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Abstract

We introduce TailCoR, a new measure for tail correlation that is a function of linear and non–linear correlations, the latter characterized by the tail index. TailCoR can be exploited in a number of financial applications, such as portfolio selection where the investor faces risks of a linear and tail nature. Moreover, it has the following advantages: i) it is exact for any probability level as it is not based on tail asymptotic arguments (contrary to tail dependence coefficients), ii) it can be used in all tail scenarios (fatter, equal to or thinner than those of the Gaussian distribution), iii), it is distribution free, and iv) it is simple and no optimizations are needed. Monte Carlo simulations and calibrations reveal its goodness in finite samples. An empirical illustration using a panel of Euro area sovereign bonds shows that prior to 2009 linear correlations were in the vicinity of one and non–linear correlations were inexistent. Since the beginning of the crisis the linear correlations have decreased sharply, and non–linear correlations appeared and increased significantly in 2010–2011.

Keywords: Tail correlation, quantile, ellipticity, risk.

JEL classification: C32, C51, G01.

Resumen

Introducimos TailCoR, una nueva medida de correlación en las colas. TailCoR es una función de correlaciones lineales y no lineales, esta última caracterizada por las colas. TailCoR puede ser utilizado en numerosas aplicaciones financieras, tales como selección de cartera cuando el inversor se enfrenta a riesgo de naturaleza lineal y de colas (un caso que cubrimos en detalle). Además, TailCoR tiene una serie de ventajas: i) no está basado en argumentos asintóticos en las colas (contrariamente a coeficientes de dependencia en las colas) y se puede calcular exactamente para cualquier nivel de probabilidad; ii) no se necesita hacer un supuesto sobre la distribución de probabilidad, y iii) es simple y no se necesitan optimizaciones. Simulaciones y calibraciones de Monte Carlo ofrecen las buenas propiedades de muestras finitas de TailCoR. Una ilustración a un panel de bonos soberanos de la zona del euro muestra que antes del 2009 la correlaciones lineales se encontraban cercanas a 1 y las correlaciones no lineales estaban ausentes. Sin embargo, desde el principio de la crisis las correlaciones lineales han descendido rápidamente y las no lineales han aparecido y aumentado significativamente en 2010 y 2011.

Palabras clave: correlación en las colas, cuantiles, elipticidad, riesgo.

Códigos JEL: C32, C51, G01.

1 Introduction

The 2007–2010 financial and the 2009–2012 European sovereign debt crises have highlighted the importance of tail –or rare– events. These events may have different nature, such as corporate and Government defaults, stock market crashes, or political news, to name a few. Understanding them, controlling for them, and insure against them has become of paramount importance. When they occur, their effect is spread over the system, creating tail correlation which has linear and non–linear contributions. Indeed, it may happen because either financial securities are linearly correlated (i.e. the Pearson correlation coefficients are close to one) and/or they are non–linearly correlated. Diagrammatically, the latter happens when the cloud of points in a scatter plot between the returns of two securities does not have a well defined direction for small and moderate values but the tail events co–move.

Several authors have proposed measures of correlation for the tails. The coefficients of tail dependence (also called extremal dependence structure) steaming from extreme value and copula theories are probably the most common. McNeil et al. (2005) and Hua and Joe (2011) summarize and present them nicely within copula theory, and Chollete et al. (2011) use copulas for analyzing international diversification and explore the dependencies between fourteen national stock market indexes. Poon et al. (2004) explore the tail dependence structure among risky asset returns and present a framework for identifying joint–tails in the context of extreme value theory. The coefficients of tail dependence have two drawbacks. First, they are asymptotic results (asymptotically on the tail) and hence their application is inevitably an approximation. Second, it cannot explain dependencies on the tails if the latter are thinner than the Gaussian, as by definition it implied that the joint tail probability decreases slower than the Gaussian. Moreover in the context of extreme value theory, they are not able to disentangle between the linear and non–linear correlations. This is however possible within copulas, but the analytical form heavily depends on the choice of the copula function. Moreover, in large dimensional problems only the Gaussian (which does not have tail dependence) and Student–t copulas are realistic and feasible, while in vast dimensional problems only the Gaussian is feasible.

Longin and Solnik (2001) introduce the exceedance correlation, i.e. the sample correlation between observations that are jointly beyond a given threshold. In a similar vein, Cizeau et al. (2001) introduce the quantile correlation, i.e. the sample correlation between observations that are jointly beyond a given quantile. These two measures are similar and they share the same drawback: when applied to thresholds and quantiles that are far on the tails, the number of observations is limited, which results in imprecise estimators. Moreover, these measures are not able to disentangle between the linear and non–linear contributions.

We introduce TailCoR, a new measure for tail correlations that is based on the following simple idea: if two random variables (properly standardized) are positively related (either linear and/or non–linearly), most of the times the pairs of observations have the same sign, meaning that most of the dots (that represent the pairs) concentrate in the north–east and south–west quadrants of the scatter plot. Now, consider the 45–degree line that diagonally crosses these quadrants, and project all the dots on this line. Since the two random variables are positively related, the projected dots –that are sitting on the 45–degree line– are dispersed all over the line.[1] The degree of dispersion depends on the strength of the relation between

[1] In the case of negative relation, the dots mostly concentrate in the north–west and south–east quadrants, and the projection is on the 315–degree line.

the two random variables. If weak, the cloud of dots does not have a well defined direction and the projected dots are concentrated around the origin (recall that the variables are standardized); and hence the dispersion is small. By contrast, if the relation is strong, the cloud is stretched around the 45–degree line and the projected dots are very dispersed. Therefore the interquantile range of the projection is informative about the relation, either linear or non–linear.

Moreover, under the elliptical family of distributions (i.e. the probability contours that describe the probability density function of the pairs of observations are ellipsoids) we show that the interquantile range of the projection equals the product of two components: one that only depends on the linear correlation coefficient and another that solely depends on the tail index. This is a convenient property that allows to disentangle the linear and non–linear contributions. TailCoR has three further advantages: i) it is exact for any probability level as it is not based on tail asymptotic arguments, ii) it does not depend of any specific distributional assumption, iii) it is simple and no optimizations are needed, and iv) the component that depends on the tail index may explain both heavy and thin tails (i.e. thinner than the Gaussian). Since the split of TailCoR in two components is under the elliptical family (but without imposing ant specific distribution), the tails can be fatter, equal or thinner than those of the Gaussian distribution.

We also show five extensions. i) Downside– and upside–TailCoR: it is often the case, like in risk management, that the interest lies in the tail of one side of the distribution. ii) Negative correlation: TailCoR as explained in the previous paragraph, is designed for positive relation as it projects the pairs of observations in the 45–degree line. A simple transformation makes it suitable for negative relation. iii) Dynamic TailCoR: it is easy to extend the vanilla definition to the dynamic case. Since a stylized fact of financial data is volatility clustering, allowing for dynamic volatility provides dynamic TailCoR, a more accurate measure. iv) Multivariate TailCoR: the plain definition is for a pair of random variables but it can be extended to a random vector, yielding a vector of TailCoRs. v) Multidimensional projection: TailCoR, as defined above, is a pairwise measure. However, by performing a multidimensional projection it is possible to summarize the tail correlations of the random vector into a scalar.

The analysis of tail correlations and the linear and non–linear contributions has a plethora of financial applications. Poon et al. (2004) study the implications of tail dependence for a number of financial problems, such as portfolio selection. More specifically, let two portfolios formed by securities with returns that are linear and non–linearly correlated. The investor that is faced to these portfolios is exposed to linear and non–linear sources of risk. Disentangling them allows for a more effective risk reduction: a portfolio of securities that are non–linear risk independent has thinner tails than those with securities that are non–linear risk dependent. Indeed, the investment decision does not only depend on his risk appetite, but also on his/her preferences for linear and non–linear risks.

The empirical illustration is to the euro area sovereign bond yields for the period 2002-2012. The Green Paper of the European Commission (2011) assesses the feasibility of common issuance of sovereign bonds (the Stability bonds) among the Member States of the euro area.[2] Though the document does not set guidelines for the implementation, it mentions that the new bonds should be a pool. The most natural way of pooling is by means of a linear combination.

[2] Other proposals, more elaborated, of pan–European bonds are the Blue bonds of Delpla and von Weizsäcker (2010), the European safe bonds of Brunnermeier et al. (2011), the Safe bonds of the German Council of Economic Experts (2011), the synthetic Eurobonds of Beck et al. (2011), and the Eurobills of Hellwig and Philippon (2011).

If so, the correlations between them play an important role in order to determine the risks associated with the pooling. Over the years the linear contribution to TailCoR has decreased significantly since 2008. The average linear correlation coefficient before 2008 was close to 1 and at the end of the sample it was near 0.2. In other words, while before the crisis there was an almost perfect linear relation between the sovereign bond yields of the euro area, in 2009 it started to decreased, reaching in 2011 very low values, never seen before the creation of the common currency. This is in contrast with the non–linear contribution of TailCoR. It was very low from 2002 to 2007 but since them it has increased steadily, specially during the last two years. The backbone of this analysis is that the a common issuance of sovereign bonds by pooling sovereign bonds may have unexpected negative consequences for the peripheral and core countries, as they may not be necessarily less risky and more resilient to adverse shocks. A deeper analysis of the tail risks and tail correlations in the euro area sovereign bond yields can be found in Veredas (2012).

The remaining sections are laid out as follows. Section 2 introduces the notation, assumptions, definition and representations of TailCoR. It also shows a calibration exercise and the asymptotic properties of the estimator. Section 3 covers a brief Monte Carlo study. The extensions are touched upon in Section 4. The application to euro area sovereign bonds is presented in Section 6. Section 7 concludes, and a lengthy table and the proofs are relegated to the appendixes.

