Can *GDP* measurement be improved further?

Jan P.A.M. Jacobs
University of Groningen, University of Tasmania, CAMA and CIRANO

Samad Sarferaz
KOF Swiss Economic Institute, ETH Zurich, Switzerland

Jan-Egbert Sturm
KOF Swiss Economic Institute, ETH Zurich, Switzerland and CESifo, Germany

Simon van Norden
HEC Montréal, CIRANO and CIREQ

Work in progress, June 2017

Abstract

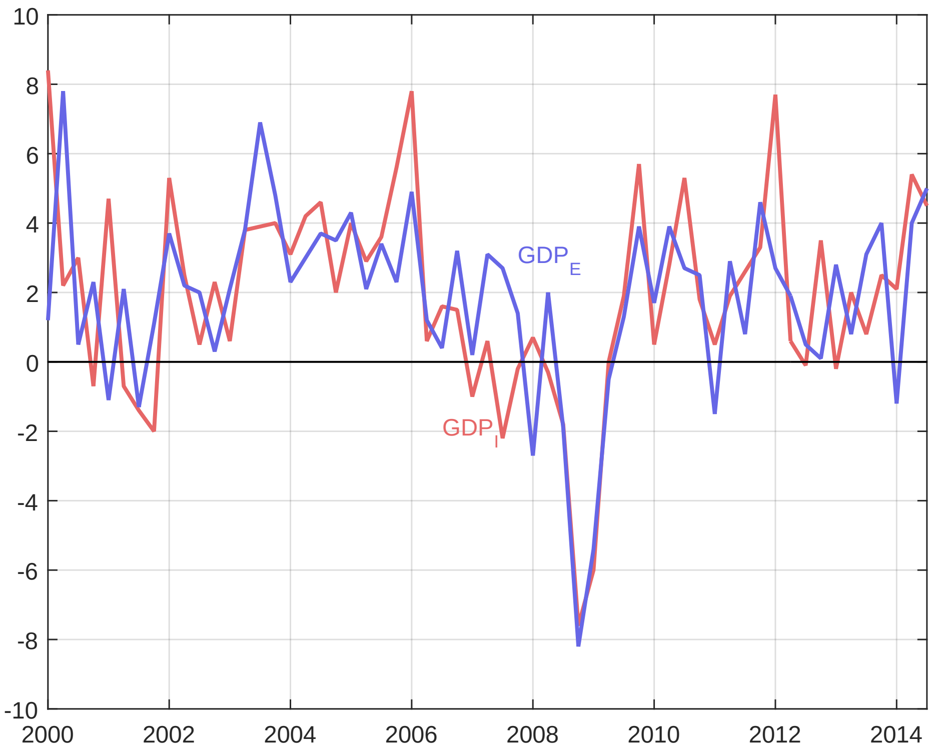
Recently, Aruoba et al. (2016) provided several estimates of historical U.S. *GDP* growth (*GDPplus*), adopting a measurement-error perspective. By distinguishing news and noise measurements errors and allowing for data revisions, we propose a new measure for U.S. *GDP* growth based on releases of expenditure-side estimates of *GDP* (*GDE*) and income-side estimates of GDP (*GDI*). Our measure is more persistent than *GDE* and *GDI* and has smaller residual variance. It has a similar autoregressive coefficient but smaller residual variance than *GDPplus*. Historical decompositions of *GDE* and *GDI* measurement errors reveal a larger news share in *GDE* than in *GDI*.

JEL classification: E01, E32
Keywords: output, income, expenditure, state space form, dynamic factor model, data revisions, news, noise

1 Introduction

Unlike most developed nations, U.S. national accounts feature distinct estimates of real output based on the expenditure approach (GDE) and the income approach (GDI), see Figure 1. While these should be identical in theory, measurement errors result in discrepancies between the two estimates, as is well-known from the data reconciliation literature initiated by Stone, Champernowne and Meade (1942).[1]

Figure 1: U.S. GDP growth: Expenditure side vs. income side

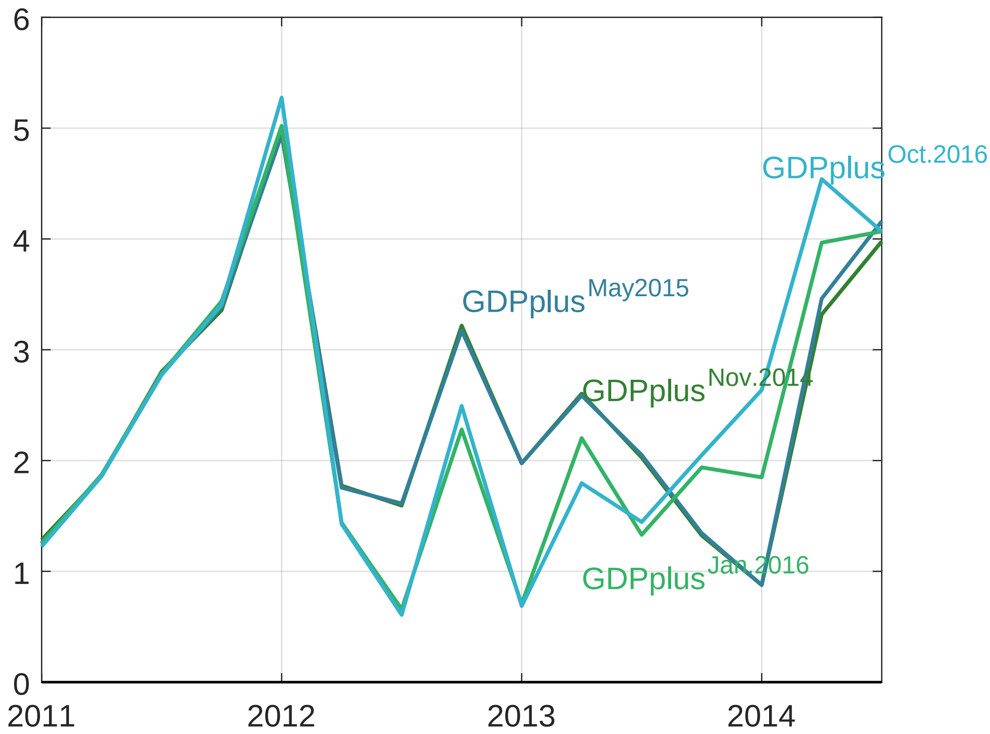


A series of recent articles has examined the extent to which future revisions in one series may be predicted by the other, as well as whether a weighted combination of the two series gives an improved estimate of real output. Key papers in this literature include Fixler and Nalewaik (2009), Nalewaik (2010, 2011a, 2011b, 2012), Greenaway-McGrevy (2011), and Aruoba et al. (2012, 2016). Underlining

[1]The same applies to the production-based estimate of output. See e.g. the study of Rees, Lancaster and Finlay (2015) on Australian GDP.

the perceived importance of this issue for forecasting and current analysis, the Federal Reserve Bank of Philadelphia draws on the above work to show estimates of a combined indicator (*GDPplus*), which they feature as an indicator of recent economic performance.[2] Yet, *GDPplus* is subject to important revision, as shown in Figure 2.

