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Abstract

We derive the conditions for the optimal portfolio choice within a constant relative risk aversion type of utility function considering alternative probability distributions that are able to capture the asymmetric and leptokurtic features of asset returns. We illustrate the role —beyond risk aversion— played by higher-order moments in the optimal decision to form a portfolio of risky assets. In particular, we show that higher-order risk attitudes such as prudence and temperance associated with the third and fourth moments of the distribution define different optimal portfolios than those constrained under risk aversion.

Keywords: decision analysis, risk management, higher-order moments and preferences, portfolio choice, weighted generalized beta two distribution.

JEL classification: C16, D81, G11.

Resumen

Derivamos las condiciones para la elección óptima de cartera bajo una utilidad con aversión al riesgo relativo constante y distribuciones de probabilidad alternativas que son capaces de capturar las caraterísticas de asimetría y curtosis de los rendimientos de los activos financieros. Ilustramos el papel —más allá de la aversión al riesgo— que desempeñan los momentos de orden superior en la decisión de formar una cartera de activos. En particular, demostramos que las actitudes de orden superior, tales como la prudencia y la temperancia, asociadas a los momentos tercero y cuarto de la distribución, definen diferentes carteras óptimas a las restringidas bajo aversión al riesgo.

Palabras clave: análisis de decisión, distribución Beta 2 generalizada ponderada, elección de cartera, gestión del riesgo, momentos y preferencias de orden superior.

Códigos JEL: C16, D81, G11.

1 Introduction

A common assumption in models of asset allocation is that agents invest one hundred percent of their liquid wealth in risky assets, see e.g. Brunnermeier and Nagel (2008). This is based on calibrated life cycle models such as those in Haliassos and Bertaut (1995), Cocco et al. (2005) and Yao and Zhang (2005). Feldstein (1969) in a classic paper analyzed the optimal allocation of wealth between a risk free and risky asset. He demonstrated that the investor's decision to *plunge,* i.e., optimally allocating all wealth in the risky asset, could occur for reasonable values of the expected and variance of the portfolio return assuming log-utility and a log-normal distribution of asset returns. This analysis was a counter example to the result of Tobin (1958) who demonstrated the sufficiency of risk aversion, under quadratic utility or two-parameter distributions, to ensure diversification. These results were extensively discussed and treated in the literature on portfolio choice, but a central aspect still remains not satisfactorily addressed. This paper contributes to this literature by providing the conditions for optimal portfolio allocation assuming a wide range of distribution functions. We formally show how the optimal condition for a CRRA type of utility function depends on higher-order moments and higher-order risk preferences which adds an explaination to Feldstein's counter-intuitive results.

Feldstein assumed a two-parameter distribution, in particular, the log-normal. This distribution is useful to fit the mean and variance of asset returns but does not capture other relevant features for the portfolio decision such as significant asymmetry and fat tails. Feldstein's derived relationship between the first two moments of log-normal returns implies a third moment that it is actually not representative of the one typically observed. Feldstein's analysis revealed that with an expected annual yield of five percent, 0.05, and standard deviation of 0.23, the optimal portfolio composition would involve allocating all wealth in the risky asset. However, individuals' observed decisions on portfolio allocation are not consistent with this theoretical optimal allocation. One explanation to this conundrum is that, in fact, those typical values of return yield and standard deviation (0.05 and 0.23, respectively) imply, under log-normality, a positive skewness of 0.66902. This value contrasts with the empirically observed skewness of asset returns that is typically negative. This example points to the limitations of using two-parameter distributions in the standard portfolio choice model. In this paper we revisit the portfolio allocation model with more general distributions of asset returns. In particular, we employ the five-parameter weighted generalized beta distribution of the second kind (WGB2), recently set out by Ye et al. (2012), as well as some of the distributions nested within it, including the log-normal, which are typically employed in the financial economics literature. Our approach addresses the research question of how the conditions for maximizing expected utility in the portfolio choice model change when consideration is given to distributions more flexible than the log-normal. Consequently our analysis will allow explicit consideration of the effect that higher-order moments, and higher-order preferences, have on the optimal allocation of wealth in the standard portfolio choice model. We show that if we allow for variation in higher order statistical moments the optimal condition on liquidity is rather different from Feldstein's results obtained under log-normality.[1]

The structure of the paper is as follows. The next section revises the related literature. In section 3 we derive the theoretical optimal conditions for portfolio choice under alternative probability distribution functions. In Section 4 we analyze the effects of higher-order moments and preferences on the optimal portfolio composition. Section 5 provides an example of our analysis for the S&P500. The final section is a brief conclusion.

2 Related literature

It is of course known that higher-order moments affect investors' decisions. However different theoretical arguments are employed in the literature to illustrate this feature. Firstly, Menezes et al. (1980) developed the concept of downside risk (DR hereafter) within a choice-theoretic framework and provided a relationship between the third derivative of the utility function and individuals' risk preferences. Their definition allows for the distinction between increasing DR and riskiness. This is because probability distribution functions (pdfs hereafter) that can be obtained as mean-variance-preserving transformations of other pdfs will exhibit more DR, whilst preserving the same expected return and riskiness. An equivalent concept to DR aversion, i.e. "prudence", has been defined using agents' optimizing behavior. The importance of the third derivative of utility ($u''' > 0$) in determining demand for precautionary savings defines prudence according to Kimball (1990); however, as noted

[1] The issue of a corner solution in portfolio choice has recently been explicitly discussed within a mean-variance (MV) model in Ormiston and Schlee (2001). They discussed the comparative statics of expected utility (EU) versus MV, and provided necessary and sufficient condition for an interior solution (no-*plunging*), acknowledging the limitations of MV analysis with regards to higher-order moments. Following these results, the MV model has been extended to include skewness; see Eichner and Wagener (2011); see Markowitz (2014) for a comprehensive review on MV approximations to expected utility for portfolio optimization.

by Eeckhoudt and Schlesinger (2008), whether prudence is a necessary or/and sufficient condition to generate an increase in demand for precautionary savings, depends on the measure of risk (n^{th}-order risk change) on future wealth. Pdfs can therefore either be comparable in terms of riskiness or prudence but not in terms of both. Besides, a distribution function that has less DR than another will also be more right skewed, although the converse is not necessarily true (see Ebert and Wiesen (2011) for an experiment that shows the difference between prudence and skewness preference, and Ebert (2013) for a characterization of how higher-order risk preferences relate to skewness seeking and kurtosis aversion in a non-EU framework).[2] Furthermore, behavioral aspects of investors have also been related to the fourth-order derivative of the utility function through the concept of "temperance" introduced by Kimball (1992), or the fifth-order derivative, "edginess" (see Lajeri-Chaherli 2004). More recently, Eeckhoudt and Schlesinger (2006) defined all those risk preference properties, and others of higher order, i.e. "risk apportionment of order n", by preferences toward particular classes of lotteries, and showed that they are equivalent to signing the n^{th}-order derivative of the utility function. These pure n^{th}-order effects can be related to stochastic dominance of order n (SD_n) even though they are not equivalent concepts, i.e. utility functions that define SD_n are a subset of those that define SD_{n-1}, however risk apportionment of order n is solely characterized by the properties of u^n, independently of the properties (sign) of u^{n-i}.[3]

