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2009

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Documentos de Trabajo
N.º 0911

BANCO DE **ESPAÑA**
Eurosistema

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[*] A previous version of this paper has been circulated under the title "Assessing the risk, return and efficiency of banks' loans portfolios". This paper is the sole responsibility of its author. The views represented here do not necessarily reflect those of the Bank of Spain. I am grateful to Max Bruche, Enrique Sentana and seminar audiences at the Bank of Austria, Bank of Spain and CEMFI for their comments and suggestions. Special thanks are due to Alfredo Martín, for his valuable suggestions as well as for his help with the interest rate database. Of course, the usual caveat applies.

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Documentos de Trabajo. N.º 0911

2009

ISSN: 0213-2710 (print)
ISSN: 1579-8666 (on line)
Depósito legal:
Unidad de Publicaciones, Banco de España

Abstract

This paper analyses the risk and return of loans portfolios in a joint setting. I develop a model to obtain the distribution of loans returns. I use this model to describe the investment opportunity set of lenders using mean-variance analysis with a Value at Risk constraint. I also obtain closed form expressions for the interest rates that banks should set in compensation for borrowers' credit risk under absence of arbitrage opportunities and I use these rates as a benchmark to interpret actual loans' prices. Finally, I study the risk-return trade-off in an empirical application to the Spanish banking system.

Keywords: Credit risk, Probability of default, Asset Pricing, Mean-Variance allocation, Stochastic Discount Factor, Value at Risk.

JEL classification: G21, G12, G11, C32, D81, G28.

1 Introduction

Standard capital market theory states that there is a risk-return trade-off in equilibrium. The more risk one is willing to take, the higher the return one will be able to get. This relationship has been extensively analysed in the context of liquid assets that trade in organised markets (see e.g. Fama and MacBeth, 1973; Ghysels, Santa-Clara, and Valkanov, 2005). However, much less is known about its implications on the behaviour of banks as risk managers and profit maximisers. Banks aggregate profits have been analysed by Behr et al. (2007) and Hayden, Porath, and von Westernhagen (2007), among others, who find that more specialisation tends to yield higher returns but also a higher level of risk. However, the optimal degree of bank specialisation is not analysed by these papers. In addition, they aggregate the different activities of banks, that is, they do not separate market and credit activities. As already explained, the features of liquid assets are well known. Thus, my goal is to focus on the less well understood features of the lending business, which is the main role of banks as transformers of short term investments (deposits) into long term ones (loans).

The risk and return of loans portfolios has generally been analysed separately. On the one hand, the Basel II framework has originated the development of many quantitative models to estimate the loss distribution of loans portfolios (see Embrechts, Frey, and McNeil, 2005, for a textbook review of the literature). On the other hand, a parallel literature that studies the determinants of interest rates has simultaneously grown during the previous years (see e.g. Martín, Salas, and Saurina, 2007; Mueller, 2008). Unfortunately, to the best of my knowledge, an specific common framework to analyse the risk and return characteristics of bank's loans portfolios is still missing in the literature, probably due to the technical difficulties that the characteristics of loans entail. In particular, banks set the interest rates of the loans that they grant. Hence, the prices of these products are not the result of trading in an open market, which implies that the lenders may be price setters rather than price takers. In addition, the returns that banks are finally able to obtain may be smaller than the required rates in the presence of default risk, but their exact value is not observable. Therefore, it is necessary to develop a model in order to infer returns from the available information.

In this paper, I propose a flexible although analytically tractable model to derive loans returns from interest rates and historical loan default rates. Since this information is generally available for banks and supervisors, my model can be readily applied either as a risk management or as an off-site supervisory tool. I consider a general structure in which defaults are driven by stochastic probabilities of default, which may be correlated with recovery rates. I make my model operational by expressing probabilities of default as a probit function of an underlying multivariate Gaussian vector of state variables. I am able to obtain closed form expressions for the expected returns, variances and covariances between different loans. The covariance matrix of returns does not only depend on the distribution of the probabilities of default, but also on the granularity of the portfolios. I analyse risk and return jointly using the mean-variance analysis of Markowitz (1952) to obtain the set of efficient portfolios. In this sense, the properties of the return distribution of loans, and in particular the absence of probability mass at the right tail, make variance a suitable measure of risk in this context. Thus, I can assume that banks would like to minimize the variance of their loans portfolio for any given target expected return, from which I can obtain the investment opportunity set on the mean-variance space. This set may be restricted by the minimum capital requirements imposed by the regulator or possibly by an even more stringent rating target. Both conditions can be interpreted as a constraint on the minimum return that the bank must obtain, which technically corresponds to a bound on the admissible Value at Risk (VaR). Sentana (2003) and Alexander and Baptista (2006) have previously considered mean-variance analysis with a VaR constraint when returns are elliptical. In this sense, I extend their approach into the non-elliptical statistical framework of this paper.

I also analyse the risk-return trade-off from a pricing point of view. In this regard, my goal is to study whether banks set interest rates taking into account the potential default risk of their borrowers. To do so, I first derive theoretical closed form expressions for the spreads over the risk-free rate that banks should require to ensure absence of arbitrage. Although the market power exercised by banks may yield significant deviations from this setting, this result provides a useful benchmark against which I can interpret the evolution of actual spreads. I obtain the model-based rates using the exponentially affine stochastic discount factor (SDF) proposed by Gourieroux and Monfort (2007). In this setting, I can fully characterise the risk-neutral measure that is equivalent to this SDF.

I consider an empirical application to Spanish loans. I use quarterly data from the Spanish credit register, from 1984.Q4 to 2008.Q4, to estimate a dynamic probit for the probabilities of default and obtain the granularity of empirical portfolios. I consider an additional database in which banks inform about the average interest rates for several classes of loans, which is available from 1990.Q1. With this information, I compute the mean-variance frontiers at different periods and analyse the historical evolution of the expected values and standard deviations of loans returns. Finally, I study the deviations of the actual mean-variance frontiers from those that would result in an arbitrage free setting.

The rest of the paper is organised as follows. I introduce a general framework to model the loan return distribution in Section 2. I discuss a simple feasible parametrisation of my model in Section 3 and introduce mean-variance analysis in Section 4. Then, I develop an arbitrage free model to price loans in Section 5. Section 6 presents the results of the empirical application. Finally, Section 7 concludes. Proofs and auxiliary results can be found in appendices.

2 General framework

Consider an economy with two periods: $t = 0, 1$. There is a risk-free asset, whose return is r, and K different types of loans. These loans, which are risky because of the presence of credit risk, may be interpreted as belonging to different economic sectors, or just as a means of classifying borrowers with different characteristics (e.g.: corporates vs. households). In each of these groups, there are N_k loans, for $k = 1, \cdots, K$. I denote the volume of loan i from group k as L_{ki}, while r_k will be the net interest rate required by the lender at $t = 1$ for each loan type. Interest rates are set at $t = 0$.

I now turn to modelling borrowers' credit risk. Borrowers may default at $t = 1$, where default is driven by a binary variable D_{ki} that takes a value of 1 if i defaults and 0 otherwise. The probability of default of any loan from group k will be given by the stochastic variable π_k. In fact, π_k is not even observable at $t = 1$ unless N_k grows to infinity.[1] In case of default, the lender will obtain a recovery rate δ_{ik}, which is a proportion

[1]Specifically, it can be shown that $\pi_k = \lim_{N_k \to \infty} \sum_i D_{ik}/N_k$ under standard regularity conditions. I

of the total amount that is owed. I will assume that δ_{ik}, which is also a stochastic variable, can be decomposed as $\delta_{ik} = \delta_k \psi_{ik}$. δ_k is the systematic component of the recovery rate, since it is common to to all the loans of the k-th category and it may also be correlated with other categories. In contrast, ψ_{ik} will be an idiosyncratic term, modelled as an *iid* positive variable with mean 1. Hence, this effect may alter the volatility of recovery rates, but it does not change their expected values nor their correlations with other types of loans. I will assume that defaults are conditionally independent given π_k and δ_{ik} for $k = 1, \cdots, K$.

Consider the portfolio of loans held by a particular bank at time 0. Its value will be the cost of the initial investment, which is the sum of the outstanding debt at this period, i.e.

$$p = \sum_{k=1}^{K} p_k,$$

where

$$p_k = \sum_{j=1}^{N_{kt-1}} L_{jk}, \tag{1}$$

denotes the outstanding debt of the k-th category of loans, for $k = 1, \cdots, K$. One period later, the pay-offs generated by each class of loans can be expressed as

$$Z_k = (1 + r_k) \sum_{j=1}^{N_k} L_{jk}(1 - D_{jk}) + \sum_{j=1}^{N_k} \delta_{jk} L_{jk} D_{jk}. \tag{2}$$

Intuitively, each borrower will either repay the principal plus interests or default, in which case the lender will only receive the recovery rate times the outstanding amount.

From (1) and (2), it is straightforward to write the return or yield generated by loans from group k as:

$$\begin{aligned} y_k &= \frac{Z_k - p_k}{p_k}, \\ &= r_k - \frac{\sum_{j=1}^{N_k}(1 + r_k - \delta_{jk}) L_{jk} D_{jk}}{N_k \bar{L}_k}, \end{aligned} \tag{3}$$

where

$$\bar{L}_k = \frac{1}{N_k} \sum_{j=1}^{N_k} L_{jk}.$$

will exploit this feature in the empirical application.

Using the conditional independence property, it can be shown that the expected value of (3) is given by

$$E(y_k) = r_k - (1 + r_k - E(\delta_k))E(\pi_k) + cov(\pi_k, \delta_k). \tag{4}$$

Hence, the expected return obtained by the lender may in practice differ from the required interest rate r_k because of the presence of credit risk. Specifically, this value will be smaller than r_k under a positive expected loss given default, which is realised with expected probability $E(\pi_k)$. This is a well known feature (see e.g. Feder, 1980). In addition, the correlation between probabilities of default and recovery rates also affects the expected return. For instance, a negative correlation would yield a further reduction in expected returns, since recoveries would then be smaller in bad times, which is precisely when probabilities of default are higher.

