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Abstract

This paper analyzes the problem of a benevolent planner wishing to control a population of heterogeneous agents subject to idiosyncratic shocks. This is equivalent to a deterministic control problem in which the state variable is the cross-sectional distribution. We show how, in continuous time, this problem can be broken down into a dynamic programming equation plus the law of motion of the distribution, and we introduce a new numerical algorithm to solve it. As an application, we analyze the constrained-efficient allocation of an Aiyagari economy with a fat-tailed wealth distribution. We find that the constrained-efficient allocation features *more* wealth inequality than the competitive equilibrium.

Keywords: Kolmogorov forward equation, wealth distribution, social welfare function, mean field control.

JEL classification: C6, D3, D5, E2.

Resumen

Este artículo analiza el problema de un planificador benevolente que desea controlar una población de agentes heterogéneos sujetos a perturbaciones idiosincrásticas. Esto es equivalente a un problema de control determinista en el que la variable de estado pertinente es la distribución de agentes. Demostramos cómo, trabajando en tiempo continuo, este problema puede descomponerse en una ecuación de programación dinámica junto a la ley de movimiento de la distribución y presentamos un nuevo algoritmo para resolverlo numéricamente. Como aplicación, analizamos la solución eficiente-restringida de una economía *à la Aiyagari* con una distribución de la riqueza que replica la observada en los datos. Nuestros resultados indican que la solución óptima se caracteriza por un nivel de desigualdad de la riqueza *mayor* que el observado en el equilibrio competitivo.

Palabras clave: ecuación de Kolmogorov, distribución de la riqueza, función de bienestar social, control óptimo.

Códigos JEL: C6, D3, D5, E2.

1 Introduction

Optimal control is an essential tool in economics and finance. In optimal control, a planner determines the evolution of a vector of control variables in order to maximize a certain optimality criterion. The state of the system is typically characterized by a finite number of state variables.[1] Some systems of interest are nevertheless composed of a very large number of heterogeneous agents; an economy, for example, is composed of millions of different households and firms and a network may contain thousands of nodes. In these cases, assuming a continuous distribution of state variables seems to be a reasonable approximation to the real problem under consideration.

The aim of this paper is to analyze optimal control problems in which there is a continuum of heterogeneous agents. The state of each of these agents is characterized by a finite set of variables which follow a controllable stochastic process, that is, there exists a vector of controls that allows the planner to modify the individual states. The state dynamics also depend on the evolution of a set of aggregate variables. We consider a benevolent social planner who maximizes an aggregate welfare criterion.

We focus on the continuous time version of the problem. The key feature of working in continuous time is that the evolution of the state distribution across agents can be characterized by a partial differential equation (PDE) known as the *Kolmogorov forward* (KF) equation.[2] Despite the random evolution of each individual state, the dynamics of the distribution are deterministic due to the Law of Large Numbers. Thanks to this, the control of an infinite number of agents subject to idiosyncratic shocks can be analyzed as the control of a deterministic distribution that evolves according to the KF equation, subject to the aggregate −or market clearing− conditions.

The main contribution of the paper is to present the necessary conditions for a solution to this problem. These conditions are characterized by a system of two coupled PDEs, the planner's *Hamilton-Jacobi-Bellman* (HJB) equation and the KF equation, plus the market clearing conditions. The planner's HJB equation is a PDE that determines the *marginal social value* of an agent being in a certain state and the KF equation describes the distributional dynamics. This characterization of the problem allows the comparison with the decentralized solution or competitive equilibrium, in which each atomistic agent chooses her own controls without taking into account their impact on the other agents. The competitive equilibrium is also composed by a system of coupled PDEs: the individual HJB and the KF equation. The difference is that the planner's HJB equation includes a term reflecting the impact of individual policies on the aggregate distribution, whereas the individual HJB does not. By comparing the two HJB equations we provide a condition to verify whether the competitive allocation and the planned economy are equal.

[1] See Bertsekas (2005, 2012) or Fleming and Soner (2006) for an introduction to optimal control problems.

[2] This equation is also known as the Fokker-Planck equation.

The second contribution of the paper is to introduce a numerical algorithm to solve optimal control problems with heterogeneous agents. Many continuous-time models with heterogeneous-agents can be solved explicitly due to the particular set of assumptions that they make.[3] In contrast, Achdou et al. (2015) provide an efficient numerical strategy based on finite difference methods in order to solve more general problems.[4] Here we extend this methodology to the case of optimal control. Our algorithm solves the planner's HJB, the KF equation and finds the Lagrange multipliers associated with the market clearing conditions.

The methodology presented here allows analyzing a variety of problems in economics. As an application, we analyze the *constrained efficient* allocation in an incomplete-market economy *à la* Aiyagari (1994) with stochastic-life agents. Recent research by Piketty and Saez (2003), Atkinson, Piketty and Saez (2011) and others has documented how since the 1970s there has been a progressive rise in top wealth inequality and in the stock of capital in several advanced economies, which in cases such as the Unites States may be close to the historical maximum. This has led to researchers such as Piketty (2014) or Atkinson (2015) to propose redistributive policies such as wealth taxation as a way to reduce wealth inequality. Notwithstanding, the increase in wealth inequality observed in the data is not necessarily negative from a social welfare perspective. It depends on which is the "optimal" amount of capital in the economy and on its distribution: it is not obvious that the society as a whole is better-off in a low-capital more equal situation compared to a high-capital more unequal one, for example.

We apply the optimal control techniques discused above to try to shed light on this issue by studying the optimal constrained-efficient wealth distribution in an Aiyagari economy with stochastic-lifetimes as in the "perpetual youth" models of Yaari (1965) and Blanchard (1985). The introduction of stochastic lifetimes generates a stationary wealth distribution with an upper tail following a power law, a well-known empirical fact.[5] The degree of wealth inequality, defined by the right tail exponent, depends on the difference between the return on capital and the long-run growth rate, $(r - g)$, as discussed by Piketty (2014).

The constrained efficient allocation is defined as the one in which a benevolent social planner chooses the individual levels of consumption, while respecting all budget constraints. As discussed by Diamond (1967) and Davila et al. (2012), this is a notion of efficiency that does not allow the planner to directly overcome the friction implied by missing markets.[6] This concept is related to

[3]Some examples are Jovanovic (1979), Luttmer (2007) or Alvarez and Shimer (2011). See Achdou et al. (2014) for a recent survey of continuous-time models in macroeconomics.

[4]Also see Rocheteau et al (2015) who propose a related incomplete-markets framework and are able to obtain a number of qualitative results.

[5]The emergence of a power-law in the wealth distribution has been analyzed in a number of previous papers, such as Wold and Whittle (1957), Benhabib and Bisin (2007), Benhabib, Bisin and Zhu (2011, 2015a, 2015b), Piketty and Zucman (2015), Jones (2015) or Acemoglu and Robinson (2015), among others. Here the power law is due to the combination of random exponential lifetimes and a lower bound to the wealth distribution. A non-microfounded approach to power laws using this mechanism is discussed in Gabaix et al. (2015).

[6]If the planner was able to fully redistribute across agents, the first-best allocation would be degenerated as a utilitarian planner would provide the same consumption level to every agent irrespective of her assets.

that of a pecuniary externality. In the model agents do not internalize the effect of their individual saving decisions on interest rates and wages. We show how the model is constrained inefficient: the market economy is undercapitalized compared to the social optimum. The optimal allocation also features *more* wealth inequality than the market economy as the reduction in interest rates due to higher capital in the social optimum is not enough to compensate for the increase in aggregate savings by wealthy households.

Related literature. Our paper relates to the recent discrete-time literature, such as Davila et al. (2012) and Acikgoz (2014), analyzing problems in which a planner has to choose the controls to be applied to a continuous population of heterogeneous agents. Davila et al. (2012) analyze the constrained efficient allocation in an Aiyagari economy with infinite lifetimes using calculus of variations. Acikgoz (2014) builds on this approach in order to solve the Ramsey problem in a similar model. The continuous-time approach presented here differs from those papers in two main aspects. The first is that we characterize the problem in terms of the planner HJB instead of the Euler equation. More precisely, we show how the planner's problem can be broken into individual HJB equations in which the value function for each person is his marginal social value under an optimal plan. The HJB equation for this marginal social value can then be compared with the HJB equation in the competitive equilibrium, thereby obtaining an easily interpretable formula that precisely characterizes the pecuniary externality causing the planner's allocation to differ from the equilibrium one. The second lies in the approach to compute the evolution of the cross-sectional distribution. Traditional discrete-time methods either simulate a large number of agents by Montecarlo methods or discretize the state-space. In contrast, in continuous time the distributional dynamics are characterized by a partial differential equation: the KF equation. We take advantage of this fact and develop an efficient and flexible computational algorithm using finite-difference methods that applies to a wide class of planning problems in which a distribution is the relevant state variable.

Our paper is also linked to a couple of recent papers that analyze optimal control problems in continuous time.[7] Lucas and Moll (2014) analyze an optimal planning problem subject to the law of motion of the aggregate distribution. Their formulation nevertheless does not consider the possibility of including aggregate constraints, such as market clearing conditions, which are prevalent in most economic problems. Here instead we analyze the general problem. This requires the use of functional analysis, in particular of optimization techniques in infinite-dimensional Hilbert spaces, in order to derive the necessary conditions for a solution. Another continuous-time paper

[7] In additon, it relates to the emerging literature in mathematics analyzing *mean field control* problems. The name is borrowed from the mean-field approximation in statistical physics, in which the effect on any given individual of all the other individuals is approximated by a single averaged effect. A survey may be found in Bensoussan, Frehse and Yam (2013).

that analyzes the optimal allocation with heterogeneous agents is Alfonso and Lagos (2015). They assume that their state variables can only take a *finite* number of values, in contrast to a continuum, and thus they can avoid the problem of optimization in infinite-dimensional spaces that we analyze here. In many applications it is more natural to work with continuous state variables, for example in models of wealth distribution like the one analyzed in the present paper.

The structure of the paper is as follows. In section 2 we discuss the problem of computing the constrained-efficient allocation in an incomplete-markets economy with stochastic lifetimes as a motivation. In section 3 we analyze the general case and present the main results. In section 4 we introduce the numerical algorithm and apply it to solve the problem posed in section 2. Finally, in section 5 we conclude.

2 The constrained-efficient allocation in an incomplete-markets economy with stochastic lifetimes

We introduce here a model that extends the incomplete-markets economy *à la* Aiyagari (1994) with stochastic-life agents as in the "perpetual youth" models of Yaari (1965) and Blanchard (1985). Our aim is to analyze the optimal constrained efficient allocation in the sense of Davila et al. (2012), as it is explained below.

2.1 Individuals

Let $(\Sigma, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space. There is a continuum of mass unity of agents that are heterogeneous in their wealth A and labor productivity z. The duration of an agent's life is uncertain. Lifetimes τ are stochastic and governed by an exponential random variable with mean $1/\eta$. At the time of death each agent is replaced by a single child so that the size of the population is constant.

Agents have standard preferences over utility flows from future consumption C_t discounted at rate $\rho \geq 0$:

$$\mathbb{E}_0 \left[\int_0^{\tau} e^{-\rho t} u(C_t) dt \right] = \mathbb{E}_0 \left[\int_0^{\infty} e^{-(\rho+\eta)t} u(C_t) dt \right], \tag{1}$$

We assume CRRA preferences, such that $u(C) = \frac{C^{1-\gamma}}{1-\gamma}$. Individuals have no intergenerational altruism and thus they purchase an annuity in a perfectly competitive insurance market that pays them a flow ηA_t in exchange of taking control of all the assets when the agent dies.[8] Each agent

[8] The amount of resources collected from expired agents is ηK_t, where K_t is the aggregate wealth, which equals the flow of payments so that insurance companies make no profits.

supplies z_t efficiency units of labor to the labor market and these get valued at wage W_t. An agent's wealth evolves according to

$$dA_t = \left(W_t z_t + \left(r_t + \eta\right) A_t - C_t\right) dt, \tag{2}$$

where r_t is the interest rate. Agents also face a borrowing limit,

$$A_t \geq \underline{A}_t, \tag{3}$$

where $\underline{A}_t \leq 0$. The agent's labor productivity evolves stochastically over time on a bounded interval $[\underline{z}, \bar{z}]$ with $\underline{z} \geq 0$, according to a bounded Ornstein–Uhlenbeck process:[9]

$$dz_t = \theta(\hat{z} - z_t)dt + \sigma dB_t,$$

where B_t is a $\mathcal{F}_t-$adapted idiosyncratic Brownian motion and θ, $\hat{z}$ and σ are positive constants.