2 TailCoR

2.1 Definition

Let $\mathbf{X}_t$ $t = 1, \ldots, T$ be a random vector of size N at time t satisfying

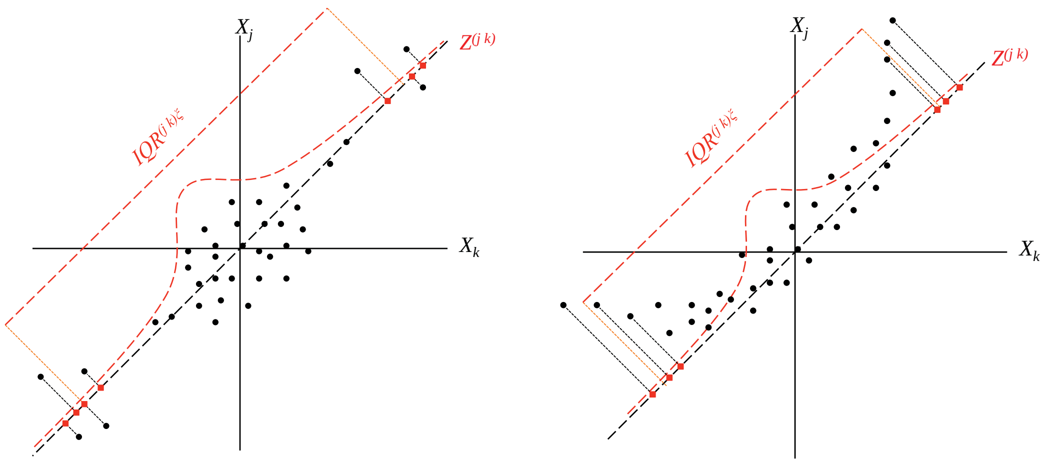
G1 The random process $\mathbf{X}_1, \ldots, \mathbf{X}_T$ is (a) a sequence of strongly stationarity random vectors, (b) a $\mathbf{X}_t$ is S–mixing, i.e. it satisfies the following two conditions: (i) for any t and m, $P(|\mathbf{X}_t - \mathbf{X}_{tm}| \geq \gamma_m) \leq \delta_m$ for some numerical sequences $\gamma_m \to 0$, $\delta_m \to 0$, (ii) for any disjoint intervals $I_1, \ldots, I_r$ of integers and any positive integers $m_1, \ldots, m_r$, the vectors $\{\mathbf{X}_{tm_1}, t \in I_1\}, \ldots, \{\mathbf{X}_{tm_r}, t \in I_r\}$ are independent provided the separation between I_k and I_l is greater than $m_k + m_l$.

Assumption **G1**(a) is standard in time series analysis. Assumption **G1**(b) specifies the time dependence of $\mathbf{X}_t$. Assuming a mixing condition instead of a particular type of dynamic model is more general and makes TailCoR applicable for a wide array of processes. Indeed, the conditions for S–mixing, introduced by Berkes et al. (2009), apply to a large number of processes used in the economics and finance, including GARCH models and its extensions, linear processes (specially ARMA models) and stochastic volatility among others. The notation **G** stands for General.

Figure 1 displays two diagrammatic representations that put forward the intuition behind TailCoR. They show scatter plots, along with the 45–degree line, where two elements of the random vector, X_j and X_k, are positively related (the pairs are depicted with circles). Projecting the observations onto the 45–degree line produces a new random variable $Z^{(j\,k)}$, depicted with squares. Because of representation purposes we show the projection only for the observations on the tails but the reader should keep in mind that the projection is done for all the observations. TailCoR is a pairwise function equal –up to a normalization– to the difference between the upper and lower tail quantiles of $Z^{(j\,k)}$. Focusing on the leftish scatter plot, the *tail interquantile range* can be large because of two reasons. First, if X_j and X_k are highly linearly correlated, then the dots and the corresponding squares are close to each other. Second, if the linear correlation between X_j and X_k only happens on the tails while the observations around the origin form a cloud with undefined direction. These two situations are not mutually exclusive and one or both may happen. In either case TailCoR is large, in a sense to be precisely defined below.[3] Moderate departures from linearity can also be handled by TailCoR, as shown in the right scatter plot in which the relation between X_j and X_k appears to be U–shaped. The case where X_j and X_k are negatively related is treated in the next Section.

Figure 1: Diagrammatic representation of TailCoR

(a) Linear **(b)** Non-linear



Scatter plots, along with the 45–degree line, where X_j and X_k are positively related (the pairs are depicted with circles). Left plot shows a linear relation while right plot shows a non–linear relation. Projecting the observations onto the 45–degree line produces the random variable $Z^{(j\,k)}$, depicted with squares. Because of representation purposes we show the projection only for the observations on the tails but the reader should keep in mind that the projection is done for all the observations.

Let $X_{j\,t}$ be the jth element of the random vector $\mathbf{X}_t$. Denote by Q_j^τ its τth quantile for $0 < \tau < 1$, and let $\mathrm{IQR}_j^\tau = \mathrm{Q}_j^\tau - \mathrm{Q}_j^{1-\tau}$ be the τth interquantile range. A typical value of τ for IQR_j^τ is 0.75, which is what is considered henceforth unless otherwise stated. Let $Y_{j\,t}$ be the standardized version of $X_{j\,t}$:

$$\mathrm{Y}_{j\,t} = \frac{X_{j\,t} - \mathrm{Q}_j^{0.50}}{\mathrm{IQR}_j^\tau}. \tag{1}$$

Likewise for $Y_{k\,t}$. In this context of heavy tails, the standardization is with respect to the median and the interquantile range, meaning that the mean $Y_{j\,t}$ is not necessarily zero and its variance is not one. This is not an issue since the aim of (1) is to have the pair $(Y_{j\,t}, Y_{k\,t})$

[3]Similar construction has been used by Dominicy et al. (2012) for estimating the dispersion matrix of an elliptical distribution.

centered at zero and with the same scale. As (1) is based on marginal quantiles, we need the following technical assumption

G2 (a) For $0 < \tau < 1$, the cumulative distribution function of $X_{j\,t}$, denoted by $F(x_{j\,t})$, is bounded and continuous in some neighborhood of Q_j^τ. (b) The probability density function, denoted by $f(x_{j\,t})$, is such that $0 < f(\mathrm{Q}_j^\tau) < \infty$. Likewise for $X_{k\,t}$.

By standard trigonometric arguments, the projection of $(Y_{j\,t}, Y_{k\,t})$ onto the 45–degree line is

$$Z_t^{(j\,k)} = \frac{1}{\sqrt{2}}(Y_{j\,t} + Y_{k\,t}), \tag{2}$$

and the tail interquantile range is

$$\mathrm{IQR}^{(j\,k)\,\xi} = \mathrm{Q}^{(j\,k)\,\xi} - \mathrm{Q}^{(j\,k)\,1-\xi},$$

where $0 < \tau < \xi < 1$ is typically close to 1 (> 0.90). The larger ξ is, the further we explore the tails. Equipped with $\mathrm{IQR}^{(j\,k)\,\xi}$ we define TailCoR as follows.

Definition 1 *Under* **G1**–**G2**, *TailCoR between $X_{j\,t}$ and $X_{k\,t}$ is*

$$\mathit{TailCoR}^{(j\,k)\,\xi} := s_g(\xi, \tau)\mathit{IQR}^{(j\,k)\,\xi},$$

where $s_g(\xi, \tau)$ is a normalization such that under Gaussianity and linear uncorrelation $\mathit{TailCoR}^{(j\,k)\,\xi} = 1$, the reference value.

Several remarks are in order. First, Gaussianity and uncorrelation implies independence, so $s_g(\xi, \tau)$ is the inverse of $\mathrm{IQR}^{(j\,k)\,\xi}$ under these conditions. A table with values of $s_g(\xi, \tau)$ for a grid of reasonable values for τ and ξ is found in Appendix T. Interpolation can be used for values of ξ and τ that are not in the table or, alternatively, a simple function can be programmed to compute exactly $s_g(\xi, \tau)$ for any value of τ and ξ. The steps for programming such function are given in Appendix T. Second, $\mathrm{TailCoR}^{(j\,k)\,\xi}$ does not depend explicitly on τ, as it is chosen a priori for the standardization. Third, for the time being we deal with the univariate definition, i.e. for the pair $(j\,k)$. The multivariate definition is treated in the next section. Last, for any S–mixing and strongly stationary process, $\mathrm{TailCoR}^{(j\,k)\,\xi}$ can be used –provided the estimator shown below– even for skewed and heavy tailed processes for which the first and second moment do not exist.

2.2 Alternative representation

An alternative, more intuitive and refined, representation of $\mathrm{TailCoR}^{(j\,k)\,\xi}$ can be obtained if we specify further structure on $\mathbf{X}_t$.

E1 The unconditional distribution of $\mathbf{X}_t$ belongs to the elliptical family, given by the stochastic representation $\mathbf{X}_t =_d \boldsymbol{\mu} + \mathcal{R}_{\alpha\,t}\boldsymbol{\Lambda}\mathbf{U}_t$.

The notation **E** stands for Elliptical. The random vector $\mathbf{U}_t$ is *i.i.d* and uniformly distributed in the unit sphere. The scaling matrix $\boldsymbol{\Lambda}$ produces the ellipticity and is such that $\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}'$, a positive definite symmetric dispersion matrix –often called the shape matrix. The non–negative and continuous random variable $\mathcal{R}_{\alpha\,t}$ generates the tail thickness through the tail index α, and is stochastically independent of $\mathbf{U}_t$. The vector $\boldsymbol{\mu}$ re–allocates the center of the distribution. Let $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\Sigma}, \alpha) \in \boldsymbol{\Theta}$ denote the vector of unknown parameters satisfying

E2 (a) The parameter space $\boldsymbol{\Theta}$ is a non–empty and compact set on $\mathbb{R}^{N+\frac{N(N+1)}{2}+1}$. (b) The true parameter value $\boldsymbol{\theta}_0$ belongs to the interior of $\boldsymbol{\Theta}$.

The elliptical family is commonly used as it nests, among others, the Gaussian, Student–t, elliptical stable (ES henceforth), Cauchy, Laplace (these four have tails heavier than the Gaussian) and Kotz (with thinner tails than the Gaussian) probability laws.[4] For a given vector of locations and a dispersion matrix, the difference between two elliptical distributions is the tail index, which plays a central role all over the remaining of the article. Note that assumption **E1** is about a family of distributions, i.e. $\mathbf{X}_t$ is assumed to belong to that family but no specific distributional assumption is made. This is very general and covers many cases encountered in practical work. It does not cover however the cases like the rightish scatter plot of Figure 1 as the probability contours are not elliptical. To obtain the alternative representation of TailCoR$^{(j\,k)\,\xi}$ we also need existence of the mean and the variance–covariance matrix, which is ensured by the following assumption

E3 The unconditional moments up to order 2 are finite, i.e. $E(\mathbf{X}_t^p) < \infty, \text{ for } p \leq 2$.

