A COMPLETE MODEL-BASED INTERPRETATION OF THE HODRICK-PRESCOTT FILTER: SPURIOUSNESS RECONSIDERED

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Abstract

The Hodrick-Prescott filter applied to seasonally adjusted series has become a paradigm for business-cycle estimation at many economic agencies and institutions. We show that the filter can be obtained from MMSE estimation of the components in an unobserved component model, where the original series is decomposed into (long-term) trend, cyclical, seasonal, and (highly-transitory) irregular components. The component models are sensible and combine desirable "ad-hoc" features with series-dependent features that guarantee consistency with the data. The model-based framework provides improvements having to do with the precision of end-point estimation and the stability of the cyclical signal.

Keywords: Business-cycle estimation; Stochastic cycles and trends; Unobserved Component Models; ARIMA models.

1. Introduction

Since originally proposed by Hodrick and Prescott (1980) in the context of business-cycle estimation, the Hodrick-Prescott (HP) filter has been the subject of considerable discussion (see, for example, King and Rebelo (1993) and Kaiser and Maravall (2001),) praise (see, for example, Kydland and Prescott (1982) and Prescott (1986),) and criticism (see, for example, Cogley and Nason (1995) and Harvey (1997).) Because it offers a simple and visually appealing solution to a very basic need of economic policy and monitoring, the HP filter has become the most widely used procedure to estimate business-cycles in applied work, including the one performed at important economic institutions (see, for example, International Monetary Fund (1993), Giorno et al. (1995) for the OECD, European Commission (1995), and European Central Bank (2000).)

The HP filter is a linear filter aimed at removing low frequency variation from a series. Originally presented as the solution to a standard "penalty function"–type problem, where a parameter λ (to be denoted the HP parameter), balances the trade-off between lack of smoothness and poor fit of the trend, it turns out to be a particular case of the Butterworth family of filters, popular in electrical engineering (see, for example, Gómez (1999), or Kaiser and Maravall (1999).) King and Rebelo (1993) show that the HP filter can also be obtained as the minimum mean square error (MMSE) estimator of the noise in a standard unobserved component (UC) model formulation. Letting B denote the backward operator, such that $B^j x_t = x_{t-j}$, ∇ denote the differencing operator (= 1 – B), such that $\nabla x_t = x_t - x_{t-1}$, and writing "w.n. (0, V)" to denote a zero-mean normally-identically-independently-distributed (white-noise) variable with variance V, the series is seen as the sum of a trend component, m_t, and a cycle component, c_t, with m_t given by the second-order random walk model $\nabla^2 m_t = a_{mt}$, $a_{mt} \sim \text{w.n.}(0, V_m)$, and the cycle being a w.n. $(0, V_c)$ variable, with a_{mt} and c_t uncorrelated. Estimates of the cycle can then be obtained with the Kalman filter; see Harvey and Jaeger (1993). Noticing that the previous trend and cycle specifications imply an IMA (2,2) model for the series, Kaiser and Maravall (2001) show how the HP filter can also be obtained as a

Wiener-Kolmogorov (WK) filter and how this approach offers a natural tool for analysis and eventual improvement of the filter. Gómez (1999) shows how these alternatives derivations of the filter yield identical results.

Given that seasonality should not contaminate the cyclical signal, when the frequency of observation of the series is higher than annual, in the vast majority of cases the filter is applied to previously seasonally adjusted (SA) series. In practice, the present paradigm for business-cycle estimation is given by the convolution of the HP and the X11 filters. This paradigm, however, presents several weaknesses; in particular,

(a) the end-point estimation is unstable;

(b) the cyclical signal may display considerable erraticity;

(c) as characterizes ad-hoc filters, it may be inadequate for certain series, raising the possibility of generating spurious results.

A natural way to reduce end-point unstability is by extending the series with ARIMA forecasts (see European Commission, (1995)), yet, as pointed out by Apel et al. (1996), uncertainty about how many forecasts are needed and the fact that the results are often sensitive to the number of forecasts limit the usefulness of the extension. The limitation, however, disappears if the extension of the series is properly applied: as Kaiser and Maravall (2001) show, using the WK procedure, only four forecasts are needed to reproduce exactly the effect of the infinite forecast extension. Further, because volatility of the cyclical signal is a result of the presence of noise in the SA series, the cyclical signal becomes smoother and more stable when the SA series is replaced by a trend-cycle component that also removes transitory noise of no cyclical interest (or, in other words, by a "noise-free" SA series). WK enforcement of the HP filter with proper forecasts extension, that uses as input the estimator of the trend-cycle is referred to by Kaiser and Maravall (1999) as the Modified Hodrick-Prescott (MHP) filter. Gómez and Bengoechea (2000) illustrate how the modified procedure clearly outperforms the method used until now at the Spanish Statistical Institute.

This paper follows the approach in Kaiser and Maravall (2001) and in Gómez (2001). We show that, for a series that follows a general ARIMA model, the complete procedure of seasonal adjustment, noise removal, and MHP estimation of the trend and cycle is identical to the problem of MMSE estimation of the trend, cycle, seasonal, and noise (or irregular) component in a complete UC model, where the components follow sensible ARIMA-type models. These models incorporate series-dependent features as well as desirable ad-hoc features of the HP filter, and aggregate into the ARIMA model identified for the observed series. In so far as the aggregate ARIMA model and the UC model are observationally equivalent, if the former is appropriate, so will the later. One may question the identification restrictions, but the results of the MHP filter cannot be properly called "spurious".

2. Wiener-Kolmogorov and Hodrick-Prescott filters

Consider a series x_t that follows the general invertible ARIMA model

$$\phi(B)\ x_t = \theta(B)\ a_t \quad , \qquad a_t \ \sim \ \text{w.n}\ (0, V_a) \quad , \qquad (2.1)$$

where $\phi(B)$ is a polynomial in *B* that includes all stationary and nonstationary roots, and $\theta(B)$ is an invertible MA polynomial. Assume we wish to estimate some unobserved component (or "signal") in x_t, say s_t, the outcome of the model

$$\phi_s(B)\ s_t = \theta_s(B)\ a_{st} \quad , \qquad a_{st} \ \sim \ \text{w.n.}\ (0, V_s)\ . \qquad (2.2)$$

We consider the additive decomposition

$$x_t = s_t + n_t \ , \qquad (2.3)$$

Where s_t and n_t are orthogonal components, with n_t denoting the "non-signal" part of x_t. Although only needed for unit roots, to simplify notation, we assume that the components AR polynomials share no root in common. We can then factorize $\phi(B)$ as

$$\phi(B) = \phi_s(B)\ \phi_n(B) \quad , \qquad (2.4)$$

where $\phi_n(B)$ denotes the polynomial in *B* made of the roots of $\phi(B)$ that are not in $\phi_s(B)$. Let $F\ (= B^{-1})$ denote the forward operator, such that

$F^j\ x_t = x_{t+j}$; for the infinite realization of $x_t : [x_{-\infty}, \dots, x_{\infty}]$, the MMSE estimator of s_t is given by the WK filter

$$\hat{s}_t = \nu_s (B, F)\ x_t \quad , \qquad (2.5)$$

where, as shown in Maravall (1995), $\nu (B, F)$ is given by the autocovariance generating function (ACF) of the ARIMA model

$$\theta (B)\ z_t = [\theta_s (B)\ \phi_n (B)]\ b_t \ , \qquad b_t \sim \text{w.n. } (0, V_s / V_a) ; \qquad (2.6)$$

thus

$$\nu_s (B, F) = \frac{V_s}{V_a} \frac{\theta_s (B)\ \phi_n (B)}{\theta (B)} \frac{\theta_s (F)\ \phi_n (F)}{\theta (F)} \ . \qquad (2.7)$$

The filter is centered, symmetric and, given that model (2.1) is invertible, it will converge in B and F. For a finite realization, expression (2.5) can still be applied replacing x_t with the observed series extended at both ends with forecasts and backcasts (Cleveland and Tiao, 1976). Moreover, application of the Burman-Wilson algorithm (Burman, 1980) reduces the number of needed forecasts and backcasts to a relatively small number.