Figure 2: GDPplus in real-time



Various vintages of GDPplus according to the estimates of the Federal Reserve Bank of Philadelphia.

In this paper we reconcile *GDE* and *GDI* in a real-time data environment using a multivariate extension of Jacobs and van Norden (2011, henceforth JvN), to decompose measurement errors into news and noise. We discuss identification through real-time data and news-noise assumptions. No additional variables or assumptions are needed like in Aruoba et al. (2016). We compute a new *GDP* series, GDP^{++},

[2]See `http://www.philadelphiafed.org/research-and-data/real-time-center/gdpplus/`

that takes data revisions into account, and compare the new GDP^{++} series to GDE, GDI and $GDPplus$. In addition we provide a historical decomposition of GDE and GDI into news and noise shocks.

Much of this work has been motivated by a desire to improve forecasts of GDP growth or turning points.[3] However, it also poses serious questions about the extent to which fluctuations in output growth have been mis-measured. One approach to assessing the severity of the measurement error has been to compare GDE growth estimates with those from a dynamic factor model which also incorporates GDI and perhaps other variables as well, see for example Aruoba et al. (2016). However, standard factor models applied in this setting typically assume that measurements are noise. This forces the estimated growth factor to be less volatile than the series upon which it is based.[4] In contrast, our framework allows for both news and noise errors, where noise implies that measurement errors are uncorrelated with the unobserved "true" value, and "news" implies that measurement errors are uncorrelated with available information. This in turn allows the latent growth factor to be more volatile or less volatile than the observed series.

The paper is structured as follows. In Section 2 we present our econometric framework, including identification. We show that our system is identified using real-time data and news-noise assumptions. In Section 3 we describe our data and estimation method. Results are shown in Section 4. Section 5 concludes.

[3]See for example Nalewaik (2011b) or Diebold's published discussion following Nalewaik (2010).

[4]As an example, consider the special case where GDI perfectly measures "true" output and GDP captures only some of this variation and is less variable than GDI. A simple factor model based only on GDI and GDP growth assumes that the additional volatility in GDI growth reflects measurement error and will interpret the reduced variability of GDP growth estimate as a sign of better accuracy. As a result, it will place more weight on GDP than GDI even though the optimal weights would be $(0, 1)$.

2 Econometric Framework

Our point of departure is the dynamic-factor measurement error model of Aruoba et al. (2016), in which GDE and GDI are measures of latent true GDP, GDP^{+}. Similar to Aruoba et al. we work with growth rates of GDE, GDI and GDP^{+} and we assume that the true GDP growth rate follows AR(1) dynamics:

$$\begin{bmatrix} GDE_t \\ GDI_t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} GDP_t^{+} + \begin{bmatrix} \epsilon_{Et} \\ \epsilon_{It} \end{bmatrix} \tag{1}$$

$$GDP_t^{+} = \mu(1-\rho) + \rho GDP_{t-1}^{+} + \epsilon_{Gt}, \tag{2}$$

where

$$[\epsilon_{Et}, \epsilon_{It}, \epsilon_{Gt},]' \sim N(0, \Sigma).$$

Moving to a real-time data environment, we have l releases on GDE_t and GDI_t, and for each release of GDE_t and GDI_t news and noise measurement errors, denoted as ν_{Et}^{j}, ν_{It}^{j} and ζ_{Et}^{j}, ζ_{It}^{j} $(j = 1, \ldots, l)$, respectively:

$$\begin{aligned} GDE_t^L &= [GDE_t^1, \ldots, GDE_t^l]', & GDI_t^L &= [GDI_t^1, \ldots, GDI_t^l]', \\ \nu_{Et}^L &= [\nu_{Et}^1, \ldots, \nu_{Et}^l]', & \nu_{It}^L &= [\nu_{It}^1, \ldots, \nu_{It}^l]' \\ \zeta_{Et}^L &= [\zeta_{Et}^1, \ldots, \zeta_{Et}^l]', & \zeta_{It}^L &= [\zeta_{It}^1, \ldots, \zeta_{It}^l]'. \end{aligned}$$

News errors are correlated with the true GDP growth rate: $E[GDP_t^{++}, \nu_t^L] \neq 0$, while noise errors are orthogonal to the true GDP growth rate $E[GDP_t^{++}, \zeta_t^L] = 0$. The measurement equation (1) can now be rewritten as

$$\begin{bmatrix} GDE_t^L \\ GDI_t^L \end{bmatrix} = \begin{bmatrix} \iota_l \\ \iota_l \end{bmatrix} GDP_t^{++} + \begin{bmatrix} \nu_{Et}^L \\ \nu_{It}^L \end{bmatrix} + \begin{bmatrix} \zeta_{Et}^L \\ \zeta_{It}^L \end{bmatrix}, \tag{3}$$

where ι_l is a $l \times 1$ vector of ones.

Following the state-space representation of Durbin and Koopman (2012), and including the news and noise measurement errors into the state vector $\boldsymbol{\alpha}$, we get the JvN model with two observed variables and one dynamic factor

$$\text{measurement equation} \quad \begin{bmatrix} GDE_t^L \\ GDI_t^L \end{bmatrix} = \boldsymbol{Z}\boldsymbol{\alpha}_t \tag{4}$$

$$\text{state equation} \quad \boldsymbol{\alpha}_t = \begin{bmatrix} \rho & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{\alpha}_{t-1} + \boldsymbol{R}\boldsymbol{\eta}_t, \tag{5}$$

where the state vector $\boldsymbol{\alpha}_t = [GDP_t^{++}, {\nu_{Et}^L}', {\nu_{It}^L}', {\zeta_{Et}^L}', {\zeta_{It}^L}']'$; $\boldsymbol{Z} = [\iota_{2l}, I_{2l}, I_{2l}]$, where I_{2l} is an $(2l \times 2l)$-identity matrix; $\boldsymbol{R} = \begin{bmatrix} 1 & \iota_l' & \iota_l' & 0 & 0 \\ 0 & -U & 0 & 0 & 0 \\ 0 & 0 & -U & 0 & 0 \\ 0 & 0 & 0 & I_l & 0 \\ 0 & 0 & 0 & 0 & I_l \end{bmatrix}$, where U is an upper triangular matrix of ones; and the errors $\boldsymbol{\eta}_t = \left[\eta_{Gt}, {\eta_{E\nu t}^L}', {\eta_{I\nu t}^L}', {\eta_{E\zeta t}^L}', {\eta_{I\zeta t}^L}'\right]' \sim N(0, D)$.