3 A simple implementation of the model

To make the framework of Section 2 operational, I need to make specific statistical assumptions. In particular, I model probabilities of default in terms of the probits

$$\pi_k = \Phi(x_k), \tag{5}$$

for $k = 1, \cdots, K$, where $\Phi(\cdot)$ is the standard normal cdf, and $\mathbf{x} = (x_1, x_2, \cdots, x_K)'$ is a vector of Gaussian state variables. For the sake of concreteness, I will express this distribution as

$$\mathbf{x} \sim N[\boldsymbol{\mu}(\boldsymbol{\theta}), \boldsymbol{\Sigma}(\boldsymbol{\theta})], \tag{6}$$

where $\boldsymbol{\theta}$ is a vector of p free parameters, and $\boldsymbol{\mu}(\boldsymbol{\theta})$ and $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ are, respectively, the mean and covariance matrix of $\mathbf{x}$. Interestingly, despite the flexibility of this framework, it is still possible to derive general closed form expressions for the moments of (5):

Proposition 1 *Consider a loan with probability of default (5). Then,*

$$E(\pi_k) = \Phi\left[\frac{\mu_k}{\sqrt{1+\sigma_{kk}}}\right], \tag{7}$$

$$\begin{aligned} E(\pi_k\pi_j) &= \omega_{kj} \\ &= \Phi_2\left[\frac{\mu_k}{\sqrt{1+\sigma_{kk}}}, \frac{\mu_j}{\sqrt{1+\sigma_{jj}}}, \frac{\sigma_{kj}}{\sqrt{(1+\sigma_{kk})(1+\sigma_{jj})}}\right] \end{aligned} \tag{8}$$

where μ_k is the k-th component of $\boldsymbol{\mu}(\boldsymbol{\theta})$, σ_{kj} is the element of $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ on row k and column j, $\Phi(\cdot)$ is the cdf of the standard normal distribution, and $\Phi_2(\cdot,\cdot,\rho)$ denotes the cdf of a bivariate normal distribution with zero means, unit variances and correlation ρ.

From Proposition 1 it is straightforward to show that the expected probability of default at $t = 0$ is $E(\pi_k)$. With this result in mind, I can express the conditional default correlation as follows.

Proposition 2 *Consider a loan with probability of default (5). Then*

$$cor(D_{ki}, D_{ji}) = \frac{\omega_{kj} - E(\pi_k)E(\pi_j)}{\sqrt{[E(\pi_k) - E^2(\pi_k)][E(\pi_j) - E^2(\pi_j)]}},$$

where $E(\pi_k), E(\pi_j)$ and ω_{kj} are defined in (7) and (8).

4 Mean-Variance frontier with a VaR constraint

In this section, I develop a framework in which banks can decide the optimal combination of different types of loans in their lending policies, such that they maximise their expected return for a given risk appetite. Hence, I need to consider an appropriate measure of risk to address this issue. In this sense, notice that, since δ_{jk} will always be smaller than the gross interest rate, the distribution of (3) is bounded on its right tail by the maximum yield that can be obtained, which is r_k. Hence, risk is always undesirable in this setting since it implies receiving a smaller return than the require rate. This is an interesting difference with equity return distributions, in which positive extreme returns may also occur. Nevertheless, the variance is still a popular measure of risk in applications to stock returns, even though it is well known that a high variance may not necessarily be undesirable if it is due to high unexpected returns (see e.g. Peñaranda, 2007). Such a concern does not apply in this paper, because there is no right tail. Hence, I can adequately assess the risk of the loans portfolios by means of their covariance matrix.

In consequence, mean-variance analysis is a natural framework to analyse efficiency in this setting, since I can assess profitability with expected returns, and use variances as a measure of risk. Let $\boldsymbol{\omega}$ denote a vector containing the proportion of credit that a particular bank allocates to each of the K different types of loans. This vector is chosen at time 0. Hence, $\boldsymbol{\omega}'\mathbf{y}$, where $\mathbf{y} = (y_1, y_2, \cdots, y_K)'$, constitutes the return of the credit portfolio of this bank at $t = 1$. The expected return and variance are given by $\boldsymbol{\omega}'E(\mathbf{y})$ and $\boldsymbol{\omega}'V(\mathbf{y})\boldsymbol{\omega}$, respectively.

I assume that banks would like to minimise the variances of their portfolios for every expected return μ_0, subject to the restriction

$$\boldsymbol{\omega}'\mathbf{y} > \tau_0. \tag{9}$$

This constraint ensures that banks will obtain a return higher than τ_0, which may be necessary to comply with the minimum capital required by the regulator. For instance, if (9) is violated, the bank may have to raise additional funds. Alternatively, if it is targeting a high rating, it may need to satisfy an even more restrictive minimum capital requirement. In general, though, it cannot be ensured that (9) will hold with probability 1. Hence, I will set the minimum probability $1 - \alpha$ with which (9) must hold, which is equivalent to specifying a maximum admissible VaR. Notice that the higher this confidence level is the more I will restrict the set of admissible portfolios.

In sum, this maximisation problem can be expressed in the following terms:

$$\min_{\boldsymbol{\omega}} \boldsymbol{\omega}'V(\mathbf{y})\boldsymbol{\omega}, \tag{10}$$

such that

$$\begin{aligned} \boldsymbol{\omega}'E(\mathbf{y}) &= \mu_0, & (11)\\ \Pr(\boldsymbol{\omega}'\mathbf{y} > \tau_0) &\geq 1 - \alpha, & (12)\\ \boldsymbol{\omega}'\boldsymbol{\iota} &= 1, & (13) \end{aligned}$$

and

$$0 \leq \omega_k \leq 1, \quad k = 1, \cdots, K, \tag{14}$$

where $\boldsymbol{\iota}$ is a vector of K ones. I have introduced (13) and (14) to ensure that banks can neither short sell nor leverage their investments. If I allow for the presence of a risk-free asset, I do not need to impose (13). In this case, the mean-standard deviation frontier would be the straight line that is tangent to the frontier without a risk-free asset and intercepts the zero standard deviation axis at the risk-free rate return.

Following Sentana (2003), this problem can be decomposed in two simpler ones. First, the region described by the VaR constraint (12) can be analysed separately by Monte

Carlo simulation techniques (see Appendix C). And secondly, the mean-variance frontier can be computed regardless of restriction (12), that is, by minimising (10) subject to (11), (13) and (14). This frontier can be easily obtained by numerical optimisation, using Proposition 1 to express the expected return in closed form, and the following Proposition to obtain the analytical formula of the covariance matrix.

Proposition 3 *Consider K portfolios of loans whose returns are given by (3) for $k = 1, \cdots, K$. Then,*

$$\begin{aligned} V(y_k) &= (1/g_k)\left[E\left[(1+r_k-\delta_k\psi_{ik})^2\right]E(\pi_k)-(1+r_k-E(\delta_k))^2\omega_{kk}\right] \\ &\quad +(1+r_k-E(\delta_k))^2[\omega_{kk}-E^2(\pi_k)], \end{aligned} \tag{15}$$

and

$$\begin{aligned} cov(y_k, y_j) &= (1+r_k-E(\delta_k))(1+r_j-E(\delta_j)) \\ &\quad \times[\omega_{kj}-E(\pi_k)E(\pi_j)], \end{aligned} \tag{16}$$

for $k \neq j$, where

$$g_k = \frac{\left(\sum_{i=1}^{N_k} L_{ik}\right)^2}{\sum_{i=1}^{N_k} L_{ik}^2}. \tag{17}$$

Therefore, both the variances and covariances depend on the moments of recovery rates and probabilities of default. In the case of the variances, (15) is the sum of two components. The first one is proportional to the reciprocal of (17), which can be interpreted as a granularity parameter. For any finite number of loans, (17) will always be finite. However, if I let N_k grow to infinity while the size of the loans remains bounded, i.e.

$$\sup\{L_{ik}, 0 \leq i \leq N_k\} \leq L_{k+} < \infty,$$

then it can be shown that g_k will tend to infinity and the first term in (15) will disappear. Therefore, g_k increases as granularity grows to infinity. Thus, the first component in (15) comprises a diversifiable risk that can be reduced by just increasing the number of borrowers in the portfolio. This risk essentially captures what is usually known as “jump to default” risk (see e.g. Collin-Dufresne et al., 2003).

In practice, banks might not be able to place their loans portfolios on the mean-variance frontier, since their lending business is constrained by several restrictions. To begin with, loans are illiquid assets that cannot be easily traded in the market. Hence, it may be costly to rebalance these portfolios. In this sense, many banks have securitised their credit portfolios until 2008 to be able to expand in the areas that they perceived as more profitable. However, rebalancing has become much more costly after the subsequent collapse of this market. In addition, the granularity of loans portfolios may also have a strong impact on the degree of efficiency that banks can obtain. From this perspective, there will be a "feasible" frontier, which takes the granularity of banks' portfolios as given, and an "infeasible" one that assumes that banks can actually get infinitely granular portfolios.

For illustrative purposes, I now consider a simple example with two categories of loans, such that (6) can be expressed as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N \left[\begin{pmatrix} \Phi^{-1}(0.05)\sqrt{1.01} \\ \Phi^{-1}(0.025)\sqrt{1.1} \end{pmatrix} ; \begin{pmatrix} 0.01 & \rho\sqrt{0.001} \\ \rho\sqrt{0.001} & 0.1 \end{pmatrix} \right], \tag{18}$$

For the sake of brevity, I will refer to these portfolios as portfolios 1 and 2. It is not clear a priory which of them is safer. On the one hand, the first type of assets have a expected probability of default of 5%, while this value is only 2.5% for the second type of assets. On the other hand, the behaviour of category 2 is more volatile, since the variance of x_2 is ten times larger than the variance of x_1. Figure 1 shows the effect of granularity on the standard deviations of portfolios 1 and 2. Both portfolios have high standard deviations for small granularity, because the smaller the granularity, the higher the idiosyncratic risk. However, portfolio 1 is riskier for small granularity due to its higher unconditional probability of default. Introducing more loans in each portfolio can reduce their risk. This reduction is smaller for portfolio 2, since the high standard deviation of x_2 dominates over the larger expected probability of default of 1 as the portfolios become more granular. For infinite granularity, only the undiversifiable risk remains, which is again higher for x_2.