Next, consider the King - Rebelo unobserved component interpretation of the HP filter. The trend, m_t , follows the model

$$\nabla^2\ m_t = a_{mt} \ , \qquad a_{mt} \sim \text{w.n. } (0, V_m) \ , \qquad (2.8)$$

and the cycle c_t is w.n. $(0, V_c)$, orthogonal to a_{mt} . The HP- parameter λ is equal to the ratio V_c / V_m , and the series is decomposed as in $x_t = m_t + c_t$. It follows that $\nabla^2\ x_t = a_{mt} + \nabla^2\ c_t$, and hence the series x_t can be expressed as an IMA (2.2) model, say

$$\nabla^2\ x_t = \theta_{HP} (B)\ a_t^{HP} =$$
$$= \left(1 + \theta_1^{HP}\ B + \theta_2^{HP}\ B^2\right) a_t^{HP} \ , \qquad a_t^{HP} \sim \text{w.n. } (0, V_{HP}) \ , \qquad (2.9)$$

where the following identity holds:

$$\theta_{HP}(B)\ a_t^{HP} = a_{mt} + \nabla^2 c_t \quad . \tag{2.10}$$

If the notation "I(d)" denotes a series with the factor ∇^d present in its AR representation (or, equivalently, with d unit AR roots at the zero frequency), in the above model formulation, the series x_t and the trend m_t are I(2) processes.

Letting $k_c = V_c / V_{HP}$ and $k_m = V_m / V_{HP}$, and equating the ACF of the r.h.s. and the l.h.s. of (2.10), it is obtained that

$$\theta_{HP}(B)\ \theta_{HP}(F) = k_m + (1-B)^2\ (1-F)^2\ k_c \quad , \tag{2.11}$$

from which the parameters θ_1^{HP}, θ_2^{HP}, and V_{HP} can be obtained. Because the right-hand-side (r.h.s.) of (2.10) is the sum of two orthogonal components, one of them white noise, $\nabla^2 x_t$ will have a strictly positive spectral minimum, which implies that $\theta_{HP}(B)\ a_t$ is an invertible process, so that $(\theta_{HP}(B))^{-1}$ converges. The MMSE estimators of m_t and c_t can now be obtained with the WK filter (2.7), by setting $s_t = c_t$ and $n_t = m_t$. This yields

$$\nu_c(B,F) = k_c\ \frac{(1-B)^2\ (1-F)^2}{\theta_{HP}(B)\ \theta_{HP}(F)}\ , \tag{2.12a}$$

$$\nu_m(B,F) = k_m\ \frac{1}{\theta_{HP}(B)\ \theta_{HP}(F)}\ . \tag{2.12b}$$

It is straightforward to verify that $\nu_c(B,F) + \nu_m(B,F) = 1$, so that $x_t = \hat{m}_t + \hat{c}_t$.

The King-Rebelo model-based interpretation may provide a useful algorithm, but the series will not follow in general model (2.9), nor will the cycle be white noise. As seen in Kaiser and Maravall (2001), the poor end-point performance of the HP-filter is due to the fact that the standard finite-sample implementation of the filter is equivalent to using model (2.9) to compute the forecasts and backcasts of x_t, needed to apply the filter at both ends of the series. The MHP filter replaces these misspecified forecasts and backcasts with the ones obtained through the appropriate ARIMA model.

3. Estimation of the cycle

Assume that the series follows the general ARIMA model (2.1), rewritten as

$$\phi(B)\ \nabla^{d}\ \nabla_{s}^{d_s}\ x_t = \theta(B)\, a_t \quad ; \qquad a_t \sim \text{w.n. } (0, V_a)\ , \qquad (3.1)$$

where s denotes the number of observations per year, ∇ and ∇_s denote the regular and seasonal differencing, d and d_s are nonnegative integers (in practice, d = 0, 1, 2, $d_s = 0, 1$), $\phi(B)$ is a stationary autoregressive polynomial in B, and $\theta(B)$ is an invertible moving average polynomial in B. Application of the HP filter to x_t requires that seasonality be removed from the series; the MHP filter requires, additionally, the removal of transitory noise. If u_t denotes the noise contained in the series, and s_t its seasonal component, we consider the decomposition of x_t into orthogonal components, as in

$$x_t = p_t + s_t + u_t \quad , \qquad (3.2)$$

where the first component p_t is the signal of interest for the posterior extraction of the cycle, namely the trend-cycle component, defined as the residual once seasonality and noise have been removed from the series. To estimate p_t we follow a model-based procedure, in the line of Burman (1980), Hillmer and Tiao (1982), Maravall (1995), and Gómez and Maravall (2001). The components will have models that will aggregate into the ARIMA model identified for the observed series. The AR polynomials of the component models are determined from the factorization of the AR polynomial of the aggregate ARIMA model according to the following rule. Let ω denote the frequency of a root expressed in radians. If $\omega \in [0, 2\pi/s)$, the root is allocated to the trend-cycle; if ω is a seasonal frequency (for example $\omega = (2\pi/s)\, j$, $j = 1, ..., 6$, for monthly series,) the root is allocated to the seasonal component; finally, when $\omega \in (2\pi/s, \pi)$ and is not a seasonal frequency, the root is allocated to the irregular component. In this way, cycles with period larger than one year will be part of the trend-cycle component, while cycles with periods shorter than a year will go to the irregular component (and will not contaminate the trend-cycle.)

Following the previous rule, the polynomial $\phi(B)$ can be factorized as $\phi(B) = \phi_p(B)\ \phi_s(B)\ \phi_u(B)$, and model (3.1) can be rewritten as

$$[(\phi_p(B)\ \nabla^D)(\phi_s(B)\ S)\ (\phi_u(B))]\ \ x_t = \theta(B)\, a_t \quad , \qquad (3.3)$$

where S is the annual aggregation operator $S = 1 + B + \ldots + B^{s-1}$, and use has been made of the identity $\nabla_s = \nabla S$. The first parenthesis groups the trend-cycle AR roots, and the second and third parenthesis group the seasonal and the irregular AR roots, respectively. Because, by construction, the components do not share AR roots in common, (3.2) implies that they will have models of the type

$$\phi_p(B)\ \nabla^D\ p_t = \theta_p(B)\, a_{pt} \quad , \qquad a_{pt} \sim \text{w.n.}(0, V_p)\,; \qquad (3.4a)$$

$$\phi_s(B)\ S^{d_s}\ s_t = \theta_s(B)\, a_{st} \quad , \qquad a_{st} \sim \text{w.n.}(0, V_s)\,; \qquad (3.4b)$$

$$\phi_u(B)\ u_t = \theta_u(B)\, a_{ut} \quad , \qquad a_{ut} \sim \text{w.n.}(0, V_u)\,; \qquad (3.4c)$$

