DEFAULT NEAR-THE-*DEFAULT*-POINT: THE VALUE OF AND THE DISTANCE TO DEFAULT

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BANKIA

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Abstract

We show that the default event defined by endogenous credit-risk models (i.e. low asset values) can likewise be described in terms of low equity prices and negative net cash-flows (high debt service and/or negative earnings). Specifically, distance-to-default (DD), a volatility-adjusted measure of leverage, is given by the ratio of equity prices to negative net cash flows. This implies that the probability of default is the probability of this ratio becoming small, which then depends on the path of these two variables. This helps to explain why just equity prices (price per share, past return, and volatility) and firm's debt and profitability are significant in reduced-form models that predict default while Merton's DD becomes redundant if we control for them [Campbell et al. (2008)]. In endogenous models, default is triggered by depressed equity prices and a negative flow to shareholders (rather than low asset value). And, inversely, default concerns are readily lessened by easing refinancing costs (e.g. sovereigns for which default is costly and which regularly roll over their debts), lowering the principal (underwater mortgages or subprime consumer loans, which increases equity value), or raising equity (troubled banks).

Keywords: credit-risk, default-risk, Merton's distance-to-default, equity prices to negative net cash-flow ratio, endogenous default.

JEL classification: G13, G21, G28, G33.

Resumen

Con respecto al riesgo de crédito corporativo, este trabajo demuestra que el «evento de *default*» que define un modelo endógeno de riesgo de crédito (i.e., un bajo valor de los activos) puede ser similarmente descrito por un bajo valor de las acciones y unos flujos de caja netos negativos (i.e., servicio de la deuda y pérdidas menos dividendos). En particular, la distancia-al-*default* (DD), una medida de apalancamiento de una empresa ajustada por volatilidad, es igual al precio por acción dividido por los flujos de caja netos (con signo negativo). Esto implica que la probabilidad de *default* es la probabilidad de que esta ratio se haga pequeño, lo cual depende de la evolución (o *path)* de estas dos variables. A su vez, ello ayuda a explicar porqué únicamente las acciones (precio, rentabilidad y volatilidad de la accion) más la deuda y pérdidas de una empresa son relevantes en modelos en forma reducida que predicen riesgo de *default,* mientras que la DD basada en el modelo de Merton (1974) se vuelve no significativa cuando controlamos por estas variables [Campbell *et al.* (2008)]. En modelos endógenos, *default* está desencadenado por precio de acciones muy bajo y flujos de caja netos negativos (más que por bajos valores de los activos subyacentes). Y, de manera inversa, el riesgo de *default* se reduce bajando los costes de refinanciación (e.g., para países soberanos, para los cuales hacer *default* es muy costoso y habitualmente hacen *rollover* de sus deudas), reduciendo el principal de la deuda (en el caso de hipotecas o préstamos al consumidor morosos, pues ello aumenta el valor del *equity),* o inyectando *equity* (en el caso de bancos con problemas).

Palabras clave: riesgo de crédito, riesgo de *default,* distancia-al-*default* de Merton, ratio del precio de las acciones sobre menos flujos de caja, *default* endógeno.

Códigos JEL: G13, G21, G28, G33.

1 Introduction

Regarding corporate default-risk, Campbell et al. (2008) derive two important empirical results: 1) Default probabilities depend largely on five covariates; i.e., price per share, past returns, equity volatility and firm's debt and profitability (besides cash-holdings), and 2) Merton's distance-to-default (DD) becomes redundant if we control for them (similar to Bharath and Shumway (2008)).[1] At the same time, however, Merton's DD is a main covariate to asses and predict financial distress in Duffie et al. (2007) and (2009)—as well as to explain CDS spreads (Bai and Wu (2013) and Doshi et al. (2013)), pure credit contract prices (American binary puts, Carr and Wu (2011)), or equity returns (Vassalou and Xing (2004)). Another robust finding is that financial distress is associated with depressed stock prices.[2]

This paper fills this gap by formally showing the consistency of these empirical results. Understanding default probably a unfortunate and last-resort decision is important by itself—financial distress looms in many places beyond firms. Two examples, Goodman (2010) stresses that 50% US mortgages are either voluntarily prepaid or go into default after four (98.5% after fifteen) years. In Spain 2015, a new law, which is in force for firms, facilitates the assets for debt exchange for distress debtors (and eventually will be extended to mortgages)—it is called a "law of second chance." In effect, it is the valuable opportunity of defaulting.

Default is triggered when the value of the underlying leveraged security (e.g., underwater house, indebted country, or distressed firm) reaches a sufficient "low" value. This is the basic intuition of when/why/how default occurs and is formalized by Merton (1974), where the leveraged security is a call option. A key byproduct of Merton's model is Merton's DD, a measure of firm's leverage standardized by volatility.

An alternative to Merton is Leland (1994) type models, where default is taken endogenously by equityholders who maximize the value of their equity stake. This paper shows that the default event defined by endogenous models can likewise be described in terms of "low" equity prices and "large" negative net cash-flows to shareholders (i.e., debt service and negative earnings minus dividends), which are both observable. Under the Leland (Merton) approach, default is an American (European) put associated to a large and negative flow rather than (a low asset value). The flow view produces many novel insights on default-risk—including Merton's DD counterpart in terms of the dynamics of equity prices and cash-flows.

[1] Subrahmanyam et al. (2014) specifically control by these five covariates to study the credit risk of reference firms upon the inception of CDS trading. See also Tian et al. (2015).

[2] Gilson et al. (1990), Shumway (2001), Garlappi and Yan (2011), or Campbell et al. (2008), among others.

Let us stress that it is essential that negative net cash-flows are paid "by equityholders" (e.g., by stock dilution), otherwise default will be optimally postponed if losses are financed by selling the firm assets. That is, nobody will default in a mortgaged house which monthly renting income covers mortgage payments by itself. It is the same in Merton's model, where equityholders will not cover the difference by which liabilities exceed assets at maturity. In the three cases, default is postponed as long as equityowners don't have to defray losses.[3]

Consider a Leland-type endogenous credit-risk model. Our trick is to study the value of equity near-the-default-point, a 2nd-order Taylor expansion is analytical and it is not necessary to observe the default point. Let V the asset value (or earnings), V_B the default boundary, and σ the asset volatility. Let E be the equity value, we show that

$$E \approx b_B \times a^2, \text{ where } a = \frac{V - V_B}{\sigma V_B}.$$

"$b_B \times dt$" is the firm's negative net cash-flows per unit of time (i.e., debt service and negative earnings minus dividends); "a" is a proper distance-to-default since it is adjusted by volatility. b_B and a represent the firm's financial needs and economic health, respectively. $b = b_B$ is evaluated at the default-point (i.e., $V = V_B$).

We then invert this relationship; i.e.,

$$a^2 \approx E/b. \tag{1}$$

The default event (i.e., a low a, literally $a \to 0$), is equally described by a low E and a large b (a low ratio E/b). It shows why default will hardly happen between coupon or maturity dates (if $b \to 0$, the ratio $E/b \to \infty$ between those dates). b depends on negative earnings, and hence, a simple way to understate default-risk is by hiding/missreporting losses. Since the dividend-yield, which is a positive cash-flow for shareholders, enters negatively in b, it delays default. Similarly, we define a firm as near distress if this ratio is small, e.g., $E/b < 1$.

Equation (1) is robust to a multi-factor model (e.g., stochastic volatility); extends to a nonzero shareholder recovery value or strategic default (where equity is replaced by its time-value); applies to different leveraged securities beyond firms (e.g., mortgages in which case E is not observable but V); and does not depend on the debt profile (e.g., whether roll-over risk is present or not), which affects equity E but no today cash-flows b. It extends to illiquidity

[3] Some may argue that equityholders hardly put any money, the firm always manages to get extra financing and funds (I thank Rafa Repullo for rising this point). Then, in that case, default will be associated to very depressed stock prices since assets will be mostly depleted. A model with cash savings solves this issue, since negative earnings and/or debt service will eat cash savings (see Section 4.2).

and cash constrains, which requires a 2nd state variable and where b depends also on the cost of cash, allowing us to distinguish between insolvency ($b > 0$) and illiquidity ($b \leq 0$)—since no firm will waste positive cash-flows by defaulting (E is small in both cases). In most cases, an insolvent firm ($b > 0$) will be illiquid too, negative cash-flows will eat all cash savings.

From equation (1) we also define and model credit-risk; i.e., the probability of default. For small $\alpha \geq 0$, these two one-period (time $t > 0$) events are alike, i.e.,

$$\left\{ \left(\frac{V_t - V_B}{\sigma V_B} \right)^2 \leq \alpha \right\} \approxeq \{E_t \leq \alpha \times b_t\}.$$

Hence, the probability of default, the probability of the event $\{E_t \leq \alpha \times b_t\}$, depends on the "path" of equity prices and net cash-flows (in addition to E_0/b_0) once that both variables are standardized (and which yields a new distance-to-default).

This is for us the definition of credit-risk and this is how we reconcile Campbell et al. (2008) and Duffie et al. (2007, 2009) results—which basically use the right- and left-hand-side, respectively, of equation (1).[4] The focus on default in the next period is not restrictive, it is the approach of hazard-rate or logit models. A multi-period model is defined by compounding the complement of one-period default (i.e., surviving) events, where the term structure of default probabilities is hard to compute analytically and is beyond the scope of this paper.

The default constraint in a discrete-time setting is likewise given by

$$E \geq b,$$

which does not depend on the dynamics of equity prices or cash-flows (such as jumps). We, however, consider a continuous-time model because there is not DD lhs equality in discrete-time for $E/b \geq 1$ (or $E - b \geq 0$) as equation (1), i.e., whether E/b is small or not in DD units. In Merton's model, $b = 0$ and $E > 0$ is an European call price ($E \geq 0$ at maturity).

This paper's main contribution is to show that Leland-type endogenous models properly explain default—the default event is likewise described by low E and large b—bridging the gap with Merton's view based on low asset values and with reduced-form econometric models of default. To the many popular credit-risk measures (e.g., CDS and corporate bond spreads, Merton's DD based KMV–Moody's model, credit ratings, Altman's Z–score, reduced-form credit-risk hazard and logistic approaches, depressed equity prices E_0, or deep out-of-the-money puts); we add the ratio "E_0/b_0" or better the probability of the event $\{E_t/b_t \leq \alpha\}$.

[4] At a portfolio level, Gordy (2000) drives the analogy between two industry credit-risk models, JP Morgan's CreditMetrics and Credit Suisse's CreditRisk$^+$, by also linking the default events underlying each model.

We do not present any empirical study, but show that our results are consistent with an extensive and rich empirical literature on default risk.

We also derive the static comparative analysis of default-risk from our analytical results. For instance, to lessen default concerns for a distress firm or underwater house (i.e., $E/b < 1$), it helps more to lower the principal (which rises equity value, E) than temporary reductions in mortgages payments b, which is a housing US policy debate (Elul et al. (2010)).

The rest is as follows. Section two presents the paper main results and implications on default-risk in an accessible way as well as links to the literature. Section three formally derives the main results in a one-factor setting and Section four considers a multi-factor setting. Section five studies mortgage default. Section six specializes to Leland and Toft (1996) model and includes a brief numerical exercise. Section seven concludes. Appendix A provides four extensions, strategic default, amortizable debt, stochastic volatility, and liquidity constraints. Appendix B derives some proofs.

2 The Default Event in Endogenous Credit-Risk Models

From equation (1), which provides the default event in terms of a "low" equity prices to negative net cash-flows ratio, we explain the value of, the distance to, and the probability of default. First, the "marginal rate of substitution" between a^2 and b is negative; i.e.,

$$\frac{da^2}{db} \approx \frac{-E}{b^2} < 0.$$

Hence, the largest the financial needs b the lowest the firm's economic health a. The value of (the option to) default depends explicitly on the negative net cash-flows associated to the leveraged security, b. The larger this cost, the larger the value of (and incentives to) default.

Second, $\frac{V-V_B}{\sigma V_B}$ and $\sqrt{E/b}$ are the formal and economic definition of DD, respectively, since near-the-default-point

$$\frac{V - V_B}{\sigma V_B} \approx \sqrt{E/b}.$$

E and b are market- and accounting-based variables, respectively, which are observable. By contrast, $\frac{V-V_B}{\sigma V_B}$ is model dependent, since V, V_B, σ are not observable. In Merton's DD, V_B is approximated by a firm short- and long-term debt mix and $\frac{V-V_B}{\sigma V_B}$ can be negative (while in a continuous-time setting, $V > V_B$ and, hence, $a > 0$).

Default is triggered by depressed equity prices and high debt service and/or negative earnings and both, financial and operational leverage (Garlappi and Yan (2011, p.796)), are default triggers. The ratio E/b is low-biased or conservative for less distress firms, yet both a^2 and E/b are highly correlated since both depend on V. For less risky firms, E/b is not a sufficient statistic and it is necessary to properly standardize this ratio (see below).

Let us illustrate these two properties. Consider that (i) a firm is near default and (ii) b is approximated by today's realized negative net cash-flows. Two applications follow in the cross-section and in the time-series, respectively:

Consider a sample of firms with the same value of equity (after scaling by size), E. The negative marginal rate of substitution implies that, in the cross-section, firms with large financial needs, large b, will default sooner. For a particular firm, if b is not increasing in the value of the underlying asset (or earnings) V, default coincides with large b and low V. Consider now a distress firm which has not defaulted (say) in the last six months and the minimum value of E_t/b_t has happened in the past months $(0 < \min\{E_{-t}/b_{-t}\}_{t=1}^{6} \leq E_0/b_0)$ no today. Then, the firm is more far away from default today than in the past and (probably) will not default in the near month. (See Duffie and Lando (2001) and Giesecke (2004) for related problems from unobservable state variables.)

The E/b ratio, which is dimensionless, is indeed close to the "price–earnings" ratio—a simple measure of how expensive a company is which fluctuates around sixteen for the S&P 500 index. Both ratios differ only in the sign if a company has little debt.

Third, let us approximate the continuous-time setting for a multi-period one. Next, if these two ratios are alike, $\frac{V_t - V_B}{\sigma V_B} \approx \sqrt{E_t/b_t}$ when they are small, these two one-period (time $t > 0$) events are similar too for small $\alpha \geq 0$, i.e.,

$$\left\{\left(\frac{V_t - V_B}{\sigma V_B}\right)^2 \leq \alpha\right\} \approxeq \{E_t \leq \alpha \times b_t\}.$$

α can be seen as the minimum time that equity E_t can support losses at a rate $b_t \times dt$ before defaulting, and allows us to consider both a continuous- and a discrete-time ($\alpha = 1$) setting at the same time. Hence, the probability of default, the probability of the event $\{E_t \leq \alpha \times b_t\}$, depends on the path of equity prices and net cash-flows. If earnings are difficult to predict, the following approximation, which uses standardized returns $\mathcal{Z}$ instead of prices, is intuitive; $b_t \approx b_0 > 0$ and

$$\{E_t \leq \alpha \times b_t\} \approxeq \left\{\underbrace{\frac{\ln(E_t/E_0) - \mu_E}{\sigma_E}}_{\mathcal{Z}(0,1)} \leq -\underbrace{\frac{\ln(E_0/b_0) + \mu_E}{\sigma_E}}_{DD^E} + \frac{\ln \alpha}{\sigma_E}\right\}.$$

This tail event depends on expected (in practice, past) equity returns μ_E and volatility σ_E, debt service and losses (negative profitability) b_0, and equity prices (price per share) E_0; similar to Campbell et al. (2008, Table III and page 2912) five most important covariates. And DD^E is a new distance-to-default based on endogenous models.

Moreover, 1) for severe distress, if $\frac{E_0}{\alpha \times b_0} \approx 1$, this "tail" event is more sensitive to the ratio $\frac{E_0}{b_0}$ than to equity volatility, σ_E. Near distress, if $\frac{E_0}{b_0} < 1$, it is more sensitive to future prospects, E_0, than today losses, b_0. 2) Correlated equity prices, the equity market, paves the way for default correlation in a downturn market (even if cash-flows don't). 3) The event $\{E_t \leq \alpha \times b_0\}$ is equivalent to the payoff of an European digital put, where $\alpha \times b_0$ is the strike price. Carr and Wu (2011) show that American digital puts are similar to pure credit contracts. And 4), the term-structure of default probabilities depends on the compound event

$$\left\{\bigcup_{t>0} \{E_t \leq \alpha \times b_{t-1}\}\right\} \equiv \overline{\left\{\bigcap_{t>0} \{E_t > \alpha \times b_{t-1}\}\right\}},$$

which is equivalent to no surviving. Computing the expectation of this multi-period event (a 1st-passage time problem), which requires to model E and b dynamics, is beyond the scope of this paper. This could be done following, e.g., Duffie et al. (2007) work.