Figure 2 shows the mapping between the underlying correlation parameter ρ and the correlation between the returns of portfolios 1 and 2. For very small granularity, the correlation is approximately zero regardless of ρ, because most of the risk is purely idiosyncratic in this case. In more granular portfolios, though, ρ becomes increasingly informative about the return correlation. Eventually, it approximately corresponds to the actual return correlation on infinitely granular portfolios ($g \to \infty$).

Following with this example, I show in Figure 3 the effect of different correlations between portfolios 1 and 2 on the mean-standard deviation frontier. I assume infinite granularity and interest rates $r_1 = 4.5\%$ and $r_2 = 5\%$. Since there are only two types of loans in this example, this figure simply shows the combination line between portfolios 1 and 2 for different values of ρ. This line is more convex when the correlation is small. Thus, it is easier to obtain diversification gains by combining several portfolios when the loans have small correlations. However, these benefits become increasingly smaller as the correlation increases. For instance, if the correlation is close to 1, then any combination of portfolios 1 and 2 will approximately lie on a straight line between these two assets.

5 Asset pricing implications

So far, I have treated the interest rates charged by lenders as exogenously given values. Nevertheless, in a competitive market banks should demand interest rates that compensate for the risk of borrowers' defaults to ensure absence of arbitrage opportunities. Of course, actual prices may substantially differ from those of this competitive pricing scheme since this is an illiquid market in which some counterparties may exercise market power. However, arbitrage free prices can provide a useful benchmark against which I can compare actual data. To formalise this issue, I introduce an asset pricing framework by modelling the stochastic discount factor (SDF). The SDF is a very useful instrument to obtain prices under absence of arbitrage. For instance, consider an asset with payoff Z at $t = 1$. Then, its price at $t = 0$ can be obtained from the pricing formula:

$$p = E(M\,Z). \tag{19}$$

where M is the SDF. The SDF can be related to the marginal rate of substitution between $t = 0$ and $t = 1$ in equilibrium models (see e.g. Gourieroux and Monfort, 2007). Pricing assets with an SDF is equivalent to working under a different statistical measure $\mathbb{Q}$ in which agents are risk-neutral. Under this new measure, (19) becomes

$$p = \frac{1}{1+r} E^{\mathbb{Q}}(Z). \tag{20}$$

Notice that, in the particular case in which the SDF is constant, the actual and the risk-neutral measures will coincide.

If I use (20) to price the portfolio with payoffs (2), then I can express the spread that banks should charge over the risk-free rate as

$$r_k - r = \frac{[1 + r - E^{\mathbb{Q}}(\delta_k)]E^{\mathbb{Q}}(\pi_k) - cov^{\mathbb{Q}}(\pi_k, \delta_k)}{1 - E^{\mathbb{Q}}(\pi_k)}. \tag{21}$$

Hence, (21) is an increasing function of the risk-adjusted probability of default and a decreasing function of the expected recovery rate under $\mathbb{Q}$. In addition, all else equal, spreads should be higher if the correlation between probabilities of default and recovery rates becomes smaller. In this sense, Altman et al. (2005) find negative correlations between recoveries and default rates in corporate bonds.

Since the market is incomplete in a discrete time model such as this one, there exists a multiplicity of SDF's that are compatible with the valuation formula (19). Hence, to operationalise this result I assume a priory an exponentially affine form for the SDF,

$$M = \nu_0 \exp(\boldsymbol{\nu}_1'\mathbf{x}), \tag{22}$$

where ν_0 and $\boldsymbol{\nu}_1$ are, respectively, a positive scalar and a $K \times 1$ vector. The exponential structure ensures that the SDF is always positive, which in turn guarantees the absence of arbitrage opportunities (see e.g. Cochrane, 2001, chapter 4). In addition, this specification corresponds to the Esscher transform used in insurance (see Esscher, 1932), as well as in derivative pricing models (Bertholon, Monfort, and Pegoraro, 2003; León, Mencía, and Sentana, 2007). As Gourieroux and Monfort (2007) argue, the SDF of several important equilibrium models, such as the consumption based CAPM, can be expressed with this structure. I can also obtain the distribution of $\mathbf{x}$ under the risk-neutral measure from the following result.

Proposition 4 *Let the SDF be given by (22). Then, the distribution of* $\mathbf{x}$ *under the risk-neutral measure* $\mathbb{Q}$ *is multivariate Gaussian with mean vector*

$$\boldsymbol{\mu}^{\mathbb{Q}}(\boldsymbol{\theta}) = \boldsymbol{\mu}(\boldsymbol{\theta}) + \boldsymbol{\Sigma}(\boldsymbol{\theta})\boldsymbol{\nu}_1 \tag{23}$$

and covariance matrix $\boldsymbol{\Sigma}(\boldsymbol{\theta})$.

Therefore, as in other financial applications (see e.g. Black and Scholes, 1973), I only have to modify the mean of $\mathbf{x}$ to change the measure, whereas the covariance matrix remains unaltered. Once I have done this change, I can price any asset by just computing its discounted expected price. This device can be used to price any combination of the original assets, as well as derivatives or collateralised debt obligations (CDO's). To obtain the required spread of a loan from the k-th category I only need to introduce the expected probability of default under $\mathbb{Q}$ in (21). From Proposition 1,

$$E^{\mathbb{Q}}(\pi_k) = \Phi\left(\frac{\mu_k + \boldsymbol{\nu}_1'\boldsymbol{\sigma}_{\bullet k}}{\sqrt{1+\sigma_{kk}}}\right), \tag{24}$$

where $\boldsymbol{\sigma}_{\bullet k}$ is the k-th column of $\boldsymbol{\Sigma}(\boldsymbol{\theta})$. In consequence, arbitrage-free spreads are the result of a combination of data-based information and preference-based factors. To begin with, they depend on variables from the actual probability measure, such as the recovery rates or the parameters of the actual probability of default. All these parameters are specific of each category of loans, except for $\boldsymbol{\sigma}_{\bullet k}$, which captures the covariance of the credit risk of loans from a given type k with the remaining groups. However, the impact of $\boldsymbol{\sigma}_{\bullet k}$ on $E^{\mathbb{Q}}(\pi_k)$ depends on $\boldsymbol{\nu}_1$, which reflects the consensus of the market about how to map systematic credit risk into prices. As already mentioned, these coefficients are closely related to the utility preferences of the agents, and in particular to their marginal rate of intertemporal substitution.

6 Empirical application

I consider an empirical application to the Spanish banking system. I distribute loans in three categories: corporate, consumption loans and mortgages. Although the empirical evidence suggests that loans from different economic sectors may not have the same exposures to common shocks (see Jiménez and Mencía, 2009), I aggregate all corporate loans into just one group due to the absence of data about discrepancies between interest rates across economic sectors.

I use data from the Spanish credit register to estimate the distribution of the probabilities of default. This database has information about every loan with a volume above €6,000. Since this threshold is very small, I can safely assume that the data is representative of the whole banking system. There are two variables in the database that are of particular interest for this paper.[2] In particular, I will obtain the volumes of the existing

[2]See Jiménez and Saurina (2004) and Jiménez, Salas, and Saurina (2006) for a thorough description

loans from the outstanding amounts that are reported in the credit register. In addition, the database also has extremely useful information about the default situation of each loan. It indicates which loans are overdue, and how long they have been in this situation. Based on this information, I have computed historical quarterly series of the default frequencies for each type of loans from 1984.Q4 to 2008.Q4. Default frequencies are defined as the ratio of the number of loans that defaulted during a particular quarter over the total number of loans. Following the definition given by the Basel II framework, I assume that a loan is in default if it has been overdue for more than 90 days. I will use default frequencies as proxies of π_{kt}, which I model with a Gaussian distribution as in (6).[3] In what follows, I introduce a t subindex to take into account the time dimension. In order to estimate the dynamics of π_{kt}, I transform them with the inverse of the standard normal cdf. Then, I consider the following vector autoregressive model for the transformed series:

$$\mathbf{x}_t = [\mathbf{I}_k - diag(\boldsymbol{\alpha})]^{-1}\boldsymbol{\mu} + diag(\boldsymbol{\alpha})\mathbf{x}_{t-1} + \boldsymbol{\beta} f_t + diag(\boldsymbol{\gamma})^{1/2}\boldsymbol{\varepsilon}_t, \tag{25}$$

where $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are $k \times 1$ vectors, $diag(\boldsymbol{\alpha})$ yields a diagonal matrix whose diagonal terms are given by $\boldsymbol{\alpha}$, f_t is a latent factor such that

$$f_t = \varphi f_{t-1} + \sqrt{1-\varphi}\nu_t,$$

while $\boldsymbol{\varepsilon}_t$ and ν_t are independent standard Gaussian vectors. Hence, correlations in this model are driven by a common latent factor that may have time series autocorrelation. In this way, I can account for two important stylised facts: persistence in default rates and correlation between the rates of default from different categories. Therefore, this model provides an equilibrium between flexibility in the time series and cross-sectional correlation structures and parsimony in terms of the number of parameters employed.

Figure 4a shows the historical evolution of default frequencies. The three series display a highly cyclical pattern, although corporate loans seem to be the category that was more sensitive to the 1993 recession. Corporate loans are also the category with higher default rates at the 2008 crisis. Table 1 shows the maximum likelihood parameter estimates of

of the database.

[3]This estimation approach relies on the property that π_{kt} can be recovered from default rates if N_{kt} is sufficiently large (see footnote 1). This is a reasonable approximation in this case because I am considering the whole loans portfolio of Spanish banks. Hence, I have about one million loans on average in each group.

the model specified in (25). The high estimates of $\boldsymbol{\alpha}$ show that there is strong time series autocorrelation.[4] In addition, the autocorrelation of f_t, combined with the positive and significant factor loadings in $\boldsymbol{\beta}$, indicate a strong cross-sectional correlation in which common shocks tend to have lasting effects on the evolution of risks.