with the variables a_{pt} , a_{st} , a_{nt} mutually uncorrelated at all lags, and $D = d + d_s$. Given that the SA series (n_t) is the sum of the trend-cycle and the irregular component, from (3.4a) and (3.4c) it is straightforward to see that the model for n_t will be of the type

$$\phi_n(B)\ \nabla^D\ n_t = \theta_n(B)\, a_{nt} \quad , \qquad (3.4d)$$

where $\phi_n(B) = \phi_p(B)\ \phi_u(B)$. Consistency between the “reduced form” model (3.1) and the “structural model” (3.4) , implied by the identity (3.2) , requires that the MA polynomials $\theta_p(B)$, $\theta_s(B)$, $\theta_n(B)$, $\theta_u(B)$, and the variances V_p, V_s, V_n, V_u , satisfy the two identities

$$\theta(B)\, a_t = \phi_s(B)\, S^{d_s}\, \phi_u(B)\, \theta_p(B)\, a_{pt} + \\ + \phi_p(B)\, \nabla^D\, \phi_u(B)\, \theta_s(B)\, a_{st} + \\ + \phi_p(B)\, \phi_s(B)\, \nabla^d\, \nabla_s^{d_s}\, \theta_u(B)\, a_{ut} \quad . \qquad (3.5)$$

$$\theta_n(B)\, a_{nt} = \phi_u(B)\, \theta_p(B)\, a_{pt} + \phi_p(B)\, \nabla^D\, \theta_u(B)\, a_{ut} \quad . \qquad (3.6)$$

From the ARIMA model identified for the observed series, the left-hand-side (l.h.s.) of each equation in (3.4) is known. The right-hand-side (r.h.s.) has to be determined for the identities (3.5) and (3.6) . It is well-known (see, for example, Hillmer and Tiao, 1982, and Maravall, 1985) that those identities do not uniquely identify model (3.4), and some additional assumptions are therefore needed. One possible solution is to “a priori” set equal to zero some of the MA parameters (as implied by the Structural Time Series Model approach of Harvey, 1989). Alternatively, as suggested by Box, Hillmer, and Tiao (1978) and Pierce (1978), one can maximize the variance of the noise component, removing in this way all additive noise from the other components; this is the solution adopted in the ARIMA Model Based (AMB) approach, originally developed by Hillmer and Tiao (1982) and Burman (1980). Although the examples will use the AMB approach, the general discussion that follows is independent of which identification assumption has been chosen.

For a polynomial in B, say H (B), let $\| H(B) \|^2$ denote the product H(B) H(F). Expression (2.7) applied to (3.3) and (3.4) yields the MMSE estimators of the trend-cycle, seasonal and irregular components, and SA series, given by

$$\hat{p}_t = \frac{V_p}{V_a} \left\| \frac{\theta_p(B)\, \phi_s(B)\, \phi_u(B)\, S^{d_s}}{\theta(B)} \right\|^2 x_t \quad , \qquad (3.7a)$$

$$\hat{s}_t = \frac{V_s}{V_a} \left\| \frac{\theta_s(B)\,\phi_p(B)\,\phi_u(B)\,\nabla^D}{\theta(B)} \right\|^2 x_t \quad , \tag{3.7b}$$

$$\hat{u}_t = \frac{V_u}{V_a} \left\| \frac{\theta_u(B)\,\phi_p(B)\,\phi_s(B)\,\nabla^d\,\nabla_s^{d_s}}{\theta(B)} \right\|^2 x_t \quad , \tag{3.7c}$$

$$\hat{n}_t = \frac{V_n}{V_a} \left\| \frac{\theta_n(B)\,\phi_s(B)\,S^{d_s}}{\theta(B)} \right\|^2 x_t \quad , \tag{3.7d}$$

It is straightforward to verify that $x_t = \hat{p}_t + \hat{s}_t + \hat{u}_t$, and that $\hat{n}_t = \hat{p}_t + \hat{u}_t$.

In the MHP procedure, in order to obtain the estimator of the cycle, first we remove from the series x_t the seasonal and irregular noise. This is the same as using the trend-cycle estimator $\hat{p}_t$ as input to the HP filter. Applying (2.11a) to $\hat{p}_t$ it is obtained that

$$\hat{c}_t = k_c \frac{(1-B)^2\,(1-F)^2}{\theta_{HP}(B)\,\theta_{HP}(F)} \hat{p}_t =$$

$$= k_c \frac{V_p}{V_a} \left\| \frac{\theta_p(B)\,\phi_s(B)\,\phi_u(B)\,\nabla^2\,S^{d_s}}{\theta_{HP}(B)\;\theta(B)} \right\|^2 x_t \quad , \tag{3.8a}$$

and, likewise,

$$\hat{m}_t = k_m \frac{V_p}{V_a} \left\| \frac{\theta_p(B)\,\phi_s(B)\,\phi_u(B)\,S^{d_s}}{\theta_{HP}(B)\;\theta(B)} \right\|^2 x_t \quad . \tag{3.8b}$$

For a finite sample, extending the series x_t with backcasts and forecasts computed with the correct model (3.1), expressions (3.8a) and (3.8b) provide the MHP estimators of the cycle (c_t) and trend (m_t), respectively.

4. A complete unobserved component model

In the two-step procedure followed in the previous section, whereby the first step is given by the AMB decomposition of the series into a trend-cycle, a seasonal, and an irregular component, and the second step by the MHP filter applied to the trend-cycle, a full decomposition of the series is obtained, namely

$$x_t = \hat{m}_t + \hat{c}_t + \hat{s}_t + \hat{u}_t \quad , \qquad (4.1)$$

where the estimators in the r.h.s. of the equation are given by the corresponding expressions in (3.7) and (3.8). The question is: can these estimators be rationalized as the MMSE estimators of the UCs in a decomposition of the series of the type

$$x_t = m_t + c_t + s_t + u_t \quad , \qquad (4.2)$$

where m_t, c_t, s_t, and u_t are the trend, cycle, seasonal, and irregular components, all of which follow sensible models and aggregate into the ARIMA model identified for the series x_t ?

The answer is yes, as we proceed to show. Consider a series that follows the general ARIMA model (3.1), and its AMB decomposition into trend-cycle, seasonal, and irregular components, as in (3.2) and (3.4).

Let $\psi_p(B) = \theta_p(B) / \phi_p(B)$. Consider now the complete UC model given by equation (4.2), with the seasonal and irregular components following models (3.4b) and (3.4c), and the trend and cycle components given by the models

$$\theta_{HP}(B)\ \nabla^D m_t = \psi_p(B)\, a_{mt} \quad , \qquad a_{mt} \sim \text{w.n.}(0, k_m\ V_p / V_a)\ ; \qquad (4.3a)$$

$$\theta_{HP}(B)\ c_t = \psi_p(B)\ \nabla^{2-D} a_{ct} \quad , \qquad a_{ct} \sim \text{w.n.}(0, k_c\ V_p / V_a)\ ; \qquad (4.3b)$$

where a_{st}, a_{ut}, a_{mt}, and a_{ct} are mutually uncorrelated. In section 2 it was seen that the MMSE estimator of the signal is fully determined from the models for the signal and for the series. Therefore, the estimators (3.7b) and (3.7c) are still the MMSE of s_t and u_t in the complete UC model. Further, direct application of (2.7) to the trend (4.3a) and to the cycle (4.3b) yields expressions (3.8a) and (3.8b). Thus the estimators obtained in the 2-step procedure can be seen as the (one-step) MMSE of the components in the complete UC model.