We impose an additional restriction so that the "natural borrowing limit" is not binding:

$$\underline{A}_t > -\underline{z} \int_t^{\infty} e^{-\int_t^s r_\tau d\tau} W_s ds, \quad \forall t \geq 0.$$

The optimal value function results in

$$V(t, A, z) = \max_{\{C_s\}_{s=t}^{\infty}} \mathbb{E}_t \left[\int_t^{\infty} e^{-(\rho+\eta)(s-t)} u(C_s) ds \right], \tag{4}$$

subject to evolution of individual wealth (2).

2.2 Firms

There is a representative firm with a constant returns to scale production function $Y = F(K, L) = K^{\alpha} \left(ZL\right)^{1-\alpha}$, where K is the aggregate capital, L is aggregate labor and Z is TFP. The lattest evolves deterministically according to

$$Z_t = e^{gt},$$

where g is the constant long-run growth rate of the economy. Capital depreciates at rate δ_K. Since factor markets are competitive, the interest rate and wage are given by

$$\begin{aligned} r_t &= \frac{\partial F\left(K_t, 1\right)}{\partial K} - \delta_K = \alpha \frac{Y_t}{K_t} - \delta_K, \\ W_t &= \frac{\partial F\left(K_t, 1\right)}{\partial L} = \left(1 - \alpha\right) \frac{Y_t}{L_t}. \end{aligned} \tag{5}$$

[9] This is the continuous-time counterpart of the AR(1).

2.3 Competitive equilibrium

As described above, agents leave no bequest. New agents begin with an initial debt $\underline{A}_t$ as we assume that they had to borrow in order to finance their education.[10] They are also born with a labor productivity level of $\underline{z}$. The state of the economy is the joint distribution of wealth and labor, $f(t, A, z)$. The dynamics of the distribution are given by the Kolmogorov Forward (KF) equation

$$\begin{aligned} \frac{\partial f(t,A,z)}{\partial t} &= -\frac{\partial}{\partial A}\left[\left(wz + (r+\eta)A - C(t,A,z)\right) f(t,A,z)\right] \\ &\quad -\frac{\partial}{\partial z}\left[\theta(\hat{z}-z) f(t,A,z)\right] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2 f\right) - \eta f(t,A,z) + \eta\delta_0, \end{aligned} \tag{6}$$

$\forall A > \underline{A}_t$. The term $-\eta f(t,A,z)$ is the outflow of agents due to death and the term $\eta\delta_0 = \eta\delta\left(A - \underline{A}_t\right)\delta\left(z - \underline{z}\right)$ is the inflow of newborn agents with wealth $\underline{A}_t$ and productivity $\underline{z}$.[11] The distribution should satisfy the normalization

$$\int_{\underline{A}_t}^{\infty}\int_{\underline{z}}^{\bar{z}} f(t,A,z)\,dA\,dz = 1.$$

The total amount of capital supplied in the economy equals the total amount of wealth

$$K_t = \int_{\underline{A}_t}^{\infty}\int_{\underline{z}}^{\bar{z}} A f(t,A,z)\,dA\,dz, \tag{7}$$

and the total amount of labor supplied in the economy equals one, $L = 1$.

We define a competitive equilibrium in this economy.

Definition 1 (Competitive equilibrium) *A competitive equilibrium is composed by a pair of factor prices $W(t)$, $r(t)$, an aggregate capital stock $K(t)$, a value function $V(t,A,z)$, a consumption policy $C(t,A,z)$ and a distribution $f(t,A,z)$ such that*

[10] We could have assumed as well that newborns have zero initial assets. It would make no qualitative difference and only a minor quantitative difference for the numerical results presented below.

[11] $\delta(\cdot)$ is the Dirac delta, not to confound with the depreciation rate. The Dirac delta is a *distribution* or generalized function such that

$$\delta[f] = \int_{-\varepsilon}^{\varepsilon} f(x)\delta(x)\,dx = f(0), \quad \forall\varepsilon > 0,\ f \in L^1(-\varepsilon,\varepsilon).$$

A heuristic characterization is that

$$\int_{\infty}^{\infty} \delta(x)\,dx = 1, \quad \delta(x) = \begin{cases} \infty, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

1. *Given* W, r *and* f, V *is the solution of the individual's problem (4) and the optimal control is* C.

2. *Given* K, *firms maximize their profits and prices are given by (5).*

3. *Given* W, r *and* C, f *is the solution of the KF equation (6).*

4. *Given* f *and* K, *the capital market (7) clears.*

2.4 Stationary distribution

Let's analyze the ergodic case in which the aggregate distribution is time-independent. In this case, non-stationary variables such as capital, wages, wealth or consumption grow at a rate g.[12] We detrend them by dividing by e^{gt}. Let $c_t \equiv C_t e^{-gt}$, preferences can be expressed as

$$\mathbb{E}_0 \left[\int_0^\infty e^{-[(\rho+\eta)-(1-\gamma)g]t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right],$$

and the evolution of individual detrended wealth $a_t \equiv A_t e^{-gt}$ follows

$$da_t = \left[w z_t + (r - g + \eta) a_t - C_t \right] dt,$$

where $w \equiv W_t e^{-gt}$ is the constant detrended wage. Notice the term $(r-g)$ in the drift of the wealth, as underlined by Piketty (2014). We assume that the borrowing constraint is of the form

$$\underline{A}_t = -\phi e^{gt}, \tag{8}$$

with $0 \leq \phi \leq \underline{z}\frac{w}{r}$.

The Hamilton-Jacobi-Bellman (HJB) equation of the individual problem is

$$\begin{aligned} \left[(\rho+\eta) - (1-\gamma) g\right] V(a,z) \;=\; & \max_c \frac{c^{1-\gamma}}{1-\gamma} + \left[wz + (r-g+\eta) a - c(t,a,z) \right] \frac{\partial V}{\partial a} \\ & + \theta(\hat{z} - z) \frac{\partial V}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial z^2}. \end{aligned} \tag{9}$$

And the KF equation

$$\begin{aligned} 0 \;=\; & -\frac{\partial}{\partial a} \left[\left(wz + (r-g+\eta) a - c(a,z) \right) f(a,z) \right] \\ & -\frac{\partial}{\partial z} \left[\theta(\hat{z} - z) f(a,z) \right] + \frac{1}{2} \frac{\partial^2}{\partial z^2} \left(\sigma^2 f(a,z) \right) - \eta f(a,z) + \eta \delta_0, \end{aligned} \tag{10}$$

where $\delta_0 = \delta(a+\phi)\,\delta(z-\underline{z})$.

[12] Non-stationary variables were previously defined with capital letters.

Factor prices are given by

$$r = \alpha k^{\alpha-1} - \delta_K, \qquad w = (1-\alpha)\, k^{\alpha}, \tag{11}$$

where $k \equiv K_t e^{-gt}$ is the constant detrended capital.

Finally, the market clearing condition results in

$$k = \int_{-\phi}^{\infty} \int_{\underline{z}}^{\bar{z}} a f(a,z) da dz. \tag{12}$$

The proposition below shows how the Aiyagari model with exogenous deaths features a fat-tail ergodic wealth distribution.

Proposition 1 *Provided that $\rho + \gamma\eta > (1-\gamma)\, r$ and $r > r^* \equiv \rho + \gamma g$, the stationary wealth distribution is characterized by an asymptotic power law, $f(a) \sim a^{-(1+\zeta)}$, with tail exponent*

$$\zeta = \frac{\eta\gamma}{r - r^*} = \frac{\eta\gamma}{(r-g) - (\rho - (1-\gamma)\, g)}. \tag{13}$$

Proof. See Appendix A. ■

The insight that the combination of a Blanchard-Yaari setting with an optimal consumption problem produces a power-law distribution is due to Benhabib and Bisin (2007). Our result differs from theirs in three regards. First, we analyze a stochastic general equilibrium incomplete-market case whereas Benhabib and Bisin work in a complete-market deterministic environment (except for the death probability) where interest rates and wages are exogenous. Second, we assume that agents only care about their own utility and hence we do not consider any "joy of giving" preferences for bequests. Finally, we work with general CRRA utility instead of the particular case of log-utility.

2.5 Constrained efficiency

We investigate the optimal allocation of wealth in this economy. Due to heterogeneity, our optimality criterion requires some degree of interpersonal utility comparison. In line with most of the literature, we consider a utilitarian social welfare function (SWF) so that the objective of the social planner is ex-ante expected utility. This amounts to a probability-weighted average: the planner is "behind the veil of ignorance." In addition, it is necessary to specify which degree of redistribution is possible for the planner. If the planner was able to fully redistribute across agents, the first-best allocation would be degenerated as a utilitarian planner would provide the same consumption level to every agent irrespective of his or her assets. This allocation does not seem too interesting as a practical benchmark. Instead, we follow Davila et al. (2012) and focus on the study of the *constrained-efficient* allocation. In this case the planner is constrained to consider allocations with zero net transfers across individuals. The question is whether the planner can improve on the market allocation by simply commanding different levels of consumption, while respecting all individual budget constraints. This issue is closely related to the existence of a *pecuniary externality,* typically present in this kind of models: individual agents do not internalize that their saving decisions affect the aggregate amount of capital, which affects the rest of agents through wages and interest rates. The planner does take this effect into account when computing the optimal individual saving decision and thus the optimal wealth allocation.

The problem of the planner is to choose individual consumption $c(t,\cdot)$ across agents in order to maximize the discounted aggregate utility

$$\begin{aligned} J(f(0,\cdot)) &= \max_{\{c(t,\cdot)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t}\left[\int_{-\phi}^{\infty}\int_{\underline{z}}^{\bar{z}} u(c)\, f(t,a,z)\, dzda\right] dt, \qquad (14)\\ &= \max_{\{c(t,\cdot)\}_{t=0}^{\infty}} \int_0^{\infty}\int_{-\phi}^{\infty}\int_{\underline{z}}^{\bar{z}} e^{-[\rho-(1-\gamma)g]t}\frac{(c(t,a,z))^{1-\gamma}}{1-\gamma} f(t,a,z)\, dzdadt, \end{aligned}$$

subject to the law of motion of the aggregate distribution (10), to the factor prices (11) and to the market clearing condition (12). Here $J(f(0,\cdot))$ is the optimal value *functional*, as its state variable is a distribution $f(0,\cdot)$.

Notice that the planner gives the same weight to every agent irrespective of its age. This contrasts with the SWF chosen in Benhabib and Bisin (2007) which only considers the welfare of the agents alive at an arbitrary time. Notice also that the planner discounts future aggregate utility flows at the same rate of individual agents ρ, not at rate $(\rho+\eta)$. The theoretical and numerical approach to solve problem (14) will be described in the next two sections.

3 General approach

In this section we analyze a general optimal control problem with heterogeneous agents and provide the necessary conditions for a solution.

3.1 Competitive equilibrium

3.1.1 Individual problem.

We consider a continuous-time infinite-horizon economy. Let $(\Sigma,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ be a filtered probability space. There is a continuum of unit mass of ex-ante identical agents indexed by $j\in[0,1]$. The duration of an agent's life is uncertain. Death is governed by a Poisson random variable with rate

η. At the time of death each agent is replaced by a single child so that the size of the population is constant.