Secondly, an alternative approach is based on the relation between individual preferences for risk and moments of the distribution through approximations of the utility function. Levy (1969) extended the EU model in Tobin (1958) and Feldstein (1969) to show that for linear utility functions of order n, only the first n moments matter for the investor's liquidity decision, irrespective of the number of parameters of the pdf. For the particular case of a cubic utility function, Horvath and Scott (1985) showed that an EU maximizer investor is more likely to change drastically the composition of the portfolio towards the riskier asset when the skewness of the distribution of returns consistently increases relative to the variance. More recently, Jurczenko and Maillet (2006) presented the theoretical framework of utility specifications and multi-moment decision criteria in an EU model where they determined

[2] See also Deck and Schlesinger (2010) for empirical evidence on prudent and temperate behavior and its relation to aversion to negative skewness and kurtosis.

[3] In a recent paper Levy and Kaplanski (2015) employ the SD criterion to generalize the portfolio selection MV rule for the case of Mixed Normal pdfs. They show that the former is more appropriate than the latter as it better accounts for the departures from normality in the overall distribution of returns.

the preference and distributional restrictions needed to ensure that utility approximations, written in terms of moments, do preserve the individual's preference ranking.[4]

Thridly, a more recent strand of literature apply uncertainty theory (Liu 2007) to the standard MV analysis for hybrid (random and uncertain) portfolio returns; see Qin (2015) and references therein.

Our paper contributes to the extant literature in two ways. First, it provides the theoretical conditions for the allocation of wealth within the standard portfolio choice model under a wide range of pdfs and, second, it shows how those conditions are related to pure higher-order risk preferences, such as, prudence and temperance. Boyle and Conniffe (2005) discussed alternative two-parameter pdfs together with different utility functions and showed that the likelihood of a risky-asset-only portfolio is higher with some distributions than others, whilst the core of this paper presents exact conditions for optimal *plunging* behavior, providing a formal approach. In particular, we examine the effect of higher-order moments on portfolio choice through parametric pdfs widely used in the literature to account for the asymmetric and leptokurtic distribution of asset returns. We consider the five-parameter WGB2 of Ye et al. (2012), which nests the four-parameter generalized beta type 2 (GB2) pdf, which, in turns, nests three-parameter pdfs such as the generalized gamma (GG) and two-parameter pdfs such as the log-normal (LN), gamma (g), Weibull (W) and many other distributions (see McDonald 1984, Bookstaber and McDonald 1987, McDonald and Xu 1995, and Ye et al. 2012, for the theoretical properties of these densities and applications to economic data).

3 Standard portfolio choice model

Consider a two-asset (risky/riskless) economy in which an investor with initial wealth ω_0 decides to invest a proportion, $0 \leq \theta \leq 1$ (ruling out short selling), in the risky asset so that after one period expected wealth becomes

$$\overline{\omega} = (1-\theta)\omega_0 + \theta\omega_0 E(r), \tag{1}$$

[4]Empirical studies on the effect of higher-order moments in EU models can be found in Brandt et al. (2005), and Jondeau and Rockinger (2006).

where $E(r)$ is the expected gross rate of return of the risky asset.[5] We relax the traditional MV two-parameter distribution assumption and study the effect of alternative five-, four- and three-parameter pdfs (hereafter denoted by f) on the optimal solution.

For the investor's preferences we assume a log-utility function, $u_1(\omega) = \ln(\omega)$, which presents constant relatives risk aversion (CRRA) of 1.[6] In Appendix A we provide an extension to the discussion in this section by considering an alternative (power) utility function, which can exhibit smaller degree of relative risk aversion.[7] These two utility functions display the features that characterize prudence (or DR) and temperance, that is $u''' > 0$, and $u^{iv} < 0$, respectively (see Eeckhoudt and Schlesinger, 2006).

The representative agent maximizes her EU by choosing the proportion θ to invest in the risky asset, so her objective program is (2),

$$\begin{aligned} \max_{\{\theta\}} E_f\left[u(\omega)\right] &= \max_{\{\theta\}} E_f\left[u\left((1-\theta)\omega_0 + \theta\omega_0 r\right)\right] \\ &= \max_{\{\theta\}} E_f\left\{u\left(\omega_0\left[1+\theta(r-1)\right]\right)\right\}. \end{aligned} \tag{2}$$

For simplicity, let denote the first derivative of $E_f\left[u(\omega)\right]$ with respect to θ as $\xi_f(u(\omega);\theta) = \frac{\partial E_f[u(\omega)]}{\partial\theta}$. Following the analysis in Feldstein, the conditions under which the investor will maximize $E_f\left[u(\omega)\right]$ by holding only the risky asset are,

$$\xi_f(u(\omega);\theta) > 0 \;\forall\theta \in [0,1) \tag{3}$$

$$\left.\xi_f(u(\omega);\theta)\right|_{\theta=1} \geq 0. \tag{4}$$

For a log-utility function, (2) becomes[8]

$$\max_{\{\theta\}}\left(E_f\left[u_1(\omega)\right]\right) = \max_{\{\theta\}}\left(\ln(\omega_0) + E_f\left\{\ln\left[1+\theta\left(r-1\right)\right]\right\}\right). \tag{5}$$

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[5] The gross return r equals one plus the net return. The investor's expected value of the gross return under her believe on r's probability distribution f must be larger than 1, i.e., $E_f(r) > 1$, so that she holds some proportion of her wealth on risky assets.

[6] Amongst others, we note that the empirical evidence reported by Chetty (2006), Bombardini and Trebbi (2012) and Hartley et al. (2014), in the context of labor supply and attitudes to risk in a game show, respectively, suggests that log-utility is a good approximation to agents' utility function. Brunnermeier and Nagel (2008) in an study of the microeconomics of household's asset allocation provide evidence on that the relationship between wealth and asset allocation seems best described by CRRA rather than by time-varying risk aversion preferences.

[7] In that case, however, the first-order condition for the optimum will depend on two parameters, risk aversion and the proportion of wealth invested in the risky asset, θ.

[8] This model could easily be extended to a dynamic framework because for log-utility the optimal one-period θ is the same as the multi-period θ (see Brandt 2010).

