Divergent Risk-Attitudes and Endogenous Collateral Constraints [*]

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First draft: January 2015. This draft: April 2017.

Abstract

Consensus exists that financial crises are triggered by excessive leverage and low risk-sensitivity at the tails, together leading to endogenous risk-formation. A large body of literature explored the first determinant. None combined the two. We build a model with heterogenous preference borrowers and lenders, occasionally binding collateral constraints and *loss-averse* borrowers with kinked preferences (diminishing risk-sensitivity on the tails). Analytically we show that: the shadow price of debt (the tightness of the borrowing limit) endogenously increases in the distance between lenders'/borrowers' pricing kernels and decreases in borrowers' risk-tolerance; the borrowers' Sharpe ratio raises with respect to the shadow price of debt and declines with respect to his risk-tolerance. We quantify the transmission channels by simulating the model with a global solution method, which account for anticipatory effects (endogenous switches) in the occasionally binding borrowing limit and in the preference status around the kink. Among other things we find that our model matches well several facts about asset prices, returns and debt.

JEL: E0, E5, G01

Keywords: *loss averse borrowers, risk-tolerance, endogenous price of risk, excessive leverage, occasionally binding constraints.*

1 Introduction

There is by now consensus on the narrative describing the determinants of the financial crisis as well as its unfolding. Excessive leverage coupled with excessive risk-taking fostered excessive volatility and the boom-burst cycle of the asset prices[1]. In this context non-linearities play a crucial role in explaining the heightened dynamic. Less consensus exists on the main (albeit not necessarily the sole) determinants of the channel just described. On the one side, some link it to limited enforcement in credit and financial markets and to collateral-based incentive mechanisms (see Mendoza[50]). Both are responsible for shock multipliers and exacerbated dynamics in asset markets due to the anticipatory effects from the occasionally binding collateral constraint and to the ensuing pecuniary/fire-sale externalities (see Jeanne and Korinek[35] or Lorenzoni[47]). The one just described is the institutional finance explanation. On the other side, others consider beliefs' dis-agreement (see Simsek[62][2]) as a main driver of excessive leverage and of volatile asset price dynamics. Intuitively, borrowers (or more generally buyers in financial markets) hold either optimistic beliefs or assign low weight (they are more risk-tolerant) to tail events. This induces them to lever up and invest in risky assets. As lenders need to be compensated for providing liquidity into risky investment and for bearing the risk of future collateral values, borrowers must pay growing and volatile debt premia. Hence they need to extract higher asset returns premia. The one just described is the behavioral explanation. The two narratives differ in that the first relies on agency problems, albeit linked to rational pricing of the fundamentals, while the second is inspired by behavioral theories. We believe that both elements played a role[3]. In debt markets where limited enforcement ties leverage to risky collateral and where leveraged buyers value risky assets differently than sellers price dynamics are amplified by fluctuations in leverage and also reflect differences in risk-tolerance. The idea that market forces determine financial instability and boom-bust cycles in presence of agents' heterogeneity in risk-attitudes dates back to Minsky[54]. We engage in dissecting the link between leverage (and its excessive volatility), risk-taking and excess

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[1]See Morris and Shin [55], Hanson, Kashyap and Stein [29], Gorton and Metrick [25], Angeloni and Faia[3], Mian and Sufi [52], Lo and Rogoff [46], among many others.

[2]Earlier literature on this channel includes Miller[53], Harrison and Kreps[30] and Ganakoplos[24].

[3]See also a recent paper by Cochrane [13] arguing that institutional finance mechanisms shall be complemented by behavioral explanations to obtain a full picture of crisis-like events, but also of the pattern of asset prices in normal times.

asset market volatility by examining the joint role of limited enforcement and of heterogenous risk-attitudes.

To this purpose we construct a model with heterogenous agents, who can be on two sides of the debt market and exhibit different risk-attitudes. Borrowers, who lever up to invest in risky assets, have loss averse reference-dependent preferences, hence exhibit diminishing risk-sensitivities on the tails relatively to lenders[4]. This makes them prone to risk-taking when they approach the tails, namely when they are highly exposed to the risky asset. Furthermore, borrowers are subject to endogenous and occasionally binding collateral constraints. Indeed they lever up to invest in a risky asset, which also serves as collateral to secure debt. Collateral serves to mitigate asymmetric information[5] or to discipline lack of commitment[6]. However the presence of limited enforcement and of the collateral constraint induces shock amplification and risk-shifting behavior. First, agency problems foster borrowers' incentives of risk-shifting[7] onto external financiers and this in turn increases their risk-taking attitudes. Second, in markets for debt transactions (REPOS, mortgages, asset purchases on the margin) default is disciplined through collateral, which has a future resale option value[8]. This induces inter-temporal and intra-temporal multipliers effects of shocks transmission. Finally, occasionally binding collateral constraints produce anticipatory effects (see recently Mendoza[50], Korinek and Simsek[41] among others): the fear of future tightening of the debt constraint induces borrowers to engage into heightened deleveraging, thereby exacerbating volatility. In this context, the divergence in risk-attitudes between lenders and borrowers contributes to intensify those mechanisms. Debt contracts are akin to commodities[9], whose price is determined endogenously by the arbitrage condition between the two agents' pricing kernels. The higher the distance in risk-tolerance between borrowers and lenders, the higher is the shadow price of debt, which in turn determines the tightness of the borrowing limit. To fix ideas consider a boom. Borrowers wish to lever up to invest in risky assets, possibly approaching the tails if the shock is large enough. In collateralized debt markets borrowers shall compensate lenders for bearing the

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[4]Loss averse preferences and diminishing risk-sensitivities on the tails for traders in risky assets are well documented in experimental evidence (see Haigh and List [27])

[5]See Townsend [65] or Gale and Hellwig [23].

[6]See Hart and Moore[31].

[7]See Stiglitz and Weiss[63].

[8]See Kiyotaki and Moore[40].

[9]See Prescott and Townsend[58] and Geanakoplos[24] among others.

risk of future fluctuations in the value of collateral, the higher is the difference in risk-tolerance between borrowers and lenders and the higher is the premium (proxied by the shadow price of debt) that borrowers wish to transfer to lenders in exchange of liquid funds. To generate higher risk-premia borrowers shall expose themselves to high return/risk assets, a behavior labelled as risk-taking. The larger is the gamble, namely the leveraged exposure to the risky asset, the higher is their risk-taking behavior.

To capture all gradations embedded in risk behavior we work with loss-averse preferences (as pioneered by Kahneman and Tversky[37]) with state-dependent reference points (see Kõszegi and Rabin[43], Woodford [67][10]), as captured by past consumption (see also Yogo[66] and Santoro et. al.[60]). This choice captures several elements of the crisis narrative. First, loss averse agents exhibit diminishing risk sensitivity (relatively to lenders), hence lower risk-sensitivity and higher risk-taking on the tails. Second, losses in consumption resonate more than gains and this contribute to rationalize heightened de-leveraging in anticipation of tightening in the debt constraint. Third, the kink in the utility and the state-dependent nature of the reference point enhance non-linearities. Leverage pro-cyclicality (de-leveraging) is enhanced when borrowers' consumption is on the right (the left) of the reference point.

We prove analytically two sets of results. The first relates to the shadow price of debt (or tightness of the borrowing limit), which is endogenously determined. First, for given evaluation of the lender, the shadow price of debt depends negatively on borrower's stochastic discount factor. Consider the case in which borrowers experience wealth or income shocks that bring them to the right of the reference level. In this case their stochastic discount factor raises (they become less risk-sensitive) as they move toward the tails. They wish to invest and lever up more. In those boom states there is leverage growth and the collateral constraint is relaxed. Second, the shadow price of debt raises in the distance between lenders'/borrowers' stochastic discount factors. As the evaluations of risk diverge borrowers need to transfer higher risk premia to entice lenders into providing funds. The second set of results relates to asset price growth. The asset price depends positively on borrowers' stochastic discount factor and on the warrant value of the risky asset.

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[10]Prospect theory, first introduced by Kahneman and Tversky[37], is supported by experimental evidence and has been used to explain important puzzles in asset price behavior and in household portfolio decisions. See for instance Benartzi and Thaler[9], Barberis, Huang and Santos [5] and Barberis and Huang [6].

As borrowers move toward the tails, their risk sensitivity declines (their stochastic discount factor raises) and they demand more risky asset, which in turn foster asset price growth. Further, we show that borrowers' Sharpe ratio raises with respect to the shadow price of debt and declines with respect to the borrower's pricing kernel. First, as the tightness of the borrowing limit increases the borrowers' return/risk frontier becomes steeper. Borrowers need to extract higher expected returns in order to transfer higher risk-premia to lenders. Second, as the borrowers' become risk-takers (their stochastic discount factor raises) they require lower expected returns for given variance of the excess returns.