This assumption implies that the mean, the variance–covariance matrix, and the correlation matrix are $E(\mathbf{X}_t) = \boldsymbol{\mu}$, $Cov(\mathbf{X}_t) = E(\mathcal{R}_\alpha)\boldsymbol{\Sigma}$, and $Corr(\mathbf{X}_t) = diag(\boldsymbol{\Sigma})^{-1/2}\boldsymbol{\Sigma} diag(\boldsymbol{\Sigma})^{-1/2}$. Note that the correlation matrix, with $(j\,k)$ element $\rho_{j\,k}$, does not depend on the tail index. Last, under the elliptical family with finite first two moments, we substitute assumptions **G1** and **G2** for

E4 The random process $\mathbf{X}_1, \ldots, \mathbf{X}_T$ is (a) a sequence of weakly stationarity random vectors, (b) a $\mathbf{X}_t$ is S–mixing, i.e. it satisfies the following two conditions: (i) for any t and m, $P(|\mathbf{X}_t - \mathbf{X}_{tm}| \geq \gamma_m) \leq \delta_m$ for some numerical sequences $\gamma_m \to 0$, $\delta_m \to 0$, (ii) for any disjoint intervals $I_1, \ldots, I_r$ of integers and any positive integers $m_1, \ldots, m_r$, the vectors $\{\mathbf{X}_{tm_1}, t \in I_1\}, \ldots, \{\mathbf{X}_{tm_r}, t \in I_r\}$ are independent provided the separation between I_k and I_l is greater than $m_k + m_l$.

E5 (a) For $0 < \tau < 1$, the cumulative distribution function $P(\mathcal{R}_{\alpha\,t} \leq r)$ has a bounded, continuous and positive derivative.

Assumption **E5** replaces **G2** since it ensures that the marginal distributions of the elements of $\mathbf{X}_t$ fulfill the conditions on **G2**. We are now ready to announce the alternative representation for TailCoR$^{(i\,j)\,\xi}$.

Theorem 1 *Under* **E1**–**E5**

$$TailCoR^{(j\,k)\,\xi} = s_g(\xi, \tau)s(\xi, \tau, \alpha)\sqrt{1 + \rho_{j\,k}}.$$

Proof See Appendix P.

The rightmost element, $\sqrt{1 + \rho_{j\,k}}$, captures the linear contribution to TailCoR$^{(j\,k)\,\xi}$, while $s(\xi, \tau, \alpha)$ captures the non–linear contribution as it depends on the tail index α. We will denote these contributions as linear and non–linear correlations.

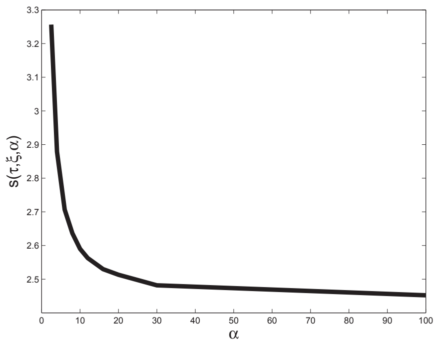
Top plots of Figure 2 displays the non–linear correlation as a function of α for the Student–t and ES distributions (and for $\xi = 0.95$ and $\tau = 0.75$). These two distributions, along with

[4]See Hashorva (2008) and Hashorva (2010) for tail analyses within the elliptical family.

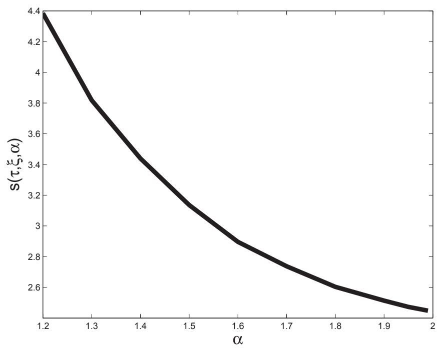
the Gaussian, are also used in the Monte Carlo study. For the left plot the tail index varies from 2.5 to 100. The non–linear correlation decreases as α increases, and it stays steady at 2.43 for $\alpha > 30$, as the distribution is indistinguishable to the Gaussian. The table in the appendix shows that $s_g(0.75, 0.95) = 0.41$, the inverse of 2.43, and hence $s_g(\xi, \tau)s(\xi, \tau, \alpha) \approx 1$ for $\alpha > 30$. Similar reading applies for the right plot, with a tail index that varies between 1.2 and 2. The decrease is more slow than for the Student–t as tails remain significantly thicker than the Gaussian even for α in the vicinity of 2. The bottom plot shows the sensitivity of $s_g(0.75, 0.95)s(0.75, 0.95, \alpha)$ to ρ for the Gaussian (solid line) and Student–t (dashed line) distributions: the non–linear correlation is not affected by ρ. This calibration exercise confirms that the two components of $\text{TailCoR}^{(j\,k)\,\xi}$ capture different aspects and are independent.

Figure 2: SENSITIVITY OF $s(\xi, \tau, \alpha)$ TO α AND ρ

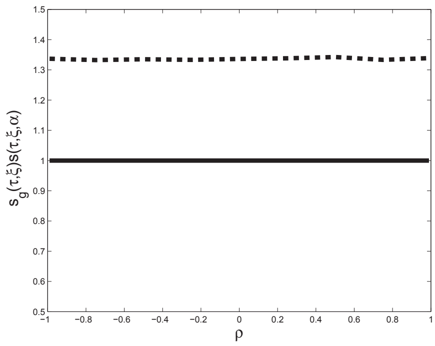
(a) SENSITIVITY TO α – STUDENT–T



(b) SENSITIVITY TO α – ES



(c) SENSITIVITY TO ρ



The top plots show the sensitivity of the non–linear correlation to α for the Student–t distribution (left) and ES (right). For the Student–t the tail index varies from 2.5 to 100 while for the ES it varies from 0 to 2. The bottom plot shows the sensitivity to ρ (which takes the full range $[-1, 1]$) for the Gaussian (solid line) and the Student–t (dashed line) distributions. All the plots are for $\tau = 0.75$ and $\xi = 0.95$.

$\text{TailCoR}^{(j\,k)\,\xi}$ has a number of interesting properties. First, it is location–scale invariant, i.e. $\text{TailCoR}^{(j\,k)\,\xi}$ between $X_{j\,t}$ and $X_{k\,t}$ is the same as between $a+bX_{j\,t}$ and $c+dX_{k\,t}$, for a, b, c, and d real numbers. Second, it can capture non–linear correlations even if tails are thinner than the Gaussian. I.e. if tails are fatter, equal or thinner than those of the Gaussian distribution,

$s(\xi, \tau, \alpha)s_g(\xi, \tau)$ is greater, equal or smaller than one. As for the remaining properties, for the ease of exposition we consider two cases: either $\mathbf{X}_t$ is thin tailed (i.e. Gaussian) or heavy tailed. If $\mathbf{X}_t$ is Gaussian, i) $s(\xi, \tau, \alpha) = s_g(\xi, \tau)^{-1}$ and $\text{TailCoR}^{(j\,k)\,\xi} = \sqrt{1 + \rho_{j\,k}}$, i.e. the only source of correlation is linear, ii) $\text{TailCoR}^{(j\,k)\,\xi} = 1$ –the aforementioned reference value– if X_j and X_k are uncorrelated (and hence independent), and iii) $\text{TailCoR}^{(j\,j)\,\xi}$, i.e. between X_j and itself, is $\sqrt{2}$. If, by contrast, $\mathbf{X}_t$ is heavy tailed, i) $s(\xi, \tau, \alpha)$ captures the non–linear correlation: the heavier tailed $\mathbf{X}_t$ is, the higher is $s(\xi, \tau, \alpha)$ and so does $\text{TailCoR}^{(i\,j)\,\xi}$, ii) the most appealing property is that, even if X_j and X_k are linearly uncorrelated, $\text{TailCoR}^{(j\,k)\,\xi}$ is larger than one as $s_g(\xi, \tau)s(\xi, \tau, \alpha) > 1$, [5] and iii) $\text{TailCoR}^{(j\,j)\,\xi} = s_g(\xi, \tau)s(\xi, \tau, \alpha)\sqrt{2}$.

2.3 Estimation

Estimation is straightforward and is divided in four simple steps that can be followed under **G1–G2**.

Step 1 Standardize $(X_{j\,t}, X_{k\,t})$ with the corresponding sample median and IQR, yielding $(\hat{Y}_{j\,t}, \hat{Y}_{k\,t})$.

Step 2 Estimate the IQR of the projection $\hat{Z}_t^{(j\,k)}$ for a given ξ: $\widehat{\text{IQR}}_{\hat{Z}\,T}^{(j\,k)\,\xi}$.

Step 3 Find the normalization $s_g(\xi, \tau)$ from the table. In practice ξ takes a limited number of large values, $\xi = \{0.90, 0.95, 0.99\}$ say, for which, if $\tau = 0.75$, $s_g(\xi, 0.75)$ equals 0.526, 0.410 and 0.290 respectively.

Step 4 Estimate $\widehat{\text{TailCoR}}_{\hat{Z}\,T}^{(j\,k)\,\xi} = s_g(\xi, 0.75)\widehat{\text{IQR}}_{\hat{Z}\,T}^{(j\,k)\,\xi}$.

$\widehat{\text{TailCoR}}_{\hat{Z}\,T}^{(j\,k)\,\xi}$ is sub–indexed by $\hat{Z}$ and T to explicitly emphasize its dependence with the estimation steps 1 and 2. The first estimation is the standardization, and hence the hat on Z, and the second is the interquantile range of the projection, and hence the hat on $\text{IQR}^{(j\,k)\,\xi}$. This double source of uncertainty affects the limiting distribution through the variance–covariance matrix, as shown below.

Under **E1–E5** we estimate the linear correlation $\hat{\rho}_{j\,k,T}$ with a robust method. Lindskog et al. (2003) introduce a robust estimator that exploits the geometry of an elliptical distribution. Let $\hat{\kappa}_{j\,k,T}$ be the estimator of the Kendall's correlation:

$$\hat{\kappa}_{j\,k,T} = \binom{T}{2}^{-1} \sum_{1 \leq t < s \leq T} \text{sign}((\hat{Y}_{j\,t} - \hat{Y}_{j\,s})(\hat{Y}_{k\,t} - \hat{Y}_{k\,s})).$$

Lindskog et al. (2003) show that $\hat{\kappa}_{j\,k,T}$ is invariant in the class of elliptical distributions. Then

$$\hat{\rho}_{j\,k,T} = \sin\left(\frac{\pi}{2}\hat{\kappa}_{j\,k,T}\right),$$

[5] This is akin to the coefficients of tail dependence steaming from copula theory. For instance, for the bivariate case, the coefficient of tail dependence of a Student–t copula is $2t_{\alpha+1}\left(-\sqrt{\frac{(\alpha+1)(1-\rho)}{1+\rho}}\right)$, where $t_{\alpha+1}(\cdot)$ is a standardized Student–t cumulative distribution function with tail index $\alpha + 1$. Even if $\rho = 0$ the tail dependence is positive.

and $\sqrt{1+\hat{\rho}_{j\,k,T}}$ follows. Second, given $\hat{\text{IQR}}^{(j\,k)\,\xi}_{\hat{Z}\,T}$ obtained step 2 above, the estimator of the non–linear correlation is

$$s(\xi, \hat{\tau}, \alpha)_T = \frac{\hat{\text{IQR}}^{(j\,k)\,\xi}_{\hat{Z}\,T}}{\sqrt{1+\hat{\rho}_{j\,k,T}}}.$$

We now see the computational advantages of $\text{TailCoR}^{(j\,k)\,\xi}$. First, it can be estimated exactly for any probability level ξ. Second, $\text{TailCoR}^{(j\,k)\,\xi}$ can be computed under the general assumptions **G1**–**G2**, i.e. it is distribution free. Third, no optimizations are needed as it is based on simple steps, each requiring no more than one line of programming code. This makes $\text{TailCoR}^{(j\,k)\,\xi}$ fast to compute. Fourth, no estimation of the tail index is required. Though it is assumed to exist under **E1**–**E5**, its estimation is not needed.