2.1 Implications on Default-Risk

The term b may depend on many other features which alter the default decision; e.g., time-varying borrowing rates. If borrowing rates are floating rates (e.g., adjusted-rates mortgages), the leveraged security will "swing" between being closer (far away) to default when variable rates rise (lower) precisely to avoid "larger" debt service payments. Time-varying rates are also linked to short-term borrowing or roll-over debt (e.g., Leland and Toft (1996)), and a lower credit quality (hence larger refinancing costs) will lead to a large b and earlier default. And the "timing" of default will coincide with payment days (e.g., maturity or coupon-days), which resembles the optimal exercise of American equity options around cash dividend days.[5]

That a large b is associated with default fits well with the empirical evidence on default triggers across asset classes. Campbell and Cocco (2012) report that Adjusted-(Fixed-)Rate Mortgages defaults tend to occur when interest rates are high (low) for moderate declines in house prices; and the clustering of balloon/interest-only loan defaults at maturity. Davydenko (2013) finds that corporate default is associated with economic distress (i.e., low assets value)

[5] See the extreme case of the developer Reyal Urbis, a €4.3 billion—the 2nd largest bankruptcy in Spain, which filed for bankruptcy one day before a €73m tax debt expiry (*Cinco Días* (Madrid), April 1, 2013).

but also negative profitability or high financial costs; and, similarly, the clustering of firm defaults around coupon days. Gopalan et al. (2013) show that default (and lower credit quality) are related to a larger proportion of maturing debt. Performance-sensitive debt also leads to earlier default (Manso et al. (2010)).

In reduced-form econometric models that predict default, most studies focus on Merton's DD, the value of assets at default (Davydenko (2013)), negative equity (Foote et al. (2008)), the gain at default which is a put payoff (Fay et al. (2002) for household bankruptcy), besides a battery of control variables. Our main result says that we should focus on the "path" of equity prices and cash-flows, with the debt payments calendar present, and these two variables explain many robust empirical findings on financial distress. Einav et al. (2012) stress that subprime consumer credit "down payments" limit defaults by screening out high-risk borrowers and lowering loan size—but, at the same time, it rises equity value.

An estimator of $\frac{V-V_B}{\sigma V_B}$ will become redundant if we control by E and b, which explains the little significance of (the alternative) Merton's DD in other studies. Depressed stock prices are closely related to financial distress if we control by b; in Garlappi and Yan (2011) stocks with per-share price less than \$5 are associated with high levels of Moody's KMV expected default frequency.[6] If past returns predict (low) equity prices, they will predict DD too; Duffie et al. (2009, p.2102) wonder why one-year trailing stock returns have a high impact on default intensities. Early papers as Gilson et al. (1990) and Asquith et al. (1994) also find that default is associated with low stock prices and poor firm-specific performance.

In strategic models of default,[7] equity E is replaced by its time-value, i.e., equity value minus equityholders recovery value, (and the effect of the dividend-yield on b, which is negative, is reduced by this recovery value). In addition, we show that the default point V_B rises with this shareholders bargaining payoff, which directly implies that a lowers—default is accelerated. Hence, shareholders residual value matters to predict default (e.g., Garlappi and Yan (2011) show that it explains equity risk and returns of firms in financial distress).

The result $E \approx b \times a^2$ extends to a multi-factor setting. We focus on stochastic volatility and cash reserves, which accounts for both insolvency and illiquidity. Bhamra et al. (2010) and Chen (2010) find that (asset values are procyclical but) earnings at default are countercyclical in a macroeconomic regime switching model. b depends on negative earnings and the negative gain of moving between regimes, hence b (the value of default) is "procyclical" in these models.

[6]The asset-pricing literature relates anomalous returns with financial distress, but these low price stocks are precisely those exclude in most empirical studies to avoid microstructure issues (Garlappi and Yan (2011)).

[7]See Fan and Sundaresam (2000), Davydenko and Strubaylev (2007)), and Favara et al. (2012).

2.2 Default in a multi-factor model: stochastic volatility and cash reserves

In the case of additional state variables (e.g., stochastic volatility), the formula $E \approx b \times a^2$ also holds, DD becomes more complex but is also embedded in the term a. It is not necessary to observe volatility nor its underlying process, though b depends on volatility.

Default can be the result of illiquidity too, and liquidity constraints impact all the financial decisions of a firm (Anderson and Carverhill (2012), Gryglewicz (2011), and Murto and Terviö (2014)). Consider a 2nd state variable (cash reserves) and liquidity constraints. This allows us to integrate insolvency and illiquidity. Now, first, in addition to negative net cash-flows, b also depends on the cost of cash, which is given by the spread between the risk-free rate and the rate earned/paid by cash. The largest this spread, the largest the value of default. Second, b is not constant *but* depends on cash reserves, which lower with earnings at default. Hence, a lower b indicates little cash savings (and less negative earnings) at default; i.e., the value of default is low because it is more related to illiquidity than insolvency.

In the extreme case of default due to illiquidity (e.g., cash is exhausted – borrowing is forbidden), the quadratic formula does not hold (i.e., no smooth-pasting condition). Consider, however, the following situation, a solvent firm *but* which is near default because of severe liquidity concerns; i.e., $-b \geq 0$ but $E > 0$ and small, respectively. Although the formula does not hold, the ratio $E/b \leq 0$, which does not make sense as a distance cannot be negative, but indicates illiquidity. Therefore, we can distinguish between the following three distress scenarios from poor economic and financial conditions: a large, smaller, and low or negative b imply insolvency, a mix of insolvency and illiquidity, and illiquidity, respectively.[8] Yet, insolvency and illiquidity must be closely related, since low cash savings depend mainly on two factors: past negative earnings (and/or a low cash initial endowment).

[8] Two remarks on cash. 1) If cash is not costly, the value of default is the same in models with or without cash constraints. We show this in Gryglewicz (2011), who consider both cases and where the intrinsic value of default is not linear in the cash constraint model. (The no linearity cancels by using the equity time-value) 2) In the last three models, the payout rate is given by earnings, which can be negative, and default is associated to a combination of illiquidity, debt service, and operating losses. In Leland or Leland-Toft, the payout rate is a dividend-yield, which is no negative, and default allows stopping debt service.

In sum, in its more general form, the default event is likewise described by

$$\left\{ \left\{ \frac{E_t - \phi_t}{E_0} \leq \alpha \times \frac{\overbrace{\text{debt service} + \text{losses} - \text{dividends} + \text{cost of cash} \times \text{cash}}^{b_t}}{E_0} \right\} \cup \left\{ \text{cash} \leq \underline{C} \right\} \right\},$$

ϕ_t is shareholders recovery value and $\underline{C} \leq 0$ a lower risk-free, if expensive, firm dependent borrowing limit. See Davydenko (2013) for an empirical study of insolvency and illiquidity of firms in financial distress and Campbell and Cocco (2012) in the case of mortgages (which are known as default dual triggers). Default probabilities also depend on cash holdings in Campbell et al. (2008).

Our paper has a similar perspective on default as Cornell, Schwartz, and Longstaff (1996), as made clear in their title "Throwing Good Money After Bad? Cash Infusions and Distressed Real State," Carr and Wu (2011) American cancellable call, or Hellwig and Lorenzoni (2009) equilibrium model with debt constraints, for whom no defaulting means "..positive levels of debt are sustainable in our environment because the interest rate is sufficiently low to provide repayment incentives." There is a literature on the determinants of default based on one-period models (Bulow and Shoven (1978), Foote et al. (2008)).[9]

2.3 Default in a discrete-time model

The "default" constraint in a discrete-time setting is simply given by

$$E \geq b.$$

Predicting default one-period ahead (of length t), distance-to-default is given by

$$DD^E = \frac{\ln(E_0/b_0) + \mu_E \times t}{\sigma_E \times \sqrt{t}}, \text{ since } \left\{ \mathcal{Z}(0,1) \leq -DD_0^E + \frac{\ln \alpha}{\sigma_E \sqrt{t}} \right\},$$

where we make explicit the dependence of equity volatility on t and $b_t \approx b_0 > 0$ (and $\alpha = 1$).

Let us show how DD^E and the associated default event, which depends on α, include both a continuous- and a discrete-time setting. If $E_0/b_0 < 1$ (i.e., $\ln(E_0/b_0) < 0$), DD^E becomes large and negative if t is small, where a small t also approximates a continuous-time setting. This is precisely our definition of default risk in continuous-time—a small E_0/b_0. On the other, if either E_0/b_0 or t are not so small (the 2nd-order approximation is not so accurate

[9] Our results apply to He and Xiong (2012) roll-over risk model and hold for a given capital structure, which is robust if leverage is not optimally chosen (in Morellec et al. (2013) leverage depends on selfish managers). In all cases, default is determined by smooth-pasting.

or a discrete-time setting, respectively), E_0/b_0 is not a substitute of DD^E. $\alpha = 1$ does not change these implications, $\alpha \to 0$ implies that E_0/b_0 must be really small in continuous-time.

Martin et al. (2014) derive two reciprocal "liquidity" and "collateral" constraints in a equilibrium model of repo runs and study (the fragility of) different repo markets from the static comparative of these two constraints; finding "...more leveraged borrowers, or less profitable ones, are more fragile (p.967)," which is the meaning of b. This is basically what we do with the default constraint in a continuous-time model. Indeed, the static comparative of the three illiquidity, collateral, and default constraints in their model is the same; hence, more fragile borrowers are also riskier (and illiquidity always precedes insolvency, which is ruled out in their model).

2.4 Mortgages and sovereigns

Mortgages and sovereigns are two cases where borrowing rates largely determine default. Mortgages are long-term (gradually amortized) loans and renting has a positive value; sovereigns regularly roll-over debt and default is costly (Gennaioli et al. (2014)).[10] In the case of mortgages, Foote et al. (2008) and Bhutta et al. (2010) show in a one-period model that default depends on mortgage minus renting payments, which is precisely the meaning of b.

For sovereigns, the borrowing rate may contain a sizable credit-spread. This is evident nowadays in Europe where the view is a country is bailout, no because is deeply indebted and in recession, *but* because the debt service is unsustainable; e.g., bailout triggered if borrowing rates hit a 7%–8% threshold (Financial Times (2011)). If $E > 0$, $E/b \to 0$ only if b hikes. In this context, b is like an "austerity cost" suffered by taxpayers. No surprisingly both the Fed and the ECB have capped interest-rates to all time lows by all means.

3 Default in Endogenous Models: the Equity Value Near-the-Deault-Point

Solving free-boundary problems requires numerical methods and static comparative is lost. Even from a explicit formula for perpetual cases, if highly non lineal, it can be difficult to get the proper intuition. American options, however, simplify near the exercise boundary. We apply this observation to endogenous credit-risk models, which allows us to understand the value of (the option to) default.

[10] Mortgage default is also costly in Spain (or Sweden)—homeowners cannot mail the keys and walk away from mortgage debt actually can be ejected from their homes. This has created social stress in recession times.

We denote by E the equity value and by V the underlying state variable, e.g., asset values or earnings. We consider a standard free-boundary problem (Duffie (2001, ch.11)), which can be time-unhomegenous and the option (i.e., equity) maturity can be finite. Under the pricing or Q–risk-neutral measure, the dynamics of V is given by

$$dV_t = \mu(V_t)\,dt + \sigma(V_t)\,dZ_t,$$

where dZ is a Wiener processes. Ito's Lemma implies that $E(t, V_t)$ solves the following PDE in the continuation (i.e., no defaulting) region

$$rE = E_t + \mu(V_t)\,E_V + \frac{1}{2}\sigma^2(V_t)\,E_{VV} + NC_t(V_t), \tag{2}$$

where $NC_t(V_t)$ is the net cash-flow to equityholders and r the risk-free rate. In endongenous credit-risk models, NC_t is the payout rate minus the net debt service, after tax-payments. The payout rate is the dividend-yield if V is the asset value instead of earnings (e.g., Leland (1994), Leland and Toft (1996), or He and Xiong (2012)).[11]

NC_t is the key variable to understand default. NC_t can be, no only time unhomogeneous, but, the result of an optimization process; e.g., optimal cash reserves, dividend policy, or consumption. This allows us to apply our results to a wide range of models, besides the standard trade-off theory, and to take NC_t as given.[12]

Let denote by $e(V)$ the intrinsic value and by $V_B(t)$ the optimal default boundary. If $V_t = V_B(t)$, E satisfies the following two boundary conditions; i.e.,

$$E(t, V_B(t)) = e(V_B(t)) \;\; \text{and} \;\; E_V(t, V_B(t)) = e_V(V_B(t)), \tag{3}$$

value-matching and smooth-pasting, respectively. It also holds that $E_t(t, V_B(t)) = 0$ (i.e., $E_t + E_V V_B' = e_V V_B'$). If $e \neq 0$, e is the equityholders residual value in default.

From equations (2) and (3), for $V = V_B$, the option Gamma at the boundary is given by

$$\frac{1}{2}E_{VV}(V_B) = \frac{-NC_t + \left(r - \mu(V_B)\frac{e_V}{e}\right)e}{\sigma^2(V_B)}, \tag{4}$$

where we omit the arguments, the numerator is equal to $-NC_t$ if $e(t, V) = 0$. Consider the continuation (i.e., no defaulting) region of equity value,

$$\{(t, V) : E(t, V) \geq e(V_B(t))\}.$$

[11] As it is well known, a similar PDE holds in real option models (Dixit and Pindyck (1994, ch.7)).

[12] The significance of NC_t has been recognized elsewhere, from Leland and Toft (1996) and Cornell, Longstaff, and Schwartz (1996) to He and Milbradt (2012) and Manso et al. (2010), we fully exploit it.

From a 2nd-order Taylor expansion in V_t, with error $\mathrm{O}(V_t - V_B)^3$,

$$E(t, V_t) \approx e(V_B) + e_V(V_B) \times (V_t - V_B) + \frac{1}{2} E_{VV}(V_B) \times (V_t - V_B)^2 . \tag{5}$$

Equity beta near default As a simple application, it is easy to derive the equity beta; i.e.,

$$\frac{E_V}{E} \approx \frac{e_V(V_B) + E_{VV}(V_B) \times (V_t - V_B)}{e(V_B) + e_V(V_B) \times (V_t - V_B) + \frac{1}{2} E_{VV}(V_B) \times (V_t - V_B)^2}.$$

If we assume that $E_{VV}(V_B) > 0$, in the limit $(\lim V_t \downarrow V_B)$,

$$\frac{E_V}{E} \approx \frac{1}{V_t - V_B} \to \infty \text{ if } e(V_B) = 0; \; \textit{but} \; \frac{E_V}{E} = \frac{e_V(V_B)}{e(V_B)} < \infty \text{ if } e(V_B) > 0,$$

and $\frac{E_V}{E} = 0$ if $e(V_B) = c \neq 0$. This radically different behavior of equity risk near default, which depends on shareholders recovery value $e(V_B)$, is also derived in Garlappi and Yan (2011) by other means and use to link financial distress and equity returns in a novel way. ■

Instead of equity prices, we consider the "time-value" of equity; i.e., equity minus the intrinsic value,

$$E^{TV}(t, V) = E(t, V_t) - \left(e(V_B) + e_V(V_B) \times (V_t - V_B) + \frac{1}{2} e_{VV}(V_B) \times (V_t - V_B)^2 \right).$$

We include (here) the case that the intrinsic value is not lineal at the boundary, $e_{VV}(V_B) \neq 0$ (e.g., Gryglewicz (2011)). Note that $E^{TV} > 0$ since, by definition, the value of equity is larger than the shareholders recovery value in the continuation region.