We then quantify the transmission channels previously discussed through numerical simulations by building an algorithm for policy function iterations augmented with Markov-switching regimes. The method allows us to account for the anticipatory effects due to the occasionally binding nature of the debt constraint and for endogenous switching in preferences around the kink (see Davig and Leeper [16], Farmer, Waggoner and Zha[20] and [21]). The presentation of the numerical results is done in three steps. First, we present and comment impulse responses, policy functions and the pattern of simulated series. This allows us to clarify the transmission channels in our model. Two main features are worth noticing. First, the comparison with equally risk-averse borrowers and lenders shows that heterogeneity in risk-attitude contributes significantly in accounting for the heightened dynamic of debt and asset markets. Second, debt policy functions show that in the context of our model both leverage growth and de-leverage materialize accounting for a leverage cycle. The dependence of the shadow price of debt upon the stochastic discount factor explains why de-leverage occurs once consumption falls below the reference (hence the stochastic discount factor falls)[11]. Next, we compute a series of model-based statistics and compare them to the data-equivalent for US and UK over two different time samples. The model does remarkably well in matching jointly several stylized facts on both the long run level and the cyclical properties of returns (on debt and risky assets), equity premia, debt and consumption. Excessive asset price growth, high and counter-cyclical equity premia as well as excess debt volatility are jointly explained. High and volatile premia emerge when investors become very afraid of bad events[12],

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[11] Notice that de-leverage is not a common feature of models with debt constraints based on future collateral values.
[12] See Cochrane [13].

something which in our model is due to the fact that losses resonate more. At last, through simulations we also quantify numerically the link between the distance in lenders'/borrowers' evaluations and leverage or asset price growth, thereby confirming our analytical results.

The rest of the paper is divided as follows. Section 2 highlights the novelty of our paper by comparing to the literature. Section 3 describes the model and derives the analytical results. Section 4 describe the simulation method, the calibration and presents numerical results. Section 5 concludes. Tables, figures and appendices follow.

2   Literature review

The idea that the build-up of leverage and asset price boom-bust dynamics, which characterize almost all financial crises, are determined by market forces in presence of heterogenous agents' risk-tolerance dates back to Minsky[54]. At large agents holding optimistic beliefs or high risk-tolerance lever up to invest in high-return/high-risk assets. As the distance in risk-valuations between borrowers and lenders increases compensating risk-premia emerge to close this gap. Such compensating risk-premia endogenously determine the market price of debt and of the risky asset.

The paper which is closer to our in the spirit is Simsek[62], who builds a static and elegant model featuring beliefs dis-agreement between optimistic borrowers, who acquire funds to invest in a risky asset by engaging in collateralized debt contracts, and pessimistic lenders, who need to be compensated for the risk of changes in re-sale collateral values. The borrowing constraint is endogenously determined in equilibrium by the relative evaluation between borrowers and lenders. Our model also features endogenous collateral constraints, whose shadow price is determined by the distance between lenders and borrowers discount factors. In our case borrowers and lenders differ in terms of risk-tolerance, rather than in their beliefs. Hence in our model the difference in asset valuation affects directly the Arrow-Pratt metrics of relative risk-aversion, rather than the expectation formation process. Furthermore, we develop a fully dynamic model embedding anticipatory effects and explore its quantitative implications by using global solution methods that can account for the occasionally binding nature of the constraint as well as for the presence of kinked utility. Hence our model also adds emphasis to the role played by non-linearities for the narrative of financial crises more generally. At last, while some past literature exists (see for

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instance Miller [53]) on the consequences of beliefs dis-agreement for asset prices, none exists on the role of risk-attitudes' divergence.

Our paper contributes to the expanding literature on the role for financial markets of demand and pecuniary externalities. Demand externalities, which generally arise in presence of limit constraints which prevent prices from adjusting, have a long tradition in economics. For a recent application to credit markets see Korinek and Simsek[41]. Pecuniary externalities were first examined in a general equilibrium context by Greenwald and Stiglitz[26], who showed that in presence of market frictions changes in excess demand do not net out, thereby producing changes in market prices that affect the welfare of all agents. Recent contributions on the role of pecuniary externalities are Lorenzoni[47], Jeanne and Korinek[35], Dávila and Korinek[17]. Our model too features fire-sale externalities emerging by the presence of the collateral constraint coupled with the general equilibrium pricing mechanism. However and contrary to some of the above-mentioned models our analysis does not take a normative stand, but is rather devoted to examine the interaction of externalities with behavioral elements in the emergence of crises and for the dynamic of leverage and asset prices.

Our paper is related to the vast literature studying the effects of collateral constraints on the real economy and the related amplification effects. The literature includes early contributions such as Kiyotaki and Moore[40] or more recent ones such as Mendoza[50], Bianchi and Mendoza[11], Korinek and Mendoza[40]. Our model is closer to the second ones since we take into account the occasionally binding nature of the collateral constraint. This has important consequences as at any time t agents form expectations on whether the constraint will be binding in the future. Those anticipatory beliefs have an impact on current decisions, they contribute to the amplification effect and determine the sudden and sharp switches between leverage/de-leverage states.

The economic rationale behind collateral constraints is the presence of some limited enforcement or agency problems (moral hazard in Hart and Moore[31] or asymmetric information in Gale and Hellwig[23]) that can be disciplined through debt contracts. In all circumstances in which the borrower can repudiate the debt contract (repo contracts, money markets, mortgages, etc.) and/or divert collateral, debt is superior to equity in protecting lenders. Collateralized debt in fact obviate the need for price discovery since the lender is assigned the right to redeem collateral and since the

contract terms result from a bargaining process that determines how total surplus is split between lenders and borrowers (see Holmstrom[34]). In our case lenders and borrowers feature different asset valuations. Hence the shadow price of debt, which also proxies the contract terms, results from equilibrating divergent lenders/borrowers evaluations.

At last, our model is related to the growing literature that connects asset price volatility and bubbles to alternative preference specifications or alternative expectation formation processes. Adam, Marcet and Nicolini [2] for instance study the effects of subjective prior beliefs on the asset price. We examine the impact of divergent risk-attitude on the economy inclination toward excessive leverage and risk-taking. We model risk-tolerance by resorting on the properties of loss-averse utilities, which were initially pioneered by Kahneman and Tversky[37], and have been recently embedded into consumers' optimization problems or in other dynamic models (see Kõszegi and Rabin[43], Dubin, Grishchenko and Kartashov [18] or Yogo[66] among others). Our paper is related to recent works that rationalize the outcomes of reference-dependent utilities (see Woodford[67]). Extensive experimental evidence established that those preferences are better suited at capturing behavior in presence of risk and embed all gradations of risk-attitudes at large and small gambles. We will exploit precisely those properties to differentiate borrowers' and lenders' risk-tolerance and to model borrowers' incentives toward inter-temporal borrowing and risk-taking. Since our model is concerned also with assessing the effects of differential risk-valuations on equilibrium market prices we build a general equilibrium model: see also Santoro, Petrella, Pfajfar and Gaffeo[60] for recent work introducing loss averse preferences in a general equilibrium model.

3   The Model

We consider an economy with lenders and borrowers. The latter acquire debt in order to invest in a risky asset. Lenders and borrowers receive labor income and, at each date, choose consumption and investment. Lenders invest in a liquid asset, B^l, which pays a gross return R^f. Borrowers lever up to invest in a risky asset, S^b, that pays a net dividend of d and has ex-dividend price p. The risky assets can be interpreted as equities. Borrowers' demand for debt is limited by an occasionally binding collateral constraint, which serves the purpose of disciplining limited enforcement. We assume that both the dividend and the labour income follow random Markov-stationary processes.

Borrowers and lenders feature heterogenous income processes. In addition lenders and borrowers feature additional sources of heterogeneity. First, they discount future differently. Borrowers are relatively more impatient, hence in equilibrium they will not accumulate enough precautionary savings to ease up the collateral constraint. Second, borrowers exhibit diminishing risk-sensitivity on the tails relatively to lenders[13].