$\xi_f(u_1(\omega);\theta) = E_f\left[\dfrac{r-1}{1+\theta\,(r-1)}\right]$ is (i) positive for $\theta = 0$ (equation (6)), and (ii) a strictly decreasing function of θ for all $\theta \in [0,1]$ (equation (7)),[9] therefore $E_f\left[u_1(\omega)\right]$ has a unique global maximum at $\theta = 1$ for all admissable θ if condition (8) $\xi_f(u_1(\omega);\theta)\big|_{\theta=1} \geq 0$ holds; see Feldstein (1969, p.9),

$$\xi_f(u_1(\omega);\theta)\big|_{\theta=0} \;=\; E_f\left[r-1\right] > 0, \tag{6}$$

$$\frac{\partial\xi_f(u_1(\omega);\theta)}{\partial\theta} \;=\; -E_f\left[\frac{(r-1)^2}{\left(1+\theta\,(r-1)\right)^2}\right] < 0 \;\forall\theta \in [0,1], \tag{7}$$

$$\xi_f(u_1(\omega);\theta)\big|_{\theta=1} \;=\; 1 - E_f\left(r^{-1}\right) \geq 0. \tag{8}$$

3.1 Conditions for optimal choice under alternative return pdfs

In this section we derive the theoretical condition for optimal choice (8) under alternative portfolio return pdfs and express it in terms of their central and raw moments. To ease our analysis and presentation of the results, Table 1 displays the density and moment generating functions (mgfs) for the five-, four- and three-parameter generalized distributions we analyze, WGB2, GB2, and GG, respectively.

TABLE 1

Density and moment generating functions of the generalized distributions

	pdf	$mgf = E[r^t]$
$WGB2(r;k,c,b,p,q)$	$\dfrac{cr^{cp+k-1}\Gamma\left(p+q\right)}{b^{cp+k}\Gamma\left(p+\frac{k}{c}\right)\Gamma\left(q-\frac{k}{c}\right)(1+\frac{r^c}{b^c})^{p+q}}$	$\dfrac{b^t\Gamma(p+\frac{k}{c}+\frac{t}{c})\Gamma(q-\frac{k}{c}-\frac{t}{c})}{\Gamma(p+\frac{k}{c})\Gamma(q-\frac{k}{c})}$
$GB2(r;c,b,p,q)$	$\dfrac{cr^{cp-1}\Gamma\left(p+q\right)}{b^{cp}\Gamma\left(p\right)\Gamma\left(q\right)(1+\frac{r^c}{b^c})^{p+q}}$	$\dfrac{b^t\Gamma\left(p+\frac{t}{c}\right)\Gamma\left(q-\frac{t}{c}\right)}{\Gamma\left(p\right)\Gamma\left(q\right)}$
$GG(r;a,p,c)$	$\dfrac{ca^{cp}r^{cp-1}e^{-(ar)^c}}{\Gamma(p)}$	$\dfrac{1}{a^t}\dfrac{\Gamma(p+\frac{t}{c})}{\Gamma(p)}$

Notes: Pdfs and mgfs of WGB2, GB2 and GG distributions. $\Gamma(p) = \int_0^\infty e^{-r}r^{p-1}dr$ denotes the gamma function. Parameter k controls the shape and skewness of the WGB2 density, which nests the GB2 when $k=0$ (Ye et al. 2012), which, in turns, nests the GG when $b = a^{-1}q^{1/c}$ as $q \to \infty$ (McDonald 1984). For convenience, the mgf specifications we use in the paper are reparemeterizations of the standard ones.

[9] Note that $\frac{\partial E_f[u(\omega)]}{\partial\theta} = \frac{\partial}{\partial\theta}\int u(\omega)f(r)dr = \int \frac{\partial u(\omega)}{\partial\theta}f(r)dr = E_f\left[\frac{\partial u(\omega)}{\partial\theta}\right]$

We begin with the most flexible distribution, the five-parameter WGB2. For this case, using its mgf and the expression for b derived from its first raw moment, we can express condition (8) as,

$$\begin{aligned}1-E_f\left(r^{-1}\right) &= 1-\frac{1}{b}\frac{\Gamma(p+\frac{k}{c}-\frac{1}{c})\Gamma(q-\frac{k}{c}+\frac{1}{c})}{\Gamma(p+\frac{k}{c})\Gamma(q-\frac{k}{c})}\geq 0\\ m_{1,WGB2} &\geq \frac{\Gamma\left(p+\frac{k}{c}+\frac{1}{c}\right)\Gamma\left(q-\frac{k}{c}-\frac{1}{c}\right)\Gamma\left(p+\frac{k}{c}-\frac{1}{c}\right)\Gamma\left(q-\frac{k}{c}+\frac{1}{c}\right)}{\Gamma^2\left(p+\frac{k}{c}\right)\Gamma^2\left(q-\frac{k}{c}\right)}. \qquad (9)\end{aligned}$$

Hereafter $m_{t,f}$ denotes the th-central moment of f. Similarly, for the four-parameter GB2 the optimal condition is met when

$$m_{1,GB2} \geq \frac{\Gamma\left(p+\frac{1}{c}\right)\Gamma\left(q-\frac{1}{c}\right)\Gamma\left(p-\frac{1}{c}\right)\Gamma\left(q+\frac{1}{c}\right)}{\Gamma\left(p\right)^2\Gamma\left(q\right)^2}. \qquad (10)$$

McDonald (1984) demonstrates that the substitution $b = q^{1/c}/a$ as $q \to \infty$ in the GB2 density function generates the GG distribution with shape parameters $a > 0$ and $c > 0$, and scale parameter $p > 0$.[10] Thus, condition (8) for the GG is given by

$$m_{1,GG} \geq \frac{\Gamma(p+\frac{1}{c})}{\Gamma(p)}\frac{\Gamma(p-\frac{1}{c})}{\Gamma(p)}. \qquad (11)$$

This expression allows us to obtain results for other distributions nested within the GG. For instance, in the case of the gamma distribution, condition (11) is obtained as

$$1-E_f\left(r^{-1}\right) = 1-a\frac{\Gamma(p-\frac{1}{1})}{\Gamma(p)} \geq 0, \qquad (12)$$

$$1-E_f\left(r^{-1}\right) = 1-\frac{\Gamma(p+1)}{m_{1,g}}\frac{\Gamma(p-1)}{\Gamma(p)^2} \geq 0, \qquad (13)$$

which can be expressed in terms of central moments as

$$1-\frac{a}{p-1} \;=\; 1-\frac{\frac{m_{2,g}}{m_1,g}}{\frac{m_{1,g}^2}{m_{2,g}}-1} \geq 0, \qquad (14)$$

$$m_{1,g} \;\geq\; 1+\frac{m_{2,g}}{m_{1,g}}. \qquad (15)$$

For the case of the Weibull the condition for the optimum is

$$m_{1,W} \geq \Gamma(1+\frac{1}{c})\Gamma(1-\frac{1}{c}) \qquad (16)$$

[10]Note that the GG family nests many other distributions as special cases. For instance, gamma ($c = 1$), exponential $(p, c) = (1, 1)$, Weibull ($p = 1$), log-normal ($p \to \infty$) and Rayleigh $(p, c) = (1, 2)$.