Below, we decompose total revisions of GDE and GDI into news and noise. We illustrate the decomposition for GDE. Suppose, we have l releases of GDE_t

$$\begin{aligned} GDE_t^1 &= \rho GDP_{t-1}^{++} + \eta_{Gt} + \eta_{E\zeta t}^1 \\ GDE_t^2 &= \rho GDP_{t-1}^{++} + \eta_{Gt} + \eta_{E\nu t}^1 + \eta_{E\zeta t}^2 \\ \vdots &= \qquad \vdots \\ GDE_t^l &= \rho GDP_{t-1}^{++} + \eta_{Gt} + \eta_{E\nu t}^1 + \ldots + \eta_{E\nu t}^{l-1} + \eta_{E\zeta t}^l. \end{aligned}$$

Then the total revision of GDE can be written as

$$GDE_t^l - GDE_t^1 = \underbrace{\eta_{E\nu t}^1 + \ldots + \eta_{E\nu t}^{l-1}}_{\text{News}} + \underbrace{\eta_{E\zeta t}^l - \eta_{E\zeta t}^1}_{\text{Noise}}. \tag{6}$$

Identification

The system is identified using real-time data and news-noise assumptions. For details see the Appendix.

3 Data and Estimation

Data

We use monthly vintages of quarterly series provided by the Bureau of Economic Analysis (BEA). For GDE we employ the Advance, the Third, the 12th and the 24th releases, and Second/Third, 12th and the 24th releases for GDI. The sample we cover is 2003Q1–2014Q3.

Estimation

We employ Markov Chain Monte Carlo (MCMC) methods to obtain posterior simulations for our model's parameters (see, e.g., Kim and Nelson 1999). We use conjugate and diffuse priors for the coefficients and the variance covariance matrix, resulting into a multivariate normal posterior for the coefficients and an inverted Wishart posterior for the variance covariance matrix. For the prior for the coefficients restricted to zero we assume the mean to be zero and variance to be close to zero.

Our Gibbs sampler has the following structure. We first initialize the sampler with values for the coefficients and the variance covariance matrix. Conditional on data, the most recent draw for the coefficients and for the variance covariance matrix, we draw the latent state variables $\boldsymbol{\alpha}_t$ for $t = 1, ..., T$ using the procedure described in Carter and Kohn (1994). In the next step, we conditional on data, the most recent draw for the latent variable $\boldsymbol{\alpha}_t$ and for the variance covariance matrix, drawing the coefficients from a multivariate normal distribution. Finally, conditional on data, the most recent draw for the latent variables and the coefficients, we draw the variance covariance matrix from an inverted Wishart distribution. We cycle through 100K Gibbs iterations, discarding the first 90K as burn-in. Of those 10K draws we save only every 10th draw, which gives us in total 1000 draws on which we base our inference. Convergence of the sampler was checked by studying recursive mean plots and by varying the starting values of the sampler and comparing results.

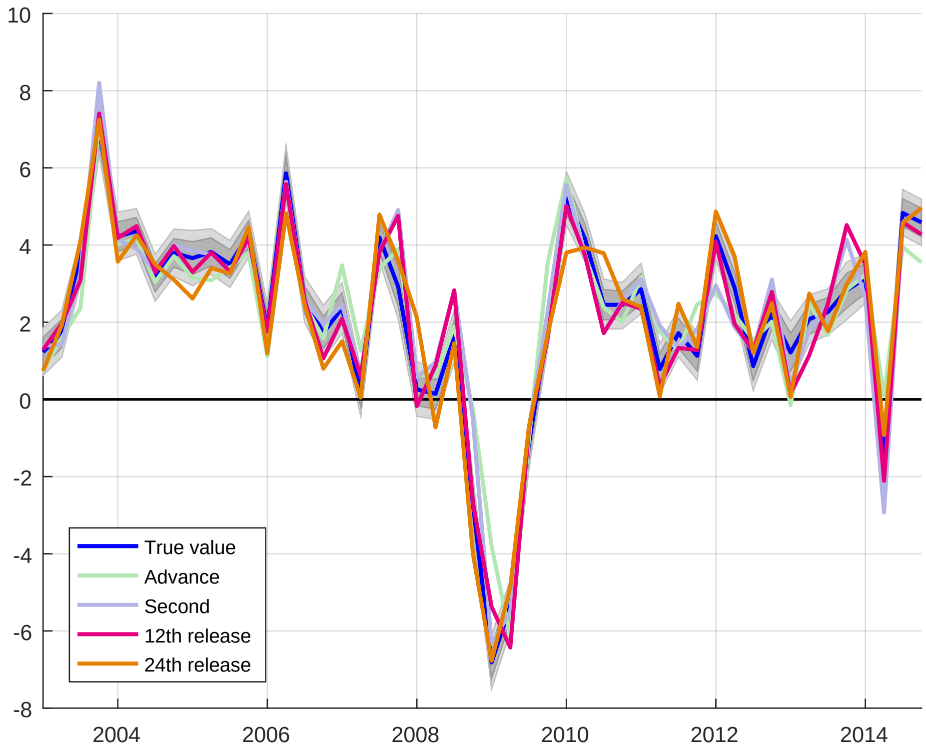
4 Results

Here we compare our measure of GDP to releases of GDE and GDI in four different ways: (i) in graphs, (ii) looking at historical decompositions, (iii) by investigating dynamics, and (iv) by calculating relative contributions.

Comparison of GDP^{++} and releases of GDE and GDI

In Figure 3 we compare our new GDP measure, GDP^{++}, with shaded posterior ranges (90% of probability mass) to the four releases of GDE we employed in the estimation, the Advance, third, the 12th and the 24th release. There is some evidence that the releases are more volatile than the true values GDP^{++}. We observe that the releases are outside the posterior bounds for some periods. This observation holds especially for the Advanced release and the 24th release; in some periods, like e.g. 2010Q1, the Advance release and the 24th release are on different sides of the posterior range.