Let B_t^j be a n-dimensional $\mathcal{F}_t$-adapted Brownian motion and $X_t^j \in \Omega$ denote the state of an agent j at time $t \in [0, \infty)$. The state evolves according to a multidimensional Itô process of the form

$$dX_t^j = b\left(X_t^j, \mu(t, X_t^j), Z_t\right) dt + \sigma\left(X_t^j\right) dB_t^j, \tag{15}$$

where X_t^j is a reflected process bounded in the domain $\Omega \subset \mathbb{R}^n$, $\mu \in \mathbb{R}^m$ is a $m-$dimensional vector of policy variables and $Z_t \in \mathbb{R}^p$ is a deterministic p-dimensional vector of aggregate variables. Here the instantaneous drift $b(\cdot)$ and volatility $\sigma(\cdot)$ are measurable functions, $b \in C^1(\Omega \times \mathbb{R}^m \times \mathbb{R}^p)$ and $\sigma \in C^2(\Omega)$.[13] In the Appendix B we provide some technical assumptions to ensure the existence of a solution of the stochastic differential equation (15).

The policy vector μ is an $\mathcal{F}_t$- adapted Markov control. The control $\mu(t, x)$ is admissible if for any initial point (t, x) such that $X_t^j = x$ the stochastic differential equation (15) has a unique solution.[14] We denote $\mathcal{M}$ as the space of all admissible controls contained in the set of all Markov controls. The control strategy is the same for every agent, but it depends on time and on the specific state of the agent.

Agents maximize their discounted utility. The optimal value function $V(t, x)$ is defined as

$$V(t, x) = \max_{\mu \in \mathcal{M}} \mathbb{E}_t \left[\int_t^{\infty} e^{-(\rho+\eta)(s-t)} u(X(t), \mu) ds | X_t = x \right], \tag{16}$$

subject to (15),[15] where utility $u(x, \mu) \in C^1(\Omega \times \mathbb{R}^m)$ is strictly increasing and strictly concave and $\rho > 0$ is a constant. The transversality condition is

$$\lim_{t \uparrow \infty} e^{-\rho t} V(t, x) = 0. \tag{17}$$

The solution to this problem is given by a value function $V(t, x)$ and a control strategy $\mu(t, x)$ that satisfy the HJB equation

$$\rho V(t, x) = \frac{\partial V}{\partial t} + \max_{\mu \in \mathcal{M}} \left\{ u(x, \mu) + \mathcal{A} V \right\}, \tag{18}$$

[13] $C^k(\Omega)$ is the set of all k-times continuously differentiable functions on Ω.

[14] This is guaranteed if $\mathbb{E}\left[\int_0^t \left\|\mu(s, X_s^i)\right\|^j ds\right] < \infty$, $\forall t < \infty$, for $j \in \mathbb{N}$.

[15] We have dropped the superindex j as there is no possibility of confusion.

where $\mathcal{A}$ is the *infinitesimal generator* of process (15):

$$\mathcal{A}V = \sum_{i=1}^{n} b_i(x,\mu,Z)\frac{\partial V}{\partial x_i} + \sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\left(\sigma(x)\sigma(x)^{\top}\right)_{i,k}}{2}\frac{\partial^2 V}{\partial x_i \partial x_k} - \eta V(t,x). \tag{19}$$

The infinitesimal generator of a stochastic process is a partial differential operator that encodes the main information about the process. It is typically defined as $\mathcal{A}f(x) = \lim_{t\downarrow 0}\frac{\mathbb{E}_0[f(X_t)|X_0=x]-f(x)}{t}$.[16]

3.1.2 Aggregate distribution and aggregate variables.

Assume that the transition measure of X_t with initial value $\tilde{x}_0$ has a density $f(t,x;\tilde{x}_0) \in C^2([0,\infty)\times\Omega)$, i.e., that $\forall\varphi \in C^2(\Omega)$:

$$\mathbb{E}_0[\varphi(X_t)|X_0=\tilde{x}_0] = \int_{\Omega}\varphi(x)f(t,x;\tilde{x}_0)dx.$$

The initial distribution of X_t at time $t=0$ is $f(0,x)=f_0(x)$. We assume than the new cohorts of agents are born with an initial state x_0. The dynamics of the distribution of agents $f(t,x)$ are given by the Kolmogorov Forward (KF) equation

$$\frac{\partial f}{\partial t} = \mathcal{A}^* f + \eta\delta_{x_0}, \tag{20}$$

$$\int_{\Omega} f(t,x)dx = 1, \tag{21}$$

where $\mathcal{A}^*$ is the *adjoint operator* of $\mathcal{A}$:

$$\mathcal{A}^* f = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[b_i(x,\mu,Z)f(t,x)\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2}{\partial x_i\partial x_k}\left[\left(\sigma(x)\sigma(x)^{\top}\right)_{i,k}f(t,x)\right] - \eta f(t,x),$$

and $\delta_{x_0} = \delta(x-x_0)$ is the Dirac delta centered at x_0.[17]

The vector of aggregate variables is determined by a system of p equations:

$$Z_k(t) = \int_{\Omega} g_k(x,\mu)f(t,x)dx, \quad k=1,..,p. \tag{22}$$

where $g_k \in C^1(\Omega\times\mathbb{R}^m)$. These equations are typically the *market clearing* conditions.

We may define a competitive equilibrium in this economy.

Definition 2 (Competitive equilibrium) *A competitive equilibrium is a vector of aggregate variables Z_t, a value function $V(t,x)$, a control $\mu(t,x)$ and a distribution $f(t,x)$ such that*

[16] See, for example, Øksendal (2010).

[17] The adjoint operator generalizes the concept of matrix transpose for infinite-dimensional operators.

1. *Given Z_t and $f(t,x)$, $V(t,x)$ is the solution of the HJB equation (18) and the optimal control is $\mu(t,x)$.*

2. *Given $\mu(t,x)$ and Z_t, $f(t,x)$ is the solution of the KF equation (20, 21).*

3. *Given $\mu(t,x)$ and $f(t,x)$, the aggregate variables Z_t satisfy the market clearing conditions (22).*

Example 1 *In the notation of this section, the example of section 2 can be expressed as*

$$\begin{aligned}
\Omega &= (\underline{a},\infty)\times(\underline{z},\bar{z})\subset\mathbb{R}^2,\\
x &= \begin{bmatrix} a\\ z\end{bmatrix},\\
b(x,\mu,Z) &= \begin{bmatrix} (1-\alpha)\,k^{\alpha}z+\left[\left(\alpha k^{\alpha-1}-\delta_K\right)-g+\eta\right]a-c\\ \theta(\hat{z}-z)\end{bmatrix},\\
\sigma(x) &= \begin{bmatrix} 0 & 0\\ 0 & \sigma z\end{bmatrix},\\
g(x,\mu) &= a,\\
u(x,\mu) &= \frac{c^{1-\gamma}}{1-\gamma}+\kappa\boldsymbol{\xi}_{\phi}(a),\\
Z &= k,\\
\mu(t,x) &= c(a,z),
\end{aligned}$$

where $\underline{a}<$ -ϕ and $\kappa\boldsymbol{\xi}_{\phi}(a)\in C^1(\underline{a},\infty)$ is a penalty function that equals zero if $a\geq-\phi$ and approximates κ if $a<-\phi$. We let $0<\kappa<\infty$ be large enough so that $\kappa\boldsymbol{\xi}_{\phi}$ approximates the borrowing constraint (3, 8).

3.2 Planner's problem

3.2.1 Statement

We now study the allocation of a planner who chooses a vector of control variables $\mu(t,x)$ to be applied to every agent $j\in[0,1]$ with state dynamics (15). The planner also chooses the vector of aggregate variables Z_t given the constraints (22). The planner chooses the controls and the aggregate variables in order to maximize the discounted aggregate utility

$$J(f(0,\cdot))\equiv\max_{Z(\cdot),\,\mu\in\mathcal{M}}\int_0^{\infty}\int_{\Omega}e^{-\rho t}\omega(t,x)u(x,\mu)f(t,x)dxdt, \tag{23}$$

subject to law of motion of the distribution (20, 21) and to the market clearing conditions (22). Notice the inclusion of the state-dependent Pareto weights $\omega(t,x)$. If $\omega(t,x) = 1$ then we have a purely utilitarian social welfare function. Notice also that the planner discounts future utility flows at rate ρ, not at rate $(\rho+\eta)$.

Notice that J is the optimal value *functional* as it maps from the space of initial densities $f(0,\cdot)$ into the real numbers. The planner's problem with heterogeneous agents is an extension of the classical optimal control problem to an infinite dimensional setting, in which the state is the whole distribution of individual states $f(t,x)$. The problem is deterministic, as so is the KF equation.

3.2.2 Solution

We provide necessary conditions to the problem (23).

Proposition 2 (Necessary conditions) *If a solution to problem (23) exists, the optimal value functional* $J(f(0,\cdot))$ *can be expressed as*

$$J(f(0,\cdot)) = \int_\Omega j(0,x) f(0,x)dx, \tag{24}$$

where $j(t,x)$ *is the marginal social value function, which represents the social value of an agent at time* t *with an state* x*. The social value function satisfies the HJB*

$$\rho j(t,x) = \frac{\partial j}{\partial t} + \max_{\mu\in\mathcal{M}} \left\{ \omega(t,x)u(x,\mu) + \sum_{k=1}^{p} \lambda_k(t)\left[g_k(x,\mu) - Z_k\right] + \mathcal{A}j \right\} + \eta j \delta_{x_0}, \tag{25}$$

where the Lagrange multipliers $\lambda_k(t)$, $k = 1,..,p$ *are given by*

$$\lambda_k(t) = -\int_\Omega j(t,x)\left\{\sum_{i=1}^{n}\left[\frac{\partial^2 b_i}{\partial Z_k \partial x_i} f(t,x) + \sum_{j=1}^{m} \frac{\partial^2 b_i}{\partial Z_k \partial \mu_j}\frac{\partial \mu_j}{\partial x_i} f(t,x) + \frac{\partial b_i}{\partial Z_k}\frac{\partial f}{\partial x_i}\right]\right\} dx. \tag{26}$$

The proof can be found in the Appendix C. This proposition is the central result of the paper. It provides a system of partial differential equations consisting of the HJB (25), the KF (20, 21) and the market clearing conditions (22) which link the dynamics of the social value function j, the policies μ, the aggregate variables Z and the distribution f.[18] The Lagrance multipliers in (26) reflect the 'shadow prices' of the market clearing condition. They price, in utility terms, the deviation of an agent from the value of the aggregate variable: $g_k(x,\mu) - Z_k$.

[18] We have implicitly assumed that $j \in C^2(\Omega)$. We do not provide any theoretical result if this is not the case although our numerical procedure is able to accommodate *viscosity solutions,* as described below.

Notice that the necessary conditions in the planner's problem with $\omega = 1$ are the same as those in the competitive equilibrium, with the exception of the term $\sum_{k=1}^{p} \lambda_k \left[g_k(x,\mu) - Z_k\right] + \eta j \delta_{x_0}$ in the planner's HJB equation (25). Therefore, it is trivial to check the following corollary.