Finally, we analyze the standard case of the log-normal pdf, that is, we assume that the logarithm of the risky asset (gross) return, $\ln(r)$, follows a Normal distribution with parameters m and v as

$$\Phi(r;m,v)=\frac{1}{rv\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln(r)-m}{v}\right)^2},\ 0<r<\infty. \tag{17}$$

The raw moments (mgf) of this distribution are given by

$$\mu_{t,LN}=E_\Phi[r^t]=\int r^t\Phi(r;m,v)dr=e^{tm+\frac{1}{2}t^2v^2},\ \forall t\in\mathbb{R} \text{ or } \forall t\in\mathbb{C}. \tag{18}$$

$\theta=1$ is optimum if the condition below holds

$$\xi_\Phi(u_1(\omega);\theta;m,\nu)|_{\theta=1}=1-E_\Phi\left(r^{-1}\right)\geq 0, \tag{19}$$

which is expressed as

$$1 \geq e^{-m+\frac{1}{2}v^2}, \tag{20}$$

$$m \geq \frac{1}{2}v^2. \tag{21}$$

Given that $\mu_{1,LN}=e^{m+\frac{1}{2}v^2}$ and $\mu_{2,LN}=e^{2m+2v^2}$ the condition above is: $2\ln\mu_{1,LN}-\frac{1}{2}\ln\mu_{2,LN}\geq\frac{1}{2}\ln\mu_{2,LN}-\ln\mu_{1,LN}$, or $3\ln\mu_{1,LN}\geq\ln\mu_{2,LN}$, so we can write the condition for the optimum in terms of either the parameters (equation (21)), the first two raw moments (equation (22)) or the central moments (equation (23))[11]

$$\mu_{1,LN} \geq \left(1+\frac{\mu_{2,LN}-\mu_{1,LN}^2}{\mu_{1,LN}^2}\right), \tag{22}$$

$$m_{1,LN} \geq 1+\frac{m_{2,LN}}{m_{1,LN}^2}. \tag{23}$$

4 Pedagogical examples

The theoretical results derived above are general and do not allow to explicitly observe the trade-off between the first four moments of the distribution and the investor's risk preferences up to the fourth order. We therefore now resort to a calibration exercise and our analysis in this section illustrates that: (i) the optimal portfolio choice decision (i.e. *plunging*) depends

[11] It is worth noting that in Boyle and Conniffe (2005) the same expression is obtained through the Taylor approximation for r^{-1}.

not only on the relationship between the first two moments (as in the case of two-parameter pdfs) but also on the whole parametric structure of the pdf when more-than-two-parameter pdfs are considered, and hence on higher-order moments; (ii) the optimal conditions obtained depend on pure third- or fourth-order properties of investors' preferences; (iii) the parametric restriction that determines the optimal choice does vary depending on alternate pdfs nested in a more general density.

As a baseline for our comparative statics analysis on optimal portfolio choice under alternative pdfs, we use the seminal example in Feldstein (1969). Let us therefore proceed by assuming a log-normal distribution ($p \rightarrow \infty$ in expression for GG, Table 1) with $m_{1,LN} = 1.05$, i.e., a net return of 5 per cent. Table 2 below summarizes all results and examples presented in this section.

According to equation (23), investors would (optimally) form a portfolio of only risky assets if $m_{2,LN} \leq 0.055125$, or similarly, unless the standard deviation is more than four times the expected net return, i.e., $m_{2,LN}^{1/2} > 0.23479$.[12] This threshold value is not unreasonable, hence Feldstein's original question as to why we do not appear to observe more investors allocating all their wealth in risky assets.

In the previous section we show how an alternative two-parameter pdf yields a different lower bound for the risky-asset-only portfolio. Next, we provide two examples. In the first one, using the gamma distribution ($c = 1$ in expression for GG, Table 1) and assuming $m_{1,g} = 1.05$, the optimal portfolio consisting of only the risky asset would occur for a lower upper bound of the variance but also of the third central moment (see Table 2, Panel A). Thus, no-diversification for a gamma distribution in relation to a log-normal requires lower riskiness (variance) and lower (positive) third central moment.

In the second example with a two-parameter pdf, for the Weibull distribution ($p = 1$ in expression for GG, Table 1), the optimal solution holds when parameter $c \geq 5.83493$ (note that we continue assuming $m_{1,W} = 1.05$). In this case, the upper bound for the second and third central moments are $(m_{2,W}, m_{3,W}) = (0.04358, -0.0032383)$.[13] Thus, a risky-asset-only portfolio for a Weibull relative to a gamma and a log-normal pdf requires both riskiness and third central moment to be smaller and this can occur even in the case that m_3 is negative. This result adds evidence to the previous case of a gamma in that the assumption about the specific pdf determines different bounds between the moments that define the optimum.

[12] Throughout the paper, for the sake of easing the replication of our results, we present parameter and moments values with different decimal points as results depend crucially on the rounding.

[13] It is worth noting that the variance decreases as parameter c is increased for a given mean so that $m_{2,W} = 0.04358$ is the highest variance for which a risky-asset-only portfolio can occur.

The implication that follows from the analysis so far of the GG (with a mean of 1.05), is that investors optimally allocate all the wealth in the risky asset if the variance is less than 0.055125 depending on both the particular nested distribution considered and the precise number for m_3 (see Table 2, Panel A).[14] We find that within the GG-class, the optimal condition under the log-normal allows the highest variance for a given mean. That is why the optimal condition under the log-normal is more probable if one were to restrict the analysis to a mean-variance framework.

We now illustrate the argument that investors' preferences of higher-order than riskiness matter for the decision to diversify within this framework. In particular, we find parameter values for two GG distributions (displayed in Table 2, Panel B) that yield the same mean and variance as the log-normal presented above under which the investor would switch her optimal allocation. However, if returns were to follow the former distribution, investors would diversify rather than allocating all the wealth in the risky asset. The EU for those two GG distributions is lower than the EU of the log-normal.[15] This critical shift in the investors' optimal portfolio allocation decision is due to DR aversion (see Theorem 2 p. 926 in Menezes et al. 1980) or, equivalently, prudence, rather than riskiness (see also Eeckhoudt and Schlesinger 2006). The GG pdfs in this example imply more DR than the log-normal does, that is, they involve a transfer of probability weight leftward in the distribution preserving its mean and variance, making the individual worse off by such a change and therefore willing to diversify.