Once I have estimated the model for the probabilities of default, I need to consider the interest rates of loans in order to compute the moments of their return distribution. Although the Spanish credit register does not include data about interest rates, I can use the values reported by banks for corporate loans, consumption loans and mortgages. Banks declared the average interest rate for each type of loans from 2003.Q1 to the end of the sample. Before 2003.Q1, though, they only informed about average marginal interest rates, that is, the interest rates of the new operations. Thus, to obtain a consistent database for the whole sample period, I have estimated the age of each loan in the database at each quarter prior to 2003.Q1. With this information, I have constructed series of average interest rates by considering the appropriate marginal rates for each cohort of loans, and weighting them by the proportion of the total credit in the economy that they represent. This procedure yields a consistent time series of interest rates from 1990.Q1 to 2008.Q4, as shown in Figure 4b. This Figure also shows the evolution of treasury rates throughout this period, which will be used as a proxy for the risk-free rate. Notice the presence of important jumps in the treasury rates due to devaluations in the Spanish peseta (see e.g. Gimeno and Marqués, 2008, for more detailed information).

As for recovery rates, I assume that they are constant. Specifically, I set them at 0.65, 0.75 and 0.85 for corporate loans, consumption loans and mortgages, respectively, which are consistent with the results reported by Spanish banks for the QIS5.[5] The assumption of constant recovery rates can of course be relaxed, but it will be rather difficult calibrate their distribution in practice due to the absence of time series data about this variable in the case of loans.

Based on these estimates I consider the mean-variance frontier in Figure 5. Since the granularity parameter g of the whole portfolio of Spanish loans is higher than 10^4

[4]It could be argued that these coefficient might actually be 1. In this sense, although it would be possible to estimate the model in first differences, I prefer using levels because the stationarity of the series ensures the existence of unconditional probabilities of default.

[5]Fifth Quantitative Impact Survey.

in all cases, the diversifiable risk due to lack of granularity will be negligible in this application.[6] I consider four panels for the frontiers at 2003.Q4, 2005.Q4, 2008.Q1 and 2008.Q4. Each panel also includes the line above which a non-negative excess return over the treasury rate will be obtained with 99.9% probability. In general, consumption loans yield the highest expected excess return, although at the cost of a slightly higher standard deviation than mortgages. In contrast, corporate loans yield an even higher standard deviation and a smaller excess return. It can also be observed that the mean-standard deviation frontier has moved down (lower expected values) and to the right (higher standard deviations) since 2003, although the most important change has occurred within 2008. In this sense, the aggregate position of the loans portfolios was well above the border of the VaR constraint in all periods except in last one, in which it is slightly below. Hence, the probability that the Spanish loans portfolio would yield a non-negative excess return in 2008.Q4 was smaller than 99.9%, albeit higher than 99.5% (also shown in Figure 5d).

In Figure 6a, I consider the evolution over the sample period of the expected returns and standard deviations of the aggregate portfolio of Spanish loans. Both moments have decreased from the beginning of the 1990's until 2005, but the value for 2008.Q4 shows that the standard deviation has sharply risen at the end of the sample. Thus, high risk is positively correlated with high gross expected returns. However, the risk-return trade-off, if it exists, should affect excess returns rather than pure returns, because the risk-free rate is mainly driven by monetary policy and not by the underlying risk of borrowers. In this sense, I consider returns in excess of the treasury rate in Figure 6b. It can be clearly observed that banks obtained high spreads on periods with very high standard deviations, such as those close to the 1993 recession. Similarly, there were small expected excess returns and standard deviations on the last decade of the sample. However, despite this broad picture there is not such a clear pattern when I compare specific periods that are not at the opposite extremes of the credit cycle.

In order to understand the deviations of actual interest rates from an arbitrage free asset pricing model, I compare the actual mean-standard deviation frontiers with those that would result from the pricing model of Section 5. To do so, I estimate the coefficients

[6]Nevertheless, the granularity of the loans portfolios of individual banks may be much smaller, depending on the size of the portfolio.

of the SDF (22) by minimising the sum of the squared differences between the actual interest rates and those implied by (21). Hence, these estimates yield the arbitrage free pricing model that is closest to the data. Figure 7 compares the actual and model-based interest rates that I have obtained. These rates do reflect some of the main characteristics of the data, such as higher interest rates and spreads in the first half of the 1990's and a rise in spreads from 2006 to the end of the sample. However, there are still important differences. In particular, from around 1996 until 2006 actual rates have been higher that the model-based ones. Finally, the model can obviously not reproduce the effect of the currency devaluations that I have already mentioned.

Figure 8a shows the implied arbitrage-free mean-standard deviation frontiers at several periods. I can clearly observe a risk-return trade-off in this case. The more risky the investment opportunity set is on one particular period, the higher the expected returns that are demanded. In fact, if I compute the envelope of the investment opportunity set for all the quarters in the sample, I obtain a line with positive slope that is also shown on the graph. The efficient part of all the frontiers is tangent to this line at some point, as can be checked for the 6 frontiers that I have represented on the graph.[7] I compare the actual mean-standard deviation frontiers with this envelope on Figure 8b. Many of these frontiers are to the left of the envelope. This implies that lenders have been able to obtain higher expected returns than in the competitive setting, probably because of their market power. However, when the risks of the loss distribution increased as quickly as in 2008, they were not able to increase the spreads accordingly at the same speed, as illustrated by the fact that the 2008.Q4 frontier is to the right of the envelope. In consequence, the movements of the mean-variance frontier for small changes in borrowers' risks do not seem to follow risk-return trade-off considerations. Nevertheless, the position of the frontier on 1993.Q4 shows that banks are eventually able to raise spreads to compensate for increases in credit risk when the situation becomes extremely uncertain.

[7] I have only plotted 6 periods to avoid the cluttering of the pictures. The envelop summarises the results for all periods but the frontiers are also available on request.

7 Conclusions

This paper provides an analytical model to compare the risk and return of loans portfolios in a joint framework. I propose the use of mean-variance asset allocation techniques to carry out this analysis, allowing for a Value at Risk constraint in order to take into account regulatory requirements. The behaviour of the return yielded by loans portfolios is described by means of a flexible albeit easily implementable model, which is driven by a vector of Gaussian state variables. In this sense, I obtain closed form expressions for the first two moments of the distribution of returns. I interpret these formulas and show that the variance of returns can be decomposed in diversifiable and non-diversifiable risk, where the diversifiable component can be eliminated by increasing portfolio granularity. In addition, I illustrate how this model can capture default correlations between different loans. These correlations are introduced by means of the state variables. In this sense, the correlations of the underlying Gaussian distribution can be approximately interpreted as the default correlations under infinite granularity.

I develop an asset pricing model as a benchmark against which I can compare the actual data. Specifically, I obtain closed form expressions under absence of arbitrage, in which spreads are increasing functions of the risk-adjusted probabilities of default and decreasing functions of expected recovery rates. In addition, they depend negatively on the covariance between recovery rates and default probabilities. If I assume an exponentially affine stochastic discount factor, I can show that the distribution of the state variables under the risk-neutral measure is another Gaussian distribution with the same covariance matrix but different means.

Finally, I consider an empirical application in which I study the loans portfolio of the Spanish banking system. Mortgages are the safest category of loans, although they yield smaller returns than consumption loans. In contrast, corporate loans are generally riskier and more sensitive to recessions. My model also reflects how the investment opportunity set on the mean-variance space has quickly moved to riskier values on the 2008 crisis. When I compare the actual mean-standard deviation frontiers with those based on the closest arbitrage-free setting, I find that banks are able to move the mean-standard deviation frontier to regions of higher expected returns in good times. This may probably be due to their market power. However, I can still observe a risk-return trade-off when I compare periods with very different standard deviations in the loans portfolios, such as the peak and the trough of the last credit cycle.

An interesting avenue for future research would be to explore the effect of macroeconomic variables on the mean-variance frontier, extending the idea of Jiménez and Mencía (2009) into an analysis of risk and return. It would also be interesting to consider the economic sectors of corporate loans to take into account any potential heterogeneity within this group. Lastly, it would be helpful to study the risk-return trade-off at multiperiod horizons.

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A Auxiliary results

Proposition 5 *Let $\Phi(\cdot)$ be the cdf of the standard normal distribution. Then, if $z \sim N(0,1)$,*

$$E\left[\Phi\left(a+bz\right)\right]=\Phi\left(\frac{a}{\sqrt{1+b^2}}\right). \tag{A1}$$

Proposition 6 *Let $\Phi(\cdot)$ be the cdf of the standard normal distribution. Then, if f, z_1, z_2 are independent standard normal variables,*

$$E\left[\prod_{i=1}^{2}\Phi\left(a_i+b_if+c_iz_i\right)\right]=\Phi_2\left(\frac{a_1}{\sqrt{1+b_1^2+c_1^2}},\frac{a_2}{\sqrt{1+b_2^2+c_2^2}};\prod_{i=1}^{2}\frac{b_i}{\sqrt{1+b_i^2+c_i^2}}\right) \tag{A2}$$

where $\Phi_2\left(p_1,p_2|\,\rho\right)$ is the cdf of a bivariate normal distribution with zero means, unit variances and correlation ρ.

Proposition 7 *Let $\Phi(\cdot)$ be the cdf of the standard normal distribution. Then, if $x \sim N(\mu,\sigma^2)$,*

$$E[\exp(ax)\Phi(x)]=\exp\left(\frac{1}{2}a^2\sigma^2+\mu a\right)\Phi\left(\frac{\mu+a\sigma^2}{\sqrt{1+\sigma^2}}\right) \tag{A3}$$

B Proofs of propositions

Proposition 1

Let us first consider (7). Since $\mathbf{x}_t$ satisfies (6), I can always express π_{kt} as

$$\pi_{kt}=\Phi(\mu_{kt}+\sqrt{\sigma_{kkt}}\varepsilon_{kt}), \tag{B4}$$

where ε_{kt} is a standard normal variable. Then, I can easily obtain (7) from Proposition 5.

As for (8), I can use the properties of the Gaussian distribution to express π_{jt} as

$$\pi_{jt}=\Phi\left[\mu_{jt}+\frac{\sigma_{kjt}}{\sqrt{\sigma_{kkt}}}\varepsilon_{kt}+\sqrt{\frac{\sigma_{kkt}\sigma_{jjt}-\sigma_{kjt}^2}{\sigma_{kkt}}}\varepsilon_{jt}\right], \tag{B5}$$

where ε_{jt} is independent of ε_{kt}. Hence, from Proposition 6, I can show that the expected value of the product of (B4) and (B5) satisfies the required result.