Lemma 1 A 2nd-order Taylor approximation of the time-value of equity is given by

$$E^{TV}(t, V) \approx \left(-NC_t + \left(r - \mu(V_B) \frac{e_V(V_B)}{e(V_B)} \right) e(V_B) - \frac{1}{2} e_{VV}(V_B) \sigma^2(V_B) \right) \times \left(\frac{V_t - V_B}{\sigma(V_B)} \right)^2 . \; ■ \tag{6}$$

In Ibáñez and Paraskevopoulos (2010) equation (6) left-hand-side represents the cost of suboptimal exercise of one American option and is given by the product of two terms: (i) the sensitivity to suboptimal exercise and (ii) the bias of the exercise policy, respectively. Ibáñez and Velasco (2012) extend this result to a multi-factor setting. We now provide three major applications of the time-value of equity to default-risk.

3.1 The value of default

We rename equation (6) as follows, assuming $e_{VV} = 0$,

$$E^{TV}(t, V_t) \approx \left(-NC_t + \left(r - \mu(V_B) \frac{e_V}{e} \right) e \right) \times \left(\frac{V_t - V_B}{\sigma(V_B)} \right)^2 = b_t \times a_t^2, \tag{7}$$

where $a_t = \frac{V_t - V_B}{\sigma(V_B)} > 0$ is the volatility-adjusted distance-to-the-default-boundary and b_t is the marginal value of default. b_t and a_t represent the financial needs and the economic health of the firm, respectively. b_t depends on negative net cash-flows, $-NC_t$. If equity holders can negotiate some value at default, $e(V_B) > 0$, b_t is adjusted by the dividend yield $r - \mu$ (since μ is the drift under Q). $E^{TV} > 0$ implies that $b > 0$.

Proposition 1 The "elasticity" or "marginal rate of substitution" between a and b is negative. That is,

$$0 = \frac{\partial E^{TV}}{\partial b} + \frac{\partial E^{TV}}{\partial a} \times \frac{da}{db} = a^2 + 2ba \times \frac{da}{db} \tag{8}$$

and

$$\frac{da}{db} = \frac{-a}{2b} \quad \text{and} \quad \frac{d \ln a}{d \ln b} = \frac{da/a}{db/b} = \frac{-1}{2} < 0. \quad \blacksquare \tag{9}$$

This implies that, for a fixed time-value of equity E^{TV}, "the larger the financial needs b_t, the lower the economic health a_t." A corollary follows for a cross-section of firms.

Corollary Consider two firms with the same time-value of equity, E^{TV}. Then, the firm with larger b will (i.e., is expected to) default first/sooner because it has a lower a.[13] $\blacksquare$

b_t is not a function of the asset value V_t but a single point which is evaluated at the boundary V_B; i.e., $b_t(V_B)$. Consider a function $b_t(V_t)$ and note that equity value is defined in the continuation region, $V_t \geq V_B$. If $b_t(V_t)$ is not increasing in V_t (i.e., $\frac{db_t(V_t)}{dV_t} \leq 0$) then $b_t(V_t)$ is maximized by $V_t = V_B$. Hence, default does not only coincide with low(est) asset value $V_t = V_B$ *but* with large(st) negative cash-flows $b_t(V_B)$ within the continuation region. (In Leland model $\frac{db_t(V_t)}{dV_t} = 0$, and in Leland-Toft $\frac{db_t(V_t)}{dV_t} < 0$).

Proposition 2 Consider the continuation region, $V_t \geq V_B$. If $\frac{db_t(V_t)}{dV_t} \leq 0$, default coincides with the lowest assets/earnings value and the largest negative cash-flows, V_B and $b_t(V_B)$, respectively. $\blacksquare$

3.2 Distance-to-default and its economic counterpart

The volatility-adjusted distance-to-the-default-point is a proper distance-to-default (DD), i.e.,

$$a_t = \frac{V_t - V_B}{\sigma(V_B)}. \tag{10}$$

[13] Consider two companies x and y, which are near default (i.e., $E_x \approx b_x \times a_x^2$ and $E_y \approx b_y \times a_y^2$), and where $E_x = E_y$. Hence, $\frac{a_x^2 - a_y^2}{b_x - b_y} \approx \frac{-E_x}{b_x b_y} < 0$; and, same way, $\lim_{b_y \to b_x} \frac{a_x^2 - a_y^2}{b_x - b_y} = \frac{da^2}{db}$ and $\lim_{b_y \to b_x} \frac{-E_x}{b_x b_y} = \frac{-E_x}{b_x^2} < 0$.

For instance, this DD is valid for a lognormal (an arithmetic) process, which are commonly used when V_t is the asset value (operating revenues) of the firm, where $\sigma(V_B) = \sigma \times V_B$ (and $\sigma(V_B) = \sigma$), which, intuitively, imply the distance to V_B in relative (in absolute) terms.

The 2nd-order approximation in equation (6) (with $e_{VV}(V_B) = 0$) has the solution

$$\frac{V_t - V_B}{\sigma(V_B)} \approx \sqrt{\frac{E^{TV}(t, V_t)}{-NC_t + \left(r - \mu(V_B)\frac{e_V}{e}\right)e}}. \tag{11}$$

This solution is implicit,[14] but if equity value is zero at default (i.e., $e(V_B) = 0$ and $e_V(V_B) = 0$), it simplifies to

$$\frac{V_t - V_B}{\sigma(V_B)} \approx \sqrt{\frac{E(t, V_t)}{-NC_t}}. \tag{12}$$

On the lhs, $a_t = \frac{V_t - V_B}{\sigma(V_B)} \geq 0$, is a measure of firm's leverage (both financial and operational leverage) over volatility.[15] On the rhs, the DD's economic counterpart is given by equity prices over the minus net cash-flow, $\frac{E(t,V_t)}{b_t} = \frac{E(t,V_t)}{-NC_t}$, and is observable (but conservative).[16] This ratio depends on negative earnings in the denominator; and hence, a simple way to "understate" default-risk is by hiding or missreporting losses.

Merton's DD is adjusted by a mean and volatility. Using standard Black-Scholes notation (V_t is asset value, K strike price, σ volatility, μ drift, and τ maturity), Merton's DD is given by

$$D^M = \frac{\ln(V_t/K) + \mu\tau}{\sigma\sqrt{\tau}}.$$

Merton's DD is implicit in the price of a Black-Scholes European call and depends only on financial leverage; i.e., the call strike price and maturity are approximated by a mix of

[14] The solution is given by the largest root; i.e., $\frac{V_t - V_B}{\sigma(V_B)} \approx$

$$\frac{-e_V(V_B)\sigma(V_B)}{2\left(-NC_t + \left(r - \mu(V_B)\frac{e_V}{e}\right)e\right)} + \frac{\sqrt{\left(e_V(V_B)\sigma(V_B)\right)^2 - 4\left(-NC_t + \left(r - \mu(V_B)\frac{e_V}{e}\right)e\right) \times \left(e(V_B) - E(t, V_t)\right)}}{2\left(-NC_t + \left(r - \mu(V_B)\frac{e_V}{e}\right)e\right)}.$$

[15] A case in point is BlackBerry's CFO Thorsten Heins, after a near-\$1bn fiscal 2nd-quarter loss, "We're very disappointed with our operational and financial results this quarter (and have announced a series of major changes..) but we remain a financially strong company with \$2.6bn in cash and no debt." At the same time that it issued its profit warning last Friday, the company also announced the planned lay-off of 4,500 of its remaining 11,700 employees by the end of February. "BlackBerry eyes turnround as results disappoint" by Paul Taylor (Financial Times, September 27, 2013).

[16] Formally, because of the 2nd-order approximation, the ratio E/b is conservative or lower-biased (i.e., $\sqrt{E/b} < \frac{V - V_B}{\sigma V_B}$). However, since E/b is unbiased at the boundary ($\sqrt{E/b} = \frac{V - V_B}{\sigma V_B} = 0$, if $V \to V_B$), it follows that E/b classifies or disentangles "perfectly" between defaulting and no defaulting firms. In addition, $\frac{V - V_B}{\sigma V_B}$ and $\sqrt{E/b}$ are highly correlated, both rise with V. It is a fact that investment grade firms hardly default.

short- and long-term debt. Merton's probability of default is given by the normal cumulative probability, $\Phi\left(-D^M\right)$. Next, we show that a similar drift and volatility adjustment happens to the ratio $\frac{E(t,V_t)}{b_t}$ when it is used to predict default.

Merton's D^M is negative for high leveraged firms, but $a \geq 0$ which is due to the European- and American-style (discrete- and continuous-time, respectively) default assumption.

3.3 The probability of default

From the DD we can obtain the associated default probability. Let E^P denote expectation under a P–measure (e.g., the $\mathcal{P}$–objective or the Q–risk-neutral measures). The quantile

$$p(t) = E_0^P\left[1_{\left\{a_t^2 \leq \alpha\right\}}\right]$$

yields the probability that a firm is less than $\sqrt{\alpha} > 0$ units from default by time $t > 0$. We assume that there are not (important) cash outflows between 0 and t.

From equation (12), let $\alpha > 0$ be a small number such that these two events are alike

$$\left\{a_t^2 \leq \alpha\right\} \approxeq \left\{E_t/b_t \leq \alpha\right\},$$

where (strictly speaking) default corresponds to $\alpha \to 0$. Then,

$$p(t) \approx E_0^P\left[1_{\{E_t \leq \alpha \times b_t\}}\right].$$

To compute the probability $p(t)$, we just need to model equity prices which are observable (and for which there are many models from the option-pricing literature) and net cash-flows. This expectation, $E_0^Q\left[1_{\{E_t \leq \alpha \times b_t\}}\right] \times e^{-rt}$, is also the price of an (out-of-the-money) binary European put.

Let denote by μ_E and σ_E^2 the first two moments of equity returns, i.e.,

$$\mu_E - \frac{\sigma_E^2}{2} = \frac{1}{t}E_0^P\left[\ln\frac{E_t}{E_0}\right] \quad \text{and} \quad \sigma_E^2 = \frac{1}{t}E_0^P\left[\left(\ln\frac{E_t}{E_0} - E_0^P\left[\ln\frac{E_t}{E_0}\right]\right)^2\right],$$

then these two α–events are equivalent; i.e.,

$$\{E_t \leq \alpha \times b_t\} \equiv \left\{\underbrace{\frac{\ln\frac{E_t}{E_0} - \left(\mu_E - \frac{\sigma_E^2}{2}\right)t}{\sigma_E\sqrt{t}}}_{\mathcal{Z}(0,1)} \leq -\underbrace{\frac{\ln\frac{E_0}{b_t} + \left(\mu_E - \frac{\sigma_E^2}{2}\right)t}{\sigma_E\sqrt{t}}}_{D_{0,t}^E} + \frac{\ln\alpha}{\sigma_E\sqrt{t}}\right\}, \tag{13}$$

where $\mathcal{Z}(0,1)$ is a standardized, zero mean and unit variance, random variable (where $b_t > 0$).

In a one-factor model, b_t is evaluated at the default boundary and hence deterministic (thought, it will be random in a multi-factor model, e.g., stochastic volatility from which b depends upon). We assume that these three default events are equivalent (for $\alpha \to 0$),

$$\{E_t (V_t) \leq b_t (V_B) \times \alpha\} \equiv \{E_t (V_t) \leq b_t (V_t) \times \alpha\} \equiv \{V_t \leq V_B\} .$$

In equation (12) we define default based on the 1st event; but since V_B is unobservable, we use the 2nd event in practice (by assuming $b_t (V_t)$ is observable).

Moreover, since p is a cumulative probability (from $-\infty$ until $-D^E_{0,t}$),

$$\frac{dp}{dD^E_{0,t}} = -p' < 0,$$

and since equity returns are standardized, the larger $D^E_{0,t}$ the lower default-risk (i.e., the lower p). The new distance-to-default $D^E_{0,t}$ has a clear static comparative analysis.

3.4 The $D^E_{0,t}$ static comparative

The larger (lower) E_0 and μ_E (b_t and σ_E) the larger this distance, $D^E_{0,t}$. For σ_E this holds only if the firm is far from distress, i.e., $\frac{E_0}{\alpha \times b_t} > 1$. Because μ_E is not observable, it is intuitive to use past equity or index returns to estimate it. a_t is volatility-adjusted ($a_t = \frac{V_t - V_B}{\sigma(V_B)}$), but $D^E_{0,t}$ also depends on equity volatility σ_E; and σ_E can be estimated from realized (implied) volatilities under the $\mathcal{P}$ (Q) measures. The term b indicates the potential losses and debt service that equityholders can finance; if b is small, equity prices need to collapse so that the firm is near default. If earnings are difficult to predict, we can assume that $b_t \approx b_0$. Moreover, $D^E_{0,t}$ is also valid in a discrete-time setting; simply $\alpha = 1$ and b_t is the cash-flow at time t (instead of the flow $b_t \times dt$ in continuous-time), see Section 3.6.

Computing partial derivatives (and abstracting from the mean term, $(\mu_E - \frac{\sigma_E^2}{2}) \approx 0$),

$$\begin{aligned} \frac{dD^E_{0,t}}{d\frac{E_0}{b_t}} &= \frac{\left(\frac{E_0}{b_t}\right)^{-1}}{\sigma_E \sqrt{t}} > 0 \ , \\ \frac{dD^E_{0,t}}{d\sigma_E} &= \frac{\ln \frac{E_0}{\alpha \times b_t}}{\sigma_E \sqrt{t}} \times \frac{-1}{\sigma_E} < 0 \text{ (or } > 0) \text{ if } \frac{E_0}{\alpha \times b_t} > 1 \ (< 1). \end{aligned}$$

Three implications follow:

1) $D^E_{0,t}$ is more sensitive to the ratio $\frac{E_0}{b_t}$ than to equity volatility σ_E if

$$\left(\frac{E_0}{b_t}\right)^{-1} > \left|\ln \frac{E_0}{\alpha \times b_t}\right| \times \frac{1}{\sigma_E} \iff \sigma_E > \frac{E_0}{b_t} \times \left|\ln \frac{E_0}{\alpha \times b_t}\right| ,$$

e.g., if the firm is near distress, $\frac{E_0}{\alpha \times b_t} \approx 1$. The opposite holds if $\frac{E_0}{b_t} \to 0$ or $\frac{E_0}{b_t} \to \infty$, a severe distress or a very healthy firm, respectively; large equity volatility will help in both scenarios, to escape from and to bring a firm closer to default.

2) Consider a distress firm, $\frac{E_0}{b_t} < 1$.

$$\text{If } \frac{E_0}{b_t} < 1 \ , \ \frac{dD_{0,t}^E}{dE_0} \sim E_0^{-1} > b_t^{-1} \sim \frac{-dD_{0,t}^E}{db_t}. \tag{14}$$

For a near distress firm, it helps more to rise future prospects than to reduce costs (E and b, respectively). In the case of underwater mortgages, it calls for reducing the principal (which rises equity value) better than interest-rate payments. In the case of indebted sovereigns, long-term structural funds better than lower interest rates.

And 3), consider only financial leverage, $b_0 = B_0 \times r_b$, where B_0 is the total debt, r_b the debt service rate, and $\ln \frac{E_0}{B_0}$ is financial leverage. Then, if all debt must be repaid, $r_b = 1 + \mu_b$, where μ_b is the debt cost (and $\mu_b \approx \ln(1 + \mu_b)$),

$$\frac{\ln \frac{E_0}{b_0} + \mu_E}{\sigma_E} \approx \frac{\ln \frac{E_0}{B_0} + \mu_E - \mu_b}{\sigma_E},$$

which depends on leverage and a carry cost between equity and debt, $\mu_E - \mu_b$. But if all debt is roll over $r_b = \mu_b$ (and $\mu_E - \ln \mu_b \approx -\ln \mu_b$),

$$\frac{\ln \frac{E_0}{b_0} + \mu_E}{\sigma_E} \approx \frac{\ln \frac{E_0}{B_0} - \ln \mu_b}{\sigma_E},$$

which is very sensitive to the debt cost, μ_b.