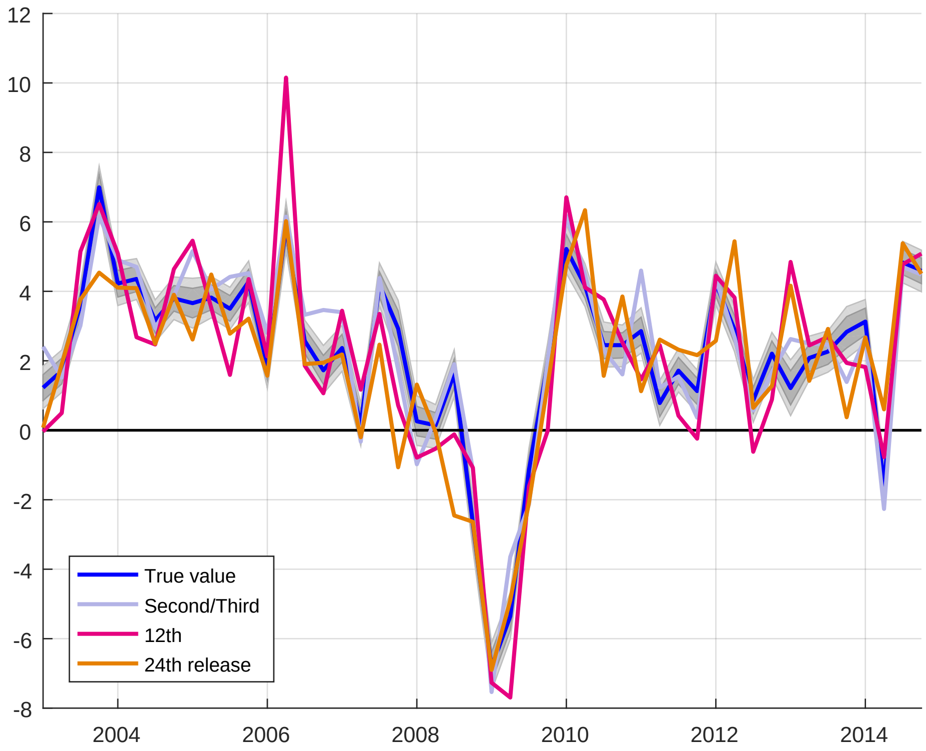
Figure 3: GDP^{++} vs. GDE



The blue line represents the posterior mean of GDP^{++}, the "true" value, and the shaded area around the blue line indicates 90% of posterior probability mass. The green line represents the advance estimate, the purple line is the second estimate, the red line the 12th release and the orange line the 24th release of expenditure side GDP growth.

Figure 4 shows our new GDP measure, GDP^{++}, together with shaded posterior ranges (90% of probability mass) and the three releases of GDI we employed in the estimation, the Second/Third, the 12th and the 24th release. The releases fluctuate around the posterior bounds of the true values. The GDI releases are more volatile than the true values GDP^{++}. The releases of GDI are also much more volatile than the releases of GDE. Note that the sample paths of GDP_M and GDE and GDI in Aruoba et al. (2016, Figure 3) show a different picture than our Figures 3 and 4. GDE differs more from their GDP measure than GDI.

Figure 4: GDP^{++} vs. GDI



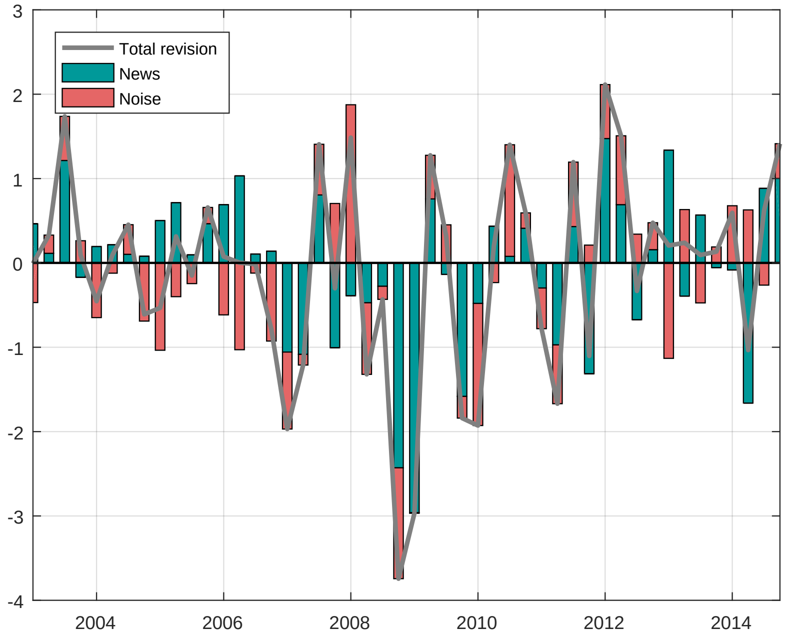
The blue line represents the posterior mean of GDP^{++}, the "true" value, and the shaded area around the blue line indicates 90% of posterior probability mass. The purple line is the second/third estimate, the red line the 12th release and the orange line the 24th release of income side GDP growth.

Historical decomposition

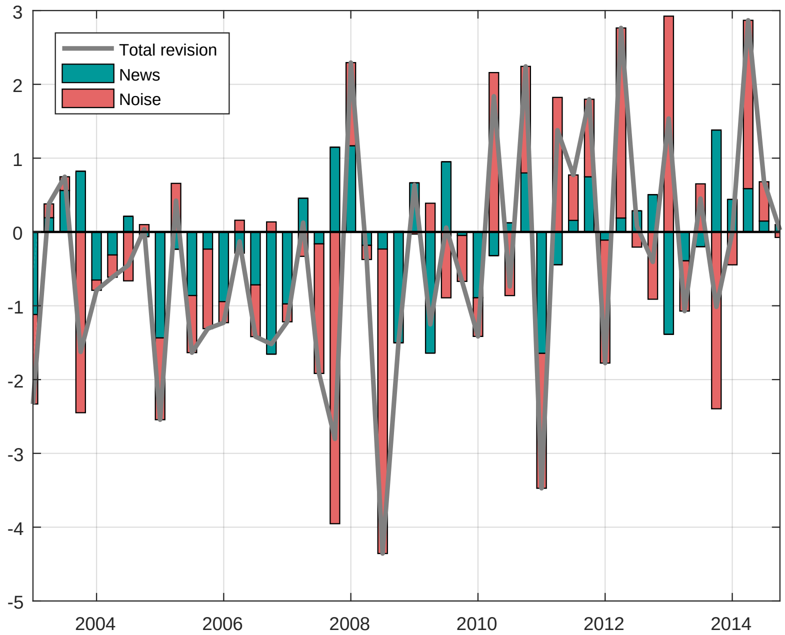
Our econometric framework (4)-(5) allows the historical decomposition of GDE and GDI in terms of news and noise measurement errors based on the decomposition described in (6). The outcomes are shown in Figure 5. The top panel shows total revisions in GDE with news and noise shares, the bottom panel total GDE revisions with news and noise shares.

Figure 5: Historical Decompositions

GDE



GDI



Historical decomposition of the total revision (24th release minus second estimate) into news and noise. The red bars depict the share of news and the green bars the share of noise in total revision (grey line). The historical decomposition is based on the decomposition described in (6).