3.1 Lenders' optimization

The economy is populated with a fraction ν of households who are endowed with labor income w_t^l and maximize a bounded[14] utility of consumption:

$$\max_{\{C_t^l, B_t^l, S_t^l\}_0^\infty} \mathbb{E}_0 \left\{ \sum_{t=0}^\infty \beta^t U^l \left(C_t^l, X_t \right) \right\} \tag{1}$$

subject to the budget constraint:

$$C_t^l = w_t^l + R_t^f B_{t-1}^l - B_t^l \tag{2}$$

where $\mathbb{E}_0$ denotes the expectation operator conditional on information at time $t = 0$, C_t^l and B_t^l denote consumption and debt by households at time t and β is their subjective discount factor. The variable X_t is the time and state dependent consumption reference level. We will return on the preferences' specification later on. The first order condition of the above optimization problem with respect B_t^l reads as follows:

$$U_C^l \left(C_t^l, X_t \right) = \beta \mathbb{E}_t \left\{ U_C^l \left(C_{t+1}^l, X_t \right) R_{t+1}^f \right\} \tag{3}$$

Equation 3 is the Euler condition on bonds. For optimality a No-Ponzi condition on both assets shall hold, so that $\lim_{k \to \infty} \mathbb{E}_t \left\{ \beta^k U_C^l \left(C_{t+k}^l, X_t \right) B_{t+k}^l \right\} = 0$.

We will discuss in a separate section the utility specification. At this stage it suffices to introduce a few considerations. Consider debt as akin to a commodity transferred from lenders to borrowers. In this context and in presence of borrowing constraints, the shadow price of debt, hence the tightness of the borrowing limit, is determined endogenously by equilibrating the lenders'

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[13]Practically this translates in a different degree of absolute risk-aversion and in a different sensitivity to the gain-loss vis-a-vis the consumption reference level.

[14]This is the minimal requirement that we impose at the moment on lenders' preferences.

marginal propensity to supply debt and the borrowers' marginal propensity to acquire debt. As distance in the risk-tolerance between the two raises larger fluctuations in the shadow price of debt (and generally in asset returns) are required to equilibrate the market. Generally speaking we will assume that borrowers are relatively more risk-tolerant on the tails. The relative higher risk-tolerance of the borrowers implies that they are more prone to risk-taking, particularly so at large gambles. Furthermore, we will assume that lenders exhibit lower desire of inter-temporal consumption substitution and are more inclined to precautionary saving. The relative stronger incentive of borrowers toward inter-temporal substitution also implies their tendency toward more inter-temporal borrowing, a feature which contributes to generate the exuberant leverage cycles.

3.2 Borrowers' optimization

A fraction $1 - \nu$ of the population is composed by identical borrowers. They consume and invest in a risky asset, such as equity. To do so they need to acquire external funds. Borrowers choose consumption, C_t^b, debt and investment in risky assets to maximize:

$$
\max_{\{C_t^b, B_t^b, S_t^b\}_0^\infty} \mathbb{E}_0 \left\{ \sum_{j=0}^\infty \rho^j U^b(C_t^b, X_t) \right\}
\tag{4}
$$

subject to the following budget constraint:

$$
C_t^b = w_t^b - R_t^f B_{t-1}^b + B_t^b + S_{t-1}^b(p_t + d_t) - S_t^b p_t
\tag{5}
$$

where w_t^b is the borrowers' labor income. B_t^b and S_t^b denote debt and stock holding of borrowers/investors at time t.

Borrowers' decisions is also limited by the following collateral constraint (in Appendix A we provide a microfoundation of the collateral constraint based on a no-default enforcement constraint):

$$
R_{t+1}^f B_t^b \leq \phi S_t^b \mathbb{E}_t \{p_{t+1}\}.
\tag{6}
$$

where $\mathbb{E}_t$ is the expectation operator given the information set at time t. Collateral constraints provide discipline device in presence of an underlying agency problems. Because of borrowers' moral hazard and or asymmetric information, the lender ensures repayment by pledging collateral in case of borrowers' bankruptcy. Under this perspective the collateral constraint can be rationalized as

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arising from the incentive compatibility condition of a standard debt contract which ensures that default never occurs in equilibrium (see Hart and Moore[31] and more recently Simsek[62]). Notice that the contractual interest rate on debt, R^f_{t+1}, albeit paid at the beginning of next period, is know in period t.

The Lagrangian for the borrowers' optimization problem is

$$\mathbf{L} = \mathbb{E}_0 \left\{ \sum_{j=0}^{\infty} \rho^j U^b(w^b_t - R^f_t B^b_{t-1} + B^b_t + S^b_{t-1}(p_t + d_t) - S^b_t p_t, X_t) - \lambda_t \left(R^f_{t+1} B^b_t - \phi S^b_t \mathbb{E}_t \{p_{t+1}\} \right) \right\} \tag{7}$$

Define λ_t as the lagrange multiplier on the collateral constraint. This variable, which also represents the shadow value of debt, plays a crucial role in our model and it is determined endogenously by the lenders'/borrowers' distance in valuations. The first order conditions, with respect to B^b_t and S^b_t, of the above Lagrangian problem read as follows:

$$U^b_C(C^b_t, X_t) = \rho \mathbb{E}_t \left\{ U^b_C(C^b_{t+1}, X_{t+1}) R^f_{t+1} \right\} + \lambda_t R^f_{t+1} \tag{8}$$

$$p_t U^b_C(C^b_t, X_t) = \rho \mathbb{E}_t \left\{ U^b_C(C^b_{t+1}, X_{t+1})(p_{t+1} + d_{t+1}) \right\} + \lambda_t \phi \mathbb{E}_t \{p_{t+1}\} \tag{9}$$

Equation 8 above is the borrowers' Euler condition with respect to debt, B^b_t, while equation 9 is the borrowers' Euler conditions with respect to the risky asset, S^b_t. The above optimality conditions will hold alongside with No-Ponzi conditions on accumulated debt and asset wealth (at the final date). Hence, $\lim_{k \to \infty} \mathbb{E}_t \left\{ \rho^k U^b_C(C^b_{t+k}, X_{t+k}) B^b_{t+k} \right\} = 0$ and $\lim_{k \to \infty} \mathbb{E}_t \left\{ \rho^k U^b_C(C^b_{t+k}, X_{t+k}) p_{t+k} S^b_{t+k} \right\} = 0$.

Notice that we have assumed $\beta > \rho$ which implies that households are more patient than borrowers. This serves two purposes. First it prevents the borrower from doing enough precautionary saving so as to ease up the collateral constraint permanently. Second, we will show below that it provides a sufficient condition for the collateral constraint to bind.

3.3 General preferences specification

Preferences are modelled based on a loss averse reference-dependent utility of consumption a' la Kõszegi and Rabin[43]. The gain' loss utility has acquired significant empirical standing since first introduced by Kahneman and Tversky[37]. Among other things this utility specification is able to

capture better the risk-behavior at both small and large gambles. Furthermore, several authors have shown its ability in explaining asset price as well as portfolio allocation puzzles (see previous literature review). Following Kõszegi and Rabin[43] the general functional form reads as follows:

$$U^i(C_t^i, X_t) = \alpha W^i(C_t^i) + (1 - \alpha)\mathcal{W}^i(C_t^i, X_t)$$

$$\mathcal{W}^i(C_t^i, X_t) = \begin{cases} \Lambda \dfrac{|\frac{(C_t^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma}|^{1-\theta}}{1-\theta}, & \text{if } C_t^i < X_t, \\ \dfrac{\left(\frac{(C_t^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma}\right)^{1-\theta}}{1-\theta}, & \text{if } C_t^i \geq X_t, \end{cases} \tag{10}$$

where C_t^i is consumption and the index $i = l, b$. First it is useful to introduce a few general considerations on gain-loss utilities. Next, we will discuss how the utilities are differentiated between borrowers and lenders. In each period t the parameter Λ captures the degree of loss aversion. Define $\Delta = C^i - X$ as the gain-loss. The function above satisfies the standard loss-gain utility assumptions, namely $\mathcal{W}(\Delta)$ is continuous and strictly increasing for all $\Delta \in R$, is twice differentiable for all $\Delta \neq 0$, and $\mathcal{W}''(\Delta) \leq 0$ for all $\Delta > 0$, $\mathcal{W}''(\Delta) \geq 0$ for all $\Delta < 0$ and $\mathcal{W}(\Delta^1) + \mathcal{W}(-\Delta^1) < \mathcal{W}(\Delta^2) + \mathcal{W}(-\Delta^2)$ for $\Delta^1 > \Delta^2 > 0$ and $\lim_{\Delta \to 0} \mathcal{W}'(-\Delta^2)/\mathcal{W}'(\Delta^2) = \Lambda > 1$. Those assumptions imply that the utility is monotonic with respect to gains, that there is diminishing sensitivity to gains (the more so the higher the Λ) and that losses resonate more than gains. At last notice that in the limit of the steady state, that is when $C_t^i \to X_t$ the utility $U^i(C_t^i, X_t) \to \alpha W^i(C_t^i)$, hence all agents exhibit standard preferences and $\theta \in [0, 1)$.