In order to isolate the effect of the 4th-order preference property of temperance, we now assume that the investor's information on the returns pdf is provided by a WGB2, which optimal condition is given in equation (9). In the same fashion as before, we find parameter values for the WGB2, e.g. $(p, c, q, b, k) = (4.92879, 2.80226, 6.5, 1.1025791, 0.7)$, that yield the same mean, variance and third central moment as the previously defined log-normal under which the investor would optimally allocate all her wealth in the risky asset. However, if returns were to follow the former distribution, investors would diversify. The fourth central moment is higher for this WGB2 while its expected utility is lower.[16]

[14] This result could also be related to the concept of 'greater central riskiness', see Gollier (1995). Gollier showed that a risk-averse EU maximizer increases her investment in the risky asset when the return distribution F is replaced by G if and only if there exists a real number m such that $\int_{-\infty}^{x} rdG(r) \geq m \int_{-\infty}^{x} rdF(r)$ for all $x \in \mathbb{R}$.

[15] In particular, $EU_{LN} = 0.0243951$, (hereafter EU_f denotes EU under density f) and for the two GG densities in Table 1 Panel B, $EU_{GG} = 0.0227413$ and $EU_{GG} = 0.0237374$.

[16] In particular, $(m_{4,WGB2}, ku_{WGB2}) = (0.012407, 4.0828)$, $(m_{4,LN}, ku_{LN}) = (0.01166, 3.83826)$ (ku denotes kurtosis) and $EU_{WGB2} = 0.024281$.

TABLE 2

GB2-class of densities and *plunging*: Range of moments and pdf specification

	GB2	GG	gamma	Weibull	log-normal
Panel A. Maximum m_2 for which PC holds within a class of pdf					
m_2^*	0.0581	0.055125	0.05250	0.043580	0.055125
m_3	0.0141	0.008826	0.00525	-0.00323	0.008826
RC	Yes	Yes	Yes	Yes	Yes
Panel B. Examples of GG distributions with same (m_1, m_2) as log-normal in Panel A					
GG that nests log-normal: $(c, p, a) = (2, 5.11592, 2.1022)$					
m_2		0.055125			0.055125
m_3		0.003034			0.008826
RC		No			Yes
GG that nests log-normal: $(c, p, a) = (0.81694, 30, 61.503)$					
m_2		0.055125			0.055125
m_3		0.006324			0.008826
RC		No			Yes
Panel C. Examples where general pdf matters for exact values of (m_2, m_3) in PC					
GB2 that nests Weibull: $(p, c, q, b) = (1, 5.855105, 90, 2.4414)$					
m_2	0.04368			0.043580	
m_3	-0.0031			-0.00323	
RC	Yes			Yes	
GB2 that nests gamma: $(p, c, q, b) = (21.55, 1, 791, 38.492)$					
m_2	0.0526		0.05250		
m_3	0.0054		0.00525		
RC	Yes		Yes		
Panel D. Example of two GB2 pdfs with same (m_1, sk) and PC holds for higher m_2					
GB2 with $(m_1, sk) = (1.05, 2.194406)$					
p	1.77451	22			
c	7.574	4.5			
q	0.85	1.5			
b	0.88208	0.52162			
m_2	0.06819	0.06836			
RC	No	Yes			

Notes: Summary of the condition under which the investor would optimally allocate all wealth in the risky asset (RC) for the examples of the GB2-class of distributions presented in this section. For all cases $m_1 = 1.05$ and m_2^* denotes the maximum variance so that the RC holds.

Finally, we show that the precise specification of a distribution (nesting case) may also matter. We illustrate this result with two examples of the GB2 distribution that nest the Weibull and the gamma pdfs admitting slightly higher variance (and skewness) than the original densities for which $\theta = 1$ is optimum (see Table 2, Panel C).

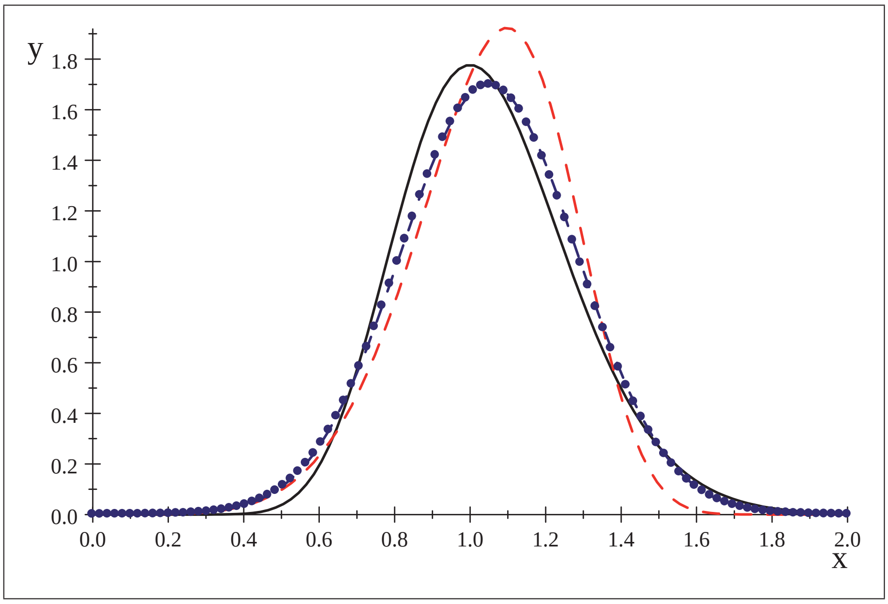

FIGURE 1. Probability density funciton of the log-normal (black solid line), Weibull (red dash line), and Gamma (blue dot-dash line).

As a graphical comparison of the distributions in terms of their pdfs, Figure 1 illustrates the differences in the tails and peaks of the two-parameter pdfs considered here (log-normal, gamma, and Weibull) with the same mean and variance $(m_{1,f}, m_{2,f}) = (1.05, 0.055125)$. These mean-variance preserving transformation pdfs differ in their higher-order moments which are all fixed once the mean and variance are given. Thus, even under utility functions that represent the risk apportionment corresponding to higher orders (e.g. log-utility), if portfolios are compared in terms of their mean and variance under two-parameter distributions, the higher-order moments would be irrelevant for the optimal portfolio decision. Figure 2 further illustrates this point, it provides plots of the (two-parameter) log-normal and (four-parameter) GB2 pdfs which are most conducive to risky-asset-only portfolio with $(m_{1,f}, \sqrt{m_{2,f}}) = (1.05, 0.23479)$. It is illustrative to observe the pdfs differences in terms of asymmetry and heavy-tails for the same mean and variance.