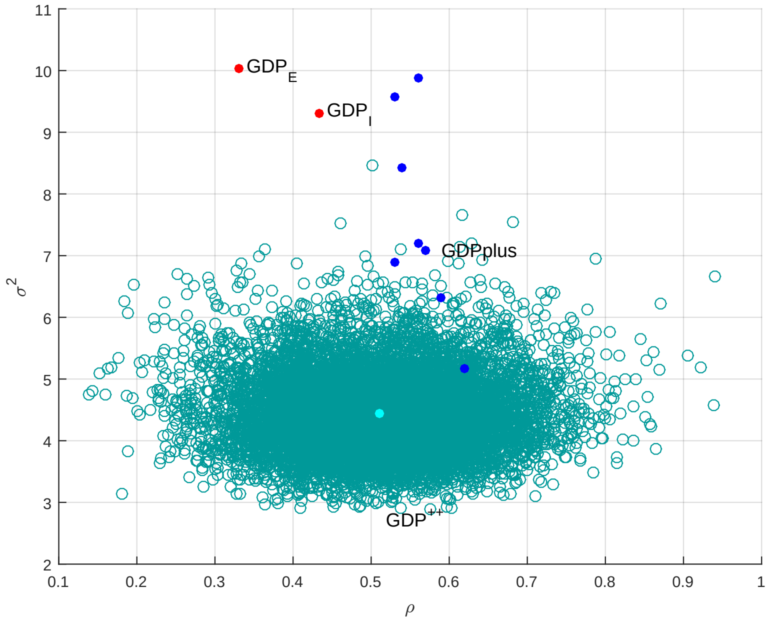
We observe that total revisions in GDI, the bottom panel, are larger than total revisions in GDE, a stylized fact which can also be distilled from the previous two figures. Eye-balling the two panels suggests that the news share in total GDE revisions is larger than the noise share. The opposite seems to hold for total revisions in GDI. This observation is in line with Fixler and Nailewaik (2009), who reject the pure noise assumption in GDI.

Dynamics of GDP^{++} and other GDP measures

In Figure 6 we depict the (ρ, σ^2) pairs summarizing the dynamics of our true GDP estimate across all draws. We contrast the (ρ, σ^2) pairs corresponding to our GDP^{++} estimate to the (ρ, σ^2) pairs obtained when fitting an AR(1) model to second releases of GDE, and GDI, and the eight different models estimated in Aruoba et al. (2016).

Figure 6 reveals that our real-time data based estimate of GDP is somewhat less persistent than the $GDPplus$ measure of Aruoba et al. (2016), but exhibits a higher persistence than the estimates for GDE and GDI. We also find that the posterior mean of the innovation variance of our GDP^{++} is much smaller than the innovation variances of GDE, GDI and the eight different models Aruoba et al. (2016). The combination of a ρ that is close to those implied by the various models estimated in Aruoba et al. (2016) and a σ^2 that is much smaller than the ones implied by Aruoba et al. (2016) leads to a higher forecastability of the GDP^{++} measure.

Figure 6: GDP Dynamics



The green shaded area consists of (ρ, σ^2) pairs across draws, the light green dot is the posterior mean of the (ρ, σ^2) pairs across draws, the blue dots represent the posterior mean estimates of the eight different models described in Aruoba et al. (2016, Figure 5) and the red dots are (ρ, σ^2) pairs, resulting from AR(1) models fitted to expenditure side GDP and income side GDP, respectively.

Relative contributions of GDE and GDI to GDP^{++}

To assess the relative importance of GDI and GDE at different releases, we use the Kalman gains. The outcomes are listed in Table 1.

Table 1: Kalman Gains

	GDE	GDI
Advance	0.0600	-
Third	0.0200	-
12th	0.2465	0.2578
24th Release	0.2178	0.0861

Annual releases are most important. Kalman gains are fairly high for the 12th releases of GDE and GDI—and the 24th release of GDE. We also find that GDE releases are more important for explaining GDP than GDI releases, in contrast to Aruoba et al. (2016).

5 Conclusion

In this paper we proposed a new measure of U.S. GDP growth using real-time data on GDE and GDI. Our measure is shown to be more persistent than GDE and GDI and has smaller residual variance. In addition it has a similar autoregressive coefficient but smaller residual variance than the GDP measure $GDPplus$ of Aruoba et al. (2016). Historical decompositions of GDE and GDI measurement errors reveal a larger news share in GDE than in GDI.

Acknowledgements

A preliminary version of this paper was presented at the 10th International Conference on Computational and Financial Econometrics (CFE 2016), Seville. We thank Dean Croushore for helpful comments.

Appendix

This appendix first analyzes the identification of the univariate state space system in Jacobs and van Norden (2011), using the procedure described in Komunjer and Ng (2011) and Aruoba et al. (2016). Thereafter, identification of the model of the present paper in discussed.

Identification in the univariate, two vintage JvN framework

The state space form of the Jacobs and van Norden (JvN) model with two vintages and no spillovers can be expressed as

$$\begin{bmatrix} y_t^1 \\ y_t^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{y}_t \\ \nu_t^1 \\ \nu_t^2 \\ \zeta_t^1 \\ \zeta_t^2 \end{bmatrix}, \tag{A.1}$$

$$\begin{bmatrix} \tilde{y}_t \\ \nu_t^1 \\ \nu_t^2 \\ \zeta_t^1 \\ \zeta_t^2 \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}_{t-1} \\ \nu_{t-1}^1 \\ \nu_{t-1}^2 \\ \zeta_{t-1}^1 \\ \zeta_{t-1}^2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_{t,\tilde{y}} \\ \eta_{t,\nu}^1 \\ \eta_{t,\nu}^2 \\ \eta_{t,\zeta}^1 \\ \eta_{t,\zeta}^2 \end{bmatrix}, \tag{A.2}$$

where y_t^i for $i = 1, 2$ denotes the different releases, $\tilde{y}_t$ is the "true" value of the variable of interest, ν_t^i and ζ_t^i for $i = 1, 2$ are the news and the noise components and $\eta_{t,\nu}^i$ and $\eta_{t,\zeta}^i$ for $i = 1, 2$ are the news and the noise shocks, $[\eta_{t,\tilde{y}}\ \eta_{t,\nu}^1\ \eta_{t,\nu}^2\ \eta_{t,\zeta}^1\ \eta_{t,\zeta}^2]' \sim N(0, H)$ with $H = \text{diag}(\sigma_{\tilde{y}}^2, \sigma_{\nu 1}^2, \sigma_{\nu 2}^2, \sigma_{\zeta 1}^2, \sigma_{\zeta 2}^2)$, where diag denotes a diagonal matrix.