Notice that we choose to embed loss-aversion in consumption rather than wealth. Some past influential literature in behavioral finance has chosen to embed loss aversion in wealth[15]. While this choice has a valid rationale for the study of risk in presence of idiosyncratic shocks, it does not serve well our purpose, which is that of assessing the response of risk to slowly decaying income and wealth shocks. Such dynamic path of the income process requires an optimizing-saving decision. Another important choice to be made on the specification of the utility concerns the modelling of the reference point. To fully exploit the role of non-linearities we choose a time-varying and state-dependent reference point, namely past consumption. This implies that risk sensitivity, hence borrowers' risk-taking, changes depending on whether income and asset shocks bring borrowers'

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[15]See Benartzi and Thaler [9] or Barberis and Huang[6].

wealth above or below the reference point[16]. We will return on this point more extensively later on.

Crucially we will assume that borrowers and lenders differ in their risk-attitude. Generally speaking we assume the lenders have larger precautionary saving motives (and less inter-temporal substitution motives) and that they are more highly risk-averse at small gambles. On the other side, borrowers exhibit diminishing sensitivity at large gambles (hence they tend to be risk-takers), are more responsive to gain-losses and engage more into inter-temporal borrowing. The latter feature will contribute to excessive leverage. In the analytical derivations we discuss those differences for general functional forms, while we pin down a specific calibration for the utilities in the simulation section.

Following Dubin, Grishchenko and Kartashov [18] or Yogo[66] we assume that the reference level of consumption X_t is given by a fraction of the per-capita consumption in the previous period.

$$X_{t+1} = bC_t \tag{11}$$

where $C_t = \nu C_t^l + (1-\nu)C_t^b$ is the per-capita consumption in the economy and $0 < b < 1$.

To get more insight into the dynamics of the reference level let $G_{t+1}^i = \frac{C_{t+1}^i}{C_t^i}$ and $S_t^i = \frac{vC_t^i}{\nu C_t^l + (1-\nu)C_t^b}$ be growth rate and the share of borrowers' consumption, respectively. Finally, denote the consumption-habit ratio as $Y_t^i = \frac{C_t^i}{X_t}$.

For borrowers, who also invest in the risky asset, we can write the log consumption-habit ratio as $\overset{\wedge b}{y}_{t+1} = \log\left(\frac{C_{t+1}^b}{X_{t+1}}\right) = -\log(b) + \overset{\wedge b}{g}_{t+1} + \overset{\wedge b}{s}_t$ where variables with a hat are in logs, $\overset{\wedge b}{g}_{t+1}$ is log consumption growth, $\overset{\wedge b}{s}_t$ represent log changes in the reference point and where ζ is the steady state level of the reference point. Note that $\overset{\wedge b}{g}_{t+1}, \overset{\wedge b}{s}_t$ have to be determined endogenously in equilibrium and thus the consumption-habit ratio is fully endogenous in our model.

It is useful to derive at this stage the Arrow-Pratt measure of absolute risk aversion for gain-loss

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[16]Notice that an alternative choice would be to model the reference point as past consistent expectations of consumption (see Kõszegi and Rabin[43]). This second alternative assigns a role to future expectations in the consumer's decision problem. However this would be redundant in our case since the role of future expectations enters the borrowers' saddle point optimization already through the future value of the collateral in the borrowing constraint.

part of the utility (namely $\mathcal{W}^i(C_t^i, X_t)$):

$$r^A(C_t^i, X_t) = \begin{cases} \Lambda \left| \frac{(C_t^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma} \right|^{-\theta}, & \text{if } C_t^i < X_t, \\ \left(\frac{(C_t^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma} \right)^{-\theta}, & \text{if } C_t^i \geq X_t, \end{cases} \tag{12}$$

The above Arrow-Pratt metric will be used in explaining the role of loss aversion. At this stage it suffices to notice that the absolute risk-aversion is different on the two sides of the references point. Specifically, since Λ is larger than one, it is higher on the left than on the right. In other words the agent becomes more risk-sensitive when consumption is below the reference point. We will return on this point later on.

3.4 The equilibrium

Market equilibrium is defined as interest rates on debt, asset prices and portfolio choices of lenders and borrowers $\{R_t^f, p_t, B_t^l, B_t^b, S_t^b\}$ such that: i) households choose B_t^l to maximize 1 subject to 2, ii) borrowers choose B_t^b, and S_t^b to maximize 12 subject to 5 and 6, iii) and the markets for equity, debt and consumption clear, that is:

$$S_t^b(1 - \nu) = 1 \tag{13}$$

$$\nu B_t^l + (1 - \nu)B_t^b = 0 \tag{14}$$

$$\nu C_t^l + (1 - \nu)C_t^b = \nu w_t^l + (1 - \nu)w_t^b + d_t \tag{15}$$

Note that conditions 14 and 15 automatically imply that 13 holds for any t. This is so by Walras law.

3.5 Endogeneity of the borrowing limit

Before proceeding with the model solution it is useful to characterize the conditions under which the collateral constraint binds in equilibrium.

Lemma. *The collateral constraint binds if and only if* $\beta \frac{\mathbb{E}_t\{U_C^l(C_{t+1}^l)\}}{U_C^l(C_t^l)} \geq \rho \frac{\mathbb{E}_t\{U_C^b(C_{t+1}^b)\}}{U_C^b(C_t^b)}$.

By merging together equations 3 and 8 we obtain the following condition:

$$\beta \frac{\mathbb{E}_t\left\{U_C^l\left(C_{t+1}^l\right)\right\}}{U_C^l\left(C_t^l\right)} = \rho \frac{\mathbb{E}_t\left\{U_C^b\left(C_{t+1}^b\right)\right\}}{U_C^b\left(C_t^b\right)} + \lambda_t' \tag{16}$$

where $\lambda_t' = \frac{\lambda_t}{U_C^b(C_t^b)}$ is the shadow price of debt adjusted for the borrowers' marginal utility.

Condition 16 states that that in expected terms the shadow price of debt is given by the difference

between the marginal evaluation of the borrowers (the ones who demand debt) and of the lenders (the one who supply debt). Consider debt as a commodity: its shadow price will be positive (that is the collateral constrain is binding) to the extent that borrowers' and lenders marginal evaluation differ. More specifically the constraint will bind to the extent that:

$$\beta \frac{\mathbb{E}_t \left\{ U_C^l \left(C_{t+1}^l \right) \right\}}{U_C^l \left(C_t^l \right)} \geq \rho \frac{\mathbb{E}_t \left\{ U_C^b \left(C_{t+1}^b \right) \right\}}{U_C^b \left(C_t^b \right)} \tag{17}$$

The above condition intuitively states that to the extent that the lenders price the risk of debt more than the borrower, they will require a risk premium in the form of a positive $\lambda_t^{'}$. In this respect the shadow price of debt represents the premium that borrowers shall transfer to lenders to convince them to supply funds in the event that their relative evaluations of risk differ. When the constraint is binding lenders bear the risk of fluctuations in the collateral value, which they will receive in case of default. Hence lenders require a premium, $\lambda_t^{'}$. The higher is the differences in the evaluation of the risk attached to debt, the higher is the compensation that lenders will require.

Evaluations might differ under various circumstances. First, consider the case in which borrowers and lenders have the same stochastic discount factor (the same preferences for risk). Given income and dividend shocks for which $\frac{\mathbb{E}_t \left\{ U_C^l \left(C_{t+1}^l \right) \right\}}{U_C^l \left(C_t^l \right)} = \frac{\mathbb{E}_t \left\{ U_C^b \left(C_{t+1}^b \right) \right\}}{U_C^b \left(C_t^b \right)}$ a sufficient condition for the constraint to bind is that $\beta > \rho$. However the above lemma also implies that differences in risk-attitudes between borrowers and lenders, as captured by the differences in the stochastic discount factor, affect the tightness of the borrowing limit. When borrowers price debt risk less than lenders, the latter wish to be compensated.

In the next sections we examine how the difference in preferences, as well as the risk-tolerance of borrowers, affect both the endogenous tightness of the borrowing constraint and the price of the risky asset in equilibrium. First, for analytical tractability and to focus on the role of loss-aversion, we momentarily set $\alpha = 0$. This parameter will be set to 0.5 in the simulations (see calibration session below). Second, since some of the analytical results will be stated in terms of the borrowers' inter-temporal marginal rate of substitution (IMRS), we characterize it below.