Corollary 1 (Constrained optimality of the competitive equilibrium) *The competitive equilibrium equals the social optimum in the utilitarian sense ($\omega = 1$) if*

$$\sum_{k=1}^{p} \tilde{\lambda}_k(t) \left(g_k(x,\mu) - Z_k\right) + \eta V\left(t,x\right) \delta_{x_0} = 0, \tag{27}$$

where $\tilde{\lambda}_k(t)$ are given by

$$\tilde{\lambda}_k(t) = -\int_{\Omega} V(t,x) \left\{ \sum_{i=1}^{n} \left[\frac{\partial^2 b_i}{\partial Z_k \partial x_i} f(t,x) + \sum_{j=1}^{m} \frac{\partial^2 b_i}{\partial Z_k \partial \mu_j} \frac{\partial \mu_j}{\partial x_i} f + \frac{\partial b_i}{\partial Z_k} \frac{\partial f}{\partial x_i} \right] \right\} dx. \tag{28}$$

Notice that we have replaced $j(t,x)$ by $V(t,x)$ in (28), that is, the marginal social value equals the individual value. Therefore, it is enough to solve the competitive equilibrium and to compute (27) to check whether it is socially optimal.

Example 2 *In the example of section 2, the HJB equation in the constrained efficient solution is*

$$\begin{aligned} \left[(\rho+\eta) - (1-\gamma) g\right] j\left(a,z\right) &= \max_c \frac{c^{1-\gamma}}{1-\gamma} + \lambda\left(a-K\right) + \left[wz + (r-g+\eta)a - c\right] \frac{\partial j}{\partial a} \\ &\quad + \theta(\hat{z} - z) \frac{\partial j}{\partial z} + \frac{\sigma^2}{2} \frac{\partial^2 j}{\partial z^2} + \eta j \delta\left(a+\phi\right) \delta\left(z - \underline{z}\right), \end{aligned} \tag{29}$$

with the Lagrange multiplier

$$\lambda = \frac{\alpha\left(1-\alpha\right)}{k^{2-\alpha}} \int_{\Omega} j(a,z) \left[f(a,z) + a\frac{\partial f}{\partial a} - kz\frac{\partial f}{\partial a} \right] dadz. \tag{30}$$

Notice that equation (29) equals the individual HJB (9) plus $\lambda\left(a-K\right) + \eta j \delta\left(a+\phi\right)\delta\left(z-\underline{z}\right)$. The term $\lambda\left(a-K\right)$ reflects the correction in the social value j compared to the individual value V due to the difference between the agent's wealth a and the average K. If the Lagrange multiplier λ is positive, then the social value of wealthy agents is higher than their private value and hence there is capital underaccumulation. If λ is negative, then the private value is higher than the social one and there is capital overaccumulation.

4 Computational algorithm

In this section we provide a numerical algorithm to solve the planning problem. First we provide a general description and then we solve the example.

4.1 General algorithm

The general idea for the solution of infinite-horizon coupled HJB-KF systems is to first solve the steady-state and then iterate backward and forward in time, as described in Achdou et al. (2015). The approach here is similar, taking into account that the problem is more complicated as we also need to find the value of the time-varying Lagrange multipliers.

4.1.1 Steady-state

The steady-state can be computed using a relaxation algorithm. Given a constant $\theta \in (0,1)$, begin with an initial guess of the aggregate variables Z^0 and the Lagrange multipliers $\lambda^0 = 0$, set $n, m = 0$:

1. Given Z^n and λ^m, solve the HJB equation (25) in the stationary case ($\frac{\partial j}{\partial t} = 0$) and obtain the social value function j^n and the optimal policies μ^n.

2. Given the optimal policies μ^n solve the KF equation (20) in the stationary case ($\frac{\partial f}{\partial t} = 0$) and obtain the distribution f^n.[19]

3. Given the optimal policies μ^n and the distribution f^n, compute the aggregate variables $\tilde{Z}^{n+1}$ using the market clearing conditions (22). If $\tilde{Z}^{n+1} \neq Z^n$, set $Z^{n+1} = \theta \tilde{Z}^{n+1} + (1-\theta) Z^n$, update $n := n+1$ and return to step 1.

4. Compute $\tilde{\lambda}^{m+1}$ using the definition (26). If $\tilde{\lambda}^{n+1} \neq \lambda^n$, set $\lambda^{n+1} = \theta \tilde{\lambda}^{n+1} + (1-\theta) \lambda^n$, update $m := m+1$ and return to step 1.

If the algorithm converges, it should produce the steady-state value function j_∞, the optimal policies μ_∞, the aggregate variables Z_∞, the Lagrange multipliers λ_∞ and the distribution f_∞.[20]

[19] Achdou et al. (2015) show how the solution of the stationary KF equation just reduces to the inversion of the matrix characterizing the infinitesimal generator of the stochastic process. This is a sparse matrix, which can be inverted using standar numerical techniques.

[20] We do not provide any proof of convergence of our numerical algorithm.

4.1.2 Dynamics

In order to solve the dynamics, we guess a time T long enough for the system to converge to the steady-state and then we iterate backward and forthward, taking as given two objects: the steady-state value function j_∞ and the initial distribution f_0. We begin with an initial path for aggregate variables $Z_t^0 = Z_\infty$, $t \in [0,T]$ and the Lagrange multipliers $\lambda_t^0 = \lambda_\infty$. Set $n, m = 0$:

1. Given $Z^n(t)$ and $\lambda^m(t)$, solve the HJB equation (25) backward beginning at $j_T = j_\infty$ and obtain the social value function j_t^n and the optimal policies μ_t^n.

2. Given the optimal policies μ_t^n solve the KF equation (20) forward beginning at f_0 and obtain the distribution f_t^n.

3. Given the optimal policies μ_t^n and the distribution f_t^n, compute the aggregate variables $\tilde{Z}_t^{n+1}$ using the market clearing conditions (22). If $\exists t \in [0,T]$ such that $\tilde{Z}_t^{n+1} \neq Z_t^n$, set $Z_t^{n+1} = \theta \tilde{Z}_t^{n+1} + (1-\theta) Z_t^n$, update $n := n+1$ and return to step 1.

4. Compute $\tilde{\lambda}_t^{m+1}$ using the definition (26). If $\exists t \in [0,T]$ such that $\tilde{\lambda}_t^{n+1} \neq \lambda_t^n$, set $\lambda_t^{n+1} = \theta \tilde{\lambda}_t^{n+1} + (1-\theta)\lambda_t^n$, update $m := m+1$ and return to step 1.

If the algorithm converges, it should produce the complete dynamics of the system.

4.2 The optimal wealth distribution with incomplete markets and stochastic lifetimes

We solve numerically the stationary problem of section 2 using the steady-state algorithm described above. In order to solve the HJB and the KF equations, we employ a finite difference method described in Appendix D. As discussed in Achdou et al. (2015), the appropriate solution concept of HJB equation with state constraints is that of a "viscosity solution" (Crandall and Lions, 1983; Crandall, Ishii and Lions, 1992) and the proposed finite difference method converges to the unique viscosity solution of this problem (Barles and Souganidis, 1991). The idea of the finite difference method is to approximate the value function $V(a,z)$ and the distribution $f(a,z)$ on a finite grid with steps Δa and Δz and to compute derivatives as differences.

4.2.1 Calibration

Let the unit of time be one year, such that all rates are in annual terms. We assume a long-run growth rate of output, g, of 1 per cent roughly close to the long-run per capita GDP growth in the US economy. We also assume a death rate, η, of 2 percent, equivalent to an average lifetime of 50 years. The capital share parameter, α, is taken to be 0.36 and the depreciation rate of capital, δ_K, is 0.10. The borrowing constraint, ϕ, in our paper is set to 5. This value is chosen such that the right-tail exponent of the distribution ζ, is roughly 1.5, a similar value to that in the United States according to Achdou et al. (2015). The mean of the income process, $\hat{z}$, is set to 1. The calibration of the rest of parameters follows Aiyagari (1994) and Davila et al. (2012), taking into account that in contrast to both papers we have stochastic lifetimes and long-run growth. The intertemporal elasticity of substitution $\frac{1}{\gamma}$ is set to 0.5 so that the risk aversion is 2. The income process is calibrated to have an autocorrelation of 0.6 and a coefficient of variation, of 0.2, so that $\theta = 0.4$ and $\sigma = 0.2$. The subjective discount rate, ρ, is set to 0.01 such that the stationary discount rate $\hat{\rho} \equiv (\rho + \eta) - (1 - \gamma) g$ is set to 0.04, which is the equivalent concept to the discount rate in the two papers. Finally, in order to solve numerically the model, we employ a grid with 500 points in wealth, ranging from -5 to 200, and 20 points in income, from 0.5 to 1.5. We introduce an upper bound to the wealth distribution of 200, equivalent to around 50 times the average wealth, in order to capture most of the dynamics at the right tail of the distribution.[21]

4.2.2 Results

The first column in Table 1 displays the steady-state values for the main aggregate variables in the competitive equilibrium. Capital is two and a half times larger than output and the interest rate, r, is 4.45 per cent, a value larger than $r^* \equiv \rho + \gamma g = 3\%$ in Proposition 1. Notice also that the adjusted interest rate $(r - g + \eta) = 5.45\%$ is larger than the adjusted discount rate $\hat{\rho} = 4\%$, but it poses no problem as agents cannot accumulate wealth indefinitely due to their random deaths. Figure 1 displays the savings policy $s(a, z) \equiv (wz + (r - g + \eta) a - c(a, z))$ and the wealth-productivity distribution $f(a, z)$. Notice how, in the case of the distribution, there is a large proportion of the agents at the borrowing limit of $-\phi = -5$ for values of z below 1. By construction the model also replicates the wealth inequality observed in the United States, with an exponent, ζ, around 1.5.[22]

[21] Result are robust to changes in the size of the domain.

[22] The exponent is computed numerically for wealth levels between the first and the last decile of the wealth distribution. The wealth distribution is defined as

$$f(a) \equiv \int_{\underline{z}}^{\bar{z}} f(a, z) dz.$$

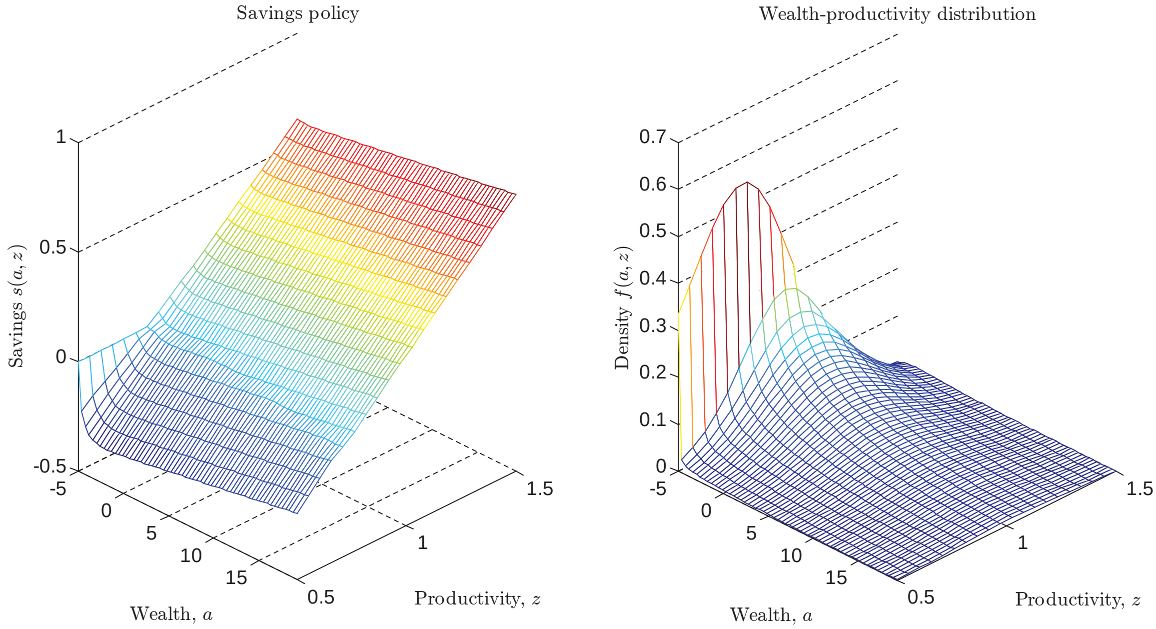

Figure 1: Savings policy and wealth-productivity distribution in the competitive equilibrium.