Let $\hat{\mathbf{Q}} = (\hat{\text{Q}}_j^{0.50}, \hat{\text{Q}}_j^{\tau}, \hat{\text{Q}}_k^{0.50}, \hat{\text{Q}}_k^{\tau})$ be the vector of sample quantiles that we use in the standardization (1).[6] We denote by $Cov(\hat{\mathbf{Q}})$ its variance–covariance matrix and by $\mathbf{Q}$ the population counterpart. The following Theorem shows the asymptotic properties of $\text{Tail}\hat{\text{Co}}\text{R}^{(j\,k)\,\xi}_{\hat{Z}\,T}$.

Theorem 2 *Under* **E1**–**E5**

$$\sqrt{T}\left(Tail\hat{C}oR^{(j\,k)\,\xi}_{\hat{Z}\,T} - TailCoR^{(j\,k)\,\xi}\right) \to \mathcal{N}\left(0, 4s_g(\xi,\tau)^2\left(\frac{\Gamma(Q^{(j\,k)\xi})}{f^2_{(j\,k)}(F^{-1}_{(j\,k)}(\xi))} + \omega\right)\right),$$

where $f_{(j\,k)}(\cdot)$ and $F_{(j\,k)}(\cdot)$ are the probability and cumulative density functions of $Z_t^{(j\,k)}$,

$$\begin{aligned}
\Gamma(Q^{(j\,k)\xi}) &= \sum_{t=-\infty}^{+\infty} E\left(Y_0(Q^{(j\,k)\xi})Y_t(Q^{(j\,k)\xi})\right), \\
Y_t(Q^{(j\,k)\xi}) &= I\{Z_t^{(j\,k)} \le Q^{(j\,k)\xi}\} - P(Z_t^{(j\,k)} \le Q^{(j\,k)\xi}), \ \textit{and} \\
\omega &= \frac{\partial Q^{(j\,k)\,\xi}(\mathbf{Q})}{\partial \mathbf{Q}} Cov(\mathbf{Q}) \frac{\partial Q^{(j\,k)\,\xi}(\mathbf{Q})}{\partial \mathbf{Q}'}.
\end{aligned}$$

Proof See Appendix P.

Five remarks to the Theorem. First, the term ω in the variance is the effect of the estimated median and IQR in the standardization of $X_{j\,t}$ and $X_{k\,t}$. Monte Carlo results indicate that its effect is negligible and in empirical work ignoring it does not have practical consequences. Second, the univariate density $f_{(j\,k)}(\cdot)$ in the denominator is elliptical and therefore easy to compute. Third, $\Gamma(Q^{(j\,k)\xi})$ is the long–run component that accounts for the time dependence. Fourth, it is possible to skip **E3** and derive the elliptical representation of TailCoR without moments. Then $\rho_{j\,k}$ becomes the $(j\,k)$ element of the standardized dispersion matrix that can be estimated with the Tyler's M–estimator (Tyler (1987)). Last, in a similar vein to the previous remark, it is possible to relax the assumptions and assume **G1**–**G2**. The limiting distribution is more involving since $\hat{\text{Q}}_j^{\tau}$ is not necessarily equal to $-\hat{\text{Q}}_j^{1-\tau}$, as shown in the following Corollary.

[6] Note that under ellipticity $\hat{\text{Q}}_j^{\tau} = -\hat{\text{Q}}_j^{1-\tau}$

Corollary *Under* **G1–G2**

$$\sqrt{T}\left(\hat{TailCoR}^{(j\,k)\,\xi}_{\hat{Z}\,T} - TailCoR^{(j\,k)\,\xi}\right) \to \mathcal{N}\left(0, s_g(\xi,\tau)^2\left(\Upsilon + \omega\right)\right),$$

where

$$\Upsilon = \frac{\Gamma(Q^{(j\,k)\xi})}{f^2_{(j\,k)}(F^{-1}_{(j\,k)}(\xi))} + \frac{\Gamma(Q^{(j\,k)1-\xi})}{f^2_{(j\,k)}(F^{-1}_{(j\,k)}(1-\xi))} - 2\frac{\Gamma(Q^{(j\,k)\xi}, Q^{(j\,k)1-\xi})}{f_{(j\,k)}(F^{-1}_{(j\,k)}(\xi))f_{(j\,k)}(F^{-1}_{(j\,k)}(1-\xi))} \quad \textit{and}$$

$$\Gamma(Q^{(j\,k)\xi}, Q^{(j\,k)1-\xi}) = \sum_{t=-\infty}^{+\infty} E\left(Y_0(Q^{(j\,k)\xi})Y_t(Q^{(j\,k)1-\xi})\right).$$

The univariate densities $f_{(j\,k)}(\cdot)$ are estimated using the Hendricks and Koenker (1992) sandwich form. See the appendix of Coroneo and Veredas (2012) for a detailed step–by–step implementation.

3 A Monte Carlo study

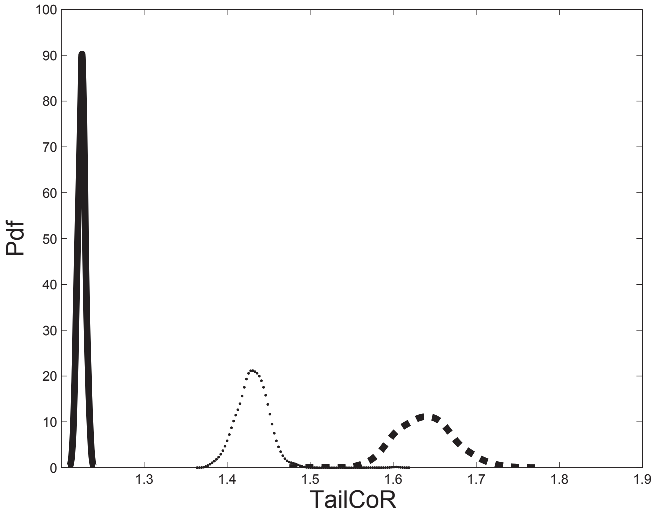
To analyze the finite sample properties of TailCoR and its behavior as a function of the linear and non–linear correlations, we proceed with a Monte Carlo study. We consider three bivariate elliptical distributions: Gaussian, Student–t with $\alpha = 2.5$ and ES with $\alpha = 1.5$. Note that the most heavy tailed is the Student–t, followed by the ES, while the Gaussian is thin tailed. The location parameters are set to 0 and the dispersion matrix has unitary diagonal elements and off–diagonal element 0.50. We consider three samples sizes $T = \{100, 1000, 5000\}$ and two number of replications $H = \{100, 500\}$. In the sequel we show results for $T = 5000$ and $H = 500$. Results for other configurations are alike and available under request.

Figure 3 shows the distributions for the TailCoR estimates for $\xi = 0.95$, for the three distributions (solid line for the Gaussian, dashed for the Student–t and dotted for the ES). In all cases, TailCoR is larger than one, the value under Gaussianity and independence. The estimated TailCoR is more precise under Gaussianity than under heavy tails, which makes sense as it only depends on the linear correlation. Moreover, the median is around 1.22, very close to the true value $\sqrt{1 + 0.50} = 1.225$. By contrast, estimators under the Student–t and the ES have medians well above, 1.43 for the ES and 1.64 for the Student–t.

Figure 4 shows the sensitivity of TailCoR to ξ for the the Gaussian (top plot), Student–t with $\alpha = 2.5$ (bottom left plot) and ES with $\alpha = 1.5$ (bottom right plot). Each line is the density of the 500 estimates of TailCoR for different values of ξ: 0.90 (solid line), 0.95 (dashed) and 0.99 (dotted). The lines overlap for the Gaussian distribution since $s(\xi, \tau, \alpha) = s_g(\xi, \tau)^{-1}$, and hence TailCoR$^{(i\,j)\,\xi} = \sqrt{1 + \rho_{i\,j}}$ and it does not depend on ξ. The median of the 500 estimates is very close to 1.225. Regarding the other distributions, results show that TailCoR increases with ξ as we explore further the tails.

Figure 5 shows the density of the estimates of $\sqrt{1 + \rho_{j\,k}}$ for $\xi = 0.95$ and for the three distributions (solid line for the Gaussian, dashed for the Student–t, and dotted for the ES). If estimated correctly, they should be invariant to the tail thickness. Indeed, the median of the estimates for the three distributions are very close to the true value 1.225, with a slight small sample bias for the Gaussian, and the precision decreases with the tail thickness.

Figure 3: TAILCOR FOR DIFFERENT DISTRIBUTIONS



Distribution of 500 estimated TailCoR for $\xi = 0.95$ and three distributions: Gaussian (solid line), Student–t with $\alpha = 2.5$ (dashed line) and ES with $\alpha = 1.5$ (dotted line).

The convergence in distribution of the estimator is shown in Figure 6 for $\xi = 0.95$. The solid line is the standardized Gaussian while the bars indicated the histogram of

$$\sqrt{T}\left(\mathrm{Tail\hat{C}oR}^{0.95}_{\hat{Z}\,T} - \mathrm{TailCoR}^{0.95}\right)$$

for different sample sizes and replications (indicated in the top of each plot). As expected, the histogram approaches to the limiting distribution as the sample size increases.

4 Extensions

Downside– & Upside–TailCoR

It is often the case, like in risk management, that the interest lies in the tail of one side of the distribution. We define downside TailCoR as follows.