Proposition 1. *The borrowers' IMRS between periods t and $t + s$ takes the form:*

$$m_{t,t+s}^b = \rho^s e^{-\gamma \hat{g}_{t+s}^b} \frac{k(\hat{y}_{t+s}^b)}{k(\hat{y}_t^b)} \tag{18}$$

The exact functional form of $k(\overset{\wedge b}{y_t})$ depends on borrowers' preferences. Under power reference-dependent utility (i.e., $\theta \in [0,1)$ and $\Lambda > 1$):

$$k(\overset{\wedge b}{y_t}) = \begin{cases} \Lambda \left| \frac{(C_t^b)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma} \right|^{-\theta}, & \text{if } \overset{\wedge b}{y_t} < 0, \\ \left(\frac{(C_t^b)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma} \right)^{-\theta}, & \text{if } \overset{\wedge b}{y_t} \geq 0, \end{cases} \tag{19}$$

Under linear reference-dependent utility (i.e.,$\theta = 0$ and $\Lambda > 1$):

$$k(\overset{\wedge b}{y_t}) = \begin{cases} \Lambda, & \text{if } \overset{\wedge b}{y_t} < 0, \\ 1, & \text{if } \overset{\wedge b}{y_t} \geq 0, \end{cases} \tag{20}$$

Under power utility (i.e.,$\theta = 0$ and $\Lambda = 1$), $k(\overset{\wedge b}{y_t}) = 1 \ \forall \overset{\wedge b}{y_t}$.

Proof. *See Appendix B.*

3.6 Shadow price of debt: limited commitment versus risk-attitudes

Given the above equilibrium characterization we are now interested in analyzing the link between the shadow price of debt, which determines the tightness of the collateral constraint, and the agents' stochastic discount factor, which measure their risk-attitudes. In this section we provide analytical results and delegate the quantitative implications to section 4 below.

For the analytical results we proceed in steps. First, we derive the model implied equilibrium condition linking the shadow price of debt, λ_t, to the differences in the stochastic discount factors between lenders and borrowers. Debt in our model is akin to a commodity which is transferred from lenders to borrowers. Lenders transfer funds to borrowers, who invest in a risky asset. However lenders shall be compensated, through the transfer of a risk premium for taking up the risk of fluctuations in the future value of collateral. In this context the shadow price of debt, λ_t, which proxies the risk-premium received by lenders, must equilibrate the lenders and borrowers debt' evaluations as captured by their respective stochastic discount factors.

Next, to highlight the role of preferences for the endogenous tightness of the collateral constraints we consider two cases. The first is a notional economy in which lenders are risk-neutral, hence they are willing to lend at the market risk-free rate, R_t^f, and do not require risk premia. In this case the shadow price of debt is determined solely by borrowers' stochastic discount factor. To be sure, we are not thinking that lenders are risk-neutral. However this case is useful since it allows

us to disentangle exactly the role of borrowers' preferences and risk-tolerance for the endogenous determination of the credit constraint. In the second case we assume that both agents have general loss-averse preferences and examine the impact of differences in risk-tolerance on the shadow price of debt.

When lenders are risk-neutral the pricing of debt is determined solely by the borrowers' first order conditions, 8 and 9, which merged together deliver:

$$\lambda_t' = \frac{\lambda_t}{U_C^b(C_t^b, X_t)} = \left[\frac{1}{R_{t+1}^f} - \mathbb{E}_t \left\{ m_{t,t+1}^b \right\} \right] \tag{21}$$

where $\frac{1}{R_{t+1}^f} - \mathbb{E}_t \left\{ m_{t,t+1}^b \right\}$ captures the difference in the price of risk between borrowers and lenders (whose stochastic discount factor is now given by the risk-free market rate). First notice that from equation 21 a sufficient and necessary condition for the collateral constraint to bind is that $\frac{1}{R_{t+1}^f} > \frac{1}{\mathbb{E}_t \left\{ m_{t,t+1}^b \right\}}$. Intuitively in the limiting case in which the expected market risk-free rate is lower than the return offered by borrowers ($\mathbb{E}_t \left\{ m_{t,t+1}^b \right\} > R_{t+1}^f$, the latter become credit constrained.

Proposition 2. *If lenders were risk-neutral and assuming that $\overset{\wedge b}{g}_t$ follows a normal distribution, $N(\mu, \sigma^2)$, the shadow price of debt, λ_t, is lower in states to the right of the reference point and higher in the opposite case.*

Proof. In the interest of analytical tractability we place the shock at the level of consumption rather than of income. Following Tallarini[64] we assume that the consumption growth, $\overset{\wedge b}{g}_t$, follows a normal distribution, $N(\mu, \sigma^2)$. In this case the expected value of the borrowers' stochastic discount factor can be written as (see Appendix C):

$$\mathbb{E}_t \left\{ m_{t,t+1}^b \right\} = \rho \exp \left\{ -\gamma\mu + \frac{(\gamma\sigma)^2}{2} \right\} \Xi_t(\gamma, \Lambda) \tag{22}$$

where:

$$\Xi_t(\gamma, \Lambda) = \begin{cases} \left[\frac{1}{\Lambda} + (1 - \frac{1}{\Lambda})F(\gamma\sigma + \frac{\hat{\kappa}_{t+1} - \mu}{\sigma}) \right] & \text{if } \overset{\wedge b}{y}_t < 0 \\ \left[1 + (\Lambda - 1)F(\gamma\sigma + \frac{\hat{\kappa}_{t+1} - \mu}{\sigma}) \right] & \text{if } \overset{\wedge b}{y}_t \geq 0 \end{cases} \tag{23}$$

and where F is the cumulative distribution function of the log-normal and $\hat{\kappa}_{t+1} = \hat{x}_{t+1} - \hat{c}_t = -\log(b) + \overset{\wedge b}{s}_t$. Equation 21 establishes (for given unitary stochastic discount factor of the lender) a

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negative relation between the shadow price of debt and the borrowers stochastic discount factor. From equation 22 and 23 we can determine that when consumption is above the reference level, hence $\overset{\wedge b}{y_t} > 0$, the borrowers' stochastic discount factor is higher. This also implies that, for given unitary stochastic discount factor of the lender, the tightness of the borrowing constraint, λ_t, is lower in booms than in recession.$Q.E.D.$

It is also worth noticing that loss averse borrowers have lower risk sensitivity on the tails. Hence at large gambles the pricing kernel $\left[\mathbb{E}_t\left\{m_{t,t+1}^b\right\}\right]^{-1}$, namely the price that borrowers attach to future risk, is lower. This implies that also the shadow price of debt would be lower were lenders willing to accept the market risk-free rate. On the one side borrowers are very risk-tolerant, hence they wish to invest and lever up more. On the other side lenders are risk-neutral, hence they are willing to transfer the funds. As a result the borrowing constraint is relaxed and leverage builds up.

It is worth examining also the other components of the borrower's IMRS. In equation 23 the precautionary saving component is given by $F(\gamma\sigma + \frac{\hat{\kappa}_{t+1}-\mu}{\sigma})$. This increases with γ, which captures the risk-aversion at large gambles (see Yogo[66]). For higher γ, when the borrower expects the constraint to bind in the future (since he experiences negative income or wealth shocks), he will de-leverage more sharply. On the contrary if he expects the constraint to be slack in the future a higher γ induces more inter-temporal borrowing (recall hat borrowers are also impatient). The value of γ therefore also contributes to explain exacerbated leverage dynamics. Second, the parameter Λ captures the sensitivity to gain and losses around the reference level. The higher the Λ, the larger is the drop in risk-sensitivity at large gambles, hence the larger is the fall in the shadow price of debt at large gambles as per Proposition 2.

Let's now examine the role of differences in agents' risk-attitudes for the endogenous tightness of the borrowing limit. To do this we consider the general case in which lenders and borrowers feature similar loss-averse preferences, but differ in their risk-tolerance (as proxied by differences in γ and Λ). The endogenous tightness of the borrowing limit is given by (this is obtained by merging :

$$\lambda_t^{'} = \mathbb{E}_t\left\{m_{t,t+1}^l\right\} - \mathbb{E}_t\left\{m_{t,t+1}^b\right\} \tag{24}$$

At first we shall notice that the shadow price of debt depends positively on the distance of the lenders/borrowers discount factors. When borrowers are more risk-tolerant than lenders, they offer higher risk-premia in exchange of liquid funds. In the quantitative simulations below we will confirm this positive relation also numerically.

