alpha_0^m\right)^{c-i-s}\left(\alpha_1^m\right)^s\sigma_{mm}(t-1)^s\right], \end{aligned} \tag{38}$$

with analogous expressions for $\sigma_{mv}(t)^e$ and $\sigma_{vv}(t)^e$.

Thus, from equation (38),

$$
\begin{aligned}
& E_{t-1}\left[\sigma_{mm}(t)^c\sigma_{mv}(t)^d\sigma_{vv}(t)^e\right] = \\
& E_{t-1}\left[\sum_{i=0}^{c}\sum_{j=0}^{d}\sum_{k=0}^{e} h_{i,m}(t-1)h_{j,mv}(t-1)h_{k,v}(t-1)m(t)^{2i+j}v(t)^{2k+j}\right] \\
= & \sum_{i=0}^{c}\sum_{j=0}^{d}\sum_{k=0}^{e} h_{i,m}(t-1)h_{j,mv}(t-1)h_{k,v}(t-1)E_{t-1}\left[m(t)^{2i+j}v(t)^{2k+j}\right] \\
= & \sum_{i=0}^{c}\sum_{j=0}^{d}\sum_{k=0}^{e}[2(i+j+k)-1]hh_{i,j,k}(t-1)\sigma_{mm}(t-1)^i\sigma_{mv}(t-1)^j\sigma_{vv}(t-1)^k. \qquad (39)
\end{aligned}
$$

where $hh_{i,j,k}(t-1) = h_{i,m}(t-1)h_{j,mv}(t-1)h_{k,v}(t-1)$ and the last equality follows from the properties of the Multivariate Normal Distribution. Substituting (38) in (39) we obtain expression (37). **Q.E.D.**

**LEMMA 2:** Under the conditions of Lemma 1

$$
\begin{aligned}
& Cov_{t-1}\left[\sigma_{mm}(t-1)^c\sigma_{mv}(t)^d\sigma_{mv}(t)^e, \sigma_{mm}(t)^{c'}\sigma_{vv}(t)^{d'}\sigma_{mv}(t)^{e'}\right] = \\
= & \sum_{i=0}^{c+c'}\sum_{j=0}^{d+d'}\sum_{k=0}^{e+e'} \psi_{c,d,e}^{c',d',e'}(i,j,k)\sigma_{mm}(t-1)^i\sigma_{mv}(t-1)^j\sigma_{vv}(t-1)^k, \qquad (40)
\end{aligned}
$$

for some family of constant terms $\{\psi_{c,d,e}^{c',d',e'}(i,j,k)\}$.

**Proof:**

From a very well known property of the covariance operator

$$
\begin{aligned}
& Cov_{t-1}\left[\sigma_{mm}(t)^c\sigma_{mv}(t)^d\sigma_{mv}(t)^e, \sigma_{mm}(t)^{c'}\sigma_{mv}(t)^{d'}\sigma_{vv}(t)^{e'}\right] = \\
= \; & E_{t-1}\left[\sigma_{mm}(t)^{c+c'}\sigma_{mv}(t)^{d+d'}\sigma_{vv}(t)^{e+e'}\right] = \\
- \; & E_{t-1}\left[\sigma_{mm}(t)^c\sigma_{mv}(t)^d\sigma_{vv}(t)^e\right]E_{t-1}\left[\sigma_{mm}(t)^{c'}\sigma_{mv}(t)^{d'}\sigma_{vv}(t)^{e'}\right]. \qquad (41)
\end{aligned}
$$

and applying Lemma 1 to both terms on the right hand side of equation (41) we find

$$
Cov_{t-1}\left[\sigma_{mm}(t-1)^c\sigma_{mv}(t)^d\sigma_{mv}(t)^e, \sigma_{mm}(t)^{c'}\sigma_{mv}(t)^{d'}\sigma_{mv}(t)^{e'}\right] =
$$

$$
\begin{aligned}
&= \sum_{i=0}^{c+c'}\sum_{j=0}^{d+d'}\sum_{k=0}^{e+e'} \varsigma_{c,d,e}(i,j,k)\sigma_{mv}(t-1)^i\sigma_{mv}(t-1)^j\sigma_{vv}(t-1)^k \\
&- \left[\sum_{i=0}^{c}\sum_{j=0}^{d}\sum_{k=0}^{e} \varsigma_{c',d',e'}(i,j,k)\sigma_{mv}(t-1)^i\sigma_{mv}(t-1)^j\sigma_{vv}(t-1)^k\right] \times \\
&\times \left[\sum_{i=0}^{c'}\sum_{j=0}^{d'}\sum_{k=0}^{e'} \varsigma_{c+c',d+d',e+e'}(i,j,k)\sigma_{mv}(t-1)^i\sigma_{mv}(t-1)^j\sigma_{vv}(t-1)^k\right]. \qquad (42)
\end{aligned}
$$