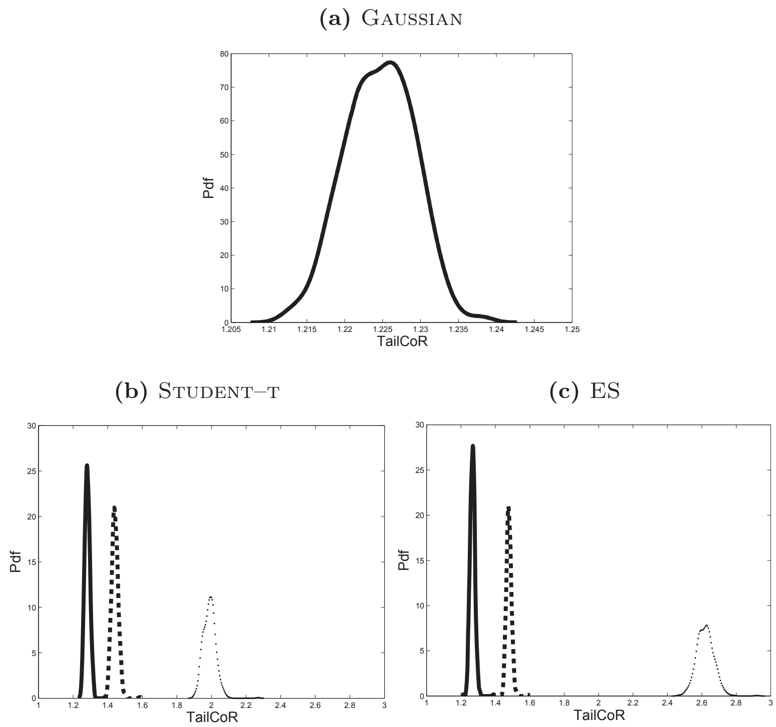
$$\mathrm{TailCoR}^{(j\,k)\,\xi-} := s_g(\xi,\tau)\mathrm{IQR}^{(j\,k)\,\xi-}$$

where $\mathrm{IQR}^{(j\,k)\,\xi-} = \mathrm{Q}^{(j\,k)\,0.50} - \mathrm{Q}^{(j\,k)\,1-\xi}$. The interquantile range of the projection is not symmetric, as it is the difference between the median and the lower tail quantile. Likewise, the upside TailCoR is defined as

$$\mathrm{TailCoR}^{(j\,k)\,\xi+} := s_g(\xi,\tau)\mathrm{IQR}^{(j\,k)\,\xi+}$$

where $\mathrm{IQR}^{(j\,k)\,\xi+} = \mathrm{Q}^{(j\,k)\,\xi} - \mathrm{Q}^{(j\,k)\,0.50}$, i.e. the difference between the upper tail quantile and the median. The estimators follow the same lines as in previous section. The limiting distribution is similar to that of $\mathrm{Tail\hat{C}oR}^{(j\,k)\,\xi}_{\hat{Z}\,T}$ except that the asymptotic variance–covariance

Figure 4: SENSITIVITY OF TAILCOR TO ξ



Sensitivity of TailCoR to ξ for the Gaussian (top plot), Student–t with $\alpha = 2.5$ (bottom left plot) and ES with $\alpha = 1.5$ (bottom right plot). Each line is the density of the 500 estimates of TailCoR for different values of ξ: 0.90 (solid line), 0.95 (dashed) and 0.99 (dotted).

matrix needs to be adapted since the interquantile range of the projection is not symmetric. The resulting expressons are similar to that of the Corollary with $\xi = 0.50$ and $1 - \xi = \xi$ for downside TailCoR, and $1 - \xi = 0.50$ for upside TailCoR.

Negative correlation

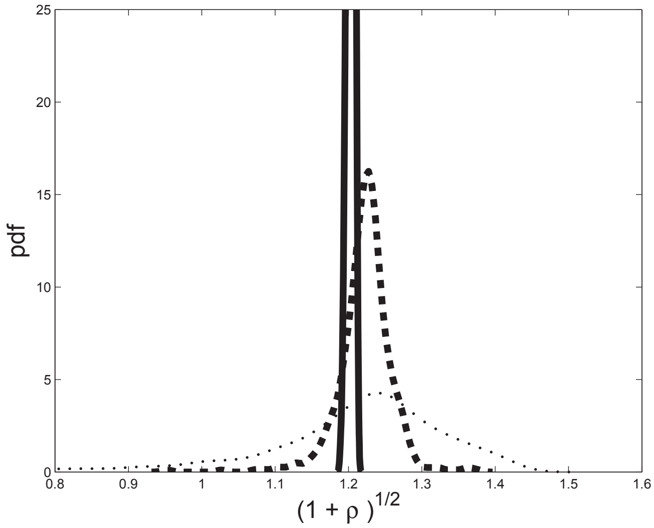
TailCoR$^{(j\,k)\,\xi}$, as defined so far, is designed for positive relations as it projects the pairs of observations onto the 45–degree line. Left plot of Figure 7 shows the projection on that line when the relation is negative: it leads to $Z^{(j\,k)}$ very concentrated around the origin. A simple transformation makes TailCoR$^{(j\,k)\,\xi}$ suitable for negative relations: instead of projecting onto the 45–degree line, we project onto the 315–degree line, as depicted in the right plot of Figure 7. Following similar trigonometric arguments as before, the projection is

$$Z_t^{(j\,k)} = \frac{1}{\sqrt{2}}(Y_{j\,t} - Y_{k\,t}).$$

Under **G1–G2**, the definition remains TailCoR$^{(j\,k)\,\xi} := s_g(\xi, \tau)IQR^{(j\,k)\,\xi}$. However, under **E1–E5**, the expression is slightly modified:

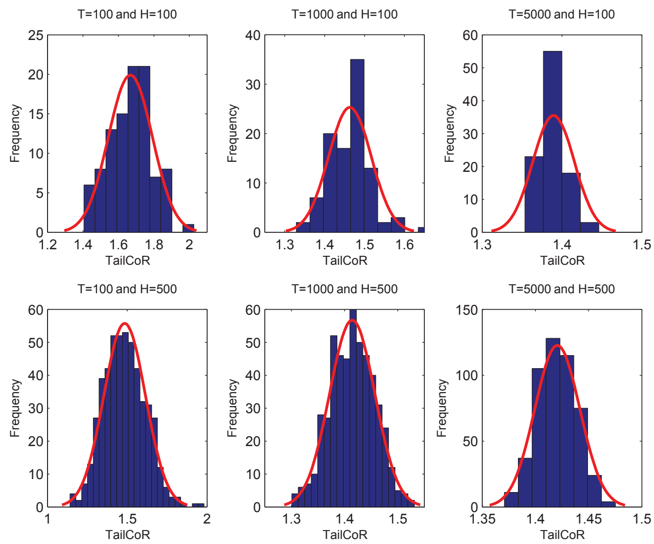
$$\text{TailCoR}^{(j\,k)\,\xi} = s_g(\xi, \tau)s(\xi, \tau, \alpha)\sqrt{1 - \rho_{j\,k}}.$$

Figure 5: THE LINEAR COMPONENT



Density of the 500 estimates of $\sqrt{1+\rho_{j\,k}}$ for $\xi = 0.95$ and for the three distributions (solid line for the Gaussian, dashed for the Student–t, and dotted for the ES).

Figure 6: CONVERGENCE IN DISTRIBUTION

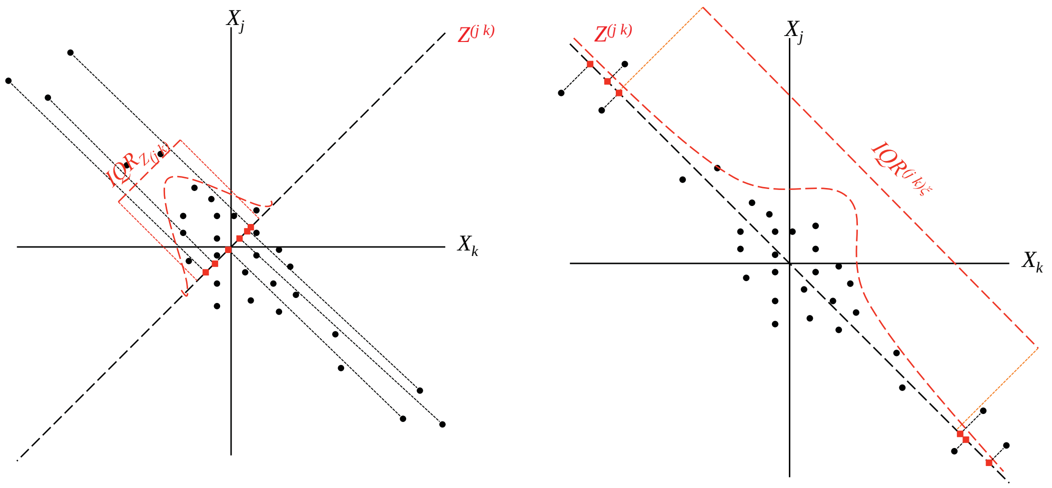


Histogram of $\sqrt{T}\left(\hat{\text{TailCoR}}^{0.95}_{\hat{Z}\,T} - \text{TailCoR}^{0.95}\right)$ as a function of the number of observations and replications (indicated in the top of each plot) against the standardized Gaussian distribution.

The asymptotic distributions of the Theorem and the Corollary above are not modified. Projecting onto the 45– or 315–degree line is a user choice. Visual inspection of the scatter plots as a mean to choose the projection is only feasible when $\mathbf{X}_t$ is small dimensional. For vast dimensions we propose the following automatized method. Let $Z_{45\,t}^{(j\,k)}$ and $Z_{315\,t}^{(j\,k)}$ be the projections onto the 45– and 315–degree lines for the pair $(j\,k)$, and let $\mathrm{IQR}_{45}^{(j\,k)\,\xi}$ and $\mathrm{IQR}_{315}^{(j\,k)\,\xi}$ be the corresponding IQR. If $\mathrm{IQR}_{45}^{(j\,k)\,\xi} > \mathrm{IQR}_{315}^{(j\,k)\,\xi}$ then use $Z_{45\,t}^{(j\,k)}$, otherwise use $Z_{315\,t}^{(j\,k)}$. If $\mathrm{IQR}_{45}^{(j\,k)\,\xi} = \mathrm{IQR}_{315}^{(j\,k)\,\xi}$ there is neither linear nor non–linear correlation and $\mathrm{TailCoR}^{(j\,k)\,\xi}$ computed in either way gives the same result.

Figure 7: A diagrammatic representation of TailCoR for negative relation

(a) Projection in the 45–degree line **(b)** Projection in the 315–degree line



Scatter plots, along with the 45– (left) and 315–degree (right) lines, where X_j and X_k are negatively related (the pairs are depicted with circles). Projecting the observations onto the 45– or 315–degree lines produces the random variable $Z^{(j\,k)}$, depicted with squares.