The same way, the term-structure of default probabilities (i.e., a first-passage time) depends on the compound event

$$\left\{ \bigcup_{t>0} \{E_t \leq \alpha \times b_t\} \right\} \equiv \overline{\left\{ \bigcap_{t>0} \{E_t > \alpha \times b_t\} \right\}},$$

which is equivalent to no surviving. A simple example, lognormal equity prices. To simplify matters, we assume that b_t is deterministic and no cash outflows between $t-1$ and t, so we focus on equity prices. We work out default probabilities one- and two-periods ahead of length Δt (where $D_{t-1,t}^E = (\ln \frac{E_{t-1}}{\alpha \times b_t} + (\mu_E - \frac{\sigma_E^2}{2})\Delta t)/(\sigma_E \sqrt{\Delta t})$).

Lognormal prices If we assume that $\ln \frac{E_t}{E_{t-1}} = \mathcal{N}\left((\mu_E - \frac{\sigma_E^2}{2})\Delta t, \sigma_E \sqrt{\Delta t}\right)$, then

$$p(1) \approx \Phi\left(-D_{0,1}^E + \frac{\ln \alpha}{\sigma_E \Delta \sqrt{t}}\right) = 1 - \Phi\left(D_{0,1}^E - \frac{\ln \alpha}{\sigma_E \Delta \sqrt{t}}\right),$$

where $\Phi\left(D_{0,1}\right)$ is the surviving probability (and $\Phi\left(-D_{0,1}\right)$ is related to a digital put price). For two periods, the probability of the defaulting α–event (i.e., one minus the surviving probability) is given by

$$p\left(2\right) = 1 - \int_{b_1\times\alpha}^{\infty} \frac{1}{E_1\sqrt{2\pi}\sigma_E\sqrt{\Delta t}} \times e^{\frac{-1}{2}\frac{\left(\ln\frac{E_1}{E_0}-\left(\mu_E-\frac{\sigma_E^2}{2}\right)\Delta t\right)^2}{\sigma_E^2\Delta t}} \times \Phi\left(D_{1,2}^{E} - \frac{\ln\alpha}{\sigma_E\Delta\sqrt{t}}\right) \times dE_1.$$

Joint default-risk We can also address the joint default-risk of two risky firms, x and y. That is, near-the-default-point, "their" joint default risk $a_{x,t} \times a_{y,t}$ is given by

$$\frac{V_{x,t}-V_{x,B}}{\sigma_x\left(V_{x,B}\right)} \times \frac{V_{y,t}-V_{y,B}}{\sigma_y\left(V_{y,B}\right)} \approx \sqrt{\frac{E_x(t,V_{x,t})}{b_{x,t}} \times \frac{E_y(t,V_{y,t})}{b_{y,t}}}$$

and their joint default α–event by

$$1_{\left\{a_{x,t}^2\leq\alpha\right\}} \times 1_{\left\{a_{y,t}^2\leq\alpha\right\}} \approxeq 1_{\{E_{x,t}\leq\alpha\times b_{x,t}\}} \times 1_{\{E_{y,t}\leq\alpha\times b_{y,t}\}}.$$

In sum, if we want to predict default, equity prices E_0, expected (or past) returns μ_E, equity volatility σ_E, and debt and past losses b_0 (with a degree of tail-event α) should predict default.[17] Equity prices E, the equity market, paves the way for a large correlation or contagion between two firms in a downturn market, even if cash-flows b do not. See Duffie et al. (2007) and (2009) for default-intensity models of a portfolio credit-risk.

3.5 DD's static comparative to structural parameters

We take into account that V_B is an endogenous variable and depends on the structural parameters. From the equality

$$d\left(a^2\right) \approx d\left(E/b\right),$$

where $d\left(\right)$ denotes differential, we can derive the DD's static comparative from either the left- or the right-hand-side. We assume that $\sigma(V_B) = \sigma \times V_B$.

The default boundary, V_B The default point is a key input but is not observable and is endogenous (see Davydenko (2012) for an empirical study). It follows that

$$\frac{d\left(\frac{V-V_B}{\sigma V_B}\right)}{dV_B} = \frac{-V}{\sigma V_B^2} < 0,$$

[17] See Doshi et al. (2013) for default-intensity models based on affine dynamics for the state variables and Bharath and Shumway (2008) and Campbell et al. (2008)) for hazard and logistic approaches. See Leland (2004) for a comparative of default probabilities in Merton (1974) and Leland and Toft (1996) models. See Duffie and Singleton (2003) for a review of credit-risk.

and if exercise boundary goes up (i.e., $dV_B > 0$), DD lowers and default-risk rises. If the exercise boundary $V_B(x)$ depends on a structural parameter x ($x \neq \sigma$), $dV_B(x) = \frac{\partial V_B}{\partial x} dx$ and $\frac{\partial V_B}{\partial x}$ determines default-risk. That is,

$$\frac{d\left(\frac{V-V_B}{\sigma V_B}\right)}{dx} = \frac{-V}{\sigma V_B^2}\frac{\partial V_B}{\partial x} \implies \operatorname{sign}\left(\frac{da}{dx}\right) = -\operatorname{sign}\left(\frac{\partial V_B}{\partial x}\right).$$

On the other hand, where $b = b\left(x, V_B\left(x\right)\right)$,

$$\frac{d\left(E/b\right)}{dx} = \frac{bE_x - E\frac{db}{dx}}{b^2} = \frac{bE_x - E\left(b_x + \frac{\partial b}{\partial V_B}\frac{\partial V_B}{\partial x}\right)}{b^2},$$

and, from $d\left(\frac{V-V_B}{\sigma V_B}\right)^2/dx \approx d\left(E/b\right)/dx$,

$$\begin{aligned}
-2a\frac{V}{\sigma V_B^2}\frac{\partial V_B}{\partial x} &\approx \frac{bE_x - E\left(b_x + \frac{\partial b}{\partial V_B}\frac{\partial V_B}{\partial x}\right)}{b^2}, \text{ and} \\
\frac{\partial V_B}{\partial x} &\approx \left(\frac{-\frac{\partial b}{\partial V_B}E}{b^2} + 2a\frac{V}{\sigma V_B^2}\right)^{-1} \times \frac{E}{b} \times \left(\frac{b_x}{b} - \frac{E_x}{E}\right).
\end{aligned}$$

Corollary Let $\frac{\partial b}{\partial V_B} \leq 0$, which is reasonable since b is given by negative net cash-flows and V are the firm's assets or earnings (e.g., Leland and Leland–Toft models). Then, if $x \neq \sigma$,

$$\operatorname{sign}\left(\frac{da}{dx}\right) = -\operatorname{sign}\left(\frac{\partial V_B}{\partial x}\right) \approx -\operatorname{sign}\left(\frac{b_x}{b} - \frac{E_x}{E}\right). \quad \blacksquare \tag{15}$$

The tradeoff between the elasticities of the "value of default, b" and "equity prices, E" determines the sensitivity of the optimal default boundary. If this tradeoff is positive (negative), the exercise boundary rises (lowers)—and the DD lowers (rises). This result is convenient since V_B is endogenous.

Two cases. 1) if $b_x = 0$ (e.g., x is interest-rates),

$$\operatorname{sign}\left(\frac{da}{dx}\right) = -\operatorname{sign}\left(\frac{\partial V_B}{\partial x}\right) \approx \operatorname{sign}\left(E_x\right),$$

and the equity price determines DD by itself.

And 2), if $\operatorname{sign}(b_x) = -\operatorname{sign}(E_x)$, e.g., in the case of cash-flow parameters,

$$\operatorname{sign}\left(\frac{da}{dx}\right) = -\operatorname{sign}\left(\frac{\partial V_B}{\partial x}\right) \approx -\operatorname{sign}\left(b_x\right),$$

which is also the intuition of the negative marginal rate of substitution between a and b.

With regard to volatility, $x = \sigma$,

$$\frac{d\left(\frac{V-V_B}{\sigma V_B}\right)}{d\sigma} = -VV_B\frac{1 + \frac{\partial \ln V_B}{\partial \ln \sigma}}{\left(\sigma V_B\right)^2} + \frac{1}{\sigma^2} = \frac{-VV_B}{\left(\sigma V_B\right)^2}\left(1 + \frac{\partial \ln V_B}{\partial \ln \sigma} - \frac{V_B}{V}\right).$$

Default-risk depends on the boundary–volatility elasticity, which is negative in Leland-type models $\frac{\partial \ln V_B}{\partial \ln \sigma} < 0$. Two clear cases, close to and far away from default, respectively,

$$\text{if } V \rightarrow V_B, \ \text{sign}\left(\frac{da}{d\sigma}\right) = -\text{sign}\left(\frac{\partial \ln V_B}{\partial \ln \sigma}\right) > 0, \ \text{and}$$
$$\text{if } V \rightarrow \infty, \ \text{sign}\left(\frac{da}{d\sigma}\right) = -\text{sign}\left(1 + \frac{\partial \ln V_B}{\partial \ln \sigma}\right).$$

For σ sufficiently large, $-1 < \frac{\partial \ln V_B}{\partial \ln \sigma} < 0$ and $d\left(\frac{V-V_B}{\sigma V_B}\right)/d\sigma < 0$. Note how we address both cases σ_E and σ_V separately, equity volatility and asset volatility.[18]

Two companies with the same value of equity, E We apply Proposition 1; i.e., $da^2/db < 0$. From $db = \frac{db}{dx}dx$, and $\frac{db}{dx}$ determines default-risk for two firms with the same equity price. For any parameter x (including $x = \sigma$), we have that

$$\frac{db}{dx} = b_x + \frac{\partial b}{\partial V_B}\frac{\partial V_B}{\partial x},$$

but $\text{sign}\left(\frac{db}{dx}\right)$ is not easy to pin down.

Let $\frac{\partial b}{\partial V_B} < 0$ as Leland-Toft model. In particular, if $b_x = 0$ (e.g., interest-rates or volatility),

$$\text{sign}\left(\frac{db}{dx}\right) = -\text{sign}\left(\frac{\partial V_B}{\partial x}\right).$$

In the case of assets volatility, if $\frac{\partial V_B}{\partial \sigma} < 0$, then $\frac{db}{d\sigma} > 0$; i.e., the firm with larger volatility is closer to default for the same equity prices. This result is relevant as we could not determine if $\frac{\partial \ln V_B}{\partial \ln \sigma} < -1$.

In Leland model, where $\frac{\partial b}{\partial V_B} = 0$ (if the dividend-yield is zero), it follows that

$$\frac{db}{d\sigma} = b_\sigma = 0.$$

Consequently, two zero-dividends Leland-type firms, which have the same equity price, have the same DD although they have different volatility. This is another example where equity prices determine default-risk. In Leland-Toft this does not necessarily hold, since the cost of roll-over debt depends on V (and ultimately on V_B).

[18] For other parameters or variables, V and E, the static comparative si as follows.

For assets value, V,

$$\frac{d\left(\frac{V-V_B}{\sigma V_B}\right)}{dV} = \frac{1}{\sigma V_B} > 0.$$

For equity value, E,

$$\frac{d\left(\frac{V-V_B}{\sigma V_B}\right)^2}{dE} \approx \frac{d\,(E/b)}{dE} = \frac{1}{b} > 0.$$

3.6 A discrete-time setting

We now consider a discrete-time setting (e.g., Merton (1974) or Geske (1978) coupon bonds). We use the same notation than in the continuous-time case. This is a Bermudan-style option problem, which requires to compare the continuation and the intrinsic values.

It is optimal to *no* default if

$$E(V_t) > -NC_t(V_t) + e(V_t),$$

and equivalently, where $E^{TV}(V_t) = E(V_t) - e(V_t)$,

$$E^{TV}(V_t) + NC_t(V_t) > 0,$$

which is a "default constraint," but the problem is to qualify this number as large or small. On the other, a discrete-time setting is robust; we do not have to specify any dynamics which, e.g., can include jumps in asset values and equity prices. In both cases, continuous-time or discrete-time, D^E provides a proper DD, see equation (13) which contains both cases, where $\alpha = 1$ in discrete-time.

In continuous-time, a small ratio $\frac{E^{TV}(V_t)}{-NC_t(V_B)} \geq 0$ is also a DD. If we freeze E^{TV} (and $r = 0$) this ratio provides a lower-bound on the maximum time that we can wait before selling the company (as an alternative to defaulting today), since NC is a cost per unit of time. Hence, an intuitive interpretation for parameter α, i.e., defaulting if $\frac{E^{TV}(V_t)}{-NC_t(V_B)} \leq \alpha$, is the minimum time than equity E_t can support losses (e.g., one week or one month) at a rate $-NC_t(V_B) \times dt$. In continuous-time, $NC_t(V_B)$ represents the sum of all net cash-flows per year (like the dividend-yield approximation from cash dividends).

3.6.1 Time unhomogenous models: coupon bonds

Structural credit-risk models assume a time homogenous capital structure for tractability. Firms, however, choose more flexible capital structures such as lumpy maturity structure and active maturity management.[19] A time unhomogenous problem holds the same PDE and free-boundary conditions, so the net cash-flow $NC(t)$ and b_t can be time-varying.

Consider that a firm pays its debt, not continuously but regularly such as coupon bonds. Recall that the marginal rate of substitution between b and a is negative (Proposition 1). If negative net cash-flows b_t approximates zero (e.g., t is not a coupon date), a_t will soar, which is consistent with it is optimal to never default when the firm does not have a negative

[19] This is a very feature of debt markets, Chen et al. (2012), Choi et al. (2012), and Mian and Santos (2012).

payout. On the other, if b_t becomes large (a coupon date t), a_t shrinks implying that the shareholders can avoid a large cash outflow by defaulting. See Davydenko's (2013) Figure 4, entitled on the timing of default relative to scheduled debt payments for a sample of defaulted firms, which is fully consistent with this.

3.7 Strategic models of default

Consider now a firm that can renegotiate some value for equityholders before going to liquidation. After all, bankruptcy is costly for bondholders (see footnote 7). This is formulated if the equity value at default is not zero (Davydenko and Strubaylev (2007)) but

$$e\left(V_B\right) = \phi V_B \text{ and } e_V\left(V_B\right) = \phi\ ,\ \ \phi > 0.$$

Equation (6) simplifies to

$$E^{TV}(t, V_t) \approx \underbrace{\left(-NC_t + \left(r - \frac{\mu\left(V_B\right)}{V_B}\right)\phi V_B\right)}_{b_t} \times \underbrace{\left(\frac{V_t - V_B}{\sigma\left(V_B\right)}\right)^2}_{a_t^2}. \tag{16}$$

Strategic default (i.e., $\phi > 0$) rises the value of default if $r - \mu > 0$, which is a dividend-yield.

Distance to default We show that $\frac{dV_B}{d\phi} > 0$ (see Appendix B), which implies that default is accelerated (delayed) if $\phi > 0$ ($\phi < 0$). That is, since $\frac{da}{dV_B} < 0$,

$$\frac{da}{d\phi} = \frac{da}{dV_B} \times \frac{dV_B}{d\phi} < 0,$$

and DD lowers (rises) if $\phi > 0$ ($\phi < 0$). In particular, a negative recovery value $\phi < 0$, which can be seen as a penalty for equityholders—costly default, implies that default is optimally delayed. We will apply this result to mortgages and households. $\phi < 0$ can also be seen as a collateral pledged to the loan—which is lost at default.

Remark. The probability of default rises if shareholders recovery value is not zero but positive, $\phi > 0$. From $E^{TV}(t, V) = E(t, V_t) - \phi V_t$,

$$\{E_t \leq \alpha \times b_t\} \subset \{E_t \leq \alpha \times b_t + \phi V_t\} \equiv \left\{E_t^{TV} \leq \alpha \times b_t\right\}$$

and

$$E_0^{\mathcal{P}}\left[1_{\{E_t \leq \alpha \times b_t\}}\right] < E_0^{\mathcal{P}}\left[1_{\left\{E_t^{TV} \leq \alpha \times b_t\right\}}\right].$$

So, for a company with the same equity value E_t but positive recovery value $\phi > 0$, default is accelerated too. Shareholders recovery value is an additional variable to forecast default.

4 Default in Multi-factor models

4.1 Stochastic volatility

The same (time-value) quadratic expression holds for a model with stochastic parameters such as r or σ. We focus on stochastic volatility (we consider cash savings in the next section).[20] The only difference is that the asset volatility $\sigma(V_B)$ is replaced by the overall multifactor volatility. It changes the standardized distance-to-default a but not b. We abstract from this overall volatility by considering the ratio E/b.