The system in (A.1) and (A.2) can also be written as

$$\begin{bmatrix} y_t^1 \\ y_t^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tilde{y}_t + \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_{t,\tilde{y}} \\ \omega_{t,\nu}^1 \\ \omega_{t,\nu}^2 \\ \omega_{t,\zeta}^1 \\ \omega_{t,\zeta}^2 \end{bmatrix}, \tag{A.3}$$

$$\tilde{y}_t = \rho \tilde{y}_{t-1} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{t,\tilde{y}} \\ \omega_{t,\nu}^1 \\ \omega_{t,\nu}^2 \\ \omega_{t,\zeta}^1 \\ \omega_{t,\zeta}^2 \end{bmatrix}, \tag{A.4}$$

where $\omega_{t,\tilde{y}} = \eta_{t,\tilde{y}} + \eta_{t,\nu}^1 + \eta_{t,\nu}^2$, $\omega_{t,\nu}^1 = -\eta_{t,\nu}^1 - \eta_{t,\nu}^2$, $\omega_{t,\nu}^2 = -\eta_{t,\nu}^2$, $\omega_{t,\zeta}^1 = \eta_{t,\zeta}^1$, $\omega_{t,\zeta}^2 = \eta_{t,\zeta}^2$ and $[\omega_{t,\tilde{y}}\ \omega_{t,\nu}^1\ \omega_{t,\nu}^2\ \omega_{t,\zeta}^1\ \omega_{t,\zeta}^2]' \sim N(0, \Sigma)$ with variance-covariance matrix Σ defined as

$$\Sigma = \begin{bmatrix} \Sigma_{\tilde{y}\tilde{y}} & \Sigma_{\tilde{y}\nu1} & \Sigma_{\tilde{y}\nu2} & 0 & 0 \\ \Sigma_{\nu1\tilde{y}} & \Sigma_{\nu1\nu1} & \Sigma_{\nu1\nu2} & 0 & 0 \\ \Sigma_{\nu2\tilde{y}} & \Sigma_{\nu2\nu1} & \Sigma_{\nu2\nu2} & 0 & 0 \\ 0 & 0 & 0 & \Sigma_{\zeta1\zeta1} & 0 \\ 0 & 0 & 0 & 0 & \Sigma_{\zeta2\zeta2} \end{bmatrix}, \tag{A.5}$$

where

$$\begin{aligned} \Sigma_{\tilde{y}\tilde{y}} &= \sigma_{\tilde{y}}^2 + \sigma_{\nu1}^2 + \sigma_{\nu2}^2, \quad \Sigma_{\tilde{y}\nu1} = -\sigma_{\nu1}^2 - \sigma_{\nu2}^2, \\ \Sigma_{\tilde{y}\nu2} &= -\sigma_{\nu2}^2, \quad \Sigma_{\nu1\nu1} = \sigma_{\nu1}^2 + \sigma_{\nu2}^2, \quad \Sigma_{\nu1\nu2} = \sigma_{\nu2}^2, \\ \Sigma_{\nu2\nu2} &= \sigma_{\nu2}^2, \quad \Sigma_{\zeta1\zeta1} = \sigma_{\zeta1}^2, \quad \Sigma_{\zeta2\zeta2} = \sigma_{\zeta2}^2, \end{aligned} \tag{A.6}$$

which implies

$$\begin{aligned}
\Sigma_{\tilde{y}\nu 1} &= -\Sigma_{\nu 1 \nu 1}, \\
\Sigma_{\tilde{y}\nu 2} &= -\Sigma_{\nu 2 \nu 2}, \\
\Sigma_{\nu 2 \nu 2} &= \Sigma_{\nu 1 \nu 2}.
\end{aligned} \tag{A.7}$$

Moreover, consider the following restriction

$$\sigma^2_{\nu 2} = 0, \tag{A.8}$$

which is justified due to the fact that the second release news shock corresponds to information outside the sample and is thus not needed.

Aruoba et al. (2016) show that a state space system described in Equations (A.3) and (A.4) is not identified with Σ unrestricted and identified under certain restrictions on elements of Σ. We now study whether the restrictions implied by JvN lead to an identified system following the procedure described in Aruoba et al. (2016).

Theorem 1. *Suppose that Assumptions 1, 2, 4-NS and 5-NS of Komunjer and Ng (2011) hold. Then according to Proposition 1-NS of Komunjer and Ng (2011), the state space model described in (A.1) and (A.2) is identified given the restrictions implied by (A.1), (A.2) and (A.8).*

Proof of Theorem 1. We begin by rewriting the state space model in (A.3) and (A.4) to match the notation used in Komunjer and Ng (2011)

$$x_{t+1} = A(\theta)x_t + B(\theta)\epsilon_{t+1} \tag{A.9}$$

$$z_{t+1} = C(\theta)x_t + D(\theta)\epsilon_{t+1}, \tag{A.10}$$

where $x_t = \tilde{y}_t$, $z_t = [y^1_t \; y^2_t]'$,$\epsilon_t = [\omega_{t,\tilde{y}} \; \omega^1_{t,\nu} \; \omega^2_{t,\nu} \; \omega^1_{t,\zeta} \; \omega^2_{t,\zeta}]'$, $A(\theta) = \rho$, $B(\theta) = [1\;0\;0\;0\;0]$,

$C(\theta) = [\rho\ \rho]'$,

$$D(\theta) = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

and $\theta = [\rho\ \sigma^2_{\tilde{y}}\ \sigma^2_{\nu 1}\ \sigma^2_{\nu 2}\ \sigma^2_{\zeta 1}\ \sigma^2_{\zeta 2}]'$.

Given that Σ is positive definite and $0 \leq \rho < 1$, Assumption 1 and 2 of Komunjer and Ng (2011) are satisfied. Given that $D(\theta)\Sigma D(\theta)'$ is nonsingular also Assumption 4-NS of Komunjer and Ng (2011) is satisfied. Rewriting the state space model in (A.9) and (A.10) into its innovation representation gives

$$\hat{x}_{t+1|t+1} = A(\theta)\hat{x}_{t|t} + K(\theta)a_{t+1} \tag{A.11}$$

$$z_{t+1} = C(\theta)\hat{x}_{t|t} + a_{t+1}, \tag{A.12}$$

where $K(\theta)$ is the Kalman gain and a_{t+1} is the one-step ahead forecast error of z_{t+1} with variance $\Sigma_a(\theta)$. The Kalman gain and the variance of the one-step ahead forecast error for this system can be expressed as

$$K(\theta) = (p\rho C' + \Sigma_{BD})(pCC' + \Sigma_{DD})^{-1} \tag{A.13}$$

$$\Sigma_a(\theta) = pCC' + \Sigma_{DD}, \tag{A.14}$$

where p is the variance of the state vector, solving the following Riccati equation

$$p = p\rho^2 + \Sigma_{BB} - (p\rho C' + \Sigma_{BD})(pCC' + \Sigma_{DD})^{-1}(p\rho C + \Sigma_{DB}). \tag{A.15}$$

and $\Sigma_{BB} = B\Sigma B'$, $\Sigma_{BD} = B\Sigma D'$, $\Sigma_{DD} = D\Sigma D'$ with

$$
\begin{aligned}
\Sigma_{BB} &= \Sigma_{\tilde{y}\tilde{y}}, \\
\Sigma_{BD} &= \begin{bmatrix} \Sigma_{\tilde{y}\tilde{y}} + \Sigma_{\tilde{y}\nu 1} & \Sigma_{\tilde{y}\tilde{y}} + \Sigma_{\tilde{y}\nu 2} \end{bmatrix}, \qquad \text{(A.16)} \\
\Sigma_{DD} &= \begin{bmatrix} \Sigma_{\tilde{y}\tilde{y}} + 2\Sigma_{\tilde{y}\nu 1} + \Sigma_{\nu 1\nu 1} + \Sigma_{\zeta 1\zeta 1} & . \\ \Sigma_{\tilde{y}\tilde{y}} + \Sigma_{\tilde{y}\nu 1} + \Sigma_{\nu 2\tilde{y}} + \Sigma_{\nu 2\nu 1} & \Sigma_{\tilde{y}\tilde{y}} + 2\Sigma_{\tilde{y}\nu 2} + \Sigma_{\nu 2\nu 2} + \Sigma_{\zeta 2\zeta 2} \end{bmatrix}.
\end{aligned}
$$