Dynamic TailCoR

TailCoR, as defined so far, is an unconditional measure. It is however possible to extend it to the dynamic case. A stylized fact of financial returns is volatility clustering, and hence the standardization (1) would be more accurate if $X_{j\,t}$ is subtracted and divided by the appropriate amounts at time t. A quantile–based measure for volatility is the dynamic IQR, or $\mathrm{IQR}_{j\,t}^{\tau} = \mathrm{Q}_{j\,t}^{\tau} - \mathrm{Q}_{j\,t}^{1-\tau}$.[7] Coroneo and Veredas (2012) show that the quantile regressions

$$\begin{aligned} \mathrm{Q}_{j\,t}^{\tau} &= \omega^{\tau} + \beta^{\tau}|X_{j\,t-1}| \text{ and} \\ \mathrm{Q}_{j\,t}^{1-\tau} &= \omega^{1-\tau} + \beta^{1-\tau}|X_{j\,t-1}| \end{aligned}$$

[7] We can also consider the dynamic median $\mathrm{Q}_{j\,t}^{0.50}$. However, another stylized fact of financial returns is its unpredictability.

provide an $\mathrm{IQR}^{\tau}_{j\,t}$ that is an accurate estimator of the marginal volatility for $X_{j\,t}$. Similarly for $X_{k\,t}$.[8] The IQR of the projection may also be time varying

$$\mathrm{IQR}_t^{(j\,k)\,\xi} = \mathrm{Q}_t^{(j\,k)\,1-\xi} - \mathrm{Q}_t^{(j\,k)\,\xi},$$

where the quantile regressions are specified similarly to above:

$$\begin{aligned} \mathrm{Q}_t^{(j\,k)\,\xi} &= \omega^{\xi} + \boldsymbol{\beta}^{\xi}\mathbf{W}_{t-1} \text{ and} \\ \mathrm{Q}_t^{(j\,k)\,1-\xi} &= \omega^{1-\xi} + \boldsymbol{\beta}^{1-\xi}\mathbf{W}_{t-1}, \end{aligned}$$

where $\mathbf{W}_{t-1}$ is a set of regressors. Similarly to the static case, under **E1**–**E5** we disentangle the dynamic contribution of the linear and non–linear correlations. The dynamic linear correlation $\rho_{i\,j,t}$ can be estimated with a robustified version of the DCC model of Engle (2002) (Boudt et al. (2012)). The dynamic non–linear correlation is computed similarly to the static case:

$$s(\xi,\tau,\alpha)_t = \frac{\mathrm{IQR}_t^{(j\,k)\,\xi}}{\sqrt{1+\rho_{j\,k,t}}}.$$

The limiting distribution of $\mathrm{Tail\hat{C}oR}^{(j\,k)\,\xi}_{\hat{Z}\,T}$ is more involving than in the static case as it is based on the asymptotic distributions for the intercept and slopes parameters of the quantile regressions $(\omega^{\xi}, \omega^{1-\xi}, \boldsymbol{\beta}^{\xi}, \boldsymbol{\beta}^{1-\xi})$, which are known since Koenker and Bassett (1978). Similar arguments to those of previous section follow nevertheless.

Multivariate

So far we have only considered the pair $(j\,k)$ of random variables while the random vector $\mathbf{X}_t$ is of dimension N. Considering all the pairs, it leads to a $N(N{+}1)/2$ vector of TailCoR (including TailCoR of a random variable with itself). For the ease of exposition let $\tilde{N} = N(N+1)/2$. We denote by $\xi_{(j\,k)}$ the probability level at which we compute the IQR for the $(j\,k)$ projection. If $\xi_{(j\,k)} = \xi\ \forall j,k$, we define the vector of TailCoR as

$$\mathbf{TailCoR}^{\xi} := s_g(\xi,\tau)\mathbf{IQR}^{\xi}, \tag{3}$$

where $\mathbf{IQR}^{\xi}$ is the vector of IQR of the $\tilde{N}\times 1$ projections. The assumption $\xi_{(j\,k)} = \xi\ \forall j,k$ is a simplification and it allows to have the above definition. It is nonetheless possible to relax it at the cost of notation. Under ellipticity, (3) becomes

$$\mathbf{TailCoR}^{\xi} = \sqrt{2}s_g(\xi,\tau)s(\xi,\tau,\alpha)\mathbf{R}, \tag{4}$$

where the matrix $\mathbf{R}$ has $(j\,k)$ element $\sqrt{\frac{1+\rho_{j\,k}}{2}}$. I.e. it is symmetric, with unitary diagonal, and off–diagonal elements bounded above and bellow by 1 and 0 respectively. This matrix has the following properties: i) similarly to the univariate case, it is invariant to location–scale shifts of $\boldsymbol{X}_t$, ii) it is semi–definite positive, iii) the minimum eigenvalue is 0 and the maximum is bounded by N, as their sum equals N (the trace of $\mathbf{R}$).

Estimation follows the same steps as in the univariate case under **G1**–**G2**:

$$\mathbf{Tail\hat{C}oR}^{\xi}_{\hat{\mathbf{Z}}\,T} = s_g(\xi,\tau)\mathbf{I\hat{Q}R}^{\xi}.$$

[8] Another possibility is to use a quantile regression similar to the CAViaR of Engle and Manganelli (2004), but it is computational more complex and time consuming.

Under the ellipticity assumptions **E1–E5**, an extra step has to be added as $s(\xi,\tau,\alpha)$ is the same for all. Let $s(\xi,\hat{\tau},\alpha)_h = s(\xi,\hat{\tau},\alpha)^{(j\,k)}$, $h = 1,\ldots,\tilde{N}$. The non–linear correlation is estimated by pooling the pairwise estimators:

$$s(\xi,\hat{\tau},\alpha)_T = \frac{1}{\tilde{N}}\sum_{h=1}^{\tilde{N}} s(\xi,\hat{\tau},\alpha)_{h\,T}.$$

Estimating by averaging estimators in a cross–sectional sense has been used in the past, see e.g. Chen et al. (2009) for efficient instrumental variable estimators, and Nolan (2010) and Dominicy et al. (2012) for the estimation of the tail index within the elliptical family of distributions.

The asymptotic distribution incorporates now the covariances between the sample quantiles of the marginal distributions. Let $\mathbf{Z}_t = (Z_t^{(1\,1)},\ldots,Z_t^{(N-1\,N)}) = (Z_{1\,t},\ldots,Z_{\tilde{N}\,t})$ be the vector of projections. Likewise, let $\mathbf{Q}^\xi = (Q_1^\xi,\ldots,Q_{\tilde{N}}^\xi)$ be the vector of quantiles and $\hat{\mathbf{Q}}^\xi$ the sample counterparts. Under **G1–G2**, Dominicy et al. (2012) show that $\sqrt{T}(\hat{\mathbf{Q}}^\xi - \mathbf{Q}^\xi) \to \mathcal{N}(\mathbf{0},\mathbf{\Omega})$ where

$$\begin{aligned}
\mathbf{\Omega}_{j\,j} &= \frac{\Gamma_{j\,j}(Q_j^\xi)}{f_j^2(F_j^{-1}(\xi))},\\
\Gamma_{j\,j}(Q_j^\xi) &= \sum_{t=-\infty}^{\infty} E(Y_0(Q_j^\xi), Y_t(Q_j^\xi)) \text{ and}\\
Y_t(Q_j^\xi) &= I\{Z_{j\,t}\le Q_j^\xi\} - P(Z_{j\,t}\le Q_j^\xi),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{\Omega}_{j\,k} &= \frac{\Gamma_{j\,k}(Q_j^\xi,Q_k^\xi)}{f_j(F_j^{-1}(\xi))f_k(F_k^{-1}(\xi))}\forall j\neq k,\\
\Gamma_{j\,k}(Q_j^\xi) &= \sum_{t=-\infty}^{\infty} E(Y_0(Q_j^\xi,Q_k^\xi), Y_t(Q_j^\xi,Q_k^\xi)) \text{ and}\\
Y_t(Q_j^\xi,Q_k^\xi) &= I\{Z_{j\,t}\le Q_j^\xi, Z_{k\,t}\le Q_k^\xi\} - P(Z_{j\,t}\le Q^\xi, Z_{k\,t}\le Q_k^\xi).
\end{aligned}$$

Let $\hat{\mathbf{Q}}^\tau = (\hat{\mathrm{Q}}_1^{0.50},\hat{\mathrm{Q}}_1^{\tau},\ldots,\hat{\mathrm{Q}}_N^{0.50},\hat{\mathrm{Q}}_N^{\tau})$ be the vector of sample quantiles used in the standardizations of $\mathbf{X}_t$. We denote by $Cov(\hat{\mathbf{Q}}^\tau)$ its variance–covariance matrix, and $\mathbf{Q}^\tau$ the population counterpart. The following Theorem shows the asymptotic distribution of $\hat{\mathbf{TailCoR}}^\xi_{\hat{\mathbf{Z}}\,T}$.

Theorem 3 *Under* **E1**–**E5**

$$\sqrt{T}\left(\hat{\mathbf{TailCoR}}^\xi_{\hat{\mathbf{Z}}\,T} - \mathbf{TailCoR}^\xi\right) \to \mathcal{N}\left(0, 4s_g(\xi,\tau)^2\left(\mathbf{\Omega}+\boldsymbol{\omega}\right)\right),$$

where $\mathbf{\Omega}$ *is defined as above and*

$$\boldsymbol{\omega} = \frac{\partial\mathbf{Q}^\xi(\mathbf{Q}^\tau)}{\partial\mathbf{Q}^\tau}Cov(\mathbf{Q}^\tau)\frac{\partial\mathbf{Q}^\xi(\mathbf{Q}^\tau)}{\partial\mathbf{Q}^{\tau\,\prime}}.$$

Proof It follows the same lines as the proof of Theorem 2.

All in one number: N–dimensional projection

It may be of interest to have a unique global tail correlation for all the elements of the random vector $\mathbf{X}_t$, instead of pairwise measures. Following similar trigonometric arguments to the bivariate case, it can be shown that the projection is

$$Z_t = \sum_{i=1}^{N} \frac{Y_{i\,t}}{\sqrt{N}}.$$

The same automatized method designed for differentiating between positive and negative relations –see above– applies here. The tail interquantile range of Z_t is $\mathrm{IQR}^{\xi} = \mathrm{Q}^{\xi} - \mathrm{Q}^{1-\xi}$, which leads to the N–dimensional definition of TailCoR.