Proposition 2

As usual, I can express $cor(D_{kit}, D_{jit}|I_{t-1})$ as

$$\begin{aligned} cor(D_{kit}, D_{jit}|I_{t-1}) &= \frac{E(D_{kit}D_{jit}|I_{t-1}) - E(D_{kit}|I_{t-1})E(D_{jit}|I_{t-1})}{\sqrt{[E(D_{kit}^2|I_{t-1}) - E^2(D_{kit}|I_{t-1})][E(D_{jit}^2|I_{t-1}) - E^2(D_{jit}|I_{t-1})]}} \\ &= \frac{E(D_{kit}D_{jit}|I_{t-1}) - \pi_{kt|t-1}\pi_{jt|t-1}}{\sqrt{(\pi_{kt|t-1} - \pi_{kt|t-1}^2)(\pi_{jt|t-1} - \pi_{jt|t-1}^2)}} \end{aligned}$$

I can easily compute the remaining cross moment by exploiting the conditional independence D_{kit} and D_{jit} given π_{kt} and π_{jt}:

$$\begin{aligned} E(D_{kit}D_{jit}|I_{t-1}) &= E[E(D_{kit}|\pi_{kt}, \pi_{jt}, I_{t-1})E(D_{jit}|\pi_{kt}, \pi_{jt}, I_{t-1})|I_{t-1}], \\ &= E[\pi_{kt}\pi_{jt}|I_{t-1}] = \omega_{kjt|t-1}. \end{aligned}$$

Proposition 3

By the law of iterated expectations, I have that

$$V[y_{kt}|I_{t-1};\boldsymbol{\theta}] = E\left[V[y_{kt}|\pi_{kt}, I_{t-1};\boldsymbol{\theta}]|I_{t-1};\boldsymbol{\theta}\right] + V\left[E[y_{kt}|\pi_{kt}, I_{t-1};\boldsymbol{\theta}]|I_{t-1};\boldsymbol{\theta}\right], \tag{B6}$$

where

$$\begin{aligned} E[y_{kt}|\pi_{kt}, I_{t-1};\boldsymbol{\theta}] &= r_{kt} - (1 + r_{kt} - \delta_{k0})\pi_{kt}, \\ V[y_{kt}|\pi_{kt}, I_{t-1};\boldsymbol{\theta}] &= (1/g_{kt})\left[E\left[(1 + r_{kt} - \delta_{ikt})^2\right]\pi_{kt} - (1 + r_{kt} - \delta_{k0})^2\pi_{kt}^2\right]. \end{aligned}$$

Hence, (15) follows directly from introducing the results of Proposition 1 in (B6). Similarly, I can also exploit the law of iterated expectations to express $cov[y_{kt}, y_{jt}|I_{t-1};\boldsymbol{\theta}]$ as

$$\begin{aligned} cov[y_{kt}, y_{jt}|I_{t-1};\boldsymbol{\theta}] &= E\left[cov[y_{kt}, y_{jt}|\pi_{kt}, \pi_{jt}, I_{t-1};\boldsymbol{\theta}]|I_{t-1};\boldsymbol{\theta}\right] \\ &\quad + cov\left[E[y_{kt}|\pi_{kt}, \pi_{jt}, I_{t-1};\boldsymbol{\theta}], E[y_{jt}|\pi_{kt}, \pi_{jt}, I_{t-1};\boldsymbol{\theta}]|I_{t-1};\boldsymbol{\theta}\right], \end{aligned}$$

where

$$cov[y_{kt}, y_{jt}|\pi_{kt}, \pi_{jt}, I_{t-1};\boldsymbol{\theta}] = 0$$

due to the conditional independence property of the model, and

$$\begin{aligned} cov\left[E[y_{kt}|\pi_{kt}, \pi_{jt}, I_{t-1};\boldsymbol{\theta}], E[y_{jt}|\pi_{kt}, \pi_{jt}, I_{t-1};\boldsymbol{\theta}]|I_{t-1};\boldsymbol{\theta}\right] &= (1 + r_{kt} - \delta_{k0})(1 + r_{jt} - \delta_{j0}) \\ &\quad \times cov(\pi_{kt}, \pi_{jt}|I_{t-1};\boldsymbol{\theta}), \end{aligned}$$

which yields (16).

Proposition 4

The risk-neutral probability density function of $\mathbf{x}_t$ can be expressed as

$$f^{\mathbb{Q}}(\mathbf{x}_t) = (1 + r_t)M_{t-1,t}f(\mathbf{x}_t), \tag{B7}$$

where $f(\mathbf{x}_t)$ is the density of $\mathbf{x}_t$ under the actual measure:

$$f(\mathbf{x}_t) = \exp[-(1/2)(\mathbf{x}_t - \boldsymbol{\mu}_t(\boldsymbol{\theta}))'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})(\mathbf{x}_t - \boldsymbol{\mu}_t(\boldsymbol{\theta}))]/\sqrt{2\pi|\boldsymbol{\Sigma}_t(\boldsymbol{\theta})|}. \tag{B8}$$

If I introduce (22) and (B8) in (B7), it is straightforward to show the required result when I substitute ν_{0t} for

$$\nu_{0t} = \frac{\exp\left[-\boldsymbol{\nu}'_{1t}\boldsymbol{\mu}_t(\boldsymbol{\theta}) - (1/2)\boldsymbol{\nu}'_{1t}\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\boldsymbol{\nu}_{1t}\right]}{1 + r_t},$$

which ensures that (19) is satisfied for the risk-free asset.

Proposition 5

I can rewrite (A1) as

$$\int_{-\infty}^{\infty} \Phi\left(a + bz\right)\phi(z)dz = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{a+bz}\phi\left(s\right)ds\right]\phi(z)dz,$$

where $\phi(\cdot)$ is the pdf of the standard normal distribution. By means of the change of variable $t = s - bz$, I can obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{a+bz}\phi\left(s\right)\phi(x)dsdx = \int_{-\infty}^{\infty}\int_{-\infty}^{a}\phi\left(t + bz\right)\phi(z)dtdz.$$

It is straightforward to show that $\phi\left(t + bz\right)\phi(z)$ is the pdf of the bivariate normal distribution:

$$\left[\begin{array}{c} z \\ t \end{array}\right] \sim N\left[\left(\begin{array}{c} 0 \\ 0 \end{array}\right);\left(\begin{array}{cc} 1 & -b \\ -b & 1 + b^2 \end{array}\right)\right].$$

This proves the result.

Proposition 6

By the law of iterated expectations, I can write

$$E\left[\prod_{i=1}^{2}\Phi\left(a_i + b_i f + c_i z_i\right)\right] = E\left[\prod_{i=1}^{2}E\left[\Phi\left(a_i + b_i f + c_i z_i\right)\right| f\right]\right]. \tag{B9}$$

Using Proposition 5, I can express (B9) as

$$\begin{aligned} E\left[\prod_{i=1}^{2} E\left[\Phi\left(a_i+b_i f+c_i z_i\right)\mid f\right]\right] &= E\left[\prod_{i=1}^{2}\Phi\left(\frac{a_i+b_i f}{\sqrt{1+c_i^2}}\right)\right] \\ &= \int_{-\infty}^{\infty}\prod_{i=1}^{2}\left[\int_{-\infty}^{\frac{a_i+b_i f}{\sqrt{1+c_i^2}}}\phi\left(s_i\right)ds_i\right]\phi(f)df \qquad \text{(B10)} \end{aligned}$$

By means of the changes of variables $t_i = s_i\sqrt{1+c_i^2} - b_i f$, for $i = 1, 2$, I rewrite (B10) as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{a_1}\int_{-\infty}^{a_2}\phi\left(\frac{t_1+b_1 f}{\sqrt{1+c_1^2}}\right)\phi\left(\frac{t_2+b_2 f}{\sqrt{1+c_2^2}}\right)\phi(f)dt_1 dt_2 df$$

It can be shown that this is the integral of the pdf of the following trivariate distribution:

$$\begin{bmatrix} f \\ t_1 \\ t_2 \end{bmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};\begin{pmatrix} 1 & -b_1 & -b_2 \\ -b_1 & 1+b_1^2+c_1^2 & b_1 b_2 \\ -b_2 & b_1 b_2 & 1+b_2^2+c_2^2 \end{pmatrix}\right].$$

Proposition 7

I can rewrite (A3) as

$$\begin{aligned} E[\exp(ax)\Phi(x)] &= \int \exp(ax)\Phi(x)\frac{\exp[-(x-\mu)^2/(2\sigma^2)]}{\sqrt{2\pi\sigma^2}}dx \\ &= \exp\left(\frac{1}{2}a^2\sigma^2+\mu a\right)\int\Phi(x)\frac{\exp[-(x-(\mu+a\sigma^2))^2/(2\sigma^2)]}{\sqrt{2\pi\sigma^2}}dx. \end{aligned}$$

If I consider the change of variable $y = (x-(\mu+a\sigma^2))/\sigma$, I can obtain

$$\begin{aligned} E[\exp(ax)\Phi(x)] &= \exp\left(\frac{1}{2}a^2\sigma^2+\mu a\right)\int\Phi(\mu+a\sigma^2+\sigma y)\phi(y)dy \\ &= \exp\left(\frac{1}{2}a^2\sigma^2+\mu a\right)E[\Phi(\mu+a\sigma^2+\sigma y)]. \end{aligned}$$

Then, if I use Proposition 5 I can obtain the required result.

C Value at Risk constraint

For each possible variance σ_0^2, I estimate the points on the mean-variance space for which (12) holds with equality by solving the following problem:

$$\min_{\boldsymbol{\omega}_t}\boldsymbol{\omega}_t' E[\mathbf{y}_t|I_{t-1};\boldsymbol{\theta}]$$

such that

$$
\begin{aligned}
\boldsymbol{\omega}_t' V[\mathbf{y}_t | I_{t-1}; \boldsymbol{\theta}] \boldsymbol{\omega}_t &= \sigma_0^2, \\
\Pr[\boldsymbol{\omega}_t' \mathbf{y}_t > \tau_0 | I_{t-1}; \boldsymbol{\theta}] &\geq 1 - \alpha,
\end{aligned}
$$

where $\Pr[\boldsymbol{\omega}_t' \mathbf{y}_t > \tau_0 | I_{t-1}; \boldsymbol{\theta}]$ is estimated for each $\boldsymbol{\omega}_t$ by computing the proportion of times that M replications of the DGP of $\boldsymbol{\omega}_t' \mathbf{y}_t$ given I_{t-1} is above τ_0. I use $M = 100000$ in the empirical application.