From (3.8a) and (3.8b),

$$\hat{m}_t + \hat{c}_t = H(B,F) \left(k_m + \nabla^2 (1-F)^2 k_c \right) x_t \quad , \tag{4.3}$$

where $H(B,F) = \frac{V_p}{V_a} \left\| \frac{\theta_p(B)\,\phi_s(B)\,\phi_u(B)\,S^{d_s}}{\theta_{HP}(B)\;\theta(B)} \right\|^2$. Since (2.11) implies that the expression in brackets in (4.3) is equal to $\| \theta_{HP}(B) \|^2$, it is obtained that

$$\hat{m}_t + \hat{c}_t = \frac{V_p}{V_a} \left\| \frac{\theta_p(B)\,\phi_s(B)\,\phi_u(B)\,S^{d_s}}{\theta(B)} \right\|^2 x_t \quad ,$$

or, considering (3.7a), $\hat{m}_t + \hat{c}_t = \hat{p}_t$. Similarly, consider the sum of the theoretical trend and cycle components $z_t = m_t + c_t$. From (4.3a) and (4.3b), $\nabla^D z_t$ is a zero-mean normal variable, with ACF equal to

$$ACF(\nabla^D z_t) = k_m \frac{V_p}{V_a} \left\| \frac{\psi_p(B)}{\theta_{HP}(B)} \right\|^2 + k_c \frac{V_p}{V_a} \left\| \frac{\psi_p(B)\,\nabla^2}{\theta_{HP}(B)} \right\|^2 =$$

$$= \frac{V_p}{V_a} \left\| \frac{\psi_p(B)}{\theta_{HP}(B)} \right\|^2 (k_m + \|\nabla^2\|^2 k_c) =$$

$$= \frac{V_p}{V_a} \left\| \frac{\theta_p(B)}{\phi_p(B)} \right\|^2 = ACF(\nabla^D p_t) \quad ,$$

where, again, use has been made of (2.11). Thus equations (4.1) and (4.2) will be satisfied, and hence aggregation of the four components or of the four estimators in the complete UC model yields the ARIMA model (3.1) for the observed series. The complete UC model turns out to be simply a way of splitting the trend-cycle component of the AMB decomposition into separate (long-term) trend and cycle components, with the split determined by the HP-parameter λ. (Recall that $\theta_{HP}(B)$, k_m, and k_c are obtained simply from λ).

The argument has been made for the historical estimators, obtained with the full filter applied to a long-enough series. Estimation of the signal at the end points of the series is equal to the application of the full filter to the series extended with forecasts and backcasts. End-point estimation of the trend and cycle in the MHP procedure requires forecasts and backcasts of the trend-cycle component, while the complete UC model requires forecasts and backcasts of the observed series x_t. The two extension procedures however can be seen to be identical because the forecasts of p_t in the AMB decomposition are obtained simply by adding more forecasts and backcasts to extend the series x_t. In both procedures, the forecasts of x_t are obtained with the proper model (3.1). Having the same filter and the same extended series, the trend and cycle estimators (preliminary or final) obtained with the MHP method are identical to those obtained with the complete UC model. (Notice that MMSE forecasts of the unobserved components can be obtained in the same way as end-point estimators are obtained: extending long enough the series x_t with forecasts.)

A similar result can be derived when the estimator of the SA series $\hat{n}_t$ is used as input of the HP filter. In this case, the complete decomposition of x_t is given by $x_t = m_t + c_t + s_t$, where s_t is the same as before, while the models for m_t and c_t are as in (4.3a) and (4.3b) with $\psi_p(B)$ replaced by $\psi_n(B) = \theta_n(B) / \phi_n(B)$, and V_p replaced by V_n . The irregular (or transitory noise) component will now be absorbed by m_t and (mostly) c_t, and the cyclical signal will be contaminated by noise.

Some relevant features of the complete UC model are worth mentioning.

1) The seasonal and irregular components are the same as the ones in the well-known standard AMB decomposition. What are new are the trend and cycle models. These two models share the polynomials $\theta_{HP}(B)$ and $\psi_p(B)$; given that the shared AR roots are stationary, the estimators MSE will be bounded and converge to a finite value (Pierce, 1979).

2) The models for the trend and cycle components incorporate “a priori” and series-dependent features. The first ones ($\theta_{HP}(B)$, k_m, and k_c) are determined by the HP-parameter λ, and reflect desirable features of the filter (broadly, which frequencies should mostly contribute to the cycle). The polynomial $\psi_p(B)$ and the variance V_p are series dependent, and guarantee consistency with the overall model identified for the series.

3) As was seen in section 2, the model-based version of the HP algorithm is based on a second-order random walk trend and a white-noise cycle, both of them invertible processes. Therefore, the HP filter does not impose the canonical condition on the trend and cycle components. In the MHP procedure, however, given that p_t is obtained from the AMB decomposition of x_t, both components will be canonical. The unit root in $\theta_p(B)$ will also be contained in $\psi_p(B)$, and hence the trend and cycle components will display the same spectral zero (typically, for the frequency $\omega = \pi$).

4) Unless the observed series is stationary, the trend will be nonstationary, with the same order of integration at the zero frequency as the observed series

(equal to *D*). The spectrum of the trend will be a narrow band around zero, with an infinite peak at zero.

5) The cycle will be stationary as long as $d < 3$, that is, practically always. Except for some ackward and complex ARIMA models, the spectrum of the cycle will have the shape of a distribution heavily skewed to the right (for quarterly or monthly series), and with a well-defined mode. Besides the spectral zero for $\omega = \pi$, when d < 2 the spectrum will contain an additional zero for $\omega = 0$.

6) **Spuriousness reconsidered**

We have concluded that the MHP procedure is the same as MMSE estimation of the components in a complete UC model, and that the reduced form of this model is the ARIMA model identified for the observed series. The UC model and the ARIMA model are observationally equivalent; they will fit equally well the data, and have the same likelihood and forecast functions. On empirical grounds, both will be equally acceptable. On a priori grounds, one may disagree with the specification of the components, but the results cannot be properly called spurious.

7) **Imprecision of end-point estimators**

Let T denote the last observed period, and t the period for which the cycle c_t is estimated ($T \geq t$). When t is far enough from 1 and T, we can assume that expression (3.8a) can be applied and the estimator obtained $\hat{c}_t$ is the final estimator. As mentioned before, when t is not far from T, in order to compute the estimator, the series needs to be extended with forecasts. This provides a preliminary estimator that will be revised as new observations become available and are used to replace and update forecasts. The full revision in this preliminary estimator represents a measurement error and it is of interest to quantify its size. Following a derivation similar to the one in Pierce (1980), from (3.1) and (3.8a), we can write

$$\hat{c}_t = \xi(B, F)\, a_t = \sum_{j=-\infty}^{\infty} \xi_j \; a_{t+j} \quad , \qquad (4.4)$$

where $\xi(B, F)$ can be expressed as $\xi(B, F) = k \;\, \xi_B(B) \;\, \xi_F(F) \quad ,$ with

$$k = (k_c \; V_p / V_a) \; ,$$

$$\xi_B(B) = \theta_p(B) \; \nabla^{2-D} / \left(\theta_{HP}(B)\, \theta(B)\right) \; ,$$

$$\xi_F(B) = \theta_p(B) \; \phi_s(B) \; \phi_u(B) \; \nabla^2 \; S^{d_s} / \left(\theta_{HP}(B)\, \theta(B)\right).$$

Thus $\xi(B, F)$ is an asymmetric polynomial, convergent in B and F, whose weights are straightforward to obtain. Given that $\hat{c}_{t|T} = E_T \; c_t = E_T \; \hat{c}_t$, and setting $E_T \; a_j = 0$ when j > T and $E_T \; a_j = a_j$ when $j \leq T$, the revision $r_{t|T} = \hat{c}_t - \hat{c}_{t|T}$ is found to be equal to

$$r_{t|T} = \sum_{j=T+1}^{\infty} \xi_{j-t} \; a_j \quad , \tag{4.5}$$

a convergent zero-mean MA process. From the variance of $r_{t|T}$ (which can be approximated as desired), confidence intervals can be built around the preliminary estimator.