Definition 2 *Under* **G1**–**G2***, the N–dimensional TailCoR between* $(X_{1\,t}, \cdots, X_{N\,t})$ *is*

$$\mathit{TailCoR}^{\xi} := s_g(\xi, \tau)\mathit{IQR}^{\xi}.$$

Under the elliptical assumptions we obtain the corresponding alternative representation:

Theorem 4 *Under* **E1**–**E5***, the N–dimensional TailCoR between* $(X_{1\,t}, \cdots, X_{N\,t})$ *can be re–written as*

$$\mathit{TailCoR}^{\xi} = s_g(\xi, \tau)s(\xi, \tau, \alpha)\sqrt{1 + \frac{2}{N} \sum_{j,k=1, j\neq k}^{\frac{N(N-1)}{2}} \rho_{j\,k}}.$$

Proof It follows the same lines as the proof of Theorem 1.

Note that for $N = 2$ we are back to Theorem 1. This result is for the case where all the relations are positive and it can be easily extended to the case of having both positive and negative relations. As in the pairwise case, the reference point is 1, i.e. this is the value of TailCoR in the case of uncorrelation and Gaussianity. Under this distribution the upper bound is $\sqrt{N}$, obtained by setting all the Pearson correlation coefficients to one. The lower bound is not straightforward since positive and negative relations have to be taken into account. Last, under another elliptical distribution than Gaussian, even if X_j and X_k are linearly uncorrelated, $\mathrm{TailCoR}^{\xi}$ is larger than one as $s_g(\xi, \tau)s(\xi, \tau, \alpha) > 1$, as in previous section.

Estimation follows the same steps as for $\mathrm{TailCoR}^{(j\,k)\,\xi}$ under **G1**–**G2**. Under the elliptical assumptions, the multivariate extension of the robust correlation estimator can be used. The asymptotic distributions under **G1**–**G2** and **E1**–**E5** are identical to those of Theorem 2 and its Corollary.

5 The risks of pooling Euro area bonds

The scholarly debate on mutualizing Euro area bonds has been active since the beginning of the Sovereign debt crisis. Their supporters have highlighted numerous advantages of such scheme: i) benefit from strong creditworthiness, ii) greater resilience to shocks, iii) reinforcing financial stability, iv) risk sharing, and v) increasing liquidity.

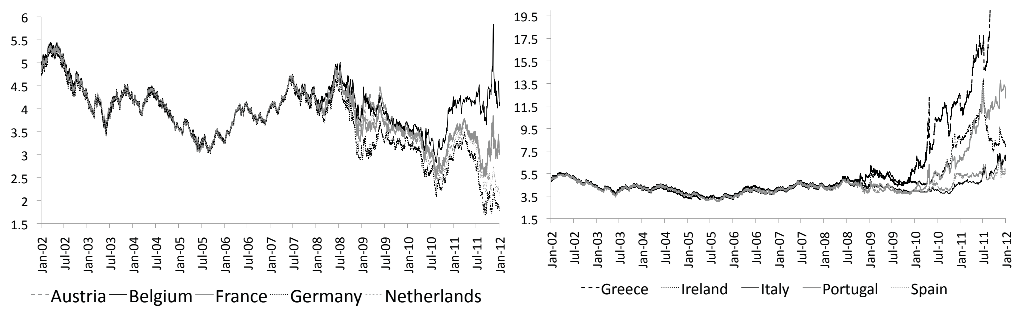
The Green Paper of the European Commission (2011) assesses the feasibility of common issuance of sovereign bonds (the Stability bonds) among the Member States of the euro area.

The Stability bonds would mean a pooling of sovereign issuance and the sharing of associated revenue flows and debt–servicing costs. The Green Paper finds that the common issuance has several potential benefits. The most relevant for us is to make the euro area financial system more resilient to future adverse shocks and so reinforce financial stability. Indeed, it would provide a source of more robust collateral for all banks in the euro area, reducing their vulnerability to the characteristics of individual Member States. The European Commission (2011) also suggests that the Stability bonds should be designed and issued such that investors consider them a very safe asset. For this to happen, it is essential reinforce fiscal surveillance and policy coordination so as ensure sustainable public finances.[9]

Though the document does not set guidelines for the implementation of the Stability bonds, it mentions that the new bonds should be a pooling of the National ones. The most natural way of pooling is by means of a linear combination. If so, the correlations between them play an important role in order to determine the risks associated with the pooling. While in tranquil periods these risks have a linear nature, in periods of turmoil non-linearities appear, namely due to extreme events, that induce non-linear risks. Therefore pooling national bonds into Euro area bonds may or may not be beneficial, depending on their signs and magnitudes of these risks.

Figure 8: Yields

(a) Core **(b)** Peripheral



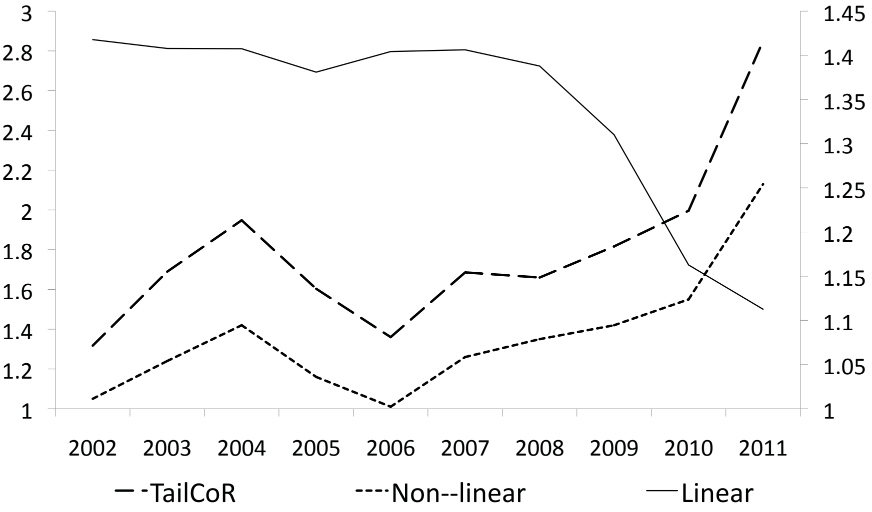
Left plot shows the yields for the core countries (Austria, Belgium, France, Germany and Netherlands) while the right plot shows the yields for the peripheral (Greece, Ireland, Italy, Portugal and Spain). The yields of Greece at the end of the sample are not plotted (they reached values beyond 30).

We estimate TailCoR and its linear and non–linear correlations for a set of the Euro area countries. Data consists of daily yields of 10–years bonds for Austria, Belgium, France, Germany, Greece, Ireland, Italy, The Netherlands, Portugal and Spain. The sample spans from January 2002 to January 2012 and the data provider is Datastream. Figure 8 shows the time series plot of the yields. Because of the large differences since the beginning of the crisis, we split the countries into two groups. The left plot shows the core countries (Belgium, France, Germany and The Netherlands) while the right plot shows the peripheral (Greece, Ireland, Italy, Portugal and Spain). Three remarks: due to the political situation during 2010-2011, Belgium experienced instability and rating downgrades that reflected into an increase of the

[9]Other proposals of pan–European bonds are the Blue bonds of Delpla and von Weizsäcker (2010), the European safe bonds of Brunnermeier et al. (2011), the Safe bonds of the German Council of Economic Experts (2011), the synthetic Eurobonds of Beck et al. (2011), and the Eurobills of Hellwig and Philippon (2011). See Claessens et al. (2012) for a discussion and comparison of all these proposals.

yields at the end of the sample. Second, because of representation purposes the yields of Greece at the end of the sample are not plotted (they reached values beyond 30%). Last, Austria, Finland and Luxembourg are excluded from the analysis either because of size or because they virtually have the same pattern of a big neighboring country. Cyprus, Estonia, Malta, Slovakia and Slovenia are excluded too because they joined the euro at the end of the sample.

Figure 9: SUMMARY



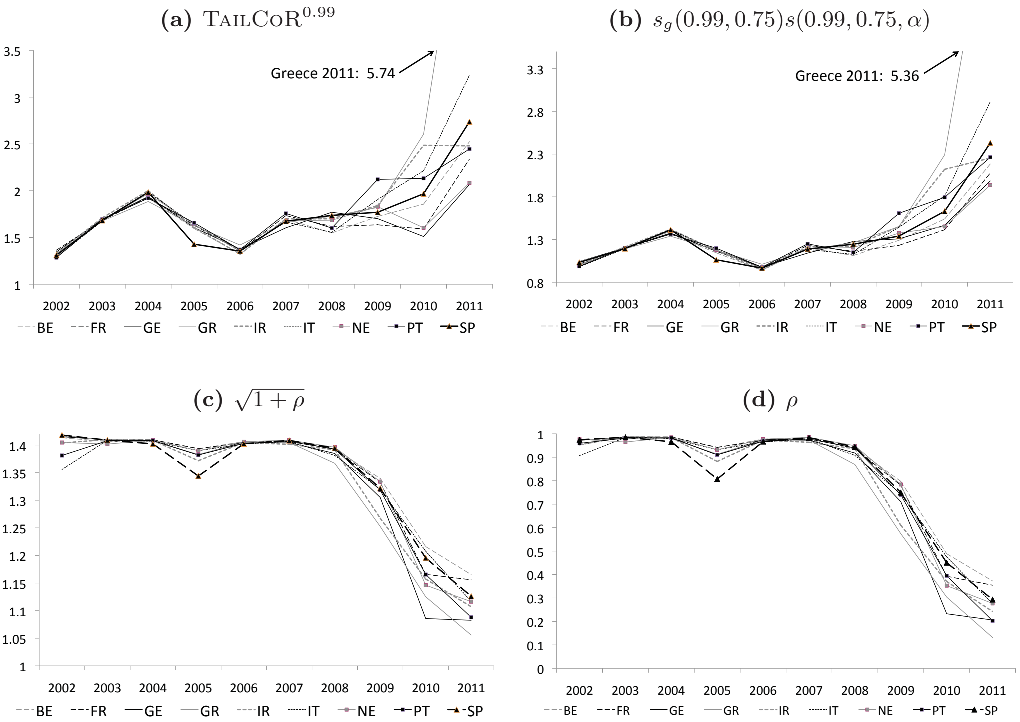
The solid line, represented in the right axis, is the average (longitudinally for all pairs of countries and for each year) of the linear correlation component of TailCoR. The thin–dashed line is the average of the non–linear correlation components, and the thick–dashed line is the average TailCoR.

We estimate daily TailCoR at 99% level (i.e. for $\xi = 0.99$) for all pairs of countries, based on a rolling window of 90 days, and, due to the non–stationary nature of the yields, on the first differences. We plot the results on annual basis (averaging the daily estimations) to obtain a neat representation.