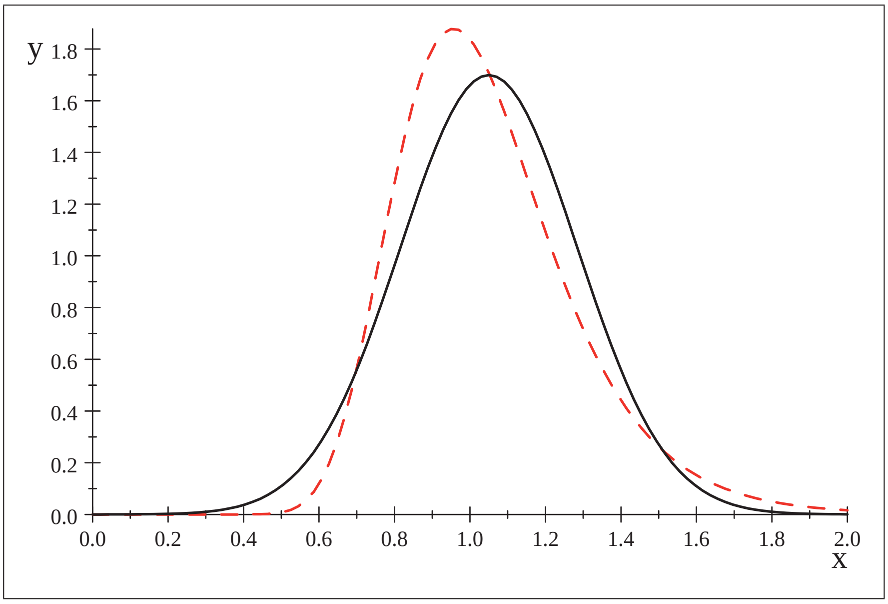

FIGURE 2. Probability density function of the log-normal (Black solid line), and GB2 (red dash line).

Our analysis has also implications for definitional issues in portfolio choice analysis. Decision theory defines agent's behavior in terms of either raw and central moments or standardized moments. The use of the latter may lead to a different optimum and could potentially result in a misinterpretation of the theory in terms of preferences on liquidity when higher-order moments are taken into account. We provide an illustration to this point (in Table 2, Panel D) by showing two different GB2 distributions with the same mean and skewness ($m_3/m_2^{3/2}$), where the agent would invest a higher proportion of her wealth under the distribution with higher variance. This seemingly counterintuitive result is due to the fact of a higher third central moment, m_3. In this example, the EU of the distribution with lower variance is lower for the same mean because it has a lower m_3, despite having the same skewness. Note that this result would contradict a classical MV argument.[17]

[17] Consequently, the GB2 distribution appears to admit cases for which an agent's choices do not meet the requirements of skewness affinity as defined in Eichner and Wagener (2011).

5 Empirical application

This section provides an application of the plunging conditions derived in Section 3 for an investor who has the choice of allocating wealth between riskless (cash) and risky assets (S&P500 index). We use data from Robert Shiller's webpage spanning the period January 1871 to February 2011 for a total of 1682 observations. Table 3 presents the descriptive statistics of the gross return series computed as $r_t = 1 + \log(P_t/P_{t-1})$, where P_t denotes the monthly real price of the S&P500.

TABLE 3

Descriptive statistics of S&P500 gross returns

Mean	Median	Max	Min	St. Dev.	Skewness	Kurtosis	J-B
1.00167	1.00525	1.41480	0.69247	0.04135	−0.30782	13.9528	8429.14

Notes: This table presents descriptive statistics of S&P500 monthly percent gross returns from February 1871 to February 2011, 1681 obs. The Jarque-Bera J-B) statistic is asymptotically distributed as a $\chi^2(2)$ under the null of normality. The critical value of $\chi^2(2)$ at 5% level is 5.99.

Table 4 provides maximum likelihood estimation results of the distributions discussed above. We observe that all distributions match rather well the first two moments of the return series (with the exception of the Weibull) but there are clear differences in the densities's fit of returns' skewness and kurtosis.[18] The distributions in the application that are most flexible (GB2, GG) display closer higher order moments to those of the data and present the best fit in terms of log-likelihood and AIC.

The last row in Table 4 indicates if risky-asset-only portfolio conditions are met for each pdf under log-utility. It turns out that for none of the distributions considered the agent would invest all her wealth in the risky asset. This result is in line with the empirical regularity that *plungers* are rarely observed. |N| measures the difference between the right and left hand sides of the conditions derived in section 3 under the assumed distributions. The estimated density that more approaches plunging is the GB2. The GG, gamma and LN give a distance |N| that is two and one orders of magnitude higher than that of the GB2, respectively. Finally, if returns are assumed to have a Weibull pdf, |N| is four orders of magnitude higher than under the GB2.[19]

[18] The WGB2 estimation yields a non-significant estimate of parameter k, thus converging to the GB2, the latter presenting a better fit according to the AIC as it has less parameters; these results are not presented in Table 2 for the sake of simplicity but are available from the authors upon request.

[19] An important avenue for further research on this topic involves the analysis of the pure effect of higher-order risk preferences and higher-order moments on the optimal proportion of wealth invested in the risky asset.

TABLE 4

Estimation results

	GB2	GG	gamma	Weibull	LN
$\widehat{b}$	1.017				
	(9.07)				
$\widehat{q}$	0.550				
	(2.86)				
$\widehat{c}$	98.71	4.086		16.15	
	(6.05)	(0.15)		(84.2)	
$\widehat{p}$	0.314	33.75	575.05		
	(6.03)	(3.03)	(70.3)		
$\widehat{a}$		2.356	574.049	0.979	
		(0.13)	(69.2)	(627.2)	
$\widehat{m}$					0.0008
					(0.78)
$\widehat{v}$					0.0420
					(57.9)
$\widehat{m}_1$	1.001590	1.001510	1.001680	0.987638	1.001687
$\widehat{m}_2$	0.001541	0.001795	0.001744	0.005663	0.001770
$\widehat{sk}$	−0.6348	−0.0452	0.0834	−0.8097	0.1260
$\widehat{m}_3$	−0.000038	−0.000003	0.000006	−0.000306	0.000009
$\widehat{ku}$	5.2034	3.0035	3.0104	4.0684	3.0282
$\widehat{m}_4$	0.0000121	0.0000096	0.0000091	0.0001112	0.0000095
LogL	3142.4	2967.9	2953.8	2464.8	2942.9
AIC	−3.7340	−3.5275	−3.5120	−2.9278	−3.4991
$\lvert$N$\rvert$	0.0000022	0.0002836	0.0000665	0.0187001	0.0000768
RC	No	No	No	No	No

Notes: Estimation results (ML t-statistics in brackets) for the GB2, GG, gamma, Weibull and log-normal (LN) distributions. $\widehat{m}_i$ denote estimated central moment of order i, $\widehat{sk}$, and $\widehat{ku}$ denote estimated skewness and kurtosis, respectively. AIC and LogL denote Akaike Information Criterion and log-likelihood, respectively. RC denotes whether condition of risky-only-portfolio is met. $|$N$|$ is a measure the distance between the right and left hand sides of the RC.