Expression (41) then follows from simple calculus. **Q.E.D.**

**LEMMA 3**: Under the conditions of Lemma 1:

$$
Cov_{t-1}\left[\sigma_{mm}(t-1)^c\sigma_{mv}(t)^d\sigma_{mv}(t)^e, m(t)\right] = 0. \qquad (43)
$$

**Proof:**

From expression (39) we have that

$$
\begin{aligned}
&Cov_{t-1}\left[\sigma_{mm}(t)^c\sigma_{mv}(t)^d\sigma_{vv}(t)^e, m(t)\right] = \\
&= E_{t-1}\left\{\left[\sum_{i=0}^{c}\sum_{j=0}^{d}\sum_{k=0}^{e} h_{i,m}(t-1)h_{j,mv}(t-1)h_{k,v}(t-1)m(t)^{2i+j}v(t)^{2k+j}\right]m(t)\right\} \\
&= \sum_{i=0}^{c}\sum_{j=0}^{d}\sum_{k=0}^{e} h_{i,m}(t-1)h_{j,mv}(t-1)h_{k,v}(t-1)E_{t-1}\left[m(t)^{2i+j+1}v(t)^{2k+j}\right] = 0, \qquad (44)
\end{aligned}
$$

where the last equality follows from the properties of the Multivariate Normal Distribution. **Q.E.D.**

## PROOF OF PROPOSITION 1

The proof proceeds by backward induction:

**t=T**

By the terminal condition,

$$
\log a_T = 0. \qquad (45)
$$

**t=T-1**

From expression (10),

$$-\log a_{T-1} = P(1) + Q_1(1)\left\{\frac{1-\rho}{\rho}E_{T-1}r_m(T) + \frac{(1-\rho)(1-\gamma)}{2\rho}\sigma_{mm}(T-1)\right\}. \tag{46}$$

**t=T-2**

First notice that by from Lemmas 1 and 2,

$$\begin{aligned}
\sigma_{aa}(T-2) &= Var_{T-2}(\log a_{T-1}) \\
&= Q_1(1)^2\left[\left(\frac{1-\rho}{\rho}\right)^2\sigma_{vv}(T-2) + \left(\frac{(1-\rho)(1-\gamma)}{2\rho}\right)^2 Var_{T-2}\sigma_{nm}(T-1)\right]^2 \\
&+ Q_1(1)^2\frac{1-\rho}{\rho}\frac{(1-\rho)(1-\gamma)}{\rho}Cov_{T-2}\left[v(T-1), \sigma_{mm}(T-1)\right] \\
&= Q_1(1)^2\left[\left(\frac{1-\rho}{\rho}\right)^2\sigma_{vv}(T-2)\right] \\
&+ \left(\frac{(1-\rho)(1-\gamma)}{2\rho}\right)^2\sum_{i=0}^{2}\psi_{2,0,0}(i,0,0)\sigma_{mm}(T-2)^i
\end{aligned}$$

and

$$\sigma_{am}(T-2) = Cov_{T-2}(\log a_{T-1}, r_m(T-1)) = -Q_1(1)\left[\frac{1-\rho}{\rho}\sigma_{mv}(T-2)\right].$$

Thus, from equation (10),

$$\begin{aligned}
-\log a_{T-2} &= P(2) + \\
&= E_{T-2}\sum_{j=1}^{T-2}Q_j(2)\left\{\frac{1-\rho}{\rho}r_m(T-2+j) + \frac{(1-\rho)(1-\gamma)}{2\rho}\sigma_{mm}(T-2+j)\right\} \\
&+ Q_1(2)\left\{-(1-\gamma)\sigma_{am}(T-2) + \frac{\rho(1-\gamma)}{2(1-\rho)}\sigma_{aa}(T-2)\right\} \\
&= P(2) + \sum_{j=1}^{T}Q_j(2)\left\{\frac{1-\rho}{\rho}E_{T-2}r_m(T-2+j)\right\} \\
&+ Q_1(2)\frac{(1-\rho)(1-\gamma)}{2\rho}\sigma_{mm}(T-2) \\
&+ Q_2(2)\frac{(1-\rho)(1-\gamma)}{2\rho}\left[\alpha_0^m + (\alpha_1^m + \alpha_2^m)\sigma_{mm}(T-2)\right] \\
&+ (1-\gamma)Q_1(2)Q_1(1)\frac{1-\rho}{\rho}\sigma_{mv}(T-2) \\
&+ Q_1(2)Q_1(1)^2\frac{\rho}{2}\frac{1-\gamma}{1-\rho}\left[\left(\frac{1-\rho}{2\rho}\right)^2\sigma_{vv}(T-2)\right.
\end{aligned}$$

$$+\left(\frac{(1-\rho)(1-\gamma)}{2\rho}\right)^2 \sum_{i=0}^{2} \psi_{i,0,0}\sigma_{mm}(T-2)^i\Bigg].$$