By using the definitions in (A.6), the expressions in (A.16) can also be written as

$$
\begin{aligned}
\Sigma_{BB} &= \sigma^2_{\tilde{y}} + \sigma^2_{\nu 1} + \sigma^2_{\nu 2}, \qquad \Sigma_{BD} = \begin{bmatrix} \sigma^2_{\tilde{y}} & \sigma^2_{\tilde{y}} + \sigma^2_{\nu 1} \end{bmatrix}, \\
\Sigma_{DD} &= \begin{bmatrix} \sigma^2_{\tilde{y}} + \sigma^2_{\zeta 1} & . \\ \sigma^2_{\tilde{y}} & \sigma^2_{\tilde{y}} + \sigma^2_{\nu 1} + \sigma^2_{\zeta 2} \end{bmatrix}. \qquad \text{(A.17)}
\end{aligned}
$$

Assumption 5-NS of Komunjer and Ng (2011) relates to the *controllability* and *observability* of state space systems. The state space system in (A.3) and (A.4) is *controllable* if the matrix $[K(\theta)\ A(\theta)K(\theta)]$ has full row rank and it is *observable* if the matrix $[C(\theta)'\ A(\theta)'C(\theta)']$ has full column rank and is thus said to be minimal.

To show that Assumption 5-NS is satisfied, first note that $\Sigma_{BB} - \Sigma_{BD}\Sigma^{-1}_{DD}\Sigma_{DB}$ is the Schur complement of Ω, the variance covariance matrix of the joint distribution of x_{t+1} and z_{t+1}, with respect to Σ_{DD} where

$$
\Omega = \begin{bmatrix} \Sigma_{BB} & \Sigma_{BD} \\ \Sigma_{DB} & \Sigma_{DD} \end{bmatrix}.
$$

Because Ω is a positive definite matrix, also its Schur complement is positive definite thus leading to $\Sigma_{BB} - \Sigma_{BD}\Sigma^{-1}_{DD}\Sigma_{DB} > 0$. Now to show that this inequality leads to $p > 0$, we use the following lemma

Lemma 1. *Assume A and $(A+B)$ are invertible and that $rank(B) = 1$, then*

$$(A+B)^{-1} = A^{-1} - \frac{1}{1+tr(BA^{-1})} A^{-1}BA^{-1}.$$

We can now use Lemma 1 to rewrite Equation (A.15) as

$$\begin{aligned} p &= p\rho^2 + \Sigma_{BB} - (p\rho C' + \Sigma_{BD})\Sigma_{DD}^{-1}(p\rho C + \Sigma_{DB}) \\ &+ \frac{p}{g}(p\rho C' + \Sigma_{BD})\Sigma_{DD}^{-1}CC'\Sigma_{DD}^{-1}(p\rho C + \Sigma_{DB}), \end{aligned} \tag{A.18}$$

where $g = 1 + ptr(CC'\Sigma_{DD}^{-1})$. After some manipulations we find the following quadratic equation

$$ap^2 + bp + c = 0, \tag{A.19}$$

with

$$\begin{aligned} a &= -tr(CC'\Sigma_{DD}^{-1}), \\ b &= (\rho - \Sigma_{BD}\Sigma_{DD}^{-1}\Sigma_{DB})^2 + tr(CC'\Sigma_{DD}^{-1})(\Sigma_{BB} - \Sigma_{BD}\Sigma_{DD}^{-1}\Sigma_{DB}) - 1, \\ c &= \Sigma_{BB} - \Sigma_{BD}\Sigma_{DD}^{-1}\Sigma_{DB}. \end{aligned}$$

The necessary and sufficient conditions for $p > 0$ are $\sqrt{b^2 - 4ac} > 0$ and $\frac{-b-\sqrt{b^2-4ac}}{2a} > 0$. The first condition leads to $b^2 + 4tr(C'\Sigma_{DD}^{-1}C)(\Sigma_{BB} - \Sigma_{BD}\Sigma_{DD}^{-1}\Sigma_{DB}) > 0$ and the second to $tr(C'\Sigma_{DD}^{-1}C)(\Sigma_{BB} - \Sigma_{BD}\Sigma_{DD}^{-1}\Sigma_{DB}) > 0$ Since Σ_{DD} is positive definite (thus $tr(C'\Sigma_{DD}^{-1}C) > 0$) both conditions are satisfied if $\Sigma_{BB} - \Sigma_{BD}\Sigma_{DD}^{-1}\Sigma_{DB} > 0$.

Given also that $A(\theta) = \rho \geq 0$ and $C(\theta) \geq 0$, we obtain $K(\theta) \neq 0$ and thus Assumption 5-NS is satisfied.

Now Proposition 1-NS of Komunjer and Ng (2011) can be applied, which implies that two vectors

$$\theta_0 = [\rho \; \sigma^2_{\tilde{y},0} \; \sigma^2_{\nu 1,0} \; \sigma^2_{\nu 2,0} \; \sigma^2_{\zeta 1,0} \; \sigma^2_{\zeta 2,0}]'$$

and

$$\theta_1 = [\rho \; \sigma^2_{\tilde{y},1} \; \sigma^2_{\nu 1,1} \; \sigma^2_{\nu 2,1} \; \sigma^2_{\zeta 1,1} \; \sigma^2_{\zeta 2,1}]'$$

are observationally equivalent iff there exists a scalar $\tau \neq 0$ such that

$$A(\theta_1) = \tau A(\theta_0)\tau^{-1} \tag{A.20}$$

$$K(\theta_1) = \tau K(\theta_0) \tag{A.21}$$

$$C(\theta_1) = C(\theta_0)\tau^{-1} \tag{A.22}$$

$$\Sigma_a(\theta_1) = \Sigma_a(\theta_0). \tag{A.23}$$

Given that $A(\theta) = \rho$, it follows from Equation (A.20) that $\rho_0 = \rho_1$ and thus we can deduce from (A.22) that $\gamma = 1$. Hence, by using Equations (A.13) and (A.14), the conditions (A.21) and (A.23) can be expressed as