Figure 9 shows a summary of the results. The solid line, represented in the right axis, is the average (longitudinally for all pairs of countries and for each year) of the linear correlation component of TailCoR. The thin–dashed line (represented in the left axis) is the average of the non–linear correlation components, and thick–dashed line (represented in the left axis) is the average TailCoR. Over the years the linear correlation has decreased significantly since 2008, as it was intuitively clear from Figure 8 that shows that yields stopped co–moving and many departed significantly as a consequence of the flight to quality and liquidity of the investors. It is worth noticing that the average linear correlation component before 2008 was of the order of 1.41 and at the end of the sample it was near 1.1, which roughly correspond to an average linear correlation coefficient of 1 ($\approx 1.41^2 - 1$) and 0.2 ($\approx 1.1^2 - 1$) respectively. In other words, while before the crisis the linear correlation between sovereign bond yields of the euro area countries was virtually 1, in 2009 it started to decreased, reaching in 2011 values never seen before the creation of the common currency.

This is in contrast with the other two lines and that show the opposite pattern. The average non–linear correlation component was very low from 2002 to 2007, even hitting the

Figure 10: Country by country



Each line is a country longitudinal average with respect to the other euro area members. The top left plot shows the TailCoRs, the top right the non–linear correlations, the bottom left the linear correlations, and the bottom right the linear correlation coefficients.

value 1, which corresponds to the Gaussian distribution (meaning that the only source of association is linear). Since them, the non–linear correlation increased steadily for few years with a marked increase over the last two years, reaching values around 2. As a consequence of this significant upward movement of the non–linear correlation, TailCoR also increased, regardless of the large decrease of the linear correlation component.

A more refined analysis is displayed in Figure 10. Each line is a country longitudinal average with respect to the other euro area members. The top left plot shows the TailCoRs, the top right the non–linear correlations, the bottom left the linear correlations, and the bottom right the linear correlation coefficients. Overall, one observes that the pattern in the linear correlations was homogeneous, i.e. during the crisis all the linear correlations decreased at roughly the same pace. At the end of the sample the range of variability in the linear correlation coefficients has approximately 0.2. The non–linear correlations were more heterogeneous however, ranging in 2012 from 1.93 for The Netherlands to 5.36 for Greece. Indeed, because of representation purposes, Greece for 2012 is off the scale. This exceptionally high non–linear correlation is in contrast with the average linear correlation coefficient for Greece: 0.13. This country was also the one with the largest longitudinal average TailCoR (5.74), followed by Italy (3.23), Spain (2.73), Belgium (2.53), Ireland (2.47), Portugal (2.44), France (2.33), The Netherlands (2.08), and Germany (2.06). The heterogeneity of TailCoR at the end of the sample is in contrast with the homogeneity during the years 2002–2007. The non–linear component was for all countries around 1, indicating a very stable period, with small movements in the yields, and with a distribution nearly Gaussian.

6 Conclusions

We have introduced TailCoR, a new measure for tail correlation that is a function of linear and non–linear correlations, the latter characterized by the tails. TailCoR can be exploited in a number of financial applications, such as portfolio selection where the investor is faced to risks of linear and tail nature. Moreover, TailCoR has the following advantages: i) it is exact for any probability level as it is not based on tail asymptotic arguments (contrary to tail dependence coefficients), ii) it does not depend of any specific distributional assumption, and iii) it is simple and no optimizations are needed. Monte Carlo simulations and calibrations reveal its goodness in finite samples. An empirical illustration to a panel of European sovereign bonds shows that prior to 2009 linear correlations were in the vicinity of one and non–linear correlations were inexistent. However, since the beginning of the crisis the linear correlations have sharply decreased and non–linear correlations appeared and increased significantly in 2010–2011.

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Appendix T: tabulation of $s_g(\xi, \tau)$

Interpolation can be used for values of ξ and τ that are not in the table. Alternatively, a function can be programmed following four simple steps: i) simulate from a bivariate standardized and uncorrelated Gaussian (we use 500 draws of 5000 observations each), ii) compute the interquantile ranges for a level τ, and standardized the variables: $Y_{1\,t} = X_{1\,t}/\mathrm{IQR}_1^\tau$ and $Y_{2\,t} = X_{2\,t}/\mathrm{IQR}_2^\tau$, iii) compute the projection $Z_t^{(1\,2)}$ and $\mathrm{IQR}^{(1\,2)\,\xi}$, and finally iv) $s_g(\xi, \tau) = 1/\mathrm{IQR}^{(1\,2)\,\xi}$.

	ξ													
τ	0.700	0.725	0.750	0.775	0.800	0.825	0.850	0.875	0.900	0.925	0.950	0.975	0.990	0.995
0.600	0.483	0.424	0.375	0.335	0.301	0.271	0.244	0.220	0.198	0.176	0.154	0.129	0.109	0.098
0.625	0.607	0.533	0.472	0.422	0.379	0.341	0.307	0.277	0.249	0.221	0.194	0.163	0.137	0.124
0.650	0.735	0.644	0.571	0.510	0.458	0.412	0.372	0.335	0.301	0.268	0.234	0.196	0.166	0.150
0.675	0.865	0.759	0.673	0.601	0.539	0.486	0.438	0.394	0.354	0.315	0.276	0.231	0.195	0.176
0.700	–	0.877	0.778	0.694	0.623	0.561	0.506	0.456	0.409	0.364	0.319	0.267	0.226	0.204
0.725	–	–	0.886	0.791	0.711	0.640	0.577	0.520	0.466	0.415	0.363	0.305	0.257	0.232
0.750	–	–	–	0.893	0.801	0.722	0.651	0.586	0.526	0.468	0.410	0.344	0.290	0.262
0.775	–	–	–	–	0.898	0.808	0.729	0.657	0.589	0.525	0.459	0.385	0.325	0.293
0.800	–	–	–	–	–	0.900	0.812	0.731	0.657	0.585	0.512	0.429	0.362	0.327
0.825	–	–	–	–	–	–	0.902	0.812	0.729	0.649	0.568	0.477	0.402	0.363
0.850	–	–	–	–	–	–	–	0.901	0.809	0.720	0.630	0.529	0.445	0.402
0.875	–	–	–	–	–	–	–	–	0.898	0.799	0.699	0.587	0.494	0.447
0.900	–	–	–	–	–	–	–	–	–	0.890	0.779	0.654	0.551	0.497

Appendix P: proofs

Proof of Theorem 1

The variance of $Y_{j\,t}$ is

$$\sigma^2_{Y_j} = \frac{\sigma^2_{X_j}}{(\mathrm{IQR}^{\tau}_j)^2},$$

and likewise for $Y_{k\,t}$. The variance of $Z_t^{(j\,k)}$ is

$$\sigma^2_{(j\,k)} = \frac{1}{2}\left(\frac{\sigma^2_{X_j}}{(\mathrm{IQR}^{\tau}_j)^2} + \frac{\sigma^2_{X_k}}{(\mathrm{IQR}^{\tau}_k)^2} + 2\sigma_{Y_j Y_k}\right),$$

where $\sigma_{Y_j Y_k}$ is the covariance between $Y_{j\,t}$ and $Y_{k\,t}$. Since $\mathrm{IQR}^{\tau}_j = k(\tau,\alpha)\sigma_{X_j}$ and $\mathrm{IQR}^{\tau}_k = k(\tau,\alpha)\sigma_{X_k}$

$$\sigma^2_{(j\,k)} = \frac{1}{2}\left(\frac{\sigma^2_{X_j}}{k(\tau,\alpha)^2\sigma^2_{X_j}} + \frac{\sigma^2_{X_k}}{k(\tau,\alpha)^2\sigma^2_{X_k}} + 2\frac{\sigma_{X_j\,X_k}}{k(\tau,\alpha)^2\sigma_{X_j}\sigma_{X_k}}\right).$$

In a more compact form

$$\sigma^2_{(j\,k)} = \frac{1}{k(\tau,\alpha)^2}\left(1+\rho_{j\,k}\right).$$

By the affine invariance of the elliptical family, $\mathrm{IQR}^{(j\,k)\,\xi} = k(\xi,\alpha)\sigma_{(j\,k)}$. Substituting in $\sigma^2_{(j\,k)}$

$$\mathrm{IQR}^{(j\,k)\,\xi} = \frac{k(\xi,\alpha)}{k(\tau,\alpha)}\sqrt{1+\rho_{j\,k}} = s(\xi,\tau,\alpha)\sqrt{1+\rho_{j\,k}}.$$

In the Gaussian case $k(\tau,\alpha) = k(\tau)$ and $k(\xi,\alpha) = k(\xi)$. We normalize $\mathrm{IQR}^{\xi}_{(j\,k)}$ by $\frac{k(\tau)}{k(\xi)} = s_g(\xi,\tau)$ yielding

$$\mathrm{TailCoR}^{(j\,k)\,\xi} = s_g(\xi,\tau)s(\xi,\tau,\alpha)\sqrt{1+\rho_{j\,k}}.$$

Q.E.D.

Proof of Theorem 2

By **E1**, $\mathrm{Tail\hat{C}oR}^{(j\,k)\,\xi}_{\hat{Z}\,T} = 2s_g(\xi,\tau)\hat{\mathrm{Q}}^{(j\,k)\,\xi}(\hat{\mathbf{Q}})$. Doing a Taylor expansion around $\mathbf{Q}$:

$$\mathrm{Tail\hat{C}oR}^{(j\,k)\,\xi}_{\hat{Z}\,T} \approx \mathrm{Tail\hat{C}oR}^{(j\,k)\,\xi}_{T} + 2s_g(\xi,\tau)\frac{\partial\hat{\mathrm{Q}}^{(j\,k)\,\xi}(\mathbf{Q})}{\partial\mathbf{Q}}(\hat{\mathbf{Q}}-\mathbf{Q}).$$

$\mathrm{Tail\hat{C}oR}^{(j\,k)\,\xi}_{T} = 2s_g(\xi,\tau)\hat{\mathrm{Q}}^{(j\,k)\,\xi}(\mathbf{Q})$ where the term $2s_g(\xi,\tau)$ is a deterministic scale shift and the only source of randomness is $\hat{\mathrm{Q}}^{(j\,k)\,\xi}(\mathbf{Q}) = \hat{\mathrm{Q}}^{(j\,k)\,\xi}$. Hence

$$\begin{aligned} E(\mathrm{Tail\hat{C}oR}^{(j\,k)\,\xi}_{T}) &= 2s_g(\xi,\tau)E(\hat{\mathrm{Q}}^{(j\,k)\,\xi}) \text{ and} \\ Var(\mathrm{Tail\hat{C}oR}^{(j\,k)\,\xi}_{T}) &= 4s_g(\xi,\tau)^2 Var(\hat{\mathrm{Q}}^{(j\,k)\,\xi}). \end{aligned}$$

By the asymptotic properties of sample quantiles under S–mixing (Dominicy et al. (2012)) and the delta method the proof is completed. Q.E.D.

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