Table 1

Model of the probabilities of default

Sample 1984.Q4-2008.Q4

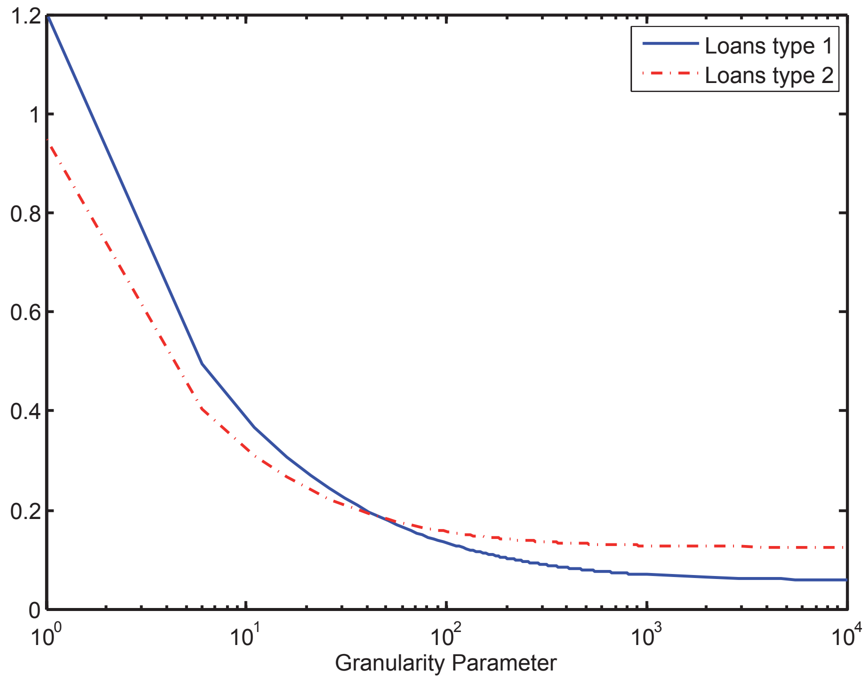
Parameter	Estimate	S.E.
$\boldsymbol{\mu}$		
Corporate	-2.309	0.046**
Consumption	-2.070	0.035**
Mortgages	-3.050	0.051**
$\boldsymbol{\alpha}$		
Corporate	0.993	0.018**
Consumption	0.997	0.007**
Mortgages	0.990	0.012**
$\boldsymbol{\beta}$		
Corporate	0.040	0.005**
Consumption	0.029	0.004**
Mortgages	0.027	0.006**
$\boldsymbol{\gamma}$		
Corporate	0.001	0.000**
Consumption	0.001	0.000**
Mortgages	0.002	0.000**
φ	0.464	0.116**

Model:

$$\begin{aligned} \mathbf{x}_t &= [\mathbf{I}_3 - diag(\boldsymbol{\alpha})]\boldsymbol{\mu} + diag(\boldsymbol{\alpha})\mathbf{x}_{t-1} + \boldsymbol{\beta} f_t + diag(\boldsymbol{\gamma})^{1/2}\boldsymbol{\varepsilon}_t \\ f_t &= \varphi f_{t-1} + \sqrt{1-\varphi^2}\nu_t \end{aligned}$$

Notes: Two asterisks indicate significance at the 5% level. $\mathbf{x}_t$ are probit transformed default frequencies for the whole Spanish banking system, where $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \mathbf{I}_3)$ and $\nu_t \sim N(0,1)$ are *iid* independent variables.

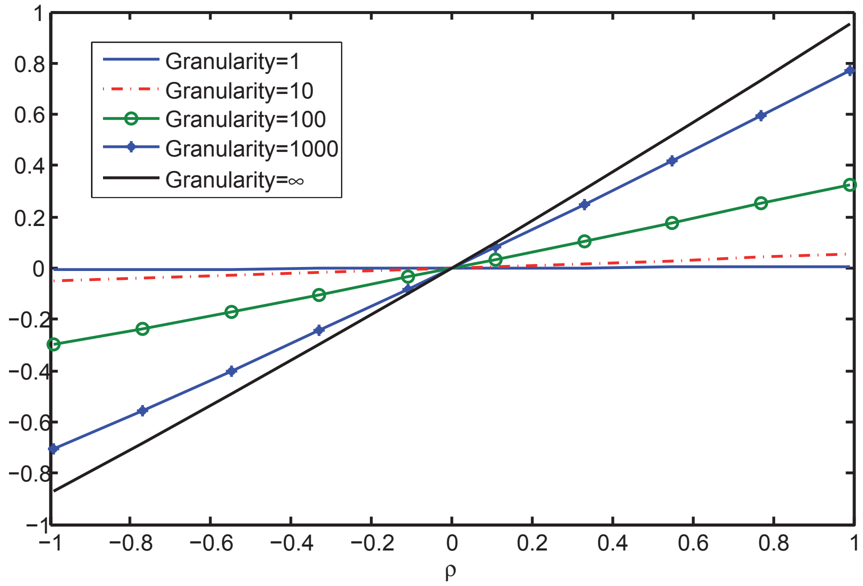
Figure 1: Standard deviations of two illustrative portfolios



Notes: Example based on two types of loans whose defaults are conditionally independent given the evolution of their respective probabilities of default $\pi_{1t} = \Phi(x_{1t})$ and $\pi_{2t} = \Phi(x_{2t})$, where

$$\mathbf{x}_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \sim N\left[\begin{pmatrix} \Phi^{-1}(0.05)\sqrt{1.01} \\ \Phi^{-1}(0.025)\sqrt{1.1} \end{pmatrix}; \begin{pmatrix} 0.01 & \rho\sqrt{0.001} \\ \rho\sqrt{0.001} & 0.1 \end{pmatrix}\right].$$

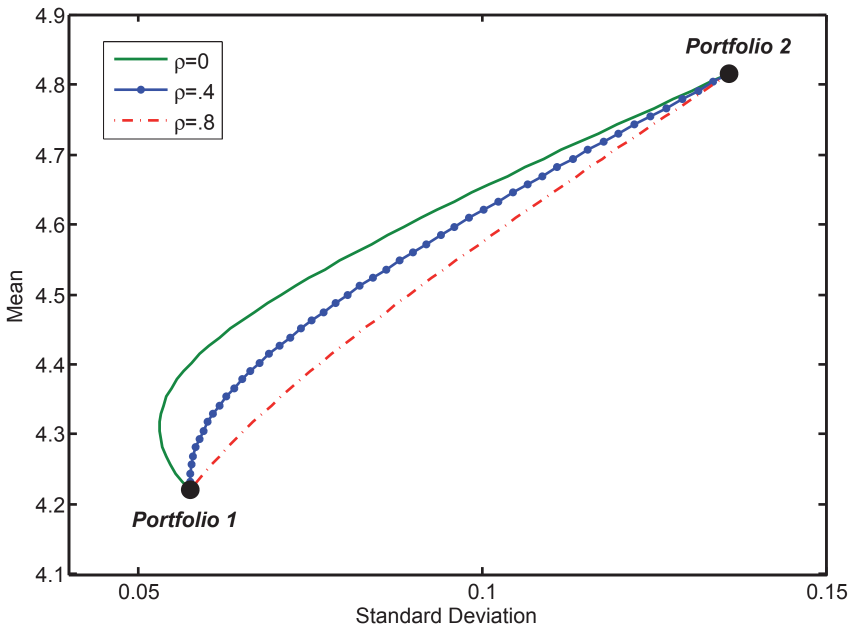
Figure 2: Correlation between two illustrative portfolios



Notes: Example based on two types of loans whose defaults are conditionally independent given the evolution of their respective probabilities of default $\pi_{1t} = \Phi(x_{1t})$ and $\pi_{2t} = \Phi(x_{2t})$, where

$$\mathbf{x}_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \sim N\left[\begin{pmatrix} \Phi^{-1}(0.05)\sqrt{1.01} \\ \Phi^{-1}(0.025)\sqrt{1.1} \end{pmatrix}; \begin{pmatrix} 0.01 & \rho\sqrt{0.001} \\ \rho\sqrt{0.001} & 0.1 \end{pmatrix}\right].$$

Figure 3: Mean-standard deviation frontier for two illustrative portfolios



Notes: Infinite granularity. Two types of loans whose defaults are conditionally independent given the evolution of their respective probabilities of default $\pi_{1t} = \Phi(x_{1t})$ and $\pi_{2t} = \Phi(x_{2t})$, where

$$\mathbf{x}_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} \sim N\left[\begin{pmatrix} \Phi^{-1}(0.05)\sqrt{1.01} \\ \Phi^{-1}(0.025)\sqrt{1.1} \end{pmatrix}; \begin{pmatrix} 0.01 & \rho\sqrt{0.001} \\ \rho\sqrt{0.001} & 0.1 \end{pmatrix}\right].$$

Interest rates are $r_1 = 4.5\%$ and $r_2 = 5\%$ for the first and second types of loans, respectively.