Table 1. Model results

	Competitive equilibrium	**Constrained optimum**
Aggregate capital, k	4.16	4.87
Output, y	1.67	1.77
Capital-output ratio, k/y	2.49	2.75
Interest rate (%), r	4.45	3.07
Pareto exponent , ζ	1.53	0.77

In order to solve the planner's problem we need to jointly solve the HJB equation (29), the KF equation (10) and the definition of the Lagrange multiplier (30), as described in Appendix D. For the calibration above, the value of λ results in 0.0044. The positiveness of λ implies that the social value of individuals with large wealth is higher than their private value. The second column in Table 1 displays the results in this case. The constrained-efficient allocation features larger levels of output, capital and of the capital-output ratio than the competitive equilibrium. This means that the market economy is undercapitalized compared to the social optimum, which necessarily implies larger levels of savings and lower interest rates in the optimum. The social optimum also

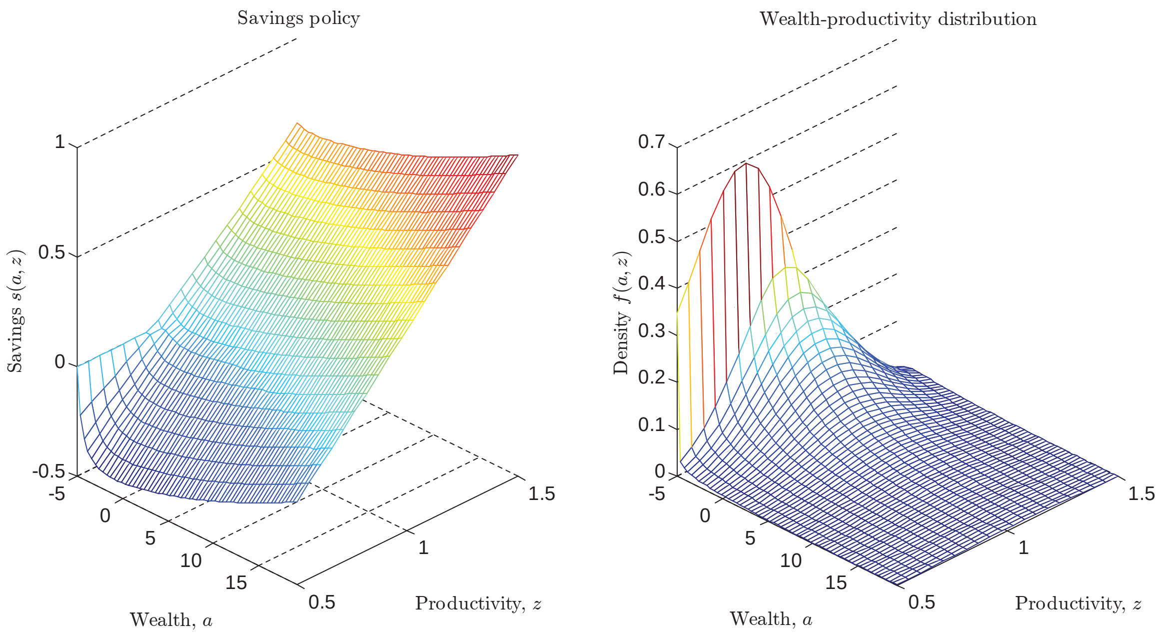

Figure 2: Savings policy and wealth-productivity distribution in the constrained-efficient allocation.

features a higher level of wealth inequality with a Pareto exponent of 0.77.[23] Notice that despite the fact that the slope is smaller than one, the wealth distribution has a well-defined mean as it is bounded by a maximum $a = 200$.[24]

Figure 2 illustrates how, despite the lower interest rates, savings $s(a, z)$ are higher in the optimal allocation than in the market economy, which explains the lower value of the tail exponent in that case. The bottom line of this analysis is that the reduction in interest rates in the social optimum is not enough to compensate for the increase in aggregate savings by wealthy households, thus resulting in an increase in wealth inequality. This is naturally linked to the fact that the competitive allocation displays less capital than the constrained-efficient allocation, and therefore a larger level of aggregate savings is welfare-improving.

Finally, we compare aggregate welfare in both allocations. Aggregate welfare in the stationary case can be defined as

$$U = \frac{1}{\hat{\rho}} \int_{-\phi}^{\infty} \int_{\underline{z}}^{\bar{z}} u\left(c\right) f\left(t, a, z\right) dadz. \tag{31}$$

[23]The planning economy also displays a power-law distribution, but with a smaller slope than the competitive equilibrium.

[24]This also explains why the interest rate may be larger than $g = 1\%$.

In order to compare both allocations, we express the ratio of welfare in consumption equivalent terms, that is, we express it as the proportion Θ of increase in the stationary consumption $c(a,z)$ of all agents such that the welfare in both allocations (constrained and competitive) is the same

$$\int_{-\phi}^{\infty}\int_{\underline{z}}^{\bar{z}} u^{const}\left(c\left(1+\Theta\right)\right) f^{const}\left(t,a,z\right) dadz = \int_{-\phi}^{\infty}\int_{\underline{z}}^{\bar{z}} u^{comp}\left(c\right) f^{comp}\left(t,a,z\right) dadz,$$

and hence

$$\Theta = \left(\frac{U^{const}}{U^{comp}}\right)^{\frac{1}{1-\gamma}} - 1 = 0.088, \tag{32}$$

that is, there is an average 8.8 percent gain in consumption-equivalent terms in the optimal allocation compared to the market one.

5 Conclusions

This paper analyzes the problem of a planner who controls a population of heterogeneous agents subject to idiosyncratic shocks in order to maximize an optimality criterion related to the distribution of states across agents. If the problem is analyzed in continuous time, the KF equation provides a deterministic law of motion of the entire distribution of state variables across agents. The problem can thus be analyzed as one of deterministic optimal control in which both the control and the state are distributions. If a solution exists, we show how it should satisfy a system of coupled PDEs composed by the planner's HJB and the KF equations. We also introduce a simple criterion to check whether a competitive equilibrium is constrained efficient.

We provide a numerical algorithm in order to find the solution to the planning problem. As an application, we employ this algorithm to analyze the welfare properties of an Aiyagari economy with stochastic lifetimes. In particular, we analyze the constrained social optimum in which a social planner maximizes the aggregate welfare subject to the same equilibrium budget constraints and competitive price setting as the individual agents. We show how the social optimum features more capital than the market economy. We also show how the level of wealth inequality is higher in the social optimum as the reduction in interest rates due to higher capital is not enough to compensate for the increase in aggregate savings by wealthy households.

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Appendix

A. Proof of Proposition 1

The proof mimics the reasoning in the proof of Proposition 6 in Achdou et al. (2015). First we show that individual consumption is asymptotically linear in a (as $a \to \infty$) and given by

$$c \sim \frac{\rho + \gamma\eta - (1-\gamma)\, r}{\gamma} a, \tag{33}$$

which has a positive solution as long as $\rho + \gamma\eta - (1-\gamma)\, r > 0$. This policy is the solution of the auxiliary problem without labor income, wz, and without borrowing constraint ($\phi = \infty$), characterized by the HJB equation

$$[(\rho + \eta) - (1-\gamma)\, g]\, V(a) = \max_c \frac{c^{1-\gamma}}{1-\gamma} + [(r - g + \eta)\, a - c(t, a, z)]\, V'(a)\,, \tag{34}$$

where $V(a) = \left(\frac{\rho+\gamma\eta-(1-\gamma)r}{\gamma}\right)^{-\gamma} \frac{a^{1-\gamma}}{1-\gamma} + \kappa$ and κ is a constant.

Second, given the HJB equation (25), for any $\xi > 0$, $V(a,z) = \xi^{1-\gamma} V_\xi(a/\xi, z)$, where $V_\xi(a,z)$ solves

$$\begin{aligned}[(\rho+\eta)-(1-\gamma)g]\,V_\xi(a,z) &= \max_c \frac{c^{1-\gamma}}{1-\gamma} + [wz/\xi + (r-g+\eta)\,a - c\,(t,a,z)]\,\frac{\partial V_\xi(a,z)}{\partial a} \\ &\quad +\theta(\hat{z}-z)\frac{\partial V_\xi(a,z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V_\xi(a,z)}{\partial z^2}, \quad a \geq -\phi/\xi.\end{aligned}$$

This can be easily verified as $V(a,z) = \xi^{1-\gamma} V_\xi(a/\xi, z)$, $\frac{\partial V(a,z)}{\partial a} = \xi^{-\gamma}\frac{\partial V_\xi(a/\xi,z)}{\partial(a/\xi)}$, $\frac{\partial V(a,z)}{\partial z} = \xi^{1-\gamma}\frac{\partial V_\xi(a/\xi,z)}{\partial z}$, $\frac{\partial^2 V_\xi(a,z)}{\partial z^2} = \xi^{1-\gamma}\frac{\partial^2 V_\xi(a,z)}{\partial z^2}$.

Third, notice how in the asymptotic limit $\lim_{\xi\to\infty} V_\xi(a,z) = V(a)$ and $\lim_{\xi\to\infty} c_\xi(a,z) = c(a)$, the latter given by equation (33). This is equivalent so state that for large a, we have $c(a,z) \approx \frac{\rho+\gamma\eta-(1-\gamma)r}{\gamma}a$.

The stationary KF equation for large a then results in

$$0 = -\frac{d}{da}\left[\left(r-g+\eta-\frac{\rho+\gamma\eta-(1-\gamma)\,r}{\gamma}\right)af\,(a)\right] - \eta f\,(a)\,.$$

We may guess and verify that $f\,(a) \sim a^{-(1+\zeta)}$ and then we get

$$\zeta\left[\frac{r-(\rho+\gamma g)}{\gamma}\right] = \eta.$$

B. Technical assumptions

These assumptions are adapted from Bensoussan, Chan and Yam (2015). Let $\mathcal{P}_2(\Omega)$ be the space of probability measures equipped with the 2^{nd} Wasserstein metric, $W_2(\cdot,\cdot)$ such that for any υ_1,$\upsilon_2 \in \mathcal{P}_2(\Omega)$,

$$W_2(\upsilon_1,\upsilon_2) = \inf_{\gamma\in\Gamma(\upsilon_1,\upsilon_2)}\left(\int_{\Omega\times\Omega}|x-y|^2\,d\gamma\,(x,y)\right)^{\frac{1}{2}},$$

where the infimum is taken over the family $\Gamma(\upsilon_1,\upsilon_2)$ of all joint measures with respective marginals υ_1 and υ_2. Let $m \in \mathcal{P}_2(\Omega)$ be such that $dm = fdx$ where $f \in L^2(\Omega)$ is a density function given by (20) and dx is the Lebesgue measure in Ω. We can combine the definition of the drift $b\,(x,\mu,Z)$ in (15) with the definition of the aggregate variable Z in (22) and the definition of measure m to represent the drift as $b\,(x,\mu,m)$. We write $M_2(m) = \left(\int_\Omega |x|^2\,dm\,(x)\right)^{\frac{1}{2}}$.