5. Some Examples

5.1 The Cycle in a Random - Walk Model

Assume the observed series follows the random-walk model

$$\nabla \; x_t = a_t \quad , \qquad (V_a = 1) \quad , \tag{5.1}$$

often found for price series in efficient markets (for example, rates of exchange, prices of stocks, or interest rates). In the AMB decomposition, the series can be decomposed into orthogonal trend-cycle (p_t) and irregular component (u_t), as in

$$x_t = p_t + u_t \quad , \tag{5.2}$$

where

$$\nabla \; p_t = (1+B) \; a_{pt} \quad , \qquad V_p = .25 \quad , \tag{5.3}$$
$$u_t = \text{w.n.}\,(0, \; V_u = .25) \quad .$$

In order to specify the complete UC model with separate trend and cycle, we need the polynomial $\theta_{HP}(B)$ and the parameters k_m and k_c that appear in (4.3). As seen in Section 2, these are determined from the HP-parameter λ. Assuming annual data, an appropriate value of λ needs to be selected. While for quarterly data there seems to be a strong consensus around the value λ = 1600, originally proposed by Hodrick and Prescott, no such consensus exists for the case of annual data applications. An appealing criterion for choosing an annual value for λ would be to select the one that would result from aggregation of the quarterly filter associated with λ = 1600. Maravall and del Río (2001) show how aggregation of HP decompositions does not yield an HP-type decomposition, but that close approximations can be found. An easy-to-apply criterion that yields values of λ approximately consistent under temporal aggregation is the following: to preserve the period associated with the frequency (ω_0) for which the gain of the filter is 50 %. This frequency ω_0, expressed in radians, is in fact the parameter used (instead of λ) when the HP filter is expressed as a Butterworth filter. The relationship between the two parameters is given by (Maravall and del Río, 2001)

$$\omega_0 = a\cos\left(1 - \frac{1}{2\sqrt{\lambda}}\right) \quad , \qquad (5.4)$$

and the period associated with ω_0 is $\tau_0 = 2\pi/\omega_0$. Setting λ = 1600 for quarterly data, τ_0 is found equal to 39.7 quarters or, approximately, 10 years. Thus, for annual data, $\omega_0 = 2\pi/10$ and, solving (5.4) for λ, it is obtained that $\lambda \cong 7$. Proceeding as in Section 2, we set $V_c = \lambda$ and $V_m = 1$, and use (2.11) to obtain $\theta_{HP}(B)$ and V_{HP} (an extremely simple algorithm is given in Kaiser and Maravall, 2001, p. 82). It is found that

$$\theta_{HP}(B) = (1 - 1.1706\,B + .4137\,B^2) \quad ,$$

and V_{HP} =16.92. Thus $k_c = V_c / V_{HP} = .414$ and $k_m = V_m / V_{HP} = .059$.

The models for the trend and cycle component (4.3a) and (4.3b) can now be fully specified as the ARIMA (2, 1, 1) and ARIMA (2, 2) models

$$\theta_{HP}(B)\, \nabla m_t = (1+B)\, a_{mt} \quad , \qquad V_m = .015$$
$$\theta_{HP}(B)\, c_t = (1+B)\, \nabla a_{ct} \quad , \qquad V_c = .103 \quad .$$

The trend is I(1), with a spectral zero for $\omega = \pi$; the cycle is stationary, with spectral zeros for $\omega = 0$ and $\omega = \pi$. Figure 1 compares the spectra of the series x_t, of the trend-cycle p_t, and of the trend m_t; Figure 2 displays the spectrum of the cycle c_t. The sum of the cycle and trend spectra yields, of course, the spectrum of p_t. While the spectrum of m_t consists of a relatively narrow peak around the zero frequency, the spectrum of c_t has the standard shape of a stochastic cyclical component, with the maximum associated with a period of 10.5 years.

Fig. 1. Spectrum series and trends: Example 1

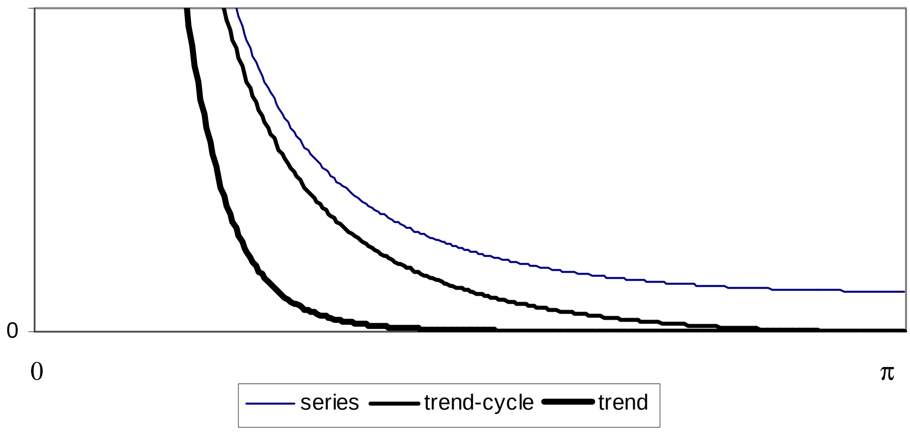
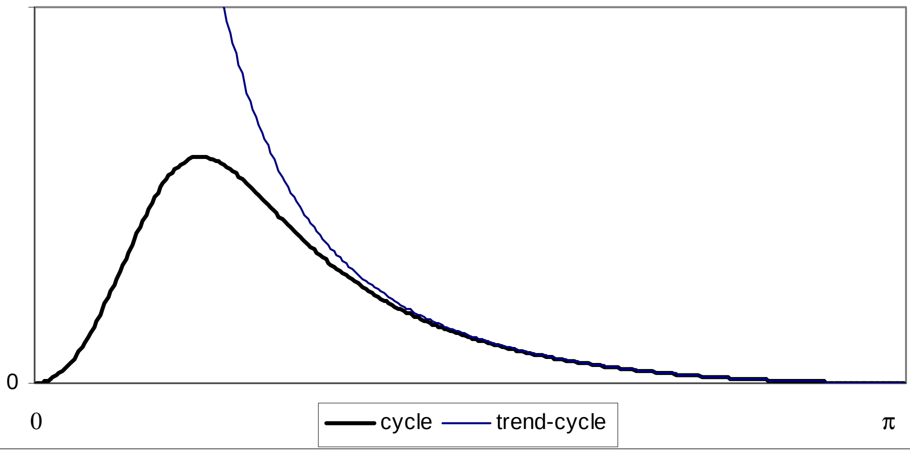


Fig. 2. Spectrum cycle: Example 1



Figures 3 and 4 present an example of a random walk, and its AMB decomposition into trend-cycle plus irregular component, for an annual series with 50 observations. (In the STSM approach, the random walk would directly provide the trend-cycle specification.) Figures 5 and 6 display the decomposition of the trend-cycle component into separate trend and cycle; for the latter, the figure also displays the 90% confidence intervals around the estimator. The sharp deterioration of the estimator at the end points of the series is clearly noticeable. The SE of the revision in the estimator for the two most recent years equals 28% and 24% of the SE of the series one-period-ahead forecast error (σ_a), and it takes about 5 years for the revision error to go below 5% of σ_a.