6 Conclusions

In this paper we examine the relationship between the statistical moments of asset returns and higher-order risk preferences within the classical standard portfolio choice model. We derive the optimal conditions for alternative two-, three-, four- and five-parameter densities and illustrate how the optimal decision on liquidity depends crucially on higher-order moments, which are associated with higher-order risk preferences such as prudence (or DR aversion) and temperance. We show that the parametric restriction that determines the optimal choice varies depending not only on the assumed non-nested pdfs (even within the two-parameter family), but also on alternate pdfs nested in a more general density. We find that under pdfs that are transformations preserving mean and variance (and skeweness) the investor switches her optimal decision on liquidity from a risky-asset-only portfolio to diversification, or viceversa, because they account for her third (and four)-order pure risk preferences of prudence (and temperance). Hence the importance of using an appropriate pdf able to accurately capture the stylized features of asset returns. Our results are relevant for portfolio choice theory and for policy makers and portfolio managers concerned with modelling demand for liquidity or allocation of wealth in risky assets.

Appendix A. Risky-asset-only portfolio with power utility under alternative distributions

We extend the analysis to the power utility function, $u_2(\omega, \lambda) = \omega^{\lambda}$, $0 < \lambda < 1$, whose risk aversion parameter, λ, is allowed to vary, and it is therefore more general than log utility, u_1.[20] Normalizing initial wealth ($\overline{\omega}_0$) to 1, the EU is given by,

$$E_f\left[u_2(\omega, \lambda)\right] = E_f(\omega^{\lambda}) = \int (1 + \theta r - \theta)^{\lambda} f(r; \mathbf{\Omega}) dr. \tag{24}$$

The condtions under wich the investor will maximize (24) by holding only the risky asset are:

$$\begin{aligned} \xi_f\left(u_2(\omega, \lambda); \theta\right) &> 0 \;\forall \theta \in [0, 1) & (25) \\ \xi_f(u_1(\omega); \theta)\big|_{\theta=1} &\geq 0 & (26) \end{aligned}$$

(see Feldstein, 1969). The first derivative of (24)

$$\begin{aligned} \xi_f\left(u_2(\omega, \lambda); \theta\right) &= \lambda \int (1 + \theta r - \theta)^{\lambda - 1} (r - 1) f(r; \mathbf{\Omega}) dr \\ &= \lambda \left\{ E_f\left[r\left(1 - \theta + \theta r\right)^{\lambda - 1} \right] - E_f\left[\left(1 - \theta + \theta r\right)^{\lambda - 1} \right] \right\} \end{aligned} \tag{27}$$

is positive for $\theta = 0$,[21]

$$\xi_f\left(u_2(\omega, \lambda); \theta\right)\big|_{\theta=0} = \lambda \left\{ E_f\left[r\right] - 1 \right\} > 0. \tag{28}$$

Unfortunately, it cannot be shown that $\xi_f\left(u_2(\omega, \lambda); \theta\right)$ is either positive for all $\theta \in [0, 1)$ or a strictly decreasing function of θ for all $\theta \in [0, 1]$ (equation (29))

$$\begin{aligned} \frac{\partial \xi_f(u_2(\omega); \theta)}{\partial \theta} &= \lambda(\lambda - 1) \int (1 + \theta r - \theta)^{\lambda - 2} (r - 1)^2 f(r; \mathbf{\Omega}) dr \\ &= \lambda(\lambda - 1) E_f\left[\left(1 + \theta(r - 1)\right)^{\lambda - 2} (r - 1)^2 \right]. \end{aligned} \tag{29}$$

[20] As the exponent of a particular version of the power utility function goes to zero, it becomes the log utility function,

$$\lim_{\lambda \to 0} \frac{\omega^{\lambda} - 1}{\lambda} = \log(\omega)$$

[21] The FOC includes two unknowns, λ and θ. It is therefore not possible to determine the optimal level of portfolio composition independently of risk aversion.

Therefore, we proceed by providing an example where higher-order moments matter for necessary (but not sufficient) condition (26),

$$\begin{aligned}\xi_f(u_2(\omega);\theta)\big|_{\theta=1} &= \lambda\left\{E_f\left[r^{\lambda}\right]-E_f\left[[r]^{\lambda-1}\right]\right\}\geq 0. \\ &= E_f\left[r^{\lambda}\right]\geq E_f\left[r^{\lambda-1}\right]. \qquad (30)\end{aligned}$$

We first consider the case of the WGB2 for which condition (30) can be written as

$$b^{\lambda}\frac{\Gamma\left(p+\frac{k}{c}+\frac{\lambda}{c}\right)\Gamma\left(q-\frac{k}{c}-\frac{\lambda}{c}\right)}{\Gamma\left(p+\frac{k}{c}\right)\Gamma\left(q-\frac{k}{c}\right)}\geq b^{\lambda-1}\frac{\Gamma\left(p+\frac{k}{c}+\frac{\lambda-1}{c}\right)\Gamma\left(q-\frac{k}{c}-\frac{(\lambda-1)}{c}\right)}{\Gamma\left(p+\frac{k}{c}\right)\Gamma\left(q-\frac{k}{c}\right)}, \qquad (31)$$

and using the expression for b obtained from $\mu_{1,WGB2}=\frac{b\Gamma\left(p+\frac{k}{c}+\frac{1}{c}\right)\Gamma\left(q-\frac{k}{c}-\frac{1}{c}\right)}{\Gamma\left(p+\frac{k}{c}\right)\Gamma\left(q-\frac{k}{c}\right)}$, we write this condition as follows

$$\mu^{p}_{1,WGB2}\geq\frac{\Gamma\left(p+\frac{k}{c}\right)\Gamma\left(q-\frac{k}{c}\right)}{\Gamma\left(p+\frac{k}{c}+\frac{1}{c}\right)\Gamma\left(q-\frac{k}{c}-\frac{1}{c}\right)}\frac{\Gamma\left(p+\frac{k}{c}+\frac{\lambda-1}{c}\right)\Gamma\left(q-\frac{k}{c}-\frac{(\lambda-1)}{c}\right)}{\Gamma\left(p+\frac{k}{c}+\frac{\lambda}{c}\right)\Gamma\left(q-\frac{k}{c}-\frac{\lambda}{c}\right)}. \qquad (32)$$