Let's now examine more in detail the role played by differences in risk-attitudes as captured by γ and Λ. Since several effects are operative at the same time we resort once again on some limiting cases. Let's first consider the case in which Λ is positive for borrowers and zero for lenders and in which both agents feature the same degree of risk-aversion. In this case lenders' stochastic discount factor reads as follows:

$$\mathbb{E}_t \left\{ m_{t,t+1}^l \right\} = \beta \exp \left\{ \gamma\mu - \frac{(\gamma\sigma)^2}{2} \right\} \tag{25}$$

the borrower's stochastic discount factor instead is still given by equations 22 and 23. Notice that contrary to the case examined in Proposition 2 here the lender is risk-averse, hence he/she needs to be compensated for bearing the risk of fluctuations in future collateral values. The tightness of the borrowing limit is determined by the distance between $\mathbb{E}_t \left\{ m_{t,t+1}^l \right\} - \mathbb{E}_t \left\{ m_{t,t+1}^b \right\}$, which ultimately changes if the borrower is above or below the consumption reference level. Let's first consider the case in which $\hat{y}_t \geq 0$. In this case the borrower discount factor contains an additional term $(\Lambda - 1)F(\gamma\sigma + \frac{\overset{\wedge}{\kappa}_{t+1}-\mu}{\sigma})$ with respect to the lender's discount factor (recall that $\Lambda > 1$). If all other parameters determining risk-aversion are left the same, this means that consumption growth[17] would determine a fall in the shadow price of debt. Intuitively in booms (when consumption is above the reference level), borrowers become optimistic (more risk-tolerant), wish to invest more and lever up more. The growth in leverage is obtained through relaxing the borrowing limit, hence a fall in λ_t. Things are different when income shocks push agents in the region in which $\hat{y}_t < 0$. Here the borrower stochastic discount factor turns lower than the lender's stochastic discount factor. This implies an increase in λ_t and a tightening of the borrowing limit. In recession sudden shifts in the status of the borrowing limit are amplified when the distance in lenders'/borrowers' evaluations becomes higher.

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[17]Notice that $F(\gamma\sigma + \frac{\kappa_{t+1}-\mu}{\sigma})$ grows with κ_{t+1}, which in turn grows with $x_{t+1} - c_t$.

3.6.1 The Link between the Shadow Price of Debt and Asset Price Growth

Having examined the link between the risk-attitudes and the shadow price of debt, let's now examine the link between the latter and the returns on the risky assets. The periods prior to a financial crisis are typically characterized by growing leverage and asset price booms. We shall examine if this correspondence exist in our model. Once again for analytical tractability we resort to the case in which lenders are risk-neutral, hence the return requested to supply funds is the risk-free rate. From 21 let's re-define the marginal utility adjusted shadow value of debt:

$$\lambda_t^{'} = \frac{\lambda_t}{U_C^b(C_t^b, X_t)} = \left[\frac{1}{R_{t+1}^f} - \mathbb{E}_t\left\{m_{t,t+1}^b\right\}\right] \tag{26}$$

From 26 we can determine a new expression for the borrowers stochastic discount factor which reads as follows:

$$\mathbb{E}_t\left\{m_{t,t+1}^b\right\} = \frac{1 - R_{t+1}^f \lambda_t^{'}}{R_{t+1}^f} \tag{27}$$

To examine the role of the shadow price of debt for asset returns we merge equation 27 with the borrowers' first order condition for risky assets. Rearranging borrowers' first order condition on risky assets, equation 9, we obtain:

$$1 = \rho \mathbb{E}_t\left\{m_{t,t+1}^b R_{t+1}^s\right\} + \lambda_t^{'}\phi \mathbb{E}_t\left\{\frac{p_{t+1}}{p_t}\right\} \tag{28}$$

where we set $R_{t+1}^s = \frac{p_{t+1}+d_{t+1}}{p_t}$. Equation 28 determines the risky asset pricing kernel based on the Harrison and Kreps[30] theorem.

Since $\mathbb{E}_t\left\{m_{t,t+1}^b R_{t+1}^s\right\} = \mathbb{E}_t\left\{m_{t,t+1}^b\right\}\mathbb{E}_t\left\{R_{t+1}^s\right\} + Cov(m_{t,t+1}^b, R_{t+1}^s)$ and upon substituting $\mathbb{E}_t\left\{m_{t,t+1}^b\right\} = \frac{1 - R_{t+1}^f \lambda_t^{'}}{R_{t+1}^f}$, we get the following expression linking the return on the risky asset and the shadow price of debt:

$$\mathbb{E}_t\left\{R_{t+1}^s\right\} = \frac{R_{t+1}^f\left[1 - \rho Cov(m_{t,t+1}^b, R_{t+1}^s) - \lambda_t^{'}\phi\mathbb{E}_t\left\{\frac{p_{t+1}}{p_t}\right\}\right]}{\rho(1 - R_{t+1}^f \lambda_t^{'})} \tag{29}$$

We can then derive the following premium between the return on the risky asset and the risk-free rate as ratio between $\mathbb{E}_t\left\{R_{t+1}^s\right\}$ and R_{t+1}^f:

$$\Psi_t = \frac{\left[1 - \rho Cov(m_{t,t+1}^b, R_{t+1}^s) - \lambda_t^{'}\phi\mathbb{E}_t\left\{\frac{p_{t+1}}{p_t}\right\}\right]}{\rho(1 - R_{t+1}^f \lambda_t^{'})} \tag{30}$$

Several considerations emerge. First, we examine the term $-\lambda_t^{'}\phi\mathbb{E}_t\left\{\frac{p_{t+1}}{p_t}\right\}$. As agents hold expectations of future capital gains, hence of future increases in the warrant value of the asset, they are willing to accept lower returns from the risky assets at the current period. If the shadow price of debt increases, which means that borrowers are more credit constrained, to the extent that borrowers can compensate lenders with future capital gains they are willing to accept lower dividends today. Through this channel there is a negative relation between the shadow price of debt and the spreads on risky assets. Second, as the expected market risk-free rate, R_{t+1}^f, raises the premium for leveraging and investing in risky assets shall raise in proportion to the tightness of the borrowing constraint, $\lambda_t^{'}$. Intuitively when borrowers are more credit constrained, they shall transfer higher premia to lenders. To do so they need to invest in higher yield/dividend assets. Through this channel there is a positive relation between the shadow price of debt and the spreads on the risky asset. Whether an increase in $\lambda_t^{'}$ produces a decrease or an increase of the current spreads depends upon whether the first or the second channel prevails. If expectations of future capital gains are very strong, borrowers are willing to accept assets with lower dividends at current periods. At last, an increase in the covariance between, $m_{t,t+1}^b$ and R_{t+1}^s, implies that borrowers are better hedged against consumption risk, hence they require lower returns on the risky investment.

3.7 Asset prices: limited commitment versus risk-attitudes

It is likely that in a general equilibrium context leverage growth affects growth in the price of the risky asset, a link which is again compatible with the narrative of most financial crises. Borrowers acquire funds to invest in the risky asset. As leverage becomes cheaper and its availability increases, the demand for the risky assets increases too. For given supply this triggers an increase in the price of the risky asset. It is worth noticing that the increase in asset price produces, in the context of the present model, a pecuniary externality[18]. The increase in the value of collateral indeed relaxes the borrowing constraints for all investors in the economy.

In this section we explore further the link between asset prices on the one side and leverage growth (as determined by the endogenous tightness of the borrowing limit) and borrowers' risk-

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[18]Greenwald and Stiglitz[26] have shown that this type of pecuniary externality is operative in all markets featuring distortions and in which changes in the excess demand of a group of agents do not net out with changes in excess supply.

attitudes on the other. First, we derive the price of the risky asset by iterating forward borrowers' first order condition for the risky asset. Using the derived expression for the asset price we discuss its dependence on the shadow price of debt (hence upon leverage growth), upon the warrant value of the asset and upon borrowers' stochastic discount factor. All those elements, as we explain below, confirmed the link tat we envisaged between leverage and asset price booms. Second, we are interested in characterizing also the volatility of asset prices. To this purpose we derive the Sharpe ratio and characterize the Hansen and Jagannathan[28] bounds. Notice that since the risky asset is held by borrowers, in the derivations below we will work primarily with borrowers' optimality conditions.

Proposition 3. *Let's denote by $\lambda'_t = \frac{\lambda_t}{U^b_C(C^b_t, X_t)}$ as the normalized Lagrange multiplier (i.e., the shadow price of the borrowing constraint expresses in unit of the marginal utility of consumption). Then, the price of the risky asset can be expressed as:*

$$
\begin{aligned}
p_t &= \mathbb{E}_t\left\{m^b_{t,t+1}d_{t+1}\right\} + \\
&\quad +\mathbb{E}_t\left\{\sum_{i=1}^{T} m^b_{t+i,t+i+1}d_{t+i+1}\prod_{j=1}^{i} K_{t+j-1,t+j}\right\} + \\
&\quad +\mathbb{E}_t\left\{\prod_{i=0}^{T} K_{t+i,t+i+1}p_{t+T}\right\}
\end{aligned}
\tag{31}
$$

where $K_{t,t+i} = m^b_{t,t+i} + \phi\lambda'_{t+i-1}$.