The assumptions are

1. *Lipschitz continuity.* There exists $K > 0$, such that

$$\begin{aligned} |b(x,\mu,m) - b(x',\mu',m')| &\leq K(|x-x'| + W_2(m,m') + |\mu-\mu'|), \\ |\sigma(x) - \sigma(x')| &\leq K(|x-x'|). \end{aligned}$$

2. *Linear growth.* There exists $K > 0$, such that

$$\begin{aligned} |b(x,\mu,m)| &\leq K(1 + |x| + M_2(m) + |\mu|), \\ |\sigma(x)| &\leq K(1+|x|). \end{aligned}$$

3. *Quadratic condition on the objective.* There exists $K > 0$, such that

$$\begin{aligned} \left| e^{-\rho t}\omega(t,x)u(x,\mu)\,m - e^{-\rho t'}\omega(t',x')u(x',\mu')\,m' \right| &\leq K \begin{pmatrix} 1 + |t| + |t'| + |x| + |x'| \\ +M_2(m) + M_2(m') + |\mu| + |\mu'| \end{pmatrix} \\ &\cdot \begin{pmatrix} |t-t'| + |x-x'| \\ +W_2(m,m') + |\mu-\mu'| \end{pmatrix}. \end{aligned}$$

C. Proof of Proposition 2.

The idea of the proof is to solve problem (23) using differentiation techniques in infinite-dimensional Hilbert spaces.

C.1. Mathematic preliminaries

First we need to introduce some mathematical concepts.[25] Let $L^2(\Phi)$ be the space of functions with a square that is Lebesgue-integrable over $\Phi \subset \mathbb{R}^n$. It is a Banach space with the norm

$$\|f\|_{L^2(\Phi)} = \sqrt{\int_\Phi |f(x)|^2\,dx},$$

that is, it is a complete normed vector space. In contrast to n-dimensional Banach spaces such as $\mathbb{R}^n$, $L^2(\Phi)$ is infinite-dimensional.

The space of Lebesgue-integrable functions $L^2(\Phi)$ with the inner product

$$\langle u, f\rangle_\Phi = \int_\Phi uf\,dx, \quad \forall u, f \in L^2(\Phi),$$

is a Hilbert space.

[25] All the contents here are adapted from texts such as Luenberger (1969), Brezis (2011), Gelfand and Fomin (1991) or Sagan (1992).

An *operator* T is a mapping from one vector space to another. For example, given the process X_t described in (15), its *infinitesimal generator* $\mathcal{A}$ is an operator in $L^2(\Phi)$ defined by (19). The *adjoint operator* T^* of a linear operator T in a Hilbert space is defined by the equation

$$\langle u, Tf \rangle_\Phi = \langle T^* u, f \rangle_\Phi .$$

Let $J(f) : L^2(\Phi) \to \mathbb{R}$ be a linear *functional* of f. A functional is *bounded* (or *continuous*) if there is a constant M such that

$$\|J\| \equiv \sup_{f \neq 0} \frac{|J(f)|}{\|f\|_{L^2(\Phi)}} \leq M.$$

The Riesz representation theorem allows us to express functionals as inner products.

Theorem 2 (Riesz representation theorem) *Let $J(f)$ be a linear continuous functional in $L^2(\Phi)$. Then there exists a unique function $j \in L^2(\Phi)$ such that*

$$J(f) = \langle j, f \rangle_\Phi = \int_\Phi j(x) f(x) dx.$$

Proof. See Brezis (2011, pp. 97-98). ■

There are two concepts of differentials in Hilbert spaces.

Definition 3 (Gateaux differential) *Let $J(f)$ be a linear continuous functional and let h be arbitrary in $L^2(\Phi)$. If the limit*

$$\delta J(f; h) = \lim_{\alpha \to 0} \frac{J(f + \alpha h) - J(f)}{\alpha} \tag{35}$$

exists, it is called the Gateaux differential *of J at f with increment h. If the limit (35) exists for each $h \in L^2(\Phi)$, the functional J is said to be* Gateaux differentiable *at f.*

If the limit exists, it can be expressed as $\delta J(f; h) = \frac{d}{d\alpha} J(f + \alpha h)|_{\alpha=0}$. A more restricted concept is that of the Fréchet differential.

Definition 4 (Fréchet differential) *Let h be arbitrary in $L^2(\Phi)$. If for fixed $f \in L^2(\Phi)$ there exists $\delta J(f; h)$ which is linear and continuous with respect to h such that*

$$\lim_{\|h\|_{L^2(\Phi)} \to 0} \frac{|J(f + h) - J(f) - \delta J(f; h)|}{\|h\|_{L^2(\Phi)}} = 0,$$

then J is said to be Fréchet differentiable *at f and $\delta J\left(f;h\right)$ is the* Fréchet differential *of J at f with increment h.*

The following proposition links both concepts.

Proposition 3 *If the Fréchet differential of J exists at f, then the Gateaux differential exists at f and they are equal.*

Proof. See Luenberger (1969, p. 173). ■

The familiar technique of maximizing a function of a single variable by ordinary calculus can be extended in infinite dimensional spaces to a similar technique based on more general differentials. We use the term *extremum* to refer to a maximum or a minimum over any set. A a function $f \in L^2\left(\Phi\right)$ is a maximum of $J(f)$ if for all functions h, $\left\|h-f\right\|_{L^2(\Phi)} < \varepsilon$ then $J(f) \geq J(h)$. The following theorem is the Fundamental Theorem of Calculus.

Theorem 3 *Let J have a Gateaux differential, a necessary condition for J to have an extremum at f is that $\delta J\left(f;h\right)=0$ for all $h \in L^2\left(\Phi\right)$.*

Proof. Luenberger (1969, p. 173), Gelfand and Fomin (1991, pp. 13-14) or Sagan (1992, p. 34). ■

Finally, we can extend this result to the case of constrained optimization.

Theorem 4 (Lagrange multipliers) *Let H be a mapping from $L^2\left(\Phi\right)$ into $\mathbb{R}^n$. If J has a continuous Fréchet differential, a necessary condition for J to have an extremum at f under the constraint $H(f)=0$ at the function f is that there exists a function $\lambda \in L^2\left(\Phi\right)$ such that the Lagrangian functional*

$$\mathcal{L}(f) = J(f) + \left\langle \lambda, H(f)\right\rangle_{\Phi} \tag{36}$$

is stationary in f, i.e., $\delta\mathcal{L}\left(f;h\right)=0$.

Proof. Luenberger (1969, p. 243). ■

C.2. Proof of the proposition

Let's define the extended domain $\Phi \equiv [0,\infty)\times\Omega$ The problem of the planner is to maximize

$$\int_{\Phi} e^{-\rho t}\omega\left(t,x\right)u(x,\mu)f\left(t,x\right)dtdx = \left\langle \left(e^{-\rho t}\omega u\right), f\right\rangle_{\Phi},$$

subject to the KF equation (20)

$$-\frac{\partial f}{\partial t} + \mathcal{A}^{*}f + \eta\delta_{x_0} = 0,\ \forall\left(t,x\right)\in\Phi$$

and the market clearing condition (22)

$$\int_{\Omega} \left[g_k(x,\mu) - Z_k(t)\right] f(t,x)dx, \quad k = 1,..,p, \ \forall t \in [0,\infty). \tag{37}$$

Assuming that the conditions above are satisfied, we can express the Lagrangian functional (36) for this problem as

$$\begin{aligned} \mathcal{L} &= \left\langle \left(e^{-\rho t}\omega u\right), f\right\rangle_\Phi + \left\langle e^{-\rho t} j, \left(-\frac{\partial f}{\partial t} + \mathcal{A}^* f + \eta \delta_{x_0}\right)\right\rangle_\Phi \\ &\quad + \sum_{k=1}^{p} \left\langle e^{-\rho t}\lambda_k, \left(g_k - Z_k\right) f\right\rangle_\Phi , \end{aligned} \tag{38}$$

where $j = j(t,x) \in L^2(\Phi)$ and $\lambda_k = \lambda_k(t) \in L^2(0,\infty)$, $k = 1,..,p$ are the Lagrange multipliers. A necessary condition for $[f, \mu_1, .., \mu_m, Z_1, .., Z_p]$ to be a maximum of (38) is that the Gateaux derivative with respect to each of these functions equals zero.

It will prove useful to modify the second term in the Lagrangian

$$\begin{aligned} \left\langle e^{-\rho t} j, \left(-\frac{\partial f}{\partial t} + \mathcal{A}^* f + \eta \delta_{x_0}\right)\right\rangle_\Phi &= \int_0^\infty \int_\Omega e^{-\rho t} j\left(t,x\right) \left(-\frac{\partial f}{\partial t} + \mathcal{A}^* f + \eta \delta_{x_0}\right) dt dx \\ &= \int_\Omega e^{-\rho t} j\left(t,x\right) f\left(t,x\right)\Big|_0^\infty dx + \int_0^\infty \int_\Omega e^{-\rho t} \left(\frac{\partial j}{\partial t} - \rho j\left(t,x\right)\right) f dt dx \\ &\quad + \eta \int_0^\infty e^{-\rho t} j\left(t,x_0\right) f\left(t,x_0\right) dt + \left\langle e^{-\rho t}\mathcal{A} j, f\right\rangle_\Phi \\ &= -\int_\Omega j\left(0,x\right) f\left(0,x\right) dx + \eta \int_0^\infty e^{-\rho t} j\left(t,x_0\right) f\left(t,x_0\right) dt \\ &\quad + \left\langle e^{-\rho t}\left(\frac{\partial j}{\partial t} - \rho j + \mathcal{A} j\right), f\right\rangle_\Phi , \end{aligned} \tag{39}$$

where we have integrated by parts with respect to time in the term $\frac{\partial f}{\partial t}$ and applied the fact that $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$. The term $\int_\Omega j\left(0,x\right) f\left(0,x\right) dx$ can be ignored as $f\left(0,x\right) = f_0(x)$, that is, the initial distribution is given.

The Gateaux derivative with respect to f is

$$\begin{aligned} &\lim_{\alpha\to 0}\frac{d}{d\alpha}\left\langle\left(e^{-\rho t}\omega u\right),f+\alpha h\right\rangle_{\Phi}+\frac{d}{d\alpha}\left\langle e^{-\rho t}\left(\frac{\partial j}{\partial t}-\rho j+\mathcal{A}j\right),f+\alpha h\right\rangle_{\Phi} \\ &+\frac{d}{d\alpha}\sum_{k=1}^{p}\left\langle e^{-\rho t}\lambda_k,\left(g_k-Z_k\right)\left(f+\alpha h\right)\right\rangle_{\Phi}+\eta\frac{d}{d\alpha}\int_0^{\infty}e^{-\rho t}j\left(t,x_0\right)\left[f\left(t,x_0\right)+\alpha h\left(t,x_0\right)\right]dt \\ =\ &\left\langle\left(e^{-\rho t}\omega u\right),h\right\rangle_{\Phi}+\left\langle e^{-\rho t}\left(\frac{\partial j}{\partial t}-\rho j+\mathcal{A}j\right),h\right\rangle_{\Phi}+\sum_{k=1}^{p}\left\langle e^{-\rho t}\lambda_k,\left(g_k-Z_k\right)h\right\rangle_{\Phi} \\ &+\eta\int_0^{\infty}e^{-\rho t}j\left(t,x_0\right)h\left(t,x_0\right)dt, \end{aligned}$$

and it equals zero according to Theorem 13. As this is satisified for any $h(t,x)$ we obtain that

$$\frac{\partial j}{\partial t}+\omega u+\sum_{k=1}^{p}\lambda_k\left(g_k-Z_k\right)+\mathcal{A}j+\eta j\delta_{x_0}=\rho j,\ \forall\left(t,x\right)\in\Phi, \tag{40}$$

which is the HJB equation of the planner (25).