$$K_1 = K_0 = (p_0 \rho C' + \Sigma_{BD0})(pCC' + \Sigma_{DD0})^{-1} \tag{A.24}$$

$$\Sigma_{a1} = \Sigma_{a0} = p_0 CC' + \Sigma_{DD0}, \tag{A.25}$$

where p_0 solves the following Riccati equation

$$p_0 = p_0 \rho^2 + \Sigma_{BB0} - K_0(p_0 \rho C + \Sigma_{DB0}). \tag{A.26}$$

Equations (A.24) to (A.26) are satisfied if and only if

$$p_1(1-\rho^2) - \Sigma_{BB1} = p_0(1-\rho^2) - \Sigma_{BB0} \tag{A.27}$$

$$p_1\rho C' + \Sigma_{BD1} = p_0\rho C' + \Sigma_{BD0} \tag{A.28}$$

$$p_1 CC' + \Sigma_{DD1} = p_0 CC' + \Sigma_{DD0}. \tag{A.29}$$

Without loss of generality let

$$\Sigma_{\tilde{y}\tilde{y},1} = \Sigma_{\tilde{y}\tilde{y},0} + \delta(1-\rho^2) \tag{A.30}$$

leading to

$$\Sigma_{BB,1} = \Sigma_{BB,0} + \delta(1-\rho^2). \tag{A.31}$$

We now proceed by splitting the analysis into two cases.

Case 1: $\delta = 0$. From (A.27) we obtain $p_1 = p_0$. (A.28) hence implies $\sigma^2_{\tilde{y},1} = \sigma^2_{\tilde{y},0}$ and $\sigma^2_{\nu1,1} = \sigma^2_{\nu1,0}$ and given that $\Sigma_{\tilde{y}\tilde{y},1} = \Sigma_{\tilde{y}\tilde{y},0}$ it follows $\sigma^2_{\nu2,1} = \sigma^2_{\nu2,0}$. (A.29) implies that $\Sigma_{DD1} = \Sigma_{DD0}$ and thus $\sigma^2_{\zeta1,1} = \sigma^2_{\zeta1,0}$ and $\sigma^2_{\zeta2,1} = \sigma^2_{\zeta2,0}$, leading to the fact that $\theta_1 = \theta_0$.

Case 2: $\delta \neq 0$. From (A.27) we obtain $p_1 = p_0 + \delta$. From (A.28) it follows

$$\sigma^2_{\tilde{y},1} = \sigma^2_{\tilde{y},0} - \delta\rho^2 \quad \text{and} \quad \sigma^2_{\nu1,1} = \sigma^2_{\nu1,0}. \tag{A.32}$$

Moreover, (A.27) gives

$$\sigma^2_{\nu2,1} = \sigma^2_{\nu2,0} + \delta. \tag{A.33}$$

Finally, the equations in (A.29) lead to

$$\sigma^2_{\zeta1,1} = \sigma^2_{\zeta1,0} \quad \text{and} \quad \sigma^2_{\zeta2,1} = \sigma^2_{\zeta2,0}. \tag{A.34}$$

Note that (A.6) and (A.32) to (A.34) result into

$$\Sigma_1 = \begin{bmatrix} \Sigma_{\tilde{y}\tilde{y},0} + \delta(1-\rho^2) & \Sigma_{\tilde{y}\nu 1,0} - \delta & \Sigma_{\tilde{y}\nu 2,0} - \delta & 0 & 0 \\ \Sigma_{\nu 1\tilde{y},0} - \delta & \Sigma_{\nu 1\nu 1,0} + \delta & \Sigma_{\nu 1\nu 2,0} + \delta & 0 & 0 \\ \Sigma_{\nu 2\tilde{y},0} - \delta & \Sigma_{\nu 2\nu 1,0} + \delta & \Sigma_{\nu 2\nu 2,0} + \delta & 0 & 0 \\ 0 & 0 & 0 & \Sigma_{\zeta 1\zeta 1} & 0 \\ 0 & 0 & 0 & 0 & \Sigma_{\zeta 2\zeta 2} \end{bmatrix}. \tag{A.35}$$

Finally, from (A.33) and (A.34) it follows that $\delta = 0$. □

Identification in the multivariate framework of the present paper

To be done.

References

Aruoba, S. Borağan, Francis X. Diebold, Jeremy Nalewaik, Frank Schorfheide, and Dongho Song (2012), "Improving GDP measurement: A forecast combination perspective", in X. Chen and N. Swanson, editors, *Causality, Prediction, and Specification Analysis: Recent Advances and Future Directions: Essays in Honor of Halbert L. White Jr*, Springer, New York, 1–26.

Aruoba, S. Borağan, Francis X. Diebold, Jeremy Nalewaik, Frank Schorfheide, and Dongho Song (2016), "Improving GDP measurement: A measurement-error perspective", *Journal of Econometrics*, **191**, 384–397.

Carter, C. and R. Kohn (1994), "On Gibbs sampling for state space models", *Biometrika*, **81**, 541–553.

Durbin, James and Siem Jan Koopman (2012), *Time Series Analysis by State Space Methods*, 2nd edition, Oxford University Press, Oxford.

Fixler, Dennis J. and Jeremy J. Nalewaik (2009), "News, noise, and estimates of the "true" unobserved state of the economy", U.S. Bureau of Economic Analysis, Department of Commerce and Federal Reserve Board Manuscript.

Greenaway-McGrevy, Ryan (2011), "Is GDP or GDI a better measure of output? A statistical approach", Bureau of Economic Analysis Manuscript.

Jacobs, Jan P.A.M and Simon van Norden (2011), "Modeling data revisions: Measurement error and dynamics of "true" values", *Journal of Econometrics*, **161**, 101–109.

Kim, Chang-Jin and Charles R. Nelson (1999), *State-Space Models with Regime Switching*, The MIT Press, Cambridge MA and London.

Komunjer, Ivana and Serena Ng (2011), "Dynamic identification of dynamic stochastic general equilibrium models", *Econometrica*, **79**, 1995–2031.

Nalewaik, Jeremy J. (2010), "On the income- and expenditure-side measures of output", *Brookings Papers on Economic Activity*, **1**, 71–27 (with discussion).

Nalewaik, Jeremy J. (2011a), "The income- and expenditure-side estimates of U.S. output growth: An update through 2011 Q2", *Brookings Papers on Economic Activity*, **2**, 385–402.

Nalewaik, Jeremy J. (2011b), "Incorporating vintage differences and forecasts into Markov Switching models", *International Journal of Forecasting*, **27**, 281–307.

Nalewaik, Jeremy J. (2012), "Estimating probabilities of recession in real time using GDP and GDI", *Journal of Money, Credit and Banking*, **44**, 235–253.

Rees, Daniel M., Lancaster, and Richard Finlay (2015), "A state-space approach to Australian Gross Domestic Product", *Australian Economic Review*, **48**, 133–149.