Similarly, for the GB2 $\theta=1$ is optimal when

$$\mu^{p}_{1,GB2}\geq\frac{\Gamma\left(\frac{pc+1}{c}\right)\Gamma\left(\frac{qc-1}{c}\right)\Gamma\left(\frac{pc+\lambda-1}{c}\right)\Gamma\left(-\frac{-qc+\lambda-1}{c}\right)}{\Gamma\left(\frac{pc+\lambda}{c}\right)\Gamma\left(-\frac{-qc+\lambda}{c}\right)\Gamma\left(p\right)\Gamma\left(q\right)}. \qquad (33)$$

For the case of the GG, using its mgf in Table 1, condition (30) can be written as,

$$1-\frac{a\Gamma(p+\frac{\lambda-1}{c})}{\Gamma(p+\frac{\lambda}{c})}\geq 0, \qquad (34)$$

and given that $a=\frac{1}{\mu_{1,GG}}\frac{\Gamma(p+\frac{1}{c})}{\Gamma(p)}$, the equation above becomes,

$$\mu^{p}_{1,GG}\geq\frac{\Gamma(p+\frac{1}{c})\Gamma(p+\frac{\lambda-1}{c})}{\Gamma(p)\Gamma(p+\frac{\lambda}{c})}. \qquad (35)$$

For the case of the two-parameter gamma distribution, condition (35) reduces to[22]

$$1-\frac{a\Gamma(p+\frac{\lambda-1}{c})}{\Gamma(p+\frac{\lambda}{c})}=1-a\frac{1}{p+\lambda-1}\geq 0, \qquad (36)$$

which can be expressed in terms of the raw moments as

$$\mu^{p}_{1,g}\geq 1+\frac{\mu_{2.g}}{\mu_{1,g}}(1-\lambda). \qquad (37)$$

[22] This expression is also obtained in Boyle and Conniffe (2005).

The conditions for a risky-asset-only portfolio above suggest that as the agent becomes more risk averse (lower λ), she is less likely to allocate all her wealth to risky assets. Log utility ($\lambda = 0$) sets an upper bound for the condition to *plunge* under power utility.

Table A.1 illustrates our results by giving an example about how the condition for non-diversifyers does depend on higher-order moments, assuming a coefficient of risk aversion of $\lambda = 0.8$. These results suggest that, if returns are characterized by a gamma distribution, condition (37) would not be met, and therefore, it would not be optimal for the agent to allocate all the wealth to the risky asset. However, under the GB2 distributions with the same first and second central moments but higher third moment than the gamma, we find that allocating all wealth to the risky asset would be optimal as the agent's risk preferences exhibit prudence.[23] Furthermore, we show that the fourth-order moment switches the agent's decision away from the corner solution by considering a WGB2 that differs from the GB2 only in m_4, because of temperance in the investor's preferences for risk.

TABLE A.1

Plunging condition under different distributions

	WGB2	GB2	gamma
k	0.02		
b	1.18947	0.5764	
q	1.685266	6.1	
c	2.93999	1.1	
p	1	10	3.42673
a			3.26355
m_1	1.05	1.05	1.05
m_2	0.32174	0.32174	0.32174
m_3	0.41694	0.41694	0.19717
m_4	2.51012	1.62705	0.49178
PC	No	Yes	No

Notes: This table presents an example for the plunging condition (PC) (30) for power utility ($\lambda = 0.8$) under WGB2, GB2 and gamma distributions. The values of parameters are calibrated so that the pdfs yield the same first and two central moments but differ on the third and/or the fourth moment.

[23] Within the four-parameter distribution GB2, it is also possible to show that a different parameterization such as $(p, c, q, b) = (1.17620698963, 2, 6.1, 2.485)$ yields the same mean and variance but lower skewness ($m_3 = 0.19171$) and condition (33) would not be met.

Finally, Figure A.1 provides the plots of the pdfs for the parameters values in Table A.1. The GB2 plot illustrates a mean-variance preserving transformation of a (non-plunging) gamma pdf. This GB2 has a higher skewness than the gamma that does not compensate its also higher leptokurtosis to prevent the investor to put all wealth in the risky asset. The plotted WGB2 is an illustration of a mean-variance-skewness preserving transformation of the GB2 with higher fourth-order risk which leads the investor to diversify.

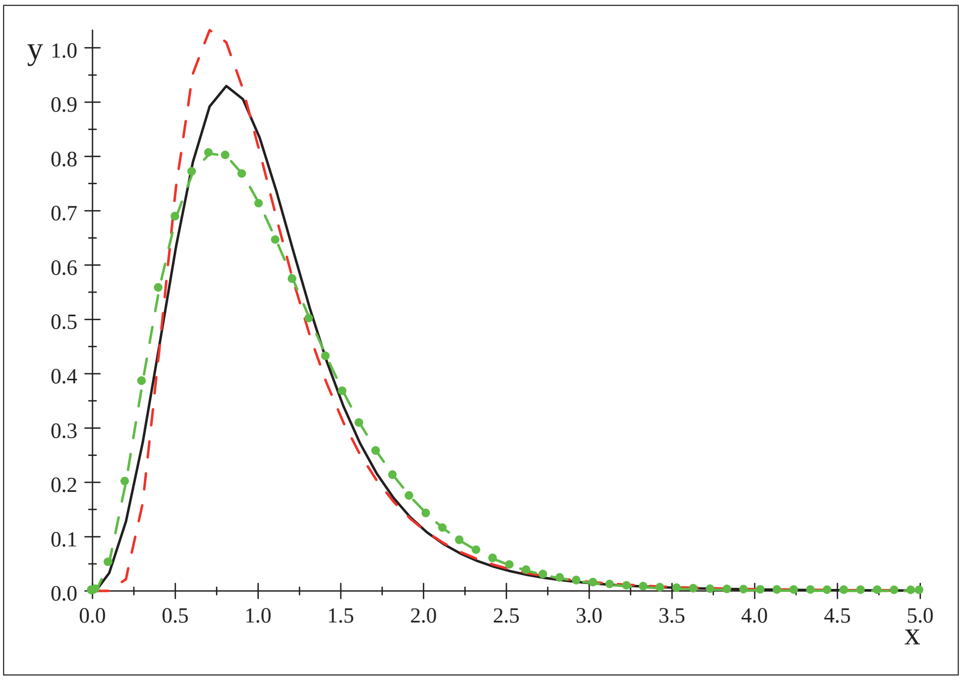

FIGURE A.1. Pdf of the WGB2 (Black solid line), GB2 (red dash line).and Gamma (green dot-dash line) for the values in Table A.1.

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