Consider that volatility is defined either, $\sigma(V_t) = \sigma_t \times V_t$ or $\sigma(V_t) = \sigma_t$, where

$$d\sigma_t = \alpha(\sigma_t)\,dt + \beta(\sigma_t)\,dW_t,$$

and ρ is the correlation between dW_t and dZ_t. The associated PDE is given by

$$rE = E_t + \mu(V_t)\,E_V + \frac{1}{2}\sigma^2(V_t)\,E_{VV} + NC_t + \alpha(\sigma_t)E_\sigma + \frac{1}{2}\beta^2(v)E_{\sigma\sigma} + \rho\sigma(V_t)\,\beta(\sigma_t)\,E_{V\sigma}, \quad (17)$$

and the exercise boundary depends also on volatility, $V_B(t,\sigma_t)$. We assume a second smooth-pasting condition

$$E_\sigma(t, V_B, \sigma_t) = 0.$$

For $V = V_B(t,\sigma_t)$ (see Ibáñez and Velasco (2012)),

$$E_{\sigma\sigma} = E_{VV} \times \left(\frac{\partial V_B}{\partial \sigma}\right)^2 \quad \text{and} \quad E_{V\sigma} = -E_{VV} \times \frac{\partial V_B}{\partial \sigma}.$$

Denoting by $\Sigma(t, V_B(t,\sigma_t), \sigma_t)$ the overall volatility at the default boundary, i.e.,

$$\Sigma(t, V_B(t,\sigma_t), \sigma_t) = \sqrt{\sigma^2(V_B) + \beta^2(\sigma_t)\left(\frac{\partial V_B}{\partial \sigma}\right)^2 - 2\rho\sigma(V_B)\,\beta(\sigma_t)\frac{\partial V_B}{\partial \sigma}},$$

which shows that the curve or boundary $V_B(t,\sigma_t)$ can be hitted from lower V_t and/or lower σ_t too (if $\frac{\partial V_B}{\partial \sigma} < 0$). Then, the time-value of equity is given by

$$E^{TV}(t, V_t, \sigma_t) \approx \underbrace{\left(-NC + \left(r - \mu(V_B)\,\frac{e_V}{e}\right)e\right)}_{b_t} \times \underbrace{\left(\frac{V_t - V_B(t,\sigma_t)}{\Sigma(t, V_B(t,\sigma_t), \sigma_t)}\right)^2}_{a_t^2}. \quad (18)$$

In this multi-factor setting, the novel term is the boundary sensitivity to the additional state variables; i.e., the term $\partial V_B/\partial\sigma$ within $\Sigma(t, V_B(t,\sigma_t), \sigma_t)$. The volatility drift does not matter (near default) since $E_\sigma(t, V_B, \sigma_t) = 0$. And the distance-to-default, $a_t^2 = \frac{V_t - V_B(t,\sigma_t)}{\Sigma(t, V_B(t,\sigma_t),\sigma_t)}$, is also given by E^{TV}/b_t.

[20] Stochastic volatility makes Leland model intractable (see McQuade (2013)), but it is of first order importance in option-pricing.

4.2 Default-risk with liquidity constraints: insolvency and illiquidity

A factor that can produce credit-risk is illiquidity and cash shortages. The theoretic models of Anderson and Carverhill (2012), Gryglewicz (2011), and Murto and Terviö (2014) consider constraints of this type. Firms "need" to hold cash and optimally decide the amount of cash and dividends; i.e., if revenues fall short of debt payments, they will incur in extra costs (e.g., new issues of equity) or simply default. The need to save cash (which is denoted by C) has two impacts on default; one if cash earns a lower rate and two if cash is constrained to be above a lower bound (e.g., $C \geq 0$). We distinguish between insolvency and illiquidity; i.e., default when $C > 0$ and $C = 0$, respectively. C can vanish and hit zero because dividends payments, negative earnings, debt service, or rollover costs of expiring debt.[21]

If saving cash is costly, this cost appears in the "financial needs" term, b. In Gryglewicz (2011), who solves the two models without and with cash constraints, we show that the time-value is the same in both models (see Appendix A, Section 4); yet the equityholders recovery value/intrinsic value is not linear in the model with cash reserves ($e_{VV}(V_B) \neq 0$). The reason is that cash reserves are not costly.

Since cash is the firm's balance equation, its dynamics must be of the type

$$dC_t = (r_C \times C + NC_t)\, dt + \xi dW_t - dD_t,$$

where $r_C = r_C\,(C)$ is the interest earned on cash and dD_t the instantaneous dividend ($dD_t \geq 0$). Now, NC_t are net earnings and dD_t is the true net cash-flow to equityholders. We add a Gaussian factor ξdW_t (where $E\,[dZdW] = \rho dt$), which allows for stochastic cash-flows and a general two-factor setting. dD_t is the result of some optimization process taken as given.

The associated PDE (which holds only in a subset of the state space) is given by

$$rE = \mu\,(V_t)\,E_V + \frac{1}{2}\sigma^2\,(V_t)\,E_{VV} + dD_t + (r_C C + NC_t - dD_t)\,E_C + \frac{1}{2}\xi^2 E_{CC} + \rho\sigma\,(V_t)\,\xi E_{VC}, \tag{19}$$

where $E = E\,(C, V)$. We denote by $V_B(C_t)$ the optimal exercise boundary, which depends on cash, and assume the following equity value at default

$$E\,(C, V_B(C)) = C + \phi V_B \ , \ \phi \geq 0,$$

[21] In the liquidity models of Acharya et al. (2012), Bolton et al. (2011), and Decamps et al. (2011), earnings follow an arithmetic process, which implies constant business conditions, and default happens only when cash exhausts. Our results do not apply in their setting, default is only because illiquidity (no insolvency).

and (assuming smooth-pasting)

$$E_C\left(C, V_B(C)\right) = 1 \text{ and } E_V\left(C, V_B(C)\right) = \phi.$$

The firm retains cash (i.e., $E_C = 1$, as it could spend all cash immediately before default) plus some strategic value ($E_V = \phi \geq 0$). The effect of the dividend policy (dD_t) depends on E_C, being irrelevant at default if $E_C = 1$.[22]

Similar to the stochastic volatility model, for $V = V_B(C_t)$,

$$E_{CC} = E_{VV}\left(\frac{\partial V_B}{\partial C}\right)^2 \text{ and } E_{VC} = -E_{VV}\frac{\partial V_B}{\partial C}.$$

Denoting by $\Sigma\left(t, V_B(t, C_t), C_t\right)$ the overall volatility at default, i.e.,

$$\Sigma\left(V_B(C_t), C_t\right) = \sqrt{\sigma^2\left(V_B\right) + \xi^2\left(\frac{\partial V_B}{\partial C}\right)^2 - 2\rho\sigma\left(V_t\right)\xi\frac{\partial V_B}{\partial C}},$$

it follows that

$$\frac{1}{2}E_{VV} = \left(-NC_t + \left(r - r_C\right)C + \left(r - \frac{\mu\left(V_B(C)\right)}{V_B(C)}\right) \times \phi V_B(C)\right) \times \frac{1}{\Sigma^2\left(V_B(C_t), C_t\right)}. \quad (20)$$

Now the value of (the option to) default depends also on the cost of cash, $r - r_C$. We assume that cash is costly; i.e., $r_C = r_C\left(C\right)$ and

$$\left(r - r_C\right) \times C \geq 0.$$

By defaulting, the firm stops paying a (negative) cash flow, costly cash, and a carry cost. The time-value of equity is given by

$$E^{TV}(t, V) \approx \underbrace{\left(-NC_t + \left(r - r_C\right)C + \left(r - \frac{\mu\left(V_B(C)\right)}{V_B(C)}\right) \times \phi V_B(C)\right)}_{b_t} \times \underbrace{\left(\frac{V_t - V_B(C)}{\Sigma\left(V_B(C), C\right)}\right)^2}_{a_t^2}. \quad (21)$$

Although a new borrowing/lending rate r_C applies to cash, we did not include any constraint on cash C. Consider that borrowing is forbidden, otherwise, the firm will default because illiquidity. A constraint such as

$$C_t \geq 0,$$

does not change equations (19) and (20), it reduces only the state space. Default now is because either insolvency ($V = V_B(C)$ and $C_t > 0$) or illiquidity ($C_t = 0$). Clearly, the value

[22] This is because dividends are paid from cash. If they are financed from the assets of the company the net effect at default will be $1 - E_V$ (instead of $1 - E_C$). See Section 6.3 application to agency problems.

of equity lowers and hence the default boundary rises, but the same formula applies to the time-value of (the option to) default near insolvency.[23]

Assume that $V_B(C)$ is decreasing in C and $C > 0$.[24] Then, a lower C lowers the value of default from $(r - r_C)\, C$; and if NC_t depends on $V_B(C)$, a lower C implies (a larger $V_B(C)$ and) a lower $-NC_t$ and lower value of defaulting too. So, defaulting when the value of default is low indicates costly cash and illiquidity concerns, which is consistent with better business conditions, i.e., larger $V_B(C)$ (where $\phi = 0$). Indeed, in the case of defaulting because illiquidity (i.e., $C = 0$), the formula does not apply.

Assume now that $C_t \geq -c$, where $c > 0$ is a borrowing limit. If $C_t < 0$, a lower C_t rises the value of default from $(r - r_C)\, C_t$. But this is intuitive too, risk-free borrowing is not forbidden in this market but is expensive.

4.3 Regime switching models

The same intuition and results carry over for regime switching models (Bhamra et al. (2010), Chen (2010)).[25] In this case, a system of PDE, value-matching, and smooth-pasting conditions hold at the default boundary. Consider two regimes: in the bad (good) one, the value of default is reduced (raised), because there is a chance to move to the 2nd regime, where prospect looks better (worse). So, a new macroeconomic term Π_t appears in b_t,

$$b_t = -NC_t + (r - r_C)\, C + \left(r - \frac{\mu\,(V_B(C))}{V_B(C)} \right) \times \phi V_B(C) - \Pi_t,$$

where b_t depends on the regime and Π_t is given by the expected gains/losses of moving between regimes (the new minus the old regime). A business cycle or macroeconomic factor may be relevant when predicting default.

[23] This is easy to see in Boyle and Guthrie (2003), who add a liquidity constraint to the standard real option model and contains a call payoff. The call is exercised earlier (compared to the unconstrained case) and hence the value is lower.

[24] Let $(c_2, V_B(c_2))$ be a point of the default boundary. If it is optimal to default at a point (c_2, V_1) for $V_1 < V_B(c_2)$, we assume the strict unequality

$$E\,(c_2, V_1) < c_2 + \phi V_1.$$

Let c_1 be such that $V_1 = V_B(c_1)$. If $c_1 \leq c_2$, we can pay $(c_2 - c_1)$ in cash and move to the boundary, where

$$E\,(c_1, V_B(c_1)) + (c_2 - c_1) = c_1 + \phi V_B(c_1) + (c_2 - c_1) = c_2 + \phi V_B(c_1) = c_2 + \phi V_1,$$

and consequently it is not optimal to default at the point (c_2, V_1). Hence, $c_1 > c_2$. And from $V_1 = V_B(c_1) < V_B(c_2)$, the default boundary is decreasing. This also holds in Figure 1 in Murto and Terviö (2013) and Anderson and Carverhill (2012), where V are earnings, and in Davydenko's (2013) data plot Figure 1.

[25] See Guo and Zhang (2004) for the pricing of a perpetual American put option in a regime switching model.

4.4 Bankruptcy and liquidation

Some models that include a bankruptcy and a liquidation process, chapter 11 and chapter 7, respectively, have associated two exercise boundaries (Broadie et al. (2007)). In the bankruptcy boundary, most models assume a new optimality condition so-called super-contact condition which requires continuity of the 2nd derivative (Dumas (1988)). In the liquidation boundary, the same, equation (7), 2nd-order approximation holds.

5 Mortgage default

Mortgage default is close to corporate default if the dividend policy is given, dividends are renting income, and there is not cash (i.e., $C = 0$). Let V_t be the value of the unlevered house. Near default, the value of the mortgaged house (i.e., equation (16)) is given by

$$\begin{aligned} E^{TV}(t, V_t) &\approx \left(-NC_t + \left(r - \frac{\mu(V_B)}{V_B}\right)\phi V_B\right) \times \left(\frac{V_t - V_B}{\sigma(V_B)}\right)^2 = b_t \times a_t^2, \\ NC_t &= \delta V_B - c_t \times (1-\tau), \end{aligned}$$

where δV_t is the renting income rate minus a maintenance cost, c_t the debt service (which may include interest and principal), and $\tau \in [0,1)$ a tax relief. $r - \mu(V_B) \geq 0$ is the depreciation rate of the house and ϕV_B a residual value for the owner (if she walks away). Here, V_t no E_t is observable (and traded) in practice.

We are interested in the following sensitivity $\text{sign}\left(\frac{da}{dx}\right)$, where from equation (15),

$$\text{sign}\left(\frac{da}{dx}\right) = -\text{sign}\left(\frac{\partial V_B}{\partial x}\right) \approx -\text{sign}\left(\frac{b_x}{b} - \frac{E_x}{E}\right).$$

We focus on cash-flow parameters, which then implies that

$$\text{sign}\left(\frac{da}{dx}\right) \sim -\text{sign}\left(\frac{\partial b}{\partial x}\right).$$

Then, it is straightforward the following static comparative analysis in Table 1. For instance, the signs of Loan-to-Value and Loan-to-Income on mortgage default-risk, + and 0, respectively (where 0 can be explained by a large loan but better home), agree with those of the rich household model in Chen et al. (2013).

Table 1: Static comparative of Mortgage Default-Risk,	
parameter, x	da/dx or $\partial b/\partial x$
Loan or Loan-to-Value, c or $c-\delta$:	$\partial b/\partial x > 0$
Floating- and Fixed-Rate Mortgage, c:	$\partial b/\partial x > 0$
Loan-to-Income, $c+\delta$:	$\partial b/\partial x \approx 0$ (indeterminate)
Renting cost, δ:	$\partial b/\partial x < 0$
Taxes (on interest payments), τ:	$\partial b/\partial x < 0$
Reduced interest payment, $-c$:	$\partial b/\partial x < 0$ (temporary, delays default)
Constant payment (amortizable debt), c:	$\partial b/\partial x > 0$**;** but after amortization $dE > 0$**,** and hence, $\frac{da^2}{dE} = \frac{1}{b} > 0$
Only-Interest-Payment, $-c$:	$\partial b/\partial x < 0$ before T (but $> \mathbf{0}$ at maturity)
Reduce principal:	$da/dx > 0$ (since $dE > 0$ and $\frac{da^2}{dE} = \frac{1}{b} > 0$)
Value, V:	$da/dV > 0$
Recovery value at default, ϕ:	$da/d\phi < 0$
Negative externality at default, $-\phi$: (# children, # years in the neighborhood,..)	$-da/d\phi > 0$
Liquidity/cash problems:	cash constraint model

In terms of loan modification, a policy debate in United States is about either reducing the principal or temporary reductions in payments (Elul et al. (2010), Foote et al (2008)). Based on the bankruptcy cost for lenders in foreclosure, Goodman (2010) proposes to lower the principal of some no-very-deep under-water mortgage holders, as the only way of solving the housing market overhang with millions of properties going into default.

Table 1 is clear in this debate, the temporary reduction mainly delays or postpones default until the household has to face debt payments again. It helps in the case of liquidity problems such as a temporary job loss or illness. Reducing the principal rises equity value (i.e., $dE > 0$), and hence, lowers the risk of default and foreclosure; i.e., $da^2 \approx \frac{dE}{b} > 0$. Indeed, this is better than lowering interest-rates for a severe underwater house (see equation (14)). Given this restructuring (i.e., reducing) of the principal, it is "in the best interest" of the debtor to stay in her house. The lender has rights and covenants (also, information on the borrower financial situation) to "induce" this process. In the United States, something similar to the separate chapters 11 and 7 processes could accommodate this situation.

The parameter ϕ can accommodate heterogeneity among homeowners, where a positive (negative) recovery value induces earlier (later) default, see Section 3.7. In the case of mortgages, default also depends on the number of kids and/or years living in the house (Campbell and Cocco (2012)), which can enter "negatively" in this residual value, $\phi < 0$.