Fig. 3. Trend-cycle

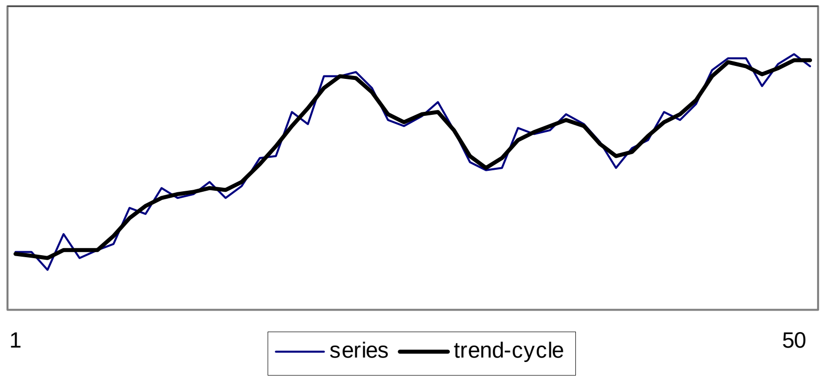
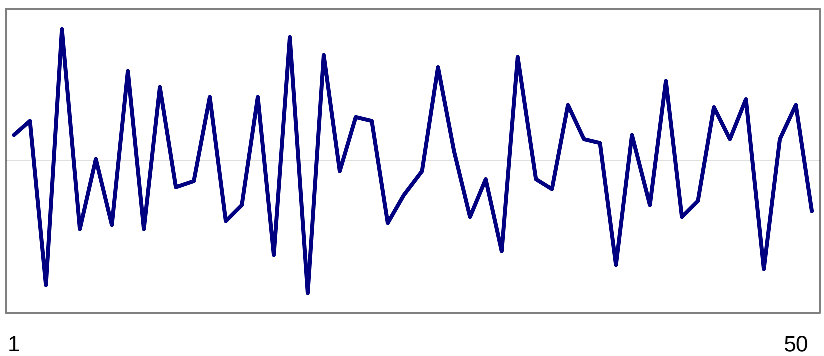
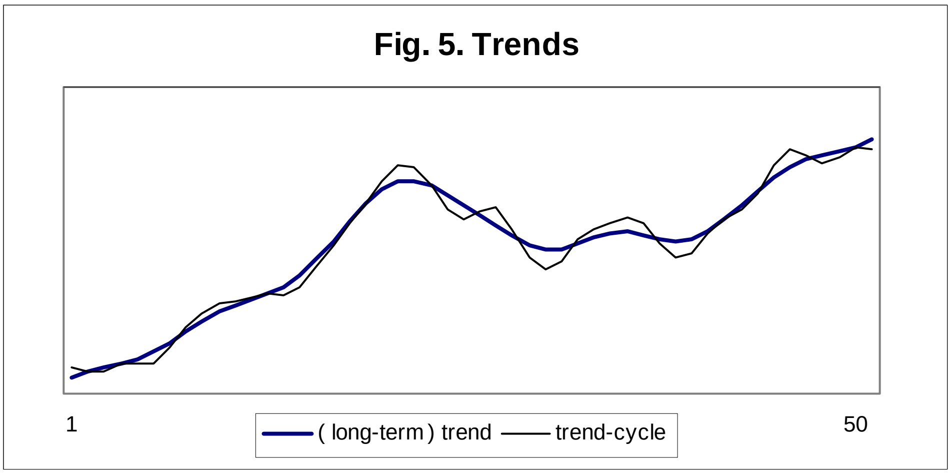
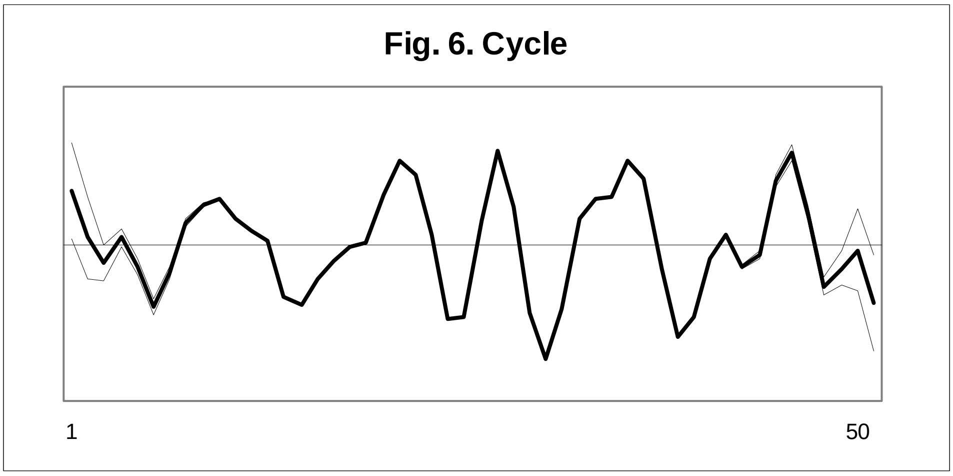


Fig. 4. Irregular







5.2 The Cycle in a Quarterly Airline Model

We consider the so-called “Airline model”, popularized by Box and Jenkins (1970), which has been found appropriate for many economic series measuring quantities or activity (see, for example, the large scale study in Fischer and Planas, 2000). For quarterly series the model is given by

$$\nabla \nabla_4 x_t = (1+ \theta_1 B)(1+ \theta_4 B^4) a_t \quad , \tag{5.5}$$

and we set $V_a = 1$, $\theta_1 = -.4$, and $\theta_4 = -.6$. The AMB decomposition of x_t is of the type (3.2) and (3.4) with the following models for the components

$$\nabla^2 p_t = (1+.119\,B-.881\,B^2)\,a_{pt} \quad , \qquad V_p = .064 \quad ,$$
$$S\,s_t = (1-.046\,B-.496\,B^2-.458\,B^3)\,a_{st} \quad , \qquad V_s = .019 \quad ,$$
$$u_t = \text{w.n.}\,(0, V_u = .305) \quad .$$

In order to split the trend-cycle (p_t) into trend (m_t) and cycle (c_t), the polynomial $\theta_{HP}(B)$, as well as k_c and k_c are needed. Setting $\lambda = 1600$, it is obtained that

$$\theta_{HP}(B) = (1-1.7771\,B+.7994\,B^2) \quad ,$$

and $V_{HP} = 2001.4$, so that $k_m = .0005$ and $k_c = .7994$. The models for the trend and cycle can now be specified as

$$(1-1.777\,B+.799\,B^2)\,\nabla^2 m_t = (1+.119\,B-.881\,B^2)\,a_{mt} \quad ,$$
$$(1-1.777\,B+.799\,B^2)\,c_t = (1+.119\,B-.881\,B^2)\,a_{ct} \quad ,$$

with $V_m = .3\,(10^{-4})$ and $V_c = .0511$. The model for m_t is I(2) while the model for c_t is stationary; both are noninvertible due to a spectral zero at $\omega = \pi$. The AMB spectral decomposition of x_t into p_t and s_t is presented in Figure 7 (the spectrum of u_t is a constant,) and the decomposition of p_t into m_t and c_t is displayed in Figure 8. Although the spectrum of p_t does not exhibit any peak for a cyclical frequency, it can be split into a smooth nonstationary peak around the zero frequency (m_t), and a stationary spectrum with a well-defined peak for a cyclical frequency. The period associated with this peak is, approximately, 13 years, relatively close to the 14-year period associated with the solution of the AR(2) polynomial in the model for the cycle. Figure 9 exhibits the squared gains of the filters to estimate the trend-cycle, trend, and cycle.