The Gateaux derivative with respect to the policy μ_j is

$$\begin{aligned} &\lim_{\alpha\to 0}\frac{d}{d\alpha}\left\langle\left(e^{-\rho t}\omega u\left(x,\mu_j+\alpha h\right)\right),f\right\rangle_{\Phi}+\frac{d}{d\alpha}\left\langle e^{-\rho t}\left(\frac{\partial j}{\partial t}-\rho j+\mathcal{A}_{\left(\mu_j+\alpha h\right)}j\right),f\right\rangle_{\Phi} \\ &+\frac{d}{d\alpha}\sum_{k=1}^{p}\left\langle e^{-\rho t}\lambda_k,\left[g_k\left(x,\mu_j+\alpha h\right)-Z_k\right]f\right\rangle_{\Phi}, \end{aligned} \tag{41}$$

where $\mathcal{A}_{\left(\mu_j+\alpha h\right)}j$ is

$$\mathcal{A}_{\left(\mu_j+\alpha h\right)}j=\sum_{i=1}^{n}b_i\left(x,\mu_1,..,\mu_j+\alpha h,..,\mu_m,Z\right)\frac{\partial j}{\partial x_i}+\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\left(\sigma(x)\sigma(x)^{\top}\right)_{i,k}}{2}\frac{\partial^2 j}{\partial x_i\partial x_k}-\eta j\left(t,x\right).$$

It is trivial to check that (41) is equal to the derivative with respect to μ_j in (40):

$$\mu=\arg\max_{\tilde{\mu}}\left\{\omega u\left(x,\tilde{\mu}\right)+\sum_{k=1}^{p}\lambda_k g_k\left(x,\tilde{\mu}\right)+\mathcal{A}_{(\tilde{\mu})}j\right\}. \tag{42}$$

Finally, the Gateaux derivative with respect to the aggregate variable Z_k is

$$\lim_{\alpha\to 0}\frac{d}{d\alpha}\left\langle e^{-\rho t}j,\left(-\frac{\partial f}{\partial t}+\mathcal{A}^{*}_{(Z_k+\alpha h)}f+\eta\delta_{x_0}\right)\right\rangle_{\Phi}+\frac{d}{d\alpha}\sum_{k=1}^{p}\left\langle e^{-\rho t}\lambda_k,\left[g_k-\left(Z_k+\alpha h\right)\right]f\right\rangle_{\Phi},$$

for any $h(t) \in L^2(0,\infty)$. Again, $\mathcal{A}^*_{(Z_k+\alpha h)}$ is

$$\begin{aligned}\mathcal{A}^*_{(Z_k+\alpha h)}f &= -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[b_i\left(x,\mu,Z_1,..,Z_k+\alpha h,..,Z_p\right)f\left(t,x\right)\right] \\ &\quad +\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}\frac{\partial^2}{\partial x_i \partial x_k}\left[\left(\sigma(x)\sigma(x)^\top\right)_{i,k}f\left(t,x\right)\right]-\eta f\left(t,x\right).\end{aligned}$$

This can be expressed as

$$\lim_{\alpha\to 0}\int_0^\infty\int_\Omega e^{-\rho t}j(t,x)\frac{d}{d\alpha}\left\{-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[b_i\left(x,\mu,Z_1,..,Z_k+\alpha h,..,Z_p\right)f\left(t,x\right)\right]-\sum_{k=1}^{p}\lambda_k\left(Z_k+\alpha h\right)f\right\},$$

and hence

$$\int_0^\infty e^{-\rho t}h(t)\left\{\int_\Omega j(t,x)\left(\sum_{i=1}^{n}\left[\frac{\partial^2 b_i}{\partial Z_k\partial x_i}f(t,x)+\sum_{j=1}^{m}\frac{\partial^2 b_i}{\partial Z_k\partial \mu_j}\frac{\partial \mu_j}{\partial x_i}f+\frac{\partial b_i}{\partial Z_k}\frac{\partial f}{\partial x_i}\right]\right)dx+\lambda_k(t)\right\}dt=0.$$

As this is satisfied for any $h(t)$, we obtain that

$$\lambda_k(t)=-\int_\Omega j(t,x)\left\{\sum_{i=1}^{n}\left[\frac{\partial^2 b_i}{\partial Z_k\partial x_i}f(t,x)+\sum_{j=1}^{m}\frac{\partial^2 b_i}{\partial Z_k\partial \mu_j}\frac{\partial \mu_j}{\partial x_i}f(t,x)+\frac{\partial b_i}{\partial Z_k}\frac{\partial f}{\partial x_i}\right]\right\}dx. \tag{43}$$

C.3. The social value

Here we show that the Lagrange multiplier j is indeed the social value function, that is, that the functional $J\left(f\left(0,\cdot\right)\right)$ can be represented as

$$J\left(f\left(0,\cdot\right)\right)=\int_\Omega j(0,x)f(0,x)dx.$$

First we check that, as long as the weighted instantaneous utility is bounded in Φ :

$$\left|\omega\left(t,x\right)u\left(x,\mu\left(t,x\right)\right)\right|\leq M_0,\quad \forall\left(t,x\right)\in\Phi,$$

the functional J is continuous:

$$\begin{aligned}\sup_{f\neq 0}\frac{\left|J\left(f\right)\right|}{\|f\|_{L^2(\Phi)}} &< \sup_{f\neq 0}\frac{1}{\|f\|_{L^2(\Phi)}}\int_0^\infty\int_\Omega e^{-\rho t}\left|\omega\left(t,x\right)u\left(x,\mu\left(t,x\right)\right)\right|f(t,x)dtdx \\ &\leq \sup_{f\neq 0}\frac{M_0}{\|f\|_{L^2(\Phi)}}\int_0^\infty e^{-\rho t}ds\leq\frac{M_0\sqrt{m\left(\Phi\right)}}{\rho},\end{aligned}$$

where the last inequality follows from the Cauchy-Schwartz inequality:

$$1 = \int_0^{b^*} f db = \langle 1, f \rangle_\Phi \leq \|f\|_{L^2(\Phi)} \|1\|_{L^2(\Phi)} = \|f\|_{L^2(\Phi)} \sqrt{\int_\Omega 1^2 dx} = \|f\|_{L^2(\Phi)} \sqrt{m(\Phi)},$$

where $m(\Phi)$ is the measure of the domain Φ.

We may then apply the Riesz representation theorem:

$$J(f(t,\cdot)) = \int_\Omega \tilde{j}(t,x) f(t,x) dx,$$

where $\tilde{j}(t,x) \in L^2(\Phi)$ is unique.

Now we proceed to show that $\tilde{j}(t,x) = j(t,x)$. For any initial condition $f(t_0,\cdot)$, we have an optimal control path μ and aggregate variables Z. We may apply *Bellman's Principle of Optimality*:

$$J(f(t_0,\cdot)) = \int_{t_0}^{t} e^{-\rho(s-t_0)} \int_\Omega \omega(s,x) u(x,\mu) f(s,b) dx ds + e^{-\rho(t-t_0)} J[f(t,\cdot)]. \tag{44}$$

Let $\Xi(\mu, f)$ be defined as

$$\Xi(\mu, f) \equiv \int_\Omega \omega u f dx.$$

Taking derivatives with respect to time in equation (44) and the limit as $t \to t_0$:

$$\begin{aligned} 0 &= \Xi(\mu,f) - \rho J(f(t,\cdot)) + \frac{\partial}{\partial t} J(f(t,\cdot)) = \Xi(\mu,f) - \rho J(f(t,\cdot)) + \frac{\partial}{\partial t} \int_\Omega \tilde{j}(t,x) f(t,x) dx \quad (45) \\ &= \Xi(\mu,f) - \rho J(f(t,\cdot)) + \int_\Omega \tilde{j}(t,x) \frac{\partial f}{\partial t} dx + \int_\Omega \frac{\partial \tilde{j}}{\partial t} f(t,x) dx \\ &= \Xi(\mu,f) - \int_\Omega \rho \tilde{j}(t,x) f(t,x) dx + \int_\Omega \tilde{j}(t,x) (\mathcal{A}^* f + \eta \delta_{x_0}) dx + \int_\Omega \frac{\partial \tilde{j}}{\partial t} f(t,x) dx. \end{aligned}$$

For the optimal controls and aggregate variables Z the market clearing conditions (37) are also satisfied, so we can add them to the expression above, to obtain:

$$0 = \Xi(\mu,f) + \int_\Omega \left\{ \sum_{k=1}^{p} \lambda_k (g_k - Z_k) f - \rho \tilde{j} f + \tilde{j} \mathcal{A}^* f + \tilde{j} \eta \delta_{x_0} f + \frac{\partial \tilde{j}}{\partial t} f \right\} dx. \tag{46}$$

If we operate on the term $\int_\Omega \tilde{j}(t,x)(\mathcal{A}^* f + \eta \delta_{x_0} f) dx = \langle \tilde{j}, \mathcal{A}^* f + \eta \delta_{x_0} f \rangle_\Phi = \langle \mathcal{A}\tilde{j} + \eta \delta_{x_0} \tilde{j}, f \rangle_\Phi - \int_\Omega \tilde{j}(0,x) f(0,x) dx$ as in (39) and take the Gateaux derivative with respect to $f(t,\cdot)$ at both sides of the expression (46), we obtain

$$\frac{\partial \tilde{j}}{\partial t} + \omega u + \sum_{k=1}^{p} \lambda_k (g_k - Z_k) + \mathcal{A}\tilde{j} + \eta \tilde{j} \delta_{x_0} = \rho \tilde{j}, \ \forall (t,x) \in \Phi,$$

which is the same expression as (25). In addition, if we take the Gateaux derivative of (46) with respect to $\mu(t,\cdot)$ and the standard derivative with respect to $Z(t)$ we obtain again the first order conditions (42) and the value of λ_k (43). Hence $\tilde{j} = j$.

D. Description of the numerical algorithm

Step 1: Solution to the Hamilton-Jacobi-Bellman equation

The HJB equation is solved by a finite difference scheme following Achdou et al. (2015). It approximates the value function $V(a,z)$ on a finite grid with steps Δa and Δz : $a \in \{a_1,...,a_I\}$, $z \in \{z_1,...,z_J\}$.[26] We use the notation $V_{i,j} \equiv V(a_i,z_j)$, $i = 1,...,I$; $j = 1,...,J$. The derivative of V with respect to a can be approximated with either a forward or a backward approximation:

$$\frac{\partial V(a_i,z_j)}{\partial a} \approx \partial_{a,F}V_{i,j} \equiv \frac{V_{i+1,j}-V_{i,j}}{\Delta a}, \tag{47}$$

$$\frac{\partial V(a_i,z_j)}{\partial a} \approx \partial_{a,B}V_{i,j} \equiv \frac{V_{i,j}-V_{i-1,j}}{\Delta a}, \tag{48}$$

where the decision between one approximation or the other depends on the sign of the savings function $s_{i,j} = wz_j + ra_i - c_{i,j}$ through an "upwind scheme" described below. The derivative of V with respect to z is approximated using a forward approximation

$$\frac{\partial V(a_i,z_j)}{\partial z} \approx \partial_z V_{i,j} \equiv \frac{V_{i,j+1}-V_{i,j}}{\Delta z}, \tag{49}$$

$$\frac{\partial^2 V(a_i,z_j)}{\partial z^2} \approx \partial_{zz} V_{i,j} \equiv \frac{V_{i,j+1}+V_{i,j-1}-2V_{i,j}}{(\Delta z)^2}. \tag{50}$$

The HJB equation (25) is

$$\hat{\rho}V = u(c) + (wz + (r-g)\,a - c)\frac{\partial V}{\partial a} + \theta(\hat{z}-z)\frac{\partial V}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial z^2},$$

where

$$c = (u')^{-1}\left(\frac{\partial V}{\partial a}\right),$$

$\hat{\rho} = [(\rho+\eta) - (1-\gamma)\,g]$ and $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ in the competitive equilibrium or $u(c) = \frac{c^{1-\gamma}}{1-\gamma} + \lambda\,(a-K)$

[26] Notice that subindexes i and j have a different meaning here than in the main text.