6 Default in Leland and Toft (1996) model

We make explicit Section 3 results using Leland-Toft model and provide a numerical exercise. We use the same notation than Leland-Toft and He and Xiong (2012), but follow He-Xiong mathematical layout. We denote by r the risk-free interest rate, δ the dividend-yield, and σ the constant volatility of the unlevered assets value V_t, which follows a geometric Brownian motion under the Q–risk-neutral probability measure,

$$\frac{dV_t}{V_t} = (r - \delta)\, dt + \sigma dZ_t, \tag{22}$$

where dZ_t is a standard Wiener process. We denote by $E(V_t)$ the equity value and by $d(V_t, m)$ the debt value (with m–maturity). Equity value solves the following differential equation

$$rE = (r-\delta)V_t E_V + \frac{1}{2}\sigma^2 V_t^2 E_{VV} + \underbrace{\delta V_t - (1-\pi)\, C + d(V_t, m) - p}_{NC_t}, \tag{23}$$

NC_t is the "net cash-flow" to equity holders. $\delta V_t - (1-\pi)\, C$ is the payout-rate less after-tax coupon payments (π is the tax-rate). $d(V_t, m) - p$ are the refinancing costs of the expiring debt, $d(V_t, m)$ is the new raised debt and p is the notional of the expiring debt which is constant. The capital letter C indicates that the total coupon paid (at time t) for the whole debt is $c \times m = C$, whereas only a bond with notional p expires at t.

The model is fully specified by providing the boundary conditions. Let V_B be the optimal default point, if $V_t = V_B$,

$$E(V_B) = 0 \;\text{ and }\; E_V(V_B) = 0, \tag{24}$$

which denote value-matching and smooth-pasting conditions, respectively. Equity is worthless if the firm defaults. The value of debt simplifies to

$$d(V_B, m) = \alpha \frac{V_B}{m}, \tag{25}$$

where α gives the firm's recovery value in default, $0 \leq \alpha \leq 1$.[26]

[26] *Remarks.* Equations (22) to (25) are the same than in He and Xiong (2012); e.g., our equation (23) is exactly their PDE (11). He and Xiong (2012) model, which also includes a rollover risk component, differs from Leland and Toft (1996) model in the value of debt, $d(V_t, m)$, and implies a different value of equity. In our case, we do no have to make explicit the value of debt, $d(V_t, m)$. The net cash-flow term, NC_t, also appears in equation (12) in Leland and Toft (1996). If $p = 0$ and $d = 0$, equity also corresponds to Leland's (1994) model, where C is the consol bond coupon. Another rollover risk model is Chen et al. (2012), see their equity pricing equations (11) to (13), but which is given by a two-state continuos-time Markov chain.

6.1 Equity value near the default-point

If $V_t = V_B$, from equations (24) and (25), equation (23) simplifies to

$$\frac{1}{2}\sigma^2 V_B^2 E_{VV}(V_B) = -\left(\delta V_B - (1-\pi)\,C + \alpha\frac{V_B}{m} - p\right), \tag{26}$$

which is the Gamma of the equity value at the boundary.

In the continuation region, $V_t \geq V_B$, we can approximate the equity value by a 2nd-order Taylor expansion at the point V_B; i.e.,

$$\begin{aligned} E(V_t) &\approx E(V_B) + E_V(V_B) \times (V_t - V_B) + \frac{1}{2}E_{VV}(V_B) \times (V_t - V_B)^2 + \mathrm{O}\,(V_t - V_B)^3 \\ &= \frac{r\frac{(1-\pi)C+p}{r} - \left(\delta + \frac{\alpha}{m}\right)V_B}{\sigma^2 V_B^2} \times (V_t - V_B)^2\,, \end{aligned} \tag{27}$$

where the 2nd equality uses equation (24).

This expression resembles the "time-value" of a perpetual American put option, where $K = \frac{(1-\pi)C+p}{r}$ is the strike price and $\gamma = \delta + \frac{\alpha}{m} \geq 0$ the dividend-yield, for a given value of the boundary level V_B. That is, if $V_t \geq V_B$,

$$E(V_t) \approx \underbrace{V_t}_{\text{“assets”}} - \underbrace{K}_{\text{“strike”}} + \underbrace{K - V_t + \frac{rK - \gamma V_B}{\sigma^2 V_B^2} \times (V_t - V_B)^2}_{\text{“perpetual American put option, with strike } K\text{”}}\,, \tag{28}$$

and equity is given by the firm's asset value V_t, less K which represents the present value of debt, plus an American put providing protection against equity becoming negative.

Finally, we write the firm's value relative to the endogenous boundary, $\frac{V_t - V_B}{\sigma V_B} \geq 0$,

$$E(V_t) \approx \left((1-\pi)\,C + p - \left(\delta + \frac{\alpha}{m}\right)V_B\right) \times \left(\frac{V_t - V_B}{\sigma V_B}\right)^2. \tag{29}$$

For a different boundary value $\widehat{V}_B$, we also have a 2nd-order approximation, with error $\mathrm{O}\Big(\Big(V_B - \widehat{V}_B\Big) \times (V_t - V_B)^2 + (V_t - V_B)^3\Big)$; i.e.,

$$E(V_t) \approx \left((1-\pi)\,C + p - \left(\delta + \frac{\alpha}{m}\right)\widehat{V}_B\right) \times \left(\frac{V_t - V_B}{\sigma V_B}\right)^2. \tag{30}$$

Remark. Leland (1994), Leland and Toft (1996), and He and Xiong (2012) are different models, but the three are closer near-the-default-point if asset values are measured in relative terms (see also footnote 26). In Leland (1994),

$$E(V_t) \approx ((1-\pi)\, c - \delta V_B) \times \left(\frac{V_t - V_B}{\sigma V_B}\right)^2 .$$

6.2 The value of default

The value of equity near default is given by the product of two components,

$$E(V_t) \approx \left((1-\pi)\, C + p - \left(\delta + \frac{\alpha}{m}\right) V_B\right) \times \left(\frac{V_t - V_B}{\sigma V_B}\right)^2 = b \times a_t^2. \tag{31}$$

The term

$$b = -NC_t = (1-\pi)\, C - \delta V_B + p - \frac{\alpha}{m} V_B,$$

is the negative net cash-flow or the financial needs of a firm, where $\frac{\alpha}{m} V_B$ is the debt value at default. $a_t = \frac{V_t - V_B}{\sigma V_B}$ is the volatility-adjusted distance to default for a lognormal process.

Economic distress vs financial distress Davydenko (2013) finds that "many" firm defaults are not only associated with economic distress (i.e., low V_t) but with financial distress too (such as illiquidity, cash shortage, or negative profitability).[27] Our Propositions 1 and 2 (see Section 3.1) show that default is consistent with low assets value and negative profitability. Default is associated to a large b, which depends on (debt service and) low or negative earnings. In Leland-Toft, the payout rate is the dividend-yield δV_t, which cannot be negative, but, δV_t and the asset value V_t reach the lowest value at default, since $V_t \geq V_B$.

Indeed, consider a company, which, given a difficult situation, decides to lower or cancel the dividend. What is the consequence? It has more incentives to default, as b will rise. For a company that pays a large dividend-yield δ, the option to default is less worth, shareholders will lose the dividend (which is paid/extracted from the firm's assets). In a different context, but in the recent attempt of Dell buyout, large stake equityholders are demanding (first of all) a large dividend payment before any action (Financial Times (2013)).

[27] Glober (2013) finds that firms with low recovery at default tend to default later, since, endogenously, are less leveraged. This implies that firms with large recovery at default will have a larger b (and a lower distance a). The minus cash flow term, b, depends on all these variables debt payments and recovery value at default, so it is compatible with Glober's results. Harford et al. (2013) report that US firms have reduced the maturity of debt but simultaneously rising cash holdings, and hence, reducing (the default risk which is associated to) roll-over risk.

6.3 The distance-to-default

If a firm equity is traded and we observe its cash flows, we can retrieve the asset value relative *to* V_B; i.e.,

$$a_t = \frac{V_t - V_B}{\sigma V_B} \approx \sqrt{\frac{E(V_t)}{(1-\pi)\, C + p - \left(\delta + \frac{\alpha}{m}\right) V_B}}, \tag{32}$$

The coupon and notional of debt, C and p, respectively, are observable as well as the dividend yield δ. Although V_B is not, what matters is the total recovery value, $\frac{\alpha}{m} V_B$, which is an input in any credit risk model (Glober (2013)).

Agency problems and bankers One of the most striking aspects of the recent financial crisis (all over the world) is that bankers keep receiving fat bonuses in spite of the large and constant losses of their own bank.[28] (Think in the recent 2013 and 2014 US authorities multibillion-dollars fines to large banks.) We rationalize this in a Leland–Toft model.

Consider the following steps. NC_t is not a payout rate to shareholders, but directly added or depleted from the assets of the company, V_t. This is not an issue for banks, which assets are most liquid (and can more easily borrow) than those of nonfinancial firms. Top managers receive a compensation, which is proportional to the firm value, δV_t, until the firm defaults. They determine default in their best interest (e.g., passive shareholders and/or hidden operational losses).

Then, managers value function, $E^{\text{man}} = E$, solves

$$rE = (rV_t - \delta V_t \underbrace{- (1-\pi)\, C + d(V_t, m) - p}_{NC_t - \delta V_t}) E_V + \frac{1}{2}\sigma^2 V_t^2 E_{VV} + \delta V_t.$$

From the boundary conditions at default (i.e., $E = 0$ and $E_V = 0$), it follows that

$$\frac{1}{2}\sigma^2 V_B^2 E_{VV} = -\delta V_B,$$

where $-\delta V_B \leq 0$ if $\delta \geq 0$ and $V_B \geq 0$. Hence, if $\delta > 0$, Gamma is zero; i.e.,

$$V_B \to 0 \quad \text{and} \quad b \to 0.$$

It agrees with some banks rolling-over operational losses, while keep paying big salaries to high managers (i.e., $\delta V_t > 0$) until the firms assets are totally depleted. If managers do no work in the firm interest, fat salaries impoverish the firm but managers will keep the firm alive (the value of default is zero, $b \to 0$) for a longest time ($V_B \to 0$).

[28] "Once Again, Punishing the Bank but Not Its Top Executives," (by Peter Eavis) New York Times (2013).

See Morellec et al. (2013) who explicitly model the manager-shareholder agency conflict, finding that it helps to reconcile some capital structure puzzles. In Duffie and Lando (2001) words, some firms default already in deep troubles. This simple model may also be used to understand LBO's (leveraged buyouts), where large dividends are paid since the beginning of a highly leveraged operation. Of course, shareholders have the same incentives to use the firm's assets to finance losses and receive another payout $\delta V_t > 0$.

6.4 Numerical exercise

To provide more intuition, we compute the value of equity in equation (31) and the terms a and b_t. We use the same parameters than He and Xiong (2012) calibration exercise for a typical US firm (during the 90's), which are based on Huang and Huang (2012) empirical work. That is, $r = 0.08$, $\pi = 0.27$, $\sigma = 0.23$, $\alpha = 1 - 0.6$, $\delta = 0.02$, $T = 1$, $V_0 = 100$, and $C = 6.39$ and $P = 61.68$. The optimal default boundary is $V_B = 65.0484 \approx 65.05$.

The marginal value of default is given by $b = 39.02$. In Table 3, we show different equity values and distance-to-default based on the exact Leland–Toft formula and based on the quadratic approximation. The closest the asset value to the default boundary, the lowest the equity value and distance-to-default errors. Equity value is convex near default, being almost linear far away from default. Hence, the 2nd-order approximation is more accurate in the most interesting part of equity. Also, given this linear relationship, the 2nd-order approximation is upper-biased and the associated distance-to-default is lower-biased (or conservative). For example, if $V_0 = 70$, then $E = 2.31$ and $\frac{V_0 - V_B}{V_B} \times 100 = 7.6\%$, and the 2nd-order approximation implies 4.27 and 5.6%, respectively.

The distance-to-default is "always" lower-biased ($\sqrt{\frac{E}{b}} < \frac{V_0 - V_B}{\sigma V_B}$ for $V_0 > V_B$) but is unbiased at the boundary ($\sqrt{\frac{E}{b}} = \frac{V_0 - V_B}{\sigma V_B} = 0$ for $V_0 \to V_B$), it follows that the distance $\sqrt{\frac{E}{b}}$ classifies or disentangles "perfectly" between defaulting and no defaulting firms. The correlation between $\frac{V_0 - V_B}{\sigma V_B}$ and $\sqrt{\frac{E}{b}}$ is above 99.6%, using the ten examples in Table 2, which are between 0 and 1 standard deviations.

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Table 2: Equity values and standardized distance to default, Leland and Toft (1996)						
$r = 0.08$, $\pi = 0.27$, $\sigma = 0.23$, $\alpha = 0.4$, $\delta = 0.02$, $T = 1$, and $C = 6.39$ and $P = 61.68$.						
Exercise boundary, $V_B = 65.05$. $E \approx b \times \left(\frac{V_0 - V_B}{\sigma V_B}\right)^2$ and $b = 39.02$.						
	Leland–Toft model			**2nd-order approximation**		
V_0	E	$\frac{V_0-V_B}{\sigma V_B}$	$100\frac{V_t-V_B}{V_B}$	$b \times \left(\frac{V_0-V_B}{\sigma V_B}\right)^2$	$\sqrt{\frac{E}{b}}$	$100\sigma\sqrt{\frac{E}{b}}$
V_B	0	0	0%	0	0	0%
65.5	0.023	0.030	0.7%	0.036	0.024	0.6%
66	0.101	0.064	1.5%	0.158	0.051	1.2%
67	0.407	0.130	3.0%	0.664	0.102	2.3%
68	0.892	0.197	4.5%	1.519	0.151	3.5%
69	1.533	0.264	6.1%	2.722	0.198	4.6%
70	2.310	0.331	7.6%	4.275	0.243	5.6%
72	4.199	0.465	10.7%	8.425	0.328	7.5%
75	7.658	0.665	15.3%	17.266	0.443	10.2%
80	14.396	0.999	23.0%	38.975	0.607	14.0%

6.5 Distance to default, the value of default, and the default boundary

Let us focus on the relationship between negative distance to default, the value of default, and the default boundary; i.e., $-a$, b, and V_B, respectively. The three are measures of default-risk; i.e., the larger they are, the larger the risk of default. That is,

$$a_t = \frac{V_t - V_B}{\sigma V_B} = \quad \text{and} \quad a_t^2 \approx \frac{E}{b},$$

since $d(-a)/dx = -da/dx$,

$$-\frac{da}{dV_B} = \frac{V_t}{\sigma V_B^2} > 0 \quad \text{and} \quad -\frac{da^2}{db} \approx \frac{E}{b^2} > 0,$$

and consequently the intuition that b and V_B are measures of default-risk is correct. b's advantage is that it is readily observable and that $\sqrt{E/b}$ is the economic counterpart to distance-to-default.

Focusing on a structural parameter x ($x \neq \sigma$), from equation (15),

$$\text{sign}\left(\frac{da}{dx}\right) = -\text{sign}\left(\frac{\partial V_B}{\partial x}\right) \approx -\text{sign}\left(\frac{b_x}{b} - \frac{E_x}{E}\right),$$

and in the case of cash-flow parameters,

$$\text{sign}\left(\frac{da}{dx}\right) \sim -\text{sign}\left(\frac{\partial b}{\partial x}\right).$$

and $\frac{-\partial b}{\partial x}$, a partial derivative, gives the DD sensitivity to a parameter x ($x \neq \sigma$).

The static comparative of V_B in Leland and Toft (1996) model is derived by Davydenko (2012) who uses a calibration exercise. We include his calibrated sensitivity $\frac{\partial}{\partial x}V_B$ and $\frac{\partial}{\partial x}b$, where $V_B = V_B\left(r, \pi, \sigma, \alpha, \delta, T, C, p\right)$ and $b = (1-\pi)\times C + p - \left(\delta + \frac{\alpha}{m}\right)\times V_B$.