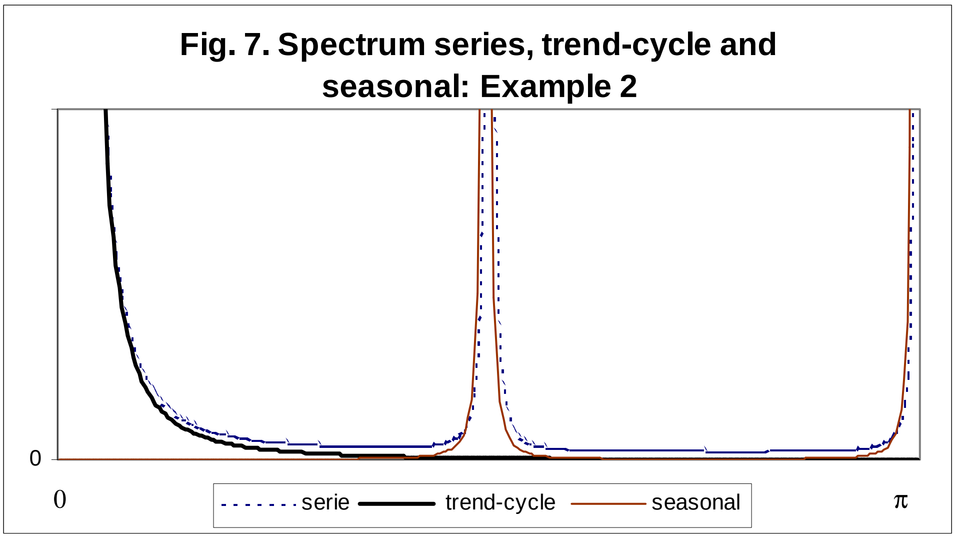


Fig. 7. Spectrum series, trend-cycle and seasonal: Example 2
0
0
π
serie
trend-cycle
seasonal

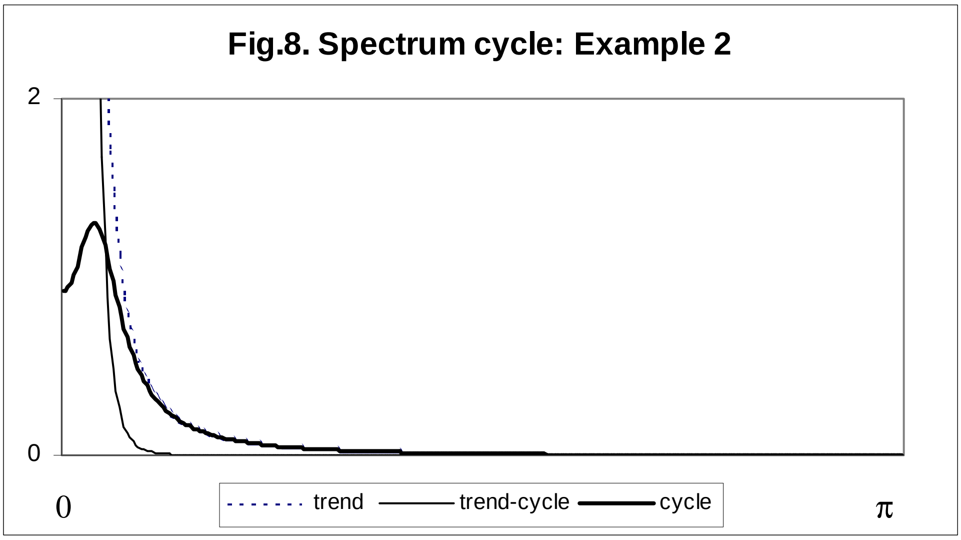
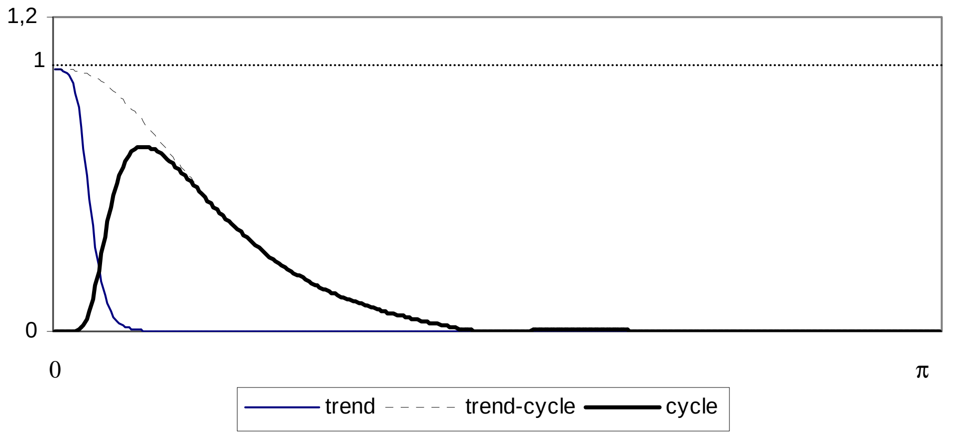


Fig.8. Spectrum cycle: Example 2
2
0
0
π
trend
trend-cycle
cycle

Fig. 9. Squared Gain Trends and cycle

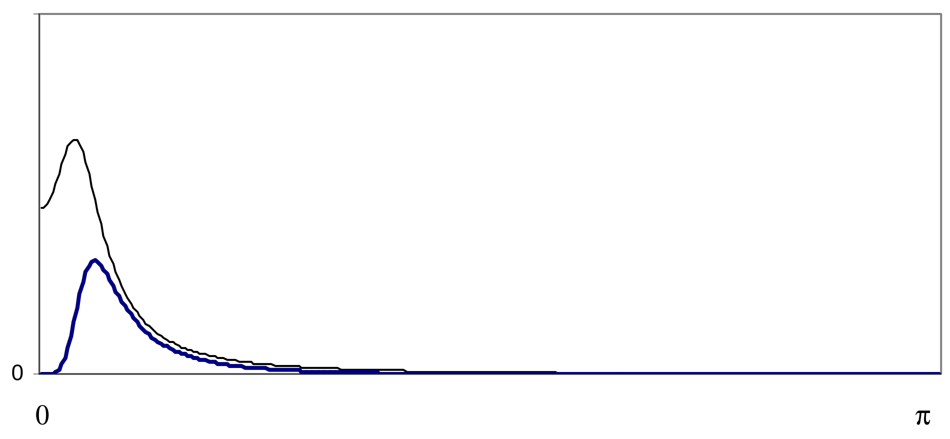


Expression (4.4), that expresses the estimator of the cycle ($\hat{c}_t$) as a moving average applied to the innovations (a_t) in the observed series, yields for this example