Figure 4a: Historical evolution of default frequencies (%)

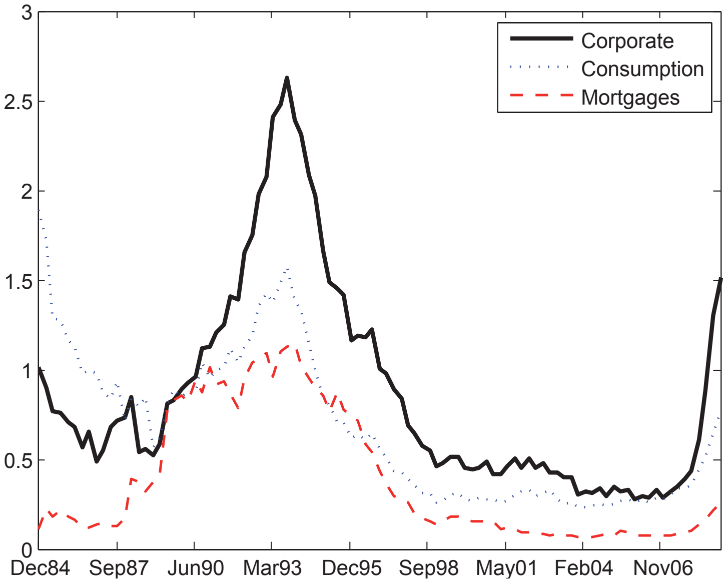


Figure 4b: Historical evolution of interest rates (%)

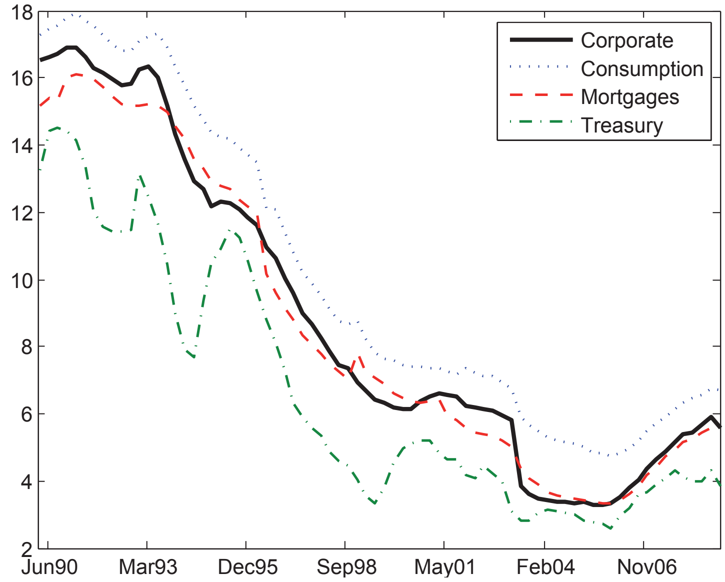
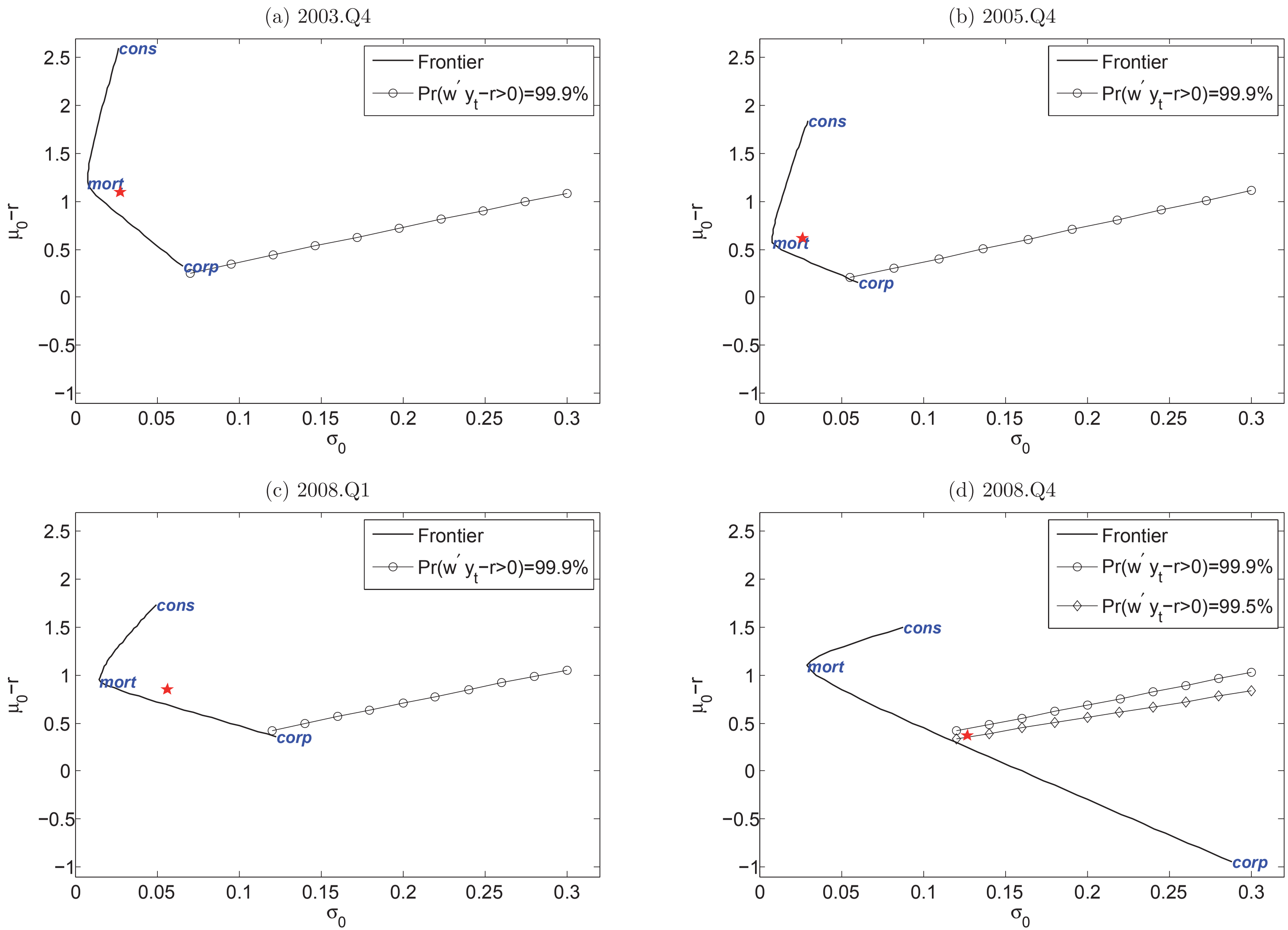


Figure 5: Mean-standard deviation frontier with a Value at Risk constraint



Notes: Excess returns with respect to the Spanish treasury rates. A star denotes the situation of the Spanish aggregate loans portfolio at the four periods. “corp”, “cons” and “mort” denote corporate loans, consumption loans and mortgages, respectively. 100,000 simulations have been employed to estimate the the border of the area that yields positive returns with 99.9% probability.

Figure 6a: Risk vs. expected returns (%)