Proof. *See Appendix D.*

It is worth discussing the economic channels unveiled by equation 31. First, the price includes the so called "warrant" value of the asset, second and third component on the right hand side of Eq 31, where the general term $K_{t+i,t+i+1}$ includes the shadow price of debt . To understand the economic nature of the warrant value we note that the risky asset plays a double role in this economy. First, the asset pays dividends that are used for consumption. The value of the asset as a consumption claim is captured primarily by the first and second component of Eq 31. In addition, the asset can be pledged as collateral to secure loans. The second and third terms on the right hand side of Eq 31 include the collateral value of the asset, $\phi\lambda'_{t+i-1}$. As the latter increases the investor can lever up more easily, hence invest more in risky asset. This in turn raises the asset

price by a factor $\lambda_{t+i-1}\phi\mathbb{E}_t\{p_{t+i}\}$ for every period $t+i$ in which the collateral constraints remains binding. The term $\lambda_{t+i-1}\phi\mathbb{E}_t\{p_{t+i}\}$ proxies the warrant value of the asset since it captures by how much the asset value increases due to its future use as collateral. Notice that the second term in the asset price equation, $\mathbb{E}_t\left\{\sum_{i=1}^{T} m_{t+i,t+i+1}^b d_{t+i+1} \prod_{j=1}^{i} K_{t+j-1,t+j}\right\}$, can also be interpreted as the expected value of future dividends paid from $t+2$ to T discounted at the entrepreneur's $IMRS$, when the borrowing constraint is slack and at the normalized Lagrange multiplier when the borrowing constraint binds. If the borrowing constraint is expected to be slack at all future dates the warrant component is zero and the asset price reduces to the familiar discounted expected value of future dividend. On the contrary, if the borrowing constraint is expected to bind at future dates fluctuations in the price of equities directly affect the collateral constraint with a feedback loop effect. Asset price growth, by increasing the value of pledgable collateral, allows the borrower to lever up more. On the other side, as debt is invested in equities, leverage growth induces asset price growth.

Notice that in our framework the credit cycle amplification is affected also by borrowers' risk-attitudes through the stochastic discount factor. To understand the role of borrowers' loss aversion consider the case in which income shocks bring the borrower on the right of the reference point. Borrowers' stochastic discount factor is higher in this region, thereby the asset price increases, since $m_{t+i,t+i+1}^b$ scale up all terms of the asset price equation. Moreover the higher is the gain in consumption (vis-a-vis the reference level), the higher is his stochastic discount factor and the higher is the asset price. In sum the loss-gain utility acts as a *leverage and risk multiplier*.

At last, notice that even when imposing the No-Ponzi condition, $lim_{T\to\infty}\mathbb{E}_t\left\{\prod_{i=0}^{T} m_{t+i,t+i+1}^b p_{t+T}\right\}$, the asset price would feature a drift if the borrowing constraint were to bind at any period between t and T. This is due to the presence of the term $\prod_{i=0}^{T} \lambda'_{t+i-1,t+i} p_{t+T}$ which is positive to the extent that the $\lambda'_{t+i-1,t+i}$ are positive.

To complete the characterization of the asset price distribution we now examine the Sharpe ratio, computed by using the Hansen and Jagannathan[28] volatility bounds computation.

Proposition 4. *The Sharpe ratio for the borrower reads as follows:*

$$\frac{\mathbb{E}_t\{Z_{t+1}\}}{\sigma_z} = \left[\frac{Var(m^b_{t,t+1})}{(\mathbb{E}_t\{m^b_{t,t+1}\})^2} + 2\lambda'_t\frac{\mathbb{E}_t\{Z_{t+1}\}}{\sigma^2_z}\frac{\mathbb{E}_t\{\Sigma_{t+1}\}}{(\mathbb{E}_t\{m^b_{t,t+1}\})^2} + \frac{(\lambda'_t)^2}{(\mathbb{E}_t\{m^b_{t,t+1}\})^2}\frac{(\mathbb{E}_t\{\Sigma_{t+1}\})^2}{\sigma^2_z}\right]^{\frac{1}{2}}$$

(32)

where $Z_t = R^s_t - R^f_t, \Sigma_t = \phi\mathbb{E}_t\left\{\frac{p_{t+1}}{p_t}\right\} - R^f_t, \sigma^2_z = Var(R^s_t - R^f_t).$

Proof. *See Appendix E.*

The Sharpe ratio in our model, equation 32, depends upon the borrowers' stochastic discount factor, the shadow price of debt, and upon the difference between the expected change in the collateral recovery value, $\phi\mathbb{E}_t\left\{\frac{p_{t+1}}{p_t}\right\}$, and the rate of return on debt. Recall that the Sharpe ratio characterizes the slope of the efficient portfolio frontier of the investor, hence it gives the expected return needed to bear the amount of risk associated with a given asset returns' volatility. Also notice that we derived the Sharpe ratio in terms of excess returns, $R^s_t - R^f_t$.

First, recall that $m^b_{t,t+1}$ (and its expected value) is higher on the right of the reference level. This means that the Sharpe ratio is lower in this region. In booms the borrower requires lower expected returns for given variance of the excess returns, since its risk-tolerance is higher in this exuberant-like region. Such channel thereby accounts well for the empirically observed counter-cyclicality of the Sharpe ratio, which is usually lower in booms and higher in recessions.

Second, notice that the third term of 32 depends positively upon λ'_t. Let's us work with the hypothesis that the borrowers' stochastic discount factor is the one featured in equation 22, namely when $\hat{g}^b_t$, follows a normal distribution, $N(\mu, \sigma^2)$. When the constraint binds (λ_t becomes positive and raises) the Sharpe ratio shall raise to compensate the borrower for the increased risk of asset expropriation. The raise is proportional to the distance between the expected recovery value of collateral and the rate of return on debt. As the distance raises the borrower's expected loss in case of default and expropriation is higher. Also recall that under loss-gain preferences, losses tend to resonate more. The result clearly shows that the occasionally binding nature of the constraint might tilt asset returns. When the constraint does not bind the Sharpe ratio, hence the required asset returns for given asset risk, are lower since all terms with λ'_t are zero. When λ'_t suddenly tilts from zero to positive, a sharp and a sudden raise in the Sharpe ratios materializes. At last notice

that in times of booms, when the $\lambda_t^{'}$ declines or turn zero, the Sharpe ratio raises. This is in line with well known evidence on its countercyclicality.

4   Quantitative results

In this section we examine the quantitative implications of the model. We will use simulation methods that allows us to account for the occasionally binding nature of the collateral constraint and for the kink in the utility. As described below the method allows us to take into account the role of non-linearities and of anticipatory effects when a change in regime is expected (binding versus non-binding constraint, above and below the reference consumption level).

In presenting quantitative results we proceed as follows. First, we show and discuss the impulse response functions. This allows us to provide a first assessment of the transmission channels operating in the model. Second, we examine the policy functions which inform us on whether the model is able to generate de-leveraging (hence the leverage cycle). Third, simulate the model in response to a large sequence of shocks and compute conditional and unconditional moments. For this case we will first discuss the pattern of simulated series graphically, also in comparison to an economy where lenders and borrowers exhibit equal CRRA utilities. Additionally we compare simulated moments for a selected group of statistics to data equivalent from the US and the UK (for different time samples) to assess the empirical validity of the model in matching main stylized facts concerning financial returns, debt and consumption. In computing model statistics we will also assess the main links highlighted in the analytical session, namely the positive dependence of the shadow price of debt and of asset price growth on lenders/borrowers distance in risk-attitudes.

4.1   Calibration and numerical method

Simulation method. We employ a simulation method that allows to deal with both, the occasionally binding nature of the collateral constraints as well as the non-linearities emerging from the kinked utility. The method is based on a policy function with a Markov-switching regime structure, which allows agents to form ex ante expectations about the future regimes based on current state variables. Specifically, at any time t we need to take into account the impact on current decisions of agents' expectations with regard to the binding nature of the constraint at all future period and to the

kinked nature of the utility. Hence, and to fix ideas, consider the following cases. At time t the agent features a consumption above the reference level and a non-binding constraint. He/she might however assign positive probability to the event that the constraint will be binding in the future and to the event that consumption will be below the reference level. Both types of beliefs would trigger anticipatory de-leveraging due to precautionary savings. Other cases can be discussed in relation to the combination of all possible state space regions in the model (binding versus non/binding constraint, above versus below the reference level). We employ Markov-switching rational expectation techniques. In such a numerical environment the agents' expectation formation process and the related switching mechanism are endogenous. They depend indeed upon the state space, which in turn includes also model's predetermined variables. To solve for endogenous regime switching we rely on the monotone mapping algorithm that finds the fixed point in decisions rules, namely a policy function iteration (Coleman[14]). In every period the model can generate the 16 regimes-states which depend on whether at time t and at time $t + 1$ the collateral constraint was binding and on whether agents' preference were on the left or right of the reference point. Effectively the study of the policy function involves Markov Switching Rational Expectations in the spirit of Davig and Leeper [16], Farmer, Waggoner and Zha[20] and [21]. In Appendix F we provide more details on the simulation method and on the combination of state regions considered.