$$\hat{c}_t = \left[\left(k_c \ V_p\right) \frac{\theta_p(B)}{\theta_{HP}(B)} \ \frac{\theta_p(F)\,\overline{\nabla}\,\overline{\nabla}_4}{\theta_{HP}(F)\,\theta(F)} \right] a_t \quad , \qquad (5.6)$$

where $\overline{\nabla} = 1 - F$, $\overline{\nabla}_4 = 1 - F^4$, and $\theta(F) = (1 - .4\,F)\ (1 - .6\,F^4)$. From (5.6) the spectrum of $\hat{c}_t$ is easily computed; it is shown in Figure 10. Comparison of the spectrum of the theoretical component with that of its estimator illustrates an interesting feature of MMSE estimation of the component. As is well-known (see, for example, Nerlove, Grether and Carvalho, 1979) the estimator underestimates the variance of the theoretical component. For the case of the cyclical component, this loss of variance affects mostly the lower frequencies. As a result, the estimator inflates the relative importance of the higher frequencies and, for example, the period associated with the spectral peak of the cycle estimator shrinks to slightly more than 8 years. This loss of power for low frequencies is a general feature of models with nonstationary trends (in the random walk series example, the period associated with the spectral peak of the estimator $\hat{c}_t$ also shrinks, to 8 $^1/_3$ years). As a consequence, when interpreting the cycle estimator in the model-based framework, one should be aware that MMSE estimation will bias downwards the period implied by the model for the cycle.

Fig. 10. Cyclical component and estimator



An example of the decomposition of a series that follows model (5.5) is displayed in figures 11 to 15. For the cycle, the 90% confidence interval implied by the revision error has also been included. The SE of the revision for the most recent period is about 35% of the SE of the one-period-ahead forecast error of the series (σ_a), and it takes 3 years to bring the SE of the revision to less than 5% of σ_a.

Fig. 11: Trend-cycle

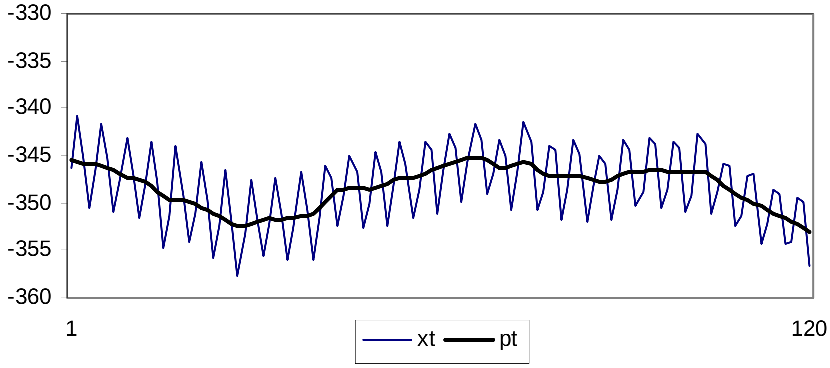


Fig. 12. Seasonal component

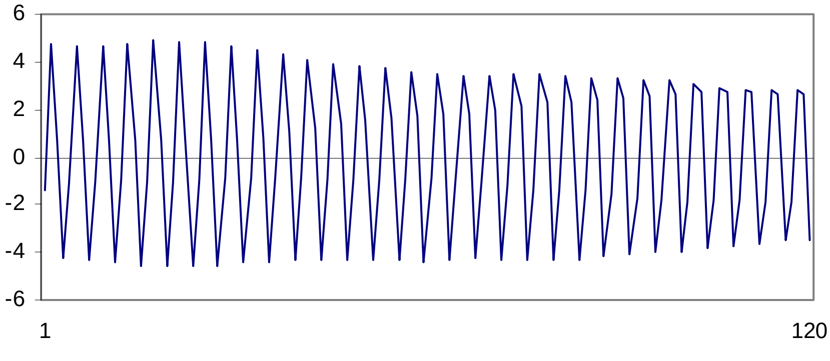


Fig. 13. Irregular component

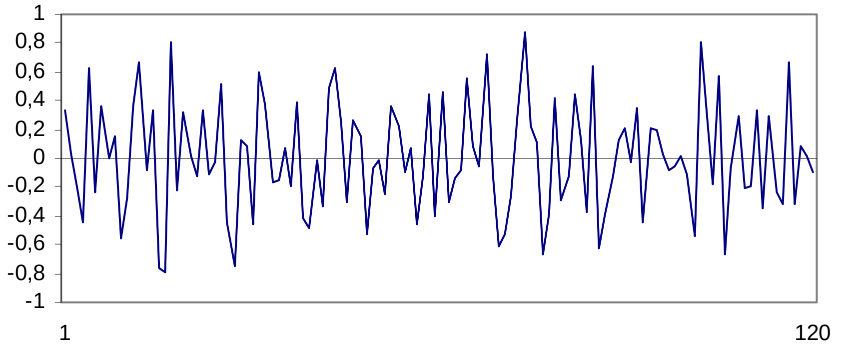


Fig. 14.Trend component

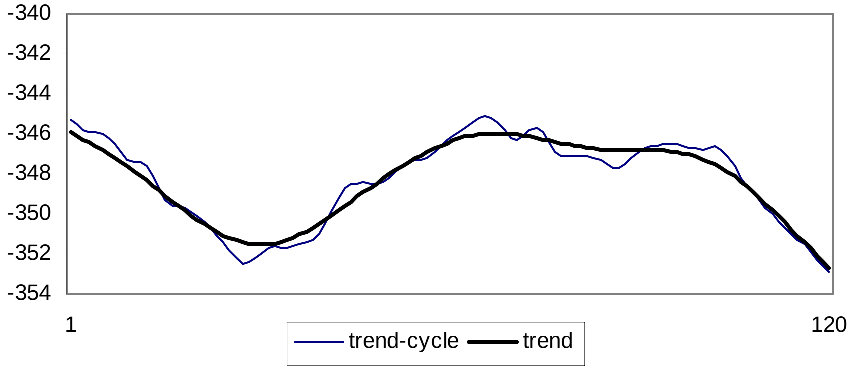
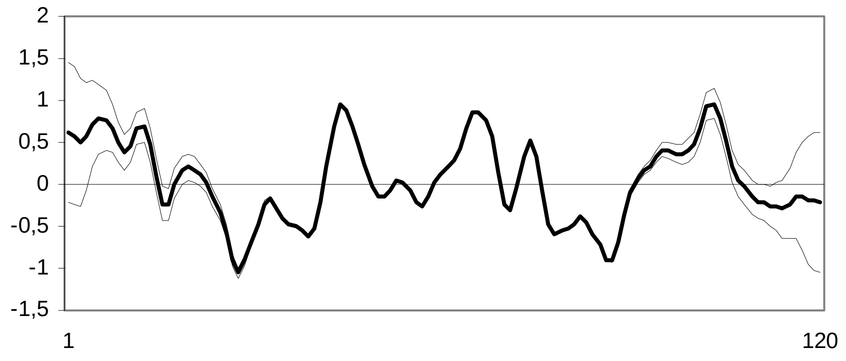


Fig.15. Cycle component



As already mentioned, forecasts of the cycle and associated SE can be straightforwardly obtained in the same way as end-point (preliminary) estimators. However, due to the size of the SE, and to the fact that the model for the cycle implies a forecast function that converges to zero, forecasts of the cycle are of limited interest in practice.

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