Therefore, the consumption-wealth ratio at period $T-2$ can be written as

$$\begin{aligned}
-\log a_{T-2} &= K(2) + E_{T-2}\sum_{j=1}^{T-2} Q_j(2)\left(\frac{1-\rho}{\rho}\right) r_m(T-2+j) \\
&+ \sum_{p=0}^{2}\sum_{q=0}^{2}\sum_{r=0}^{2} s(p,q,r,2)\sigma_{mm}(T-2)^p\sigma_{mv}(T-2)^q\sigma_{vv}(T-2)^r, \qquad (47)
\end{aligned}$$

with

$$\begin{aligned}
K(2) &= P(2) + Q_2(2)\frac{(1-\rho)(1-\gamma)}{2\rho}\alpha_0^m \\
s(1,0,0,2) &= Q_1(2)\frac{(1-\rho)(1-\gamma)}{2\rho} + Q_1(2)Q_1(1)^2\frac{\rho}{2}\frac{1-\gamma}{1-\rho}\frac{(1-\rho)(1-\gamma)}{2\rho}\psi_{2,0,0}(1,0,0) \\
&+ Q_2(2)\frac{(1-\rho)(1-\gamma)}{2\rho}(\alpha_1^m + \alpha_2^m) \\
s(0,1,0,2) &= Q_1(2)Q_1(1)(1-\gamma)\frac{1-\rho}{\rho} \\
s(0,0,1,2) &= \frac{\rho}{2}\frac{1-\gamma}{1-\rho}Q_1(1)^2Q_1(2)\left(\frac{1-\rho}{\rho}\right)^2 \\
s(2,0,0,2) &= \frac{\rho}{2}\frac{1-\gamma}{1-\rho}Q_1(1)^2Q_1(2)\left(\frac{(1-\rho)(1-\gamma)}{2\rho}\right)^2\psi_{2,0,0}(2,0,0)
\end{aligned}$$

and $s(i,j,k,2)=0$ for all $j,k>1$ and $i>2$. Thus, the result holds for $t=2$. Now, assume that the result is true for $a_t$. Then, let's prove that it also holds for $a_{t-1}$.

From, the autoregressive process (14) we can write,

$$E_t\sum_{j=1}^{T-t} Q_j(T-t)r_m(t+j) = \sum_{j=1}^{T-t} Q_j(T-t)H^{j-1}E_t r_m(t+1) \qquad (48)$$

and, therefore,

$$\begin{aligned}
\sigma_{aa}(t-1) &= \sum_{j=1}^{T-t} Q_j(T-t)H^{2(j-1)}\sigma_{vv}(t-1) \\
&+ \sum_{p,q,r=0}^{2^{T-t-1}}\ \sum_{p',q',r'=0}^{2^{T-t-1}} s(p,q,r,T-t)s(p',q',r',T-t) \\
&\quad Cov_{t-1}\left[\sigma_{mm}(t)^p\sigma_{mv}(t)^q\sigma_{mv}(t)^r, \sigma_{mm}(t)^{p'}\sigma_{mv}(t)^{q'}\sigma_{mv}(t)^{r'}\right]. \qquad (49)
\end{aligned}$$

Notice that, from Lemma 2, the second term of (49) can be written as

$$
\begin{aligned}
& \sum_{p,q,r=0}^{2^{T-t-1}} \sum_{p',q',r'=0}^{2^{T-t-1}} s(p,q,r,T-t)s(p',q',r',T-t) \times \\
& \sum_{i=0}^{2p}\sum_{j=0}^{2q}\sum_{k=0}^{2r} \psi_{p,q,r}(i,j,k)\sigma_{mm}(t-1)^i\sigma_{mv}(t-1)^j\sigma_{vv}(t-1)^k \\
= & \sum_{p,q,r=0}^{2^{T-t}} \nu(p,q,r,T-t)\sigma_{mm}(t-1)^p\sigma_{mv}(t-1)^q\sigma_{vv}(t-1)^r. \qquad (50)
\end{aligned}
$$