in the planning economy. The HJB equation is approximated by an upwind scheme

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\hat{\rho}V_{i,j}^{n+1} = u(c_{i,j}^{n})+\partial_{a,F}V_{i,j}^{n+1}s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}+\partial_{a,B}V_{i,j}^{n+1}s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}$$
$$+\theta(\hat{z}-z_j)\partial_z V_{i,j}^{n+1}+\frac{\sigma_z^2 z_j}{2}\partial_{zz}V_{i,j}^{n+1},$$

where

$$s_{i,j,F}^{n} = wz_j+(r-g)\,a_i-(u')^{-1}\left(\partial_{a,F}V_{i,j}^{n}\right),$$
$$s_{i,j,B}^{n} = wz_j+(r-g)\,a_i-(u')^{-1}\left(\partial_{a,B}V_{i,j}^{n}\right).$$

Moving all variables with $n+1$ superscripts to the left hand side and those with n superscripts to the right hand side:

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\hat{\rho}V_{i,j}^{n+1}=u(c_{i,j}^{n})+V_{i-1,j}^{n+1}\varrho_{i,j}+V_{i,j}^{n+1}\beta_{i,j}+V_{i+1,j}^{n+1}\gamma_{i,j}+V_{i,j-1}^{n+1}\chi_j+V_{i,j+1}^{n+1}\varsigma_j, \tag{51}$$

where

$$c_{i,j}^{n} = (u')^{-1}\left(\partial_{a,F}V_{i,j}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}+\partial_{a,B}V_{i,j}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}+u'(wz_j+ra_i)\mathbf{1}_{s_{i,j,F}^{n}<0,s_{i,j,B}^{n}>0}\right), \tag{52}$$
$$\varrho_{i,j} = -\frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a},$$
$$\beta_{i,j} = -\frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a}+\frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a}-\frac{\theta(\hat{z}-z_j)}{\Delta z}-\frac{\sigma^2}{(\Delta z)^2},$$
$$\gamma_{i,j} = \frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a},$$
$$\chi = \frac{\sigma^2}{2\,(\Delta z)^2},$$
$$\varsigma_j = \frac{\sigma^2}{2\,(\Delta z)^2}+\frac{\theta(\hat{z}-z_j)}{\Delta z}.$$

The state constraint (3) $a \geq -\phi$ is enforced by setting $s_{i,j,B}^{n}=0$. Similarly, $s_{I,j,F}^{n}=0$. Therefore, the values $V_{0,j}^{n+1}$ and $V_{I+1,j}^{n+1}$ are never used. The boundary conditions with respect to z are

$$\frac{\partial V(a,\underline{z})}{\partial z}=\frac{\partial V(a,\bar{z})}{\partial z}=0,$$

as the process is reflected. At the boundaries in the j dimension, equation (51) becomes

$$\begin{aligned}
\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\hat{\rho}V_{i,j}^{n+1} &= u(c_{i,1}^{n})+V_{i-1,j}^{n+1}\varrho_{i,1}+V_{i,1}^{n+1}\left(\beta_{i,1}+\chi\right)+V_{i+1,1}^{n+1}\gamma_{i,1}+V_{i,2}^{n+1}\varsigma_{1},\\
\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta}+\hat{\rho}V_{i,j}^{n+1} &= u(c_{i,J}^{n})+V_{i-1,J}^{n+1}\varrho_{i,J}+V_{i,J}^{n+1}\left(\beta_{i,J}+\varsigma_{J}\right)+V_{i+1,J}^{n+1}\gamma_{i,J}+V_{i,J-1}^{n+1}\chi_{J}.
\end{aligned}$$

Equation (51) is a system of $I\times J$ linear equations which can be written in matrix notation as:

$$\frac{\mathbf{V}^{n+1}-\mathbf{V}^{n}}{\Delta}+\hat{\rho}\mathbf{V}^{n+1}=\mathbf{u}^{n}+\mathbf{A}^{n}\mathbf{V}^{n+1},$$

where the matrix $\mathbf{A}^n$ and the vectors $\mathbf{V}^{n+1}$ and $\mathbf{u}^n$ are defined by:

$$\mathbf{A}^{n}=\begin{bmatrix}
\beta_{1,1}+\chi & \gamma_{1,1} & 0 & \cdots & 0 & \varsigma_{1} & 0 & 0 & \cdots & 0\\
\varrho_{2,1} & \beta_{2,1}+\chi & \gamma_{2,1} & 0 & \cdots & 0 & \varsigma_{1} & 0 & \cdots & 0\\
0 & \varrho_{3,1} & \beta_{3,1}+\chi & \gamma_{3,1} & 0 & \cdots & 0 & \varsigma_{1} & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & 0 & \varrho_{I,1} & \beta_{I,1}+\chi & \gamma_{I,1} & 0 & 0 & \cdots & 0\\
\chi & 0 & \cdots & 0 & \varrho_{1,2} & \beta_{1,2} & \gamma_{1,2} & 0 & \cdots & 0\\
0 & \chi & \cdots & 0 & 0 & \varrho_{2,2} & \beta_{2,2} & \gamma_{2,2} & \cdots & 0\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \varrho_{I-1,J} & \beta_{I-1,J}+\varsigma_{J} & \gamma_{I-1,J}\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \varrho_{I,J} & \beta_{I,I}+\varsigma_{J}
\end{bmatrix},$$

$$\mathbf{V}^{n+1}=\begin{bmatrix}V_{1,1}^{n+1}\\ V_{2,1}^{n+1}\\ \vdots\\ V_{1,2}^{n+1}\\ V_{2,2}^{n+1}\\ \vdots\\ V_{I-1,J}^{n+1}\\ V_{I,J}^{n+1}\end{bmatrix},\qquad \mathbf{u}^{n}=\begin{bmatrix}u(c_{1,1}^{n})\\ u(c_{2,1}^{n})\\ \vdots\\ u(c_{1,2}^{n})\\ u(c_{2,2}^{n})\\ \vdots\\ u(c_{I-1,J}^{n})\\ u(c_{I,J}^{n})\end{bmatrix}.$$

The system can in turn be written as

$$\mathbf{B}^n\mathbf{V}^{n+1} = \mathbf{d}^n, \tag{53}$$

where $\mathbf{B}^n = \left(\frac{1}{\Delta} + \hat{\rho}\right)\mathbf{I} - \mathbf{A}^n$ and $\mathbf{d}^n = \mathbf{u}^n + \frac{\mathbf{V}^n}{\Delta}$. $\mathbf{I}$ is the identity matrix.[27]

The algorithm to solve the HJB equation runs as follows. Begin with an initial guess $V_{i,j}^0 = u(ra_i + wz_j)/\rho$, set $n = 0$. Then:

1. Compute $\partial_{a,F}V_{i,j}^n$, $\partial_{a,B}V_{i,j}^n$, $\partial_z V_{i,j}^n$ and $\partial_{zz}V_{i,j}^n$ using (47)-(50).
2. Compute $c_{i,j}^n$ using (52).
3. Find $V_{i,j}^{n+1}$ solving the linear system of equations (53).
4. If $V_{i,j}^{n+1}$ is close enough to $V_{i,j}^n$, stop. If not set $n := n+1$ and go to step 1.

Step 2: Solution to the Kolmogorov Forward equation

The KF equation is also solved using an upwind finite difference scheme. The equation (6) in this case is

$$0 = -\frac{\partial}{\partial a}\left[(wz + (r-g)\,a - c)\,f\right] - \frac{\partial}{\partial z}\left[\theta(\hat{z} - z)f\right] + \frac{1}{2}\frac{\partial^2}{\partial z^2}\sigma^2 f - \eta f + \eta\delta_0, \tag{54}$$

$$\int f(a,z)\,da\,dz = 1. \tag{55}$$

This case is simpler than the previous one, as the problem is linear in f, so no iterative procedure is needed. We use the notation $f_{i,j} \equiv f(a_i, z_j)$. The system can be expressed as

$$\begin{aligned} 0 = & -\frac{f_{i,j}s_{i,j,F}^n \mathbf{1}_{s_{i,j,F}^n>0} - f_{i-1,j}s_{i-1,j,F}^n \mathbf{1}_{s_{i-1,j,F}^n>0}}{\Delta a} - \frac{f_{i+1,j}s_{i+1,j,B}^n \mathbf{1}_{s_{i+1,j,B}^n<0} - f_{i,j}s_{i,j,B}^n \mathbf{1}_{s_{i,j,B}^n<0}}{\Delta a} \\ & -\frac{f_{i,j}\eta(z_j) - f_{i,j-1}\eta(z_{j-1})}{\Delta z} + \frac{f_{i,j+1}\sigma_z^2(z_{j+1}) + f_{i,j-1}\sigma_z^2(z_{j-1}) - 2f_{i,j}\sigma_z^2(z_j)}{2\left(\Delta z\right)^2} - \eta f_{i,j} + \eta\delta_0, \end{aligned}$$

or equivalently

$$f_{i-1,j}\gamma_{i-1,j} + f_{i+1,j}\varrho_{i+1,j} + f_{i,j}\beta_{i,j} + f_{i,j+1}\chi + f_{i,j-1}\varsigma_j - \eta f_{i,j} = -\eta\delta_0, \tag{56}$$

then (56) is also a system of $I \times J$ linear equations which can be written in matrix notation as:

$$\left(\mathbf{A}^{\mathbf{T}} - \eta\mathbf{I}\right)\mathbf{f} = \mathbf{h}, \tag{57}$$

[27] In optimal planning solution the first element of the matrix $\mathbf{B}^n$ is $\left(\frac{1}{\Delta} + \hat{\rho}\right) - \left(\beta_{1,1} + \chi_1\right) - \eta$ due to the term $\eta j\delta\,(a+\phi)\,\delta\,(z - \underline{z})$ in equation (29).

where $\mathbf{A^T}$ is the transpose of $\mathbf{A} = \lim_{n\to\infty} \mathbf{A}^n$ and $\mathbf{h}$ is a vector of zeros with a -1 at the first position. We solve the system (57) and obtain a solution $\hat{\mathbf{f}}$. Then we renormalize as

$$f_{i,j} = \frac{\hat{f}_{i,j}}{\sum_{i=1}^{I}\sum_{j=1}^{J} \hat{f}_{i,j}\Delta a \Delta z}.$$

Step 3: Finding the equilibrium aggregate capital

In order to find the aggregate capital k, we employ a relaxation method. Given $\theta \in (0,1)$, begin with an initial guess of the aggregate capital k^0, set $n = 0$. Then:

1. Compute r^n and w^n using (11).
2. Given r^n and w^n, solve the planner's HJB equation as in Step 1 to obtain an estimate of the value function V^n and of the consumption c^n.
3. Given c^n, solve the KF equation as in Step 2 and compute the aggregate distribution f^n.
4. Compute the aggregate capital stock $S = \sum_{i=1}^{I}\sum_{j=1}^{J} a_i f_{i,j}\Delta a \Delta z$.
5. Compute $k^{n+1} = \theta S^n + (1-\theta)\, k^n$. If k^{n+1} is close enough to k^n, stop. If not set $n := n+1$ and go to step 1.

Step 4: Finding the Lagrange multiplier (only in the optimal allocation)

In order to find the value of the optimal Lagrange multiplier in the planning problem (38), we begin with an initial guess $\lambda^0 = 0$, then we need to find the value of λ that satisfies

$$\lambda = \frac{\alpha(1-\alpha)}{k^{2-\alpha}} \sum_{i=1}^{I}\sum_{j=1}^{J} \left[f_{i,j} + a_i \frac{f_{i+1,j} - f_{i,j}}{\Delta a} - k z_j \frac{f_{i+1,j} - f_{i,j}}{\Delta a} \right] V_{i,j}\Delta a \Delta z,$$

where $f_{i,j}$, k, and $V_{i,j}$ are obtained by solving the planner's problem with this value of λ and utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma} + \lambda(a - K)$.[28] We employ Matlab's routine `fzero` to find this value of λ.

[28] V is the value function of the planner in this case, that we denote as $j(a,z)$ in the main text.

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