Calibration. To set calibration we follows two guiding principles. First, we choose parameters for ranges which are empirically valid according to micro data and that are well established in the various reference literature. Second, where degrees of freedom exist we choose the parametrization that allows us to get closer some target data moments.

Preferences. Time is annual. In the deterministic steady state of the model the collateral constraint is binding. This also implies that the steady state return on debt is obtained from an Euler condition which features a positive lambda. In this case the annual debt rate at around 4% is compatible with a $\beta = 0.935$. The discount factor of borrower is set at lower values (0.91). Borrowers' impatience is needed to guarantee a binding borrowing constraint at least in some states and in the long run (see Becker[7] and Becker and Foias[8] or more recently Krussell and Smith[44]). The risk-aversion parameter, γ, is set equal to 3 for both borrowers and lenders. In the steady state both agents exhibit risk-aversion with respect to the level of consumption, hence in

the long-run their behavior becomes symmetric and is summarized by the degree of risk-aversion with respect to the level of consumption as captured by the parameter γ. We set the $\alpha = 0.5$ for borrowers and to 1 for lenders[19]. This implies that lenders exhibit higher precautionary saving motives, while borrowers exhibit more inter-temporal borrowing incentives. The importance of loss aversion for professional traders, namely for investors in risky assets, is well documented in experimental evidence (see Haigh and List [27]). This coupled with the diminishing risk-tolerance of the borrowers produces enhanced leverage cycles. The parametrization is also compatible with a limiting behavior (in the steady state) which is equal for lenders and borrowers (recall that in the steady state $\lim_{C-X \to 0} U^i(C_t^i, X_t) = \alpha W^i(C_t^i)$). Among other things this limiting behavior facilitates convergence and precision of the simulations. The loss-aversion parameter, Λ, is set to 2. The value obtained by Kahneman and Tversky[37] in their experimental study. Several other experimental studies confirm values around 2[20]. A positive Λ is responsible for the diminishing risk-sensitivity of the borrower at the tails and for larger sensitivity of the borrowers' stochastic discount factor to shocks. We set the inter-temporal elasticity of substitution, θ, equal 1 and equally for both lenders and borrowers. A value larger than zero for this parameter is also crucial in generating the diminishing risk-sensitivity. Also as shown in Kandel and Stambaugh[38] and Yogo[66] a high enough value for this parameter is crucial for generating the smaller insurance premium required by consumers at small gambles relatively to the one required at large gambles. The parameter b, which determines the dependence of the reference point from past consumption, is set to 0.9. Such value for the consumption persistence is needed for the model to generate the right volatility of the return on debt (see also Yogo[66], but also Santoro, Petrella, Pfajfar and Gaffeo[60]).

The parameter ν, namely the fraction of lenders/savers in the economy, is set jointly to the loan-to-value ratio, ϕ, so as to generate an aggregate ratio of debt to asset at around 0.5, value this last one compatible with the debt data for borrowers (households who own little liquid financial assets as defined in Kaplan and Violante[39]) from the U.S. Survey of Consumer Finances. Furthermore,

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[19] The assumption that lenders do not exhibit the diminishing risk-sensitivity on the tails is also compatible with evidence presented in Haigh and List [27]. They perform an experiment with professional traders and students. they find that professional traders (borrowers in our case) exhibit more loss aversion (hence diminishing risk-sensitivity on the tails) than students.

[20] Abdellaoui, Bleichrodt, and Paraschiv [1] find median coefficients between 1.53 and 2.52, Booij and van de Kuilen [12] find coefficients between 1.73 and 2.00. Lower values are found by Pennings and Smidts [57] and Schmidt and Traub[61]. Using surveys in finance Merkle[51] finds a value around 2.

we have examined the Wealth Supplement of the PSID survey from 1960 to nowadays and we computed the fraction of borrowing families to be slightly above 30% (see Appendix for details on our use of the PSID survey for measuring the fraction of borrowers and for estimating the income and dividend processes). Given those bounds we set ν to 0.65 and ϕ to 0.60. Notice that the number for the loan to value ratio is set consistently with values observed for the industrialized countries over the period 1952-2005[21]. Robustness checks are then conducted with respect to the main parameters of the utility and with respect to the loan to value ratio, within empirically admissible ranges.

Income and dividend processes. All shocks follow AR(1) processes. Specifically the lenders' income process (all in logarithms) reads as: $w_t^l = \alpha^l + \rho^l w_{t-1}^l + \sigma_t^l \varepsilon_t^l$; the borrowers' income process for borrowers reads as follows: $w_t^b = \alpha^b + \rho^b w_{t-1}^b + \sigma_t^b \varepsilon_t^b$; finally the dividend income process reads as: $d_t = \alpha^d + \rho^d w_{t-1}^d + \sigma_t^d \varepsilon_t^d$. Appendix H details the estimation of the income processes which is done using PSID data for the time span 1968 until now. The PSID survey was conducted on annual basis from 1968 until 1997 and biannually afterwards. We collect information on family income including transfers. We then divide the families into borrowers and non-borrowers using the information in the Wealth supplement. Finally we fit the AR(1) processes detailed above through OLS estimation. The parameters of the aggregate income shocks are obtained by taking sample averages of the household level estimated income processes. The dividend process is estimated also fitting AR(1) through OLS estimation and using NIPA data from 1960. The estimated process for dividends gives a persistence of 0.56 and exhibits a standard deviation of 0.1. The estimated income process for the borrower has a persistence of 0.6 and a standard deviation of 0.12 and the estimated income process for the lenders gives a persistence of 0.62 and a standard deviation of 0.13. Hence we set the following baseline calibration for the income and dividend processes: $\alpha^l = 0.35, \alpha^b = 0.35, \alpha^d = 0.15, \rho^l = 0.6, \rho^b = 0.6, \rho^d = 0.6, \sigma_t^l = 0.06, \sigma_t^b = 0.06, \sigma_t^d = 0.06$. We then perform robustness checks by changing the standard deviation and the persistence in a close interval of the estimated values. Notice that the parameters $\alpha^l, \alpha^b, \alpha^d$ are set as in their long run values.

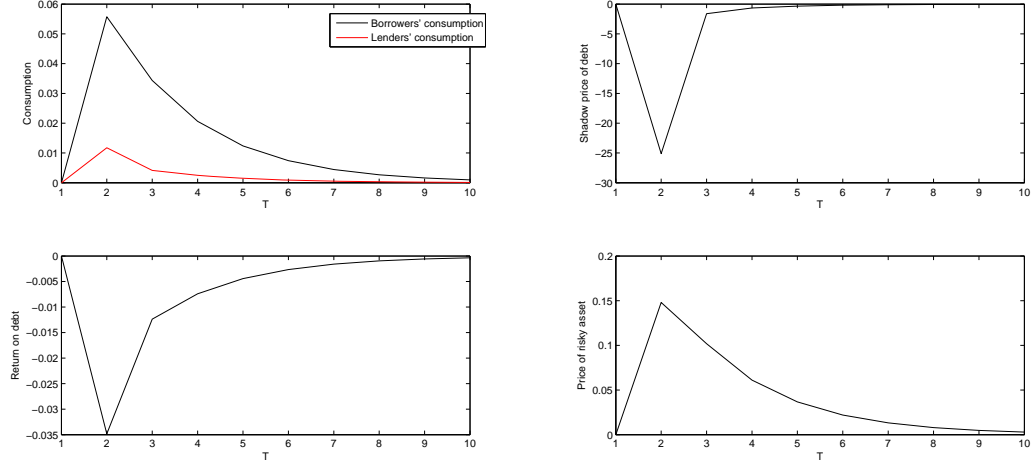[21]See IMF Report 2008 and Flow of Fund data of the U.S.

Figure 1: **Impulse responses of selected variables to one time shock to the dividend process.**

4.2 Impulse response functions

In this section we discuss the response of all variables to a one time shock to income and asset dividends respectively. Impulse responses give a first glance of the transmission mechanism operating in our model.

Figures 1, 2 and 3 below show impulse responses of selected variables to a one time positive shock respectively to dividend, to the income of the borrower and to the income of the lender.

In all cases an increase in wage or dividend income brings about an increase in consumption and in asset prices. Lenders and borrowers consume more and borrowers invest more in the risky asset. In fact the increased demand induces an increase in asset prices. The increase in asset prices is also amplified by the warrant value of the risky asset as discussed for equation 31. In all cases since we are in a boom lenders save more and the equilibrium debt rate falls. Borrowers' consumption happens to go above the reference level (consumption grows on impact compared to the past) and this increases $m_{t,t+1}^b$. As per equation 24 the shadow price of debt falls. Intuitively in booms the collateral value is expected to raise and this relaxes the borrowing limit. This is well