Table 3: Sensitivities in Leland and Toft (1996) model*

parameter, x	$\dfrac{\partial V_B}{\partial x}$	$\dfrac{\partial b}{\partial x}$	$\dfrac{db}{dx} = \dfrac{\partial b}{\partial x} + \dfrac{\partial b}{\partial V_B}\dfrac{\partial V_B}{\partial x}$
asset volatility, σ	<0	$=0$	$-\left(\delta+\dfrac{\alpha}{m}\right)\dfrac{\partial V_B}{\partial \sigma} > 0$
cost of default, $1-\alpha$	$>\mathbf{0}$	$>\mathbf{0}$	$V_B - \left(\delta+\dfrac{\alpha}{m}\right)\dfrac{\partial V_B}{-\partial \alpha} = 43.160 > 0$
dividend yield, δ	$<\mathbf{0}$	$<\mathbf{0}$	$-V_B - \left(\delta+\dfrac{\alpha}{m}\right)\dfrac{\partial V_B}{\partial \delta} = -110.40 < 0$
coupon rate, $\dfrac{C}{100}$	$>\mathbf{0}$	$>\mathbf{0}$	$\left(1 - \left(\delta+\dfrac{\alpha}{m}\right)\dfrac{\partial V_B}{\partial C}\right)\times 100 = 38 > 0$
risk-free rate, r	<0	$=0$	$-\left(\delta+\dfrac{\alpha}{m}\right)\dfrac{\partial V_B}{\partial r} > 0$
tax rate, π	$<\mathbf{0}$	$<\mathbf{0}$	$-C - \left(\delta+\dfrac{\alpha}{m}\right)\dfrac{\partial V_B}{\partial \pi} = 6.13 > 0$

* $\dfrac{\partial}{\partial x}V_B$ is from Davydenko (2012). And $\dfrac{db}{dx}$ is evaluated as Table 2 parameters (i.e., $r = 0.08, \pi = 0.27, \sigma = 0.23, \alpha = 0.4, \delta = 0.02, T = 1, C = 6.39, P = 61.68$).

Table 3 implies a close relationship between b and V_B, which have the same sensitivities to the cash-flow parameters; e.g., a firm that cancels the dividend δ has more incentives to default as well as a mortgaged house if renting costs lower.[29] When $b_x = 0$ (e.g., $x = \{r, \sigma\}$), $\text{sign}\left(\frac{db}{dx}\right) = -\text{sign}\left(\frac{\partial V_B}{\partial x}\right)$ since $\frac{\partial b}{\partial V_B} = -\left(\delta + \frac{\alpha}{m}\right) < 0$. And $\frac{\partial b}{\partial \pi} < 0$ but $\frac{db}{d\pi} > 0$ follows if the firm pays less coupon after taxes but shareholders receive also less dividends (since $\frac{\partial b}{\partial V_B} < 0$).

7 Concluding Remarks

The decision to default is of paramount importance because of the implications that entails and looms in many places—e.g., firms, mortgages, households, financial intermediaries, or sovereigns.[30] This paper provides a coherent story of the decision of going-into-default based on endogenous Leland-type structural models. By defaulting, shareholders stop from paying a

[29] Davydenko (2012) approximates the default boundary V_B by the value of the assets the month previous to default for a sample of defaulting firms. He finds that V_B is negatively (positively) related to volatility (the cost of default and macroeconomic conditions).

[30] One just has to follow the recent 2013 US debt ceiling situation, which is not due to insolvency but illiquidity, *but* which has stressed financial markets worldwide.

large and negative net cash-flow, which is given by the debt service minus operating earnings. The larger this cost, the larger the value of (and incentives to) default. Is this tradeoff, how much you can pay—*if you want to live*, which is weighted by firm owners.

We provide the economic counterpart to Merton's DD, which is given by the ratio of equity prices over minus net cash-flows, and what triggers default. We show (i) why DD is redundant if we control by these two covariates, which are observable; (ii) why financial distress is triggered by low equity prices and high debt service and/or negative profitability; (iii) why financial distress is closely related to depressed stock prices (controlling by net cash-flows); (iv) why trailing stock returns are significant (e.g., in default-intensity models) if they predict low equity prices; (v) why correlated equity prices, the equity market, paves the way for default correlation in a downturn market (even if cash-flows don't); (vi) why the timing of default will happen at scheduled deb payments—the clustering of default around coupon days; (vii) why interest-rates largely determine default for mortgages (with long-term loans) and sovereigns (which regularly roll-over debt), since the house and the country provide a positive service in both cases; (viii) why reducing the principal better (than interest-rates) helps to underwater mortgages, since it both rises equity value and lowers debt payments, which is akin to large down payments for subprime loans; (ix) why roll-over risk rises default-risk, since new borrowing rates depend on credit quality or if borrowers cannot postpone to repay a loan (i.e., illiquidity); (x) why financial costs, the cost of cash, also rises the value of default; (xi) why a positive (negative) shareholders or homeowners recovery value accelerates (delays) default; and (xii) why the value of default is procyclical in macroeconomic regime switching models—a firm in a difficult situation will be closer to (will postpone) default before the cycle turns to worst (improves) consistent with the large (low) value of default.

The determinants of the event (and hence probability) of default are "negatively" related to equity prices, earnings, expected equity returns, and cash holdings—and "positively" related to equity volatility and the debt service calendar and to, a less extent, the cost of cash, shareholders recovery, and the business cycle. Clearly, from all these variables, if we want to hide or understate default-risk, earnings/losses are the easier to manipulate (or inflate). The financial misreporting from Enron to recently Gowex are two examples of this.

Our results are robust applying to a multi-factor setting (e.g., stochastic volatility). It says that a loss-making firm when, in addition, has to make debt payments will be optimal and close to default—i.e., shareholders are indifferent between paying for debt service and/or for operating losses near default—and all leveraged assets (whether firms, mortgages, households, or sovereigns) face similar trade-offs. It is a simple, robust, and theoretically grounded measure, which can be used by policy-makers to understand default. Both E and b are in the liability side of a bank/firm. By rising capital E or moving (contingent) debt to capital, we rise the ratio E/b and the DD; the troubled bank (i.e., bankholders) have less incentives to default and more to repay the negative net cash-flow b.

We consider a model with a 2nd state variable cash reserves and with liquidity constraints (e.g., borrowing is forbidden). This allows us to integrate insolvency and illiquidity. Now, the value of default depends also on the cost of cash and the level of cash reserves, which are inversely related to earnings at default. A lower value of default is associated to lower cash-reserves (and less negative earnings), indicating that default is more because illiquidity than insolvency. Yet, insolvency and illiquidity are closely linked—low cash savings depend mainly on two factors past negative earnings (and/or a low cash initial endowment).

Olney (1999) notes that "consumer spending collapsed in 1930, turning a minor recession into the Great Depression.. Households were shouldering an unprecedent burden of installment debt. Down payments were large.. Missed installment payments triggered repossession, household lost all acquired equity. Cutting consumption was the only viable strategy in 1930 for avoiding default.." However, "Institutional changes lowered the cost of default by 1938. When recession began again, indebted households chose to default rather than reduce consumption." Our results can be applied to a better understanding of this; if $-\phi < 0$ is household recovery value at default—a penalty, $\frac{da}{d\phi} < 0$ implies that $\frac{da}{d(-\phi)} > 0$ which rises DD (and delays default). For more work on firm default in structural models, see Carr (2014).

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8 Appendix A: Firm Default, Extensions and Robustness

We add several realistic features to Leland–Toft model. We work on a case by case basis. For easy of comparison with the original papers, we use their same original notation but we will strictly limit to the needed results (see the original papers for full details).

8.1 Strategic default

Shareholders can seize some value before liquidation. The equity value at default is not zero but ϕV_B, $\phi \geq 0$. The new boundary conditions are given by

$$E(V_B) = \phi V_B \text{ and } E_V(V_B) = \phi.$$

For $V_t = V_B$, then $\left(r - (r-\delta)V_t \frac{E_V}{E}\right) E = \delta \phi V_B$, and the option Gamma is given by

$$\frac{1}{2}E_{VV}(V_B) = \frac{-NC + \delta\phi V_B}{\sigma^2 V_B^2} = \frac{-\left(\delta\,(1-\phi)\,V_B - (1-\pi)\,C + \alpha\frac{V_B}{m} - p\right)}{\sigma^2 V_B^2}. \tag{33}$$

In the non-default region, $V_t \geq V_B$, from a 2nd-order Taylor expansion at the point V_B,

$$E(V_t) \approx \phi V_t + (-NC + \delta\phi V_B) \times \left(\frac{V_t - V_B}{\sigma V_B}\right)^2, \tag{34}$$

and the time-value by

$$E^{TV}(t, V) = E(t, V_t) - \phi V_t \approx (-NC + \delta\phi V_B) \times \left(\frac{V_t - V_B}{\sigma V_B}\right)^2. \tag{35}$$

The effect of the dividend-yield on default, which is negative, is reduced to $-\delta\,(1-\phi)$.

8.2 Stochastic volatility

From equations (18) and (26), $E^{TV}(t, V_t, \sigma_t) = E(t, V_t, \sigma_t)$, and

$$
\begin{aligned}
&E(t, V_t, \sigma_t) \qquad (36)\\
\approx\; &\underbrace{\left((1-\pi)\,C + p - \left(\delta + \frac{\alpha}{m}\right) V_B(\sigma_t)\right)}_{b_t} \times \underbrace{\left(\frac{V_t - V_B(\sigma_t)}{\sqrt{\sigma_t^2 V_B^2 + b^2(\sigma_t)\left(\frac{\partial}{\partial\sigma} V_B\right)^2 - \rho\sigma_t V_B b(\sigma_t)\frac{\partial}{\partial\sigma} V_B}}\right)^2}_{a_t}.
\end{aligned}
$$

The distance-to-default, a_t, can be inverted from equity value in the same way; i.e.,

$$
a_t = \frac{V_t - V_B(\sigma_t)}{\sqrt{\sigma_t^2 V_B^2 + b^2(\sigma_t)\left(\frac{\partial}{\partial\sigma} V_B\right)^2 - \rho\sigma_t V_B b(\sigma_t)\frac{\partial}{\partial\sigma} V_B}} \approx \sqrt{\frac{E(t, V, \sigma_t)}{b_t}}. \qquad (37)
$$

8.3 Amortizable debt and rollover risk

Consider that the total debt in Leland-Toft, which consists of a portfolio of bonds with maturities ranging from t to $t+m$ (each bond paying a continuous c–coupon and p–notional), is not constantly rollover but p follows

$$
dp_t = \alpha(p_t)dt + \xi(p_t)dW_t,
$$

where $\alpha(p_t) < 0$ is the speed to which debt is continuously retired (increased if $\alpha(p_t) > 0$). We add a Gaussian factor ξdW_t (where $E\,[dZdW] = \rho dt$), which allows debt to randomly evolve. At every instant, before taxes, the firm's debt service is given by

$$
(C(p_t) + p_t - d(V_t, p_t, m)) \times dt.
$$

$d(V_t, p_t, m)$ is how much debt is raised at time t, which is associated to a $c(p_s)$–coupon bond expiring at $t+m$ and p_{t+m}–notional, $s \in [t, t+m]$. At default, all bonds pay

$$
d(V_B(p_t), p_t, m) = \alpha \frac{V_B\,(p_t)}{m}.
$$

The parameter p_t summarizes how much debt is alive. We could relate $\alpha(p_t)$ to the firm's capital structure policy and $\xi(p_t)$ to market or monetary–policy conditions faced by borrowers/lenders. $\alpha(p_t) < 0$ implies that a firm has to reduce its debt. For instance, if $\xi = 0$ and $\alpha(p_t) = -p_0/T$, then $p_t = \frac{T-t}{T} \times p_0$ represents the alive debt which is amortized in T years (compared to $\alpha(p_t) = 0$, where p is constant). In this case, for $t \geq T - m$, $p_{t+m} = 0$ a zero notional at maturity.

Then, equity value is given by

$$E(t, V_t, p_t) \tag{38}$$
$$\approx \underbrace{\left((1-\pi)\, C(p_t) + p_t - \left(\delta + \frac{\alpha}{m}\right) V_B(p_t)\right)}_{b_t} \times \underbrace{\left(\frac{V_t - V_B(P_t)}{\sqrt{\sigma_t^2 V_B^2 + \xi^2 \left(\frac{\partial}{\partial p} V_B\right)^2 - \rho\sigma_t V_B \xi \frac{\partial}{\partial p} V_B}}\right)^2}_{a_t}.$$

Initially, the value of default is larger if $\alpha(p_t) < 0$, which is how much debt is retired; i.e., $d(V_t, p_t, m)$ lowers (because its associated notional p_{t+m} will become lower since $\alpha(p_t) < 0$) and $p_t - d(V_t, p_t, m)$ rises for all V_t (included V_B). On the contrary, if most of the debt has been repaid (i.e., $(p_t, C(p_t)) \to 0$), $b_t \to -\left(\delta + \frac{\alpha}{m}\right) V_B(p_t) \leq 0$; and hence, the value of default lowers and it is not optimal to default (i.e., $V_B(p_t) \to 0$ and $a_t \to \infty$).

8.4 Liquidity: Gryglewicz (2011)

Gryglewicz (2011) model is interesting to us because the shareholders recovery value is not lineal, providing a robust test for our results. Gryglewicz provides close-form solutions in a model with cash reserves. The firm generates a stochastic flow of earnings before interest and taxes (EBIT),

$$dX_t = \overline{\mu} dt + \sigma d\overline{Z}_t, \tag{39}$$

where $\overline{\mu}$ is unknown but takes either of two values, μ_L or μ_H, $\mu_L < \mu_H$. All parties share a common prior expectation μ_0 about $\overline{\mu}$, with $\mu_0 \in (\mu_L, \mu_H)$. From optimal filtering theory,

$$dX_t = \mu_t dt + \sigma dZ_t \quad \text{and} \quad d\mu_t = \frac{(\mu_t - \mu_L)(\mu_H - \mu_t)}{\sigma} dZ_t. \tag{40}$$

We denote volatility of average earnings by $\Sigma(\mu_t) = (\mu_t - \mu_L)(\mu_H - \mu_t)/\sigma$.

The dynamic of cash reserves is given by

$$dC_t = rC_t dt + (1-\tau)[dX_t - kdt] - dDiv_t, \tag{41}$$

where k is the coupon rate of the consol bond and $dDiv_t$ is the dividend. It is required that

$$C_t \geq 0 \quad \text{and} \quad dDiv_t \geq 0, \tag{42}$$

borrowing and negative dividends are forbidden. If $C_t = 0$, illiquidity produces default.

Following Gryglewicz, the minimum level of cash reserves which is consistent with (42) is given by

$$\overline{C}(\mu_t) = (1-\tau) \times \left[\frac{\sigma^2}{\mu_H - \mu_L} \ln\left(\frac{\mu_t - \mu_L}{\mu_H - \mu_t} \frac{\mu_H - \mu^*}{\mu^* - \mu_L}\right) + \frac{1}{r}\left\{k - \frac{\mu_H + \mu_L}{2}\right\}^+\right]. \tag{43}$$

For $C_t = \overline{C}(\mu_t)$, under the optimal dividend policy, equity value depends only on the mean, μ_t, and solves

$$\begin{aligned} rE(\mu_t) &= \frac{1}{2}\Sigma^2(\mu_t)\,E''(\mu_t) + \frac{1}{dt}dDiv_t^*, \qquad (44)\\ \frac{1}{dt}dDiv_t^* &= \frac{(1-\tau)\,r\sigma^2}{\mu_H-\mu_L}\ln\left(\frac{\mu_t-\mu_L}{\mu_H-\mu_t}\frac{\mu_H-\mu^*}{\mu^*-\mu_L}\right) + (1-\tau)\left\{\frac{\mu_H+\mu_L}{2}-k\right\}^+, \end{aligned}$$

where E'' denotes a 2nd derivative, μ^* is the optimal default boundary, and $dDiv_t^*$ is the associated optimal dividend. The value-matching and smooth-pasting boundary conditions depend on cash reserves and are given by

$$E(\mu^*) = \overline{C}(\mu^*) \text{ and } E'(\mu^*) = \overline{C}'(\mu^*), \qquad (45)$$

respectively. Note that $\overline{C}(\mu_t) > 0$ and $dDiv_t^* > 0$, if $\mu_t > \mu^*$ (since $\frac{\mu_t-\mu_L}{\mu_H-\mu_t}\frac{\mu_H-\mu^*}{\mu^*-\mu_L} > 1$).