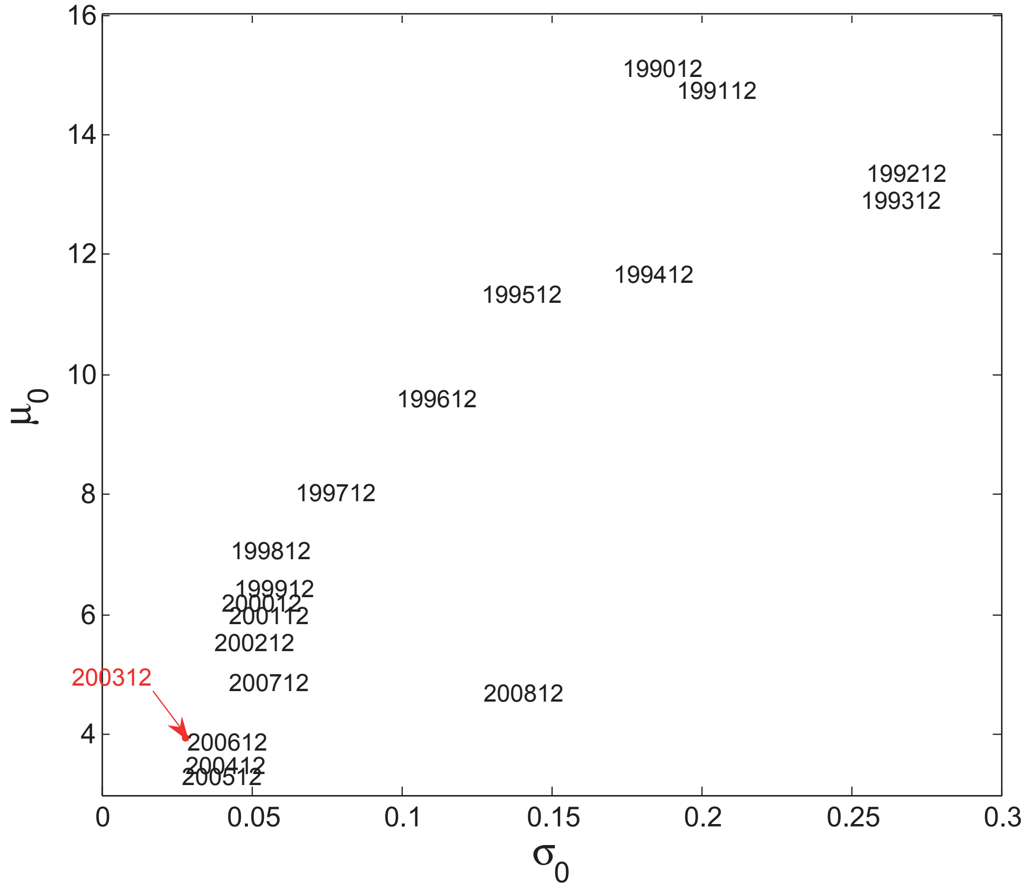
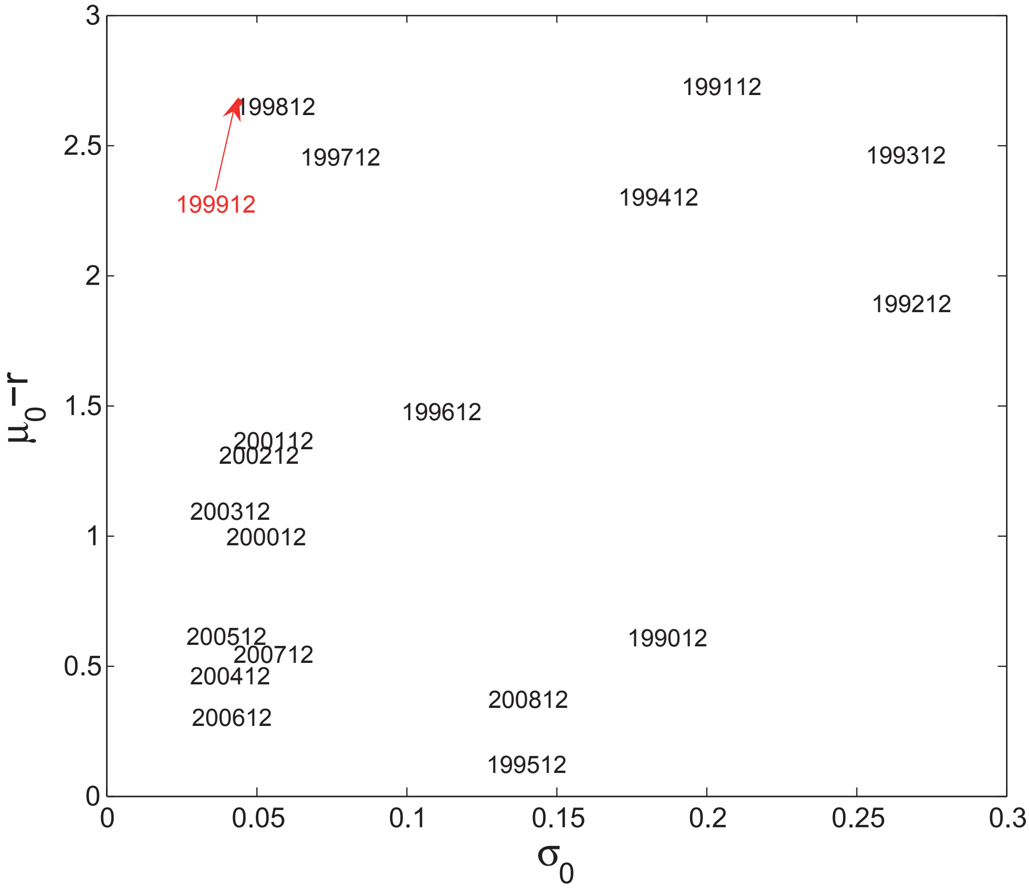
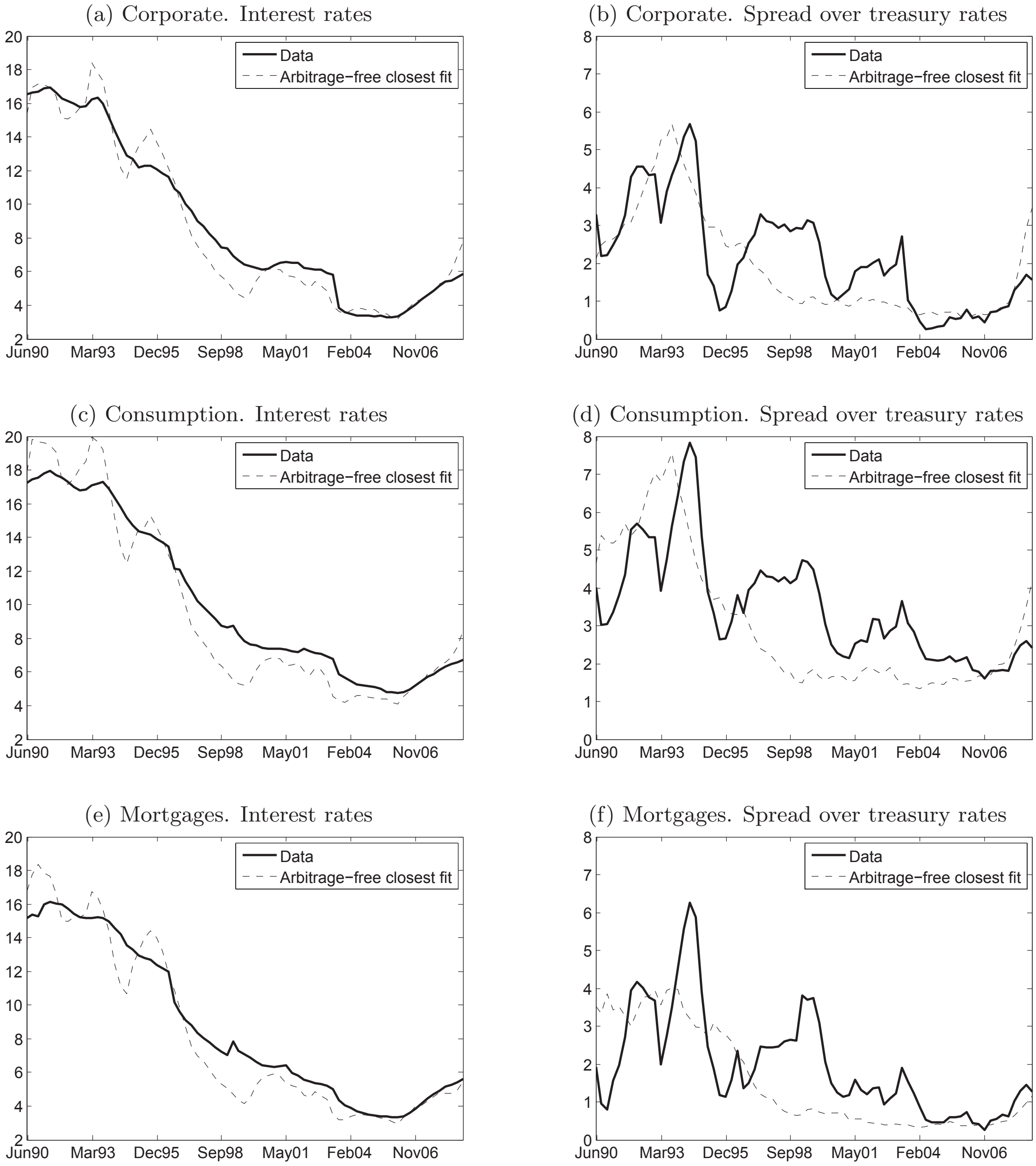


Figure 6b: Risk vs. expected excess returns (%)



Note: for the sake of clarity, only the fourth quarter of each year is plotted in the figures.

Figure 7: Historical interest rates and fit provided by the arbitrage-free pricing model



Notes: The arbitrage-free rates have been obtained from an exponentially affine stochastic discount factor whose parameters minimise the sum of square errors between the model based rates and the actual ones.

Figure 8a: Mean-standard deviation frontiers implied by the pricing model

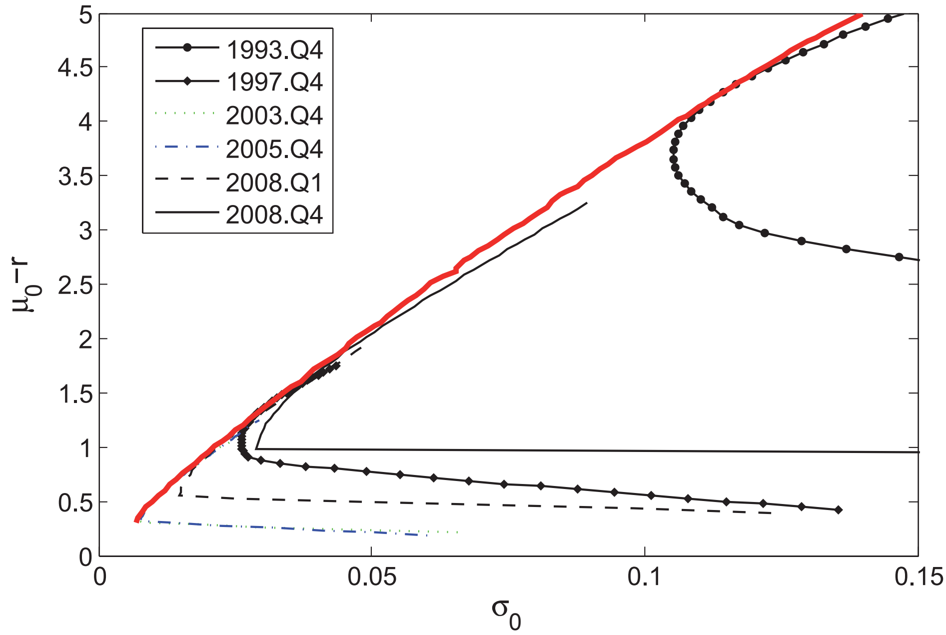
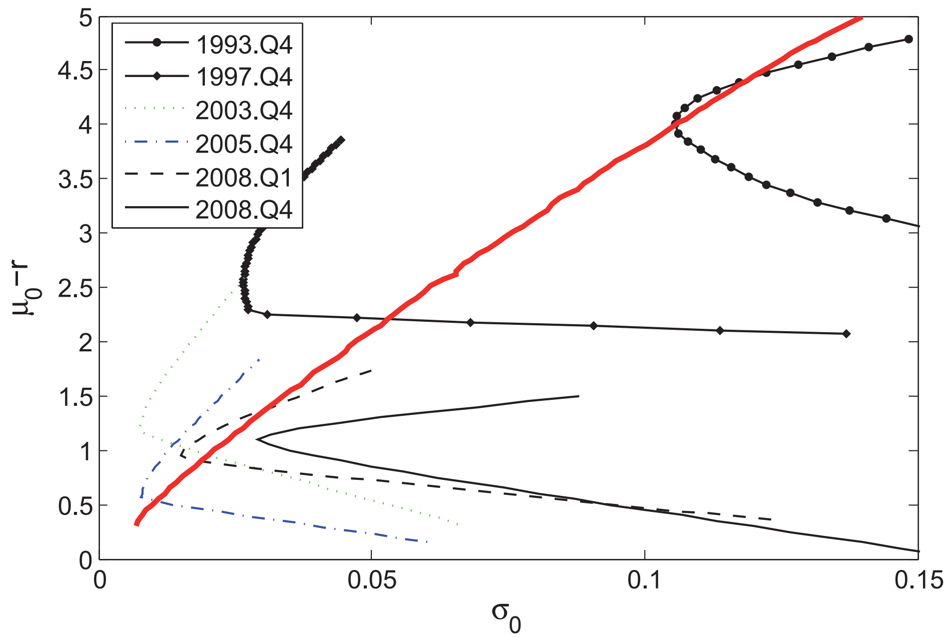


Figure 8b: Actual mean-standard deviation frontiers



Notes: The solid line indicates the envelope of all the quarterly investment opportunity sets in mean-standard deviation space implied by the pricing model.

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