Similarly,

$$
\sigma_{am}(t-1) = -\sum_{j=1}^{T-t} Q_j(T-t)H^{j-1}\sigma_{mv}(t-1) \qquad (51)
$$

Finally, from Lemma 1

$$
\begin{aligned}
-E_{t-1}\log a_t &= P(T-t) + E_{t-1}\sum_{j=1}^{T-t} Q_j(T-t)\frac{1-\rho}{\rho}r_m t+j \\
&+ E_{t-1}\sum_{p,q,r=0}^{T-t} s(p,q,r,T-t)\sigma_{mm}(t)^p\sigma_{mv}(t)^q\sigma_{vv}(t)^r \\
&= P(T-t) + E_{t-1}\sum_{j=1}^{T-t} Q_j(T-t)\frac{1-\rho}{\rho}r_m(t+j) \\
&+ \sum_{p,q,r=0}^{T-t}\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r}\zeta_{p,q,r}(i,j,k)\sigma_{mm}(t-1)^i\sigma_{mv}(t-1)^j\sigma_{vv}(t)^k \\
&= P(T-t) + E_{t-1}\sum_{j=1}^{T-t} Q_j(T-t)\frac{1-\rho}{\rho}r_m(t+j) \\
&+ \sum_{p,q,r=0}^{2^{T-t}} \epsilon(p,q,r,T-t)\sigma_{mm}(t-1)^p\sigma_{mv}(t-1)^q\sigma_{vv}(t)^r. \qquad (52)
\end{aligned}
$$

Then, substituting equations (49) to (52) in (10), we obtain that the consumption-wealth ratio satisfies

$$
\begin{aligned}
-\log(a_{t-1}) &= K(T-t+1) + E_t\sum_{j=1}^{T-t} Q_j(T-t+1)\left(\frac{1-\rho}{\rho}\right)r_m(t+j-1) \\
&+ \sum_{p=0}^{2^{T-t}}\sum_{q=0}^{2^{T-t}}\sum_{r=0}^{2^{T-t}} s(p,q,r,T-t+1)\sigma_{mm}(t-1)^p\sigma_{mv}(t-1)^q\sigma_{vv}(t-1)^r, \qquad (53)
\end{aligned}
$$

where

$$
\begin{aligned}
s(1,0,0,T-t+1) &= \nu(1,0,0,T-t)B(T-t+1)\frac{\rho}{2}\frac{1-\gamma}{1-\rho}\epsilon(1,0,0,T-t) \\
&+ B(T-t+1)\frac{(1-\rho)(1-\gamma)}{2\rho}, \\
s(0,1,0,T-t+1) &= \nu(0,1,0,T-t)B(T-t+1)\frac{\rho}{2}\frac{1-\gamma}{1-\rho} \\
&+ B(T-t+1)\epsilon(0,1,0,T-t) \\
&+ B(T-t+1)(1-\gamma)\sum_{j=1}^{T-t} Q_j(T_t)H^{j-1}, \\
s(0,0,1,T-t+1) &= \nu(0,0,1,T-t)B(T-t+1)\frac{\rho}{2}\frac{1-\gamma}{1-\rho} \\
&+ B(T-t+1)\epsilon(0,0,1,T-t) \\
&+ B(T-t+1)\left(Q_j(T-t)H^{j-1}\right)^2, \\
s(p,q,r,T-t+1) &= \nu(p,q,r,T-t)B(T-t+1)\frac{\rho}{2}\frac{1-\gamma}{1-\rho} \\
&+ B(T-t+1)\epsilon(p,q,r,T-t), \\
&\quad \text{for all p,q,r, s.t} \quad 2^{T-t-1} > \max\{p,q,r\} > 1 \qquad \text{and} \\
s(p,q,r,T-t+1) &= B(T-t+1)\epsilon(p,q,r,T-t) \quad \text{for} \quad \max\{p,q,r\} = T-t+1. \qquad (54)
\end{aligned}
$$

**Q.E.D.**

## PROOF OF PROPOSITION 2

From equation (10) and Lemma 3 we know,

$$\sigma_{am}(t) = -\frac{1-\rho}{\rho}Cov_t\left[r_m(t+1), E_{t+1}(\sum_{j=1}^{T-t} Q_j(T-t)r_m(t+j+1)\right], \qquad (55)$$

and from expressions (48) and (55) we have that

$$\sigma_{am}(t) = -\frac{1-\rho}{\rho}S(T-t)Cov_t(r_m(t+1), r_m(t+2)),$$

with $S(T-t) = \sum_{j=1}^{T-t} Q_j(T-t)H^{j-1}$. **Q.E.D.**

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