8.4.1 The value of default in Gryglewicz

Now, we explain the value of default. At the optimal default boundary, $\mu_t = \mu^*$,

$$\overline{C}(\mu^*) = \frac{1}{r}(1-\tau)\left\{k-\frac{\mu_H+\mu_L}{2}\right\}^+, \qquad (46)$$

implying that

$$(1-\tau)\left\{k-\frac{\mu_H+\mu_L}{2}\right\}^+ = \frac{1}{2}\Sigma^2(\mu^*)\,E''(\mu_t) + (1-\tau)\left\{\frac{\mu_H+\mu_L}{2}-k\right\}^+ \qquad (47)$$

and

$$\frac{1}{2}E''(\mu^*) = \frac{(1-\tau)\times\left(k-\frac{\mu_H+\mu_L}{2}\right)}{\Sigma^2(\mu^*)}. \qquad (48)$$

Since $\overline{C}$ is not lineal, we have to compute its 2nd derivative, i.e.,

$$\begin{aligned} \overline{C}'(\mu^*) &= \frac{(1-\tau)\,\sigma^2}{\mu_H-\mu_L}\times\left(\frac{1}{\mu_H-\mu^*}+\frac{1}{\mu^*-\mu_L}\right) \quad \text{and} \qquad (49)\\ \overline{C}''(\mu^*) &= \frac{(1-\tau)}{\mu_H-\mu_L}\frac{(\mu^*-\mu_L)^2-(\mu_H-\mu^*)^2}{((\mu_H-\mu^*)(\mu^*-\mu_L)/\sigma)^2} = (1-\tau)\frac{2\mu^*-(\mu_L+\mu_H)}{\Sigma^2(\mu^*)}, \qquad (50) \end{aligned}$$

Hence, in the subset $\{(\mu, C) : \mu \geq \mu^* \text{ and } C = \overline{C}(\mu)\}$, the time-value of equity is given by

$$E^{TV}(\mu_t) \approx \frac{1}{2}\left(E''(\mu^*) - \overline{C}''(\mu^*)\right)\times(\mu_t-\mu^*)^2 = (1-\tau)\times(k-\mu^*)\times\left(\frac{\mu_t-\mu^*}{\Sigma(\mu^*)}\right)^2 \qquad (51)$$

and

$$\frac{\mu_t-\mu^*}{\Sigma(\mu^*)} \approx \sqrt{\frac{E^{TV}(\mu_t)}{(1-\tau)\times(k-\mu^*)}}.$$

The value of default depends on minus the net cash-flows; i.e., the coupon rate minus average profits, $k-\mu^*$, which must be positive. Moreover, in Gryglewicz (2011) model without cash constraint, the 2nd-order approximation of the option value is also equal to the rhs of (51).[31] It turns out that both μ^*'s, equations (8) and (29) in his paper are the same, only the optimal coupon differs between both models. So, cash constraints do not affect the decision to default if cash is invested at the same rate, r.

Remark For completeness, we show the conditions under which Gamma is the same in equations (51) and (19), where (19) depends on V and C, and is given by

$$rE = \mu\left(V_t\right)E_V + \frac{1}{2}\sigma^2\left(V_t\right)E_{VV} + dD_t + \left(r_C C + NC_t - dD_t\right)E_C + \frac{1}{2}\xi^2 E_{CC} + \rho\sigma\left(V_t\right)\xi E_{VC},$$

with boundary condition at default

$$E(V,C) = C, \text{ and } E_V = 0 \text{ and } E_C = 1.$$

We have that $r = r_C$, $\mu\left(V_t\right) = 0$ (from optimal filtering), and $NC_t = (1-\tau)\times(\mu^* - k)$ and $E = \overline{C}(\mu^*)$ at default. Then, for $V_t = \mu_t$ and $\sigma\left(V_t\right) = \Sigma\left(\mu_t\right)$, equation (19) simplifies to

$$\frac{1}{2}\left(\sigma^2\left(V_B\right) + \xi^2\left(\frac{\partial V_B}{\partial C}\right)^2 - 2\rho\sigma\left(V_t\right)\xi\frac{\partial V_B}{\partial C}\right)E_{VV} = -NC_t = (1-\tau)\times(k-\mu^*),$$

which implies the same Gamma as in equation (51) if $\xi = 0$.

8.5 Liquidity: Anderson and Carverhill (2012)

Anderson and Carverhill (2012) consider a two-factor problem, which is solved by numerical methods. The rate of operating revenues of the firm, dS_t, is given by

$$dS_t = \rho_t dt + \sigma dw_t^\sigma, \tag{52}$$

$$d\rho_t = \kappa\left(\overline{\rho} - \rho_t\right)dt + \eta\sqrt{\rho_t}dw_t^\rho, \tag{53}$$

where dw_t^σ and dw_t^ρ are two orthogonal Wiener processes. The profit of the firm is $dS_t - f dt$, where f are operating costs. The firm has a "cash reserve" C with dynamics

$$dC_t = (1-\tau)\left[dS_t - \left((f+q) - r_{C_t}C_t\right)dt\right] - dD_t, \tag{54}$$

[31] The same PDE (44) holds with payout rate $dDiv_t^* = (1-\tau)\left(\mu_t - k\right)dt$, which is the net cash-flow, and boundary conditions $E(\mu^*) = 0$ and $E'(\mu^*) = 0$. See equations (5) and (23) in Gryglewicz (2011).

where τ is the tax rate, r_{C_t} a different cash borrowing/lending rate ($(r - r_C) \times C \geq 0$), and dD_t the dividend payment. $q \geq 0$ is the continuous coupon of a perpetual bond, and the firm is leveraged if $q > 0$.

Equity value $J^q(\rho, C)$, for a given coupon q, is a function of the expected revenues and cash reserves (ρ_t and C_t, respectively) defined in the state space $\{(\rho, C) : C \geq \underline{C}(\rho)\}$, where $\underline{C}(\rho)$ is an exogenous borrowing limit given by

$$\underline{C}(\rho) = -\max\left\{0, (1-\alpha) J^0(\rho, C) - \frac{q}{r}\right\},$$

α, $0 \leq \alpha \leq 1$, is the loss given default, J^0 is the value of the unlevered firm, and q/r is the value of the perpetual risk-free debt. (Intuitively, $-\underline{C}(\rho) \geq 0$ is the firm liquidation value after repaying all debt). At every instant, the firm decides "optimally" (i) the dividend policy $dD_t \geq 0$, (ii) whether to issue new stock, or (iii) whether to default. Within the so-called saving region, $\mathcal{S}$, it is optimal to no pay dividends $\frac{dD_t}{dt} = 0$, and the firm will issue new equity if the value of the firm is not negative and if it is cheaper than borrowing money.

Hence, we focus on the saving region, $(\rho, C) \in \mathcal{S}$, where $\frac{dD_t}{dt} = 0$. The firm satisfies the following PDE,

$$\frac{\partial}{\partial t} J_t^q + \kappa(\overline{\rho} - \rho_t) \frac{\partial}{\partial \rho} J_t^q + \frac{1}{2}\eta^2 \rho_t \frac{\partial^2}{\partial \rho^2} J_t^q + (1-\tau)\left[\rho_t - (f+q) - r_{C_t} C_t\right] \frac{\partial}{\partial C} J_t^q + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial C^2} J_t^q = r J_t^q, \tag{55}$$

subject to

$$1 \leq \frac{\partial}{\partial C} J_t^q(\rho, C) \leq \frac{1}{\theta}, \tag{56}$$

where $C \geq \underline{C}(\rho)$ and equity has limited liability $J_t^q(\rho, C) \geq 0$. θ is the cost of issuing new shares (costly, if $\theta < 1$). In this region, positive cash-flows are saved in cash (better than paid in dividends, $1 \leq \frac{\partial}{\partial C} J_t^q(\rho, C)$) and negative cash-flows are withdraw from the cash reserve (better than issuing new stock, $\frac{\partial}{\partial C} J_t^q(\rho, C) \leq \frac{1}{\theta}$).

Anderson and Carverhill do not specify the smooth-pasting boundary condition for default, since the problem is numerically solved; by maximizing among paying dividends, issue new shares, or default (see their Appendix A numerical techniques). They assume that shareholders get (where T is the default stopping-time),

$$J_T^q(\rho, C) = \max\left\{0, (1-\alpha) J_0^0(\rho, 0) + C - \frac{q}{r}\right\}.$$

8.5.1 The value of default in Anderson and Carverhill

Now, we explain the value of default. The default boundary, $\{B(\rho) : \rho \leq \widehat{\rho} < \infty\}$, is part of the region $\mathcal{S}$. Default happens only because insolvency, the firm can always fund losses (by issuing new stock) at a cost which is given by $\frac{1}{\theta}$. For $\rho \to \infty$, it does not make sense to default at all, the firm will rise cash to any cost (which is bounded by $\frac{1}{\theta}$). Hence, $\widehat{\rho} < \infty$.

For $\{(\rho, B(\rho)) : \rho \leq \widehat{\rho} \text{ and } B(\rho) > \underline{C}(\rho)\}$, and assuming that $\frac{\partial}{\partial t} J_t^q = 0$, the PDE simplifies to

$$\begin{aligned} & \frac{1}{2}\left(\eta^2 \rho_t \frac{\partial^2}{\partial \rho^2} + \sigma^2 \frac{\partial^2}{\partial C^2}\right) J_t^q \qquad (57) \\ = \ & \underbrace{-(1-\tau)\left[\rho_t - (f+q)\right] \times \frac{\partial}{\partial C} J_t^q} + \underbrace{\left(r - r_{C_t} \times \frac{\partial}{\partial C} J_t^q\right) B(\rho)} - \underbrace{\kappa(\overline{\rho} - \rho_t) \times \frac{\partial}{\partial \rho} J_t^q} \\ & + r \times \left(J_t^q - B(\rho)\right), \end{aligned}$$

where the first three terms in the rhs are similar to those in equation (20).

If shareholders keep cash and save some strategic value at default ($\phi \geq 0$), i.e.,

$$J_T^q(\rho, B(\rho)) = B(\rho) + \phi(\rho - \overline{\rho}), \qquad (58)$$

$$\frac{\partial}{\partial \rho} J_T^q(\rho, B(\rho)) = \phi \quad \text{and} \quad \frac{\partial}{\partial C} J_T^q(\rho, B(\rho)) = 1, \qquad (59)$$

the equity gamma in the previous PDE is obtained from

$$\frac{1}{2}\left(\eta^2 \rho_t \frac{\partial^2}{\partial \rho^2} + \sigma^2 \frac{\partial^2}{\partial C^2}\right) J_t^q = -(1-\tau)\underbrace{\left[\rho_t - (f+q)\right]} + \underbrace{(r - r_{C_t}) B(\rho)} + \underbrace{(r+\kappa)(\rho_t - \overline{\rho}) \times \phi}. \qquad (60)$$

The value of default depends on minus the operating profits $-(\rho_t - (f+q))$, the cost of cash $(r - r_{C_t})$, and the prospect of the firm if shareholders do not lost all at default $(\rho_t - \overline{\rho}) \times \phi$.

9 Appendix B

9.1 Proof that $\frac{dV_B^{(\phi)}}{d\phi} > 0$, where ϕV is shareholders recovery value at default

For simplicity, we assume a time homogeneous problem and a constant default boundary. Let $V_B^{(\phi)}$ denote the optimal default boundary associate to ϕ, where $V_B = V_B^{(0)}$ in the case of zero recovery. Let $E^{(\phi)}(V_t, B)$ denote equity value for recovery ϕ and a given default policy B, where $V_t \geq B$ and $E^{(\phi)}(V_t, B)|_{V_t = B} = \phi B$.

V_B is the optimal policy associated to $\phi = 0$, and hence,

$$\left.\frac{dE^{(0)}(V_t, B)}{dB}\right|_{B = V_B} = 0.$$

The key insight is that

$$E^{(\phi)}(V_t, B) = E^{(0)}(V_t, B) + E_t^Q\left[e^{-r\tau} \phi B | V_t\right],$$

where τ is the stopping-time associated to $V_t = B$ (i.e., $\tau = t$ if $V_t = B$). Then,

$$\begin{aligned} \left.\frac{dE^{(\phi)}(V_t,B)}{dB}\right|_{B=V_B} &= 0+\phi\times\left(E_t^Q\left[e^{-r\tau}\right]+V_B\frac{dE_t^Q\left[e^{-r\tau}|V_t\right]}{dB}\right)\\ &> 0 \text{ if } \phi>0 \text{ (and } <0 \text{ if } \phi<0\text{)}, \end{aligned}$$

where $\frac{dE_t^Q\left[e^{-r\tau}|V_t\right]}{dB} \geq 0$ since $V_t \geq B$. Consequently, the optimal default policy holds that

$$V_B^{(\phi)} > V_B \text{ if } \phi>0\text{; and } V_B^{(\phi)} < V_B \text{ if } \phi<0.$$

Next,

$$\begin{aligned} E^{(\phi+\Delta\phi)}\left(V_t,V_B^{(\phi)}\right) &= E^{(0)}\left(V_t,V_B^{(\phi)}\right)+E_t^Q\left[e^{-r\tau}\left(\phi+\Delta\phi\right)V_B^{(\phi)}|V_t\right],\\ \frac{dE^{(\phi+\Delta\phi)}\left(V_t,V_B^{(\phi)}\right)}{dV_B^{(\phi)}} &= \frac{d\left(E^{(0)}\left(V_t,V_B^{(\phi)}\right)+E_t^Q\left[e^{-r\tau}\phi V_B^{(\phi)}|V_t\right]\right)}{dV_B^{(\phi)}}+\frac{dE_t^Q\left[e^{-r\tau}\Delta\phi V_B^{(\phi)}|V_t\right]}{dV_B^{(\phi)}}\\ &= 0+\Delta\phi\times\left(E_t^Q\left[e^{-r\tau}\right]+V_B^{(\phi)}\frac{dE_t^Q\left[e^{-r\tau}|V_t\right]}{dV_B^{(\phi)}}\right)>0 \text{ , if } \Delta\phi>0. \end{aligned}$$

It follows that $V_B^{(\phi+\Delta\phi)} > V_B^{(\phi)}$, and in the limit, $\frac{dV_B^{(\phi)}}{d\phi} = \lim_{\Delta\phi\to 0}\frac{V_B^{(\phi+\Delta\phi)}-V_B^{(\phi)}}{\Delta\phi} > 0$.

■

9.2 Repo runs model – Martin et al. (2014)

In a equilibrium model of repo runs Martin et al. (2014) derive a liquidity (and collateral) constraints to show that "..more leveraged borrowers, or less profitable ones, are more fragile." The liquidity constraints is given by (see their equation (13))

$$\beta^2 R\overline{I} \geq (1-\alpha+\beta)\,b,$$

where $R\overline{I}$ and b are earnings and debt, respectively; $\beta < 1$ is a discount factor and $(1-\alpha+\beta)$ is the amount of debt to be repaid.

One can derive a similar default constraint, if profits are nonnegative (i.e., $\pi \geq 0$ and π is given their unenumerated equation in page 967),

$$\beta^2\frac{R-1}{1-\beta}\overline{I} \geq (1-\alpha+\beta)\,b.$$

First, the liquidity and default constraints have the same static comparative analysis to profitability R, size $\overline{I}$ and debt $b/\overline{I}$. Second, the default constraint is strictly weaker than the liquidity constraint. If the liquidity constraint is binding (i.e., $\beta^2 R\overline{I} = (1-\alpha+\beta)\,b$), then

$$\beta^2\overline{I} = \frac{1}{R\beta}\beta\,(1-\alpha+\beta)\,b < \frac{1}{R\beta^2}\beta\,(1-\alpha+\beta)\,b < \beta\,(1-\alpha+\beta)\,b,$$

since $\beta < 1$ and $\beta^2 R > 1$ imply that $\beta R > \beta^2 R > 1$. Next

$$\beta^2 (R - 1) \overline{I} - (1 - \beta)(1 - \alpha + \beta) b = -\beta^2 \overline{I} + \beta (1 - \alpha + \beta) b > 0.$$

Hence, illiquidity precedes default—a bankruptcy bank is also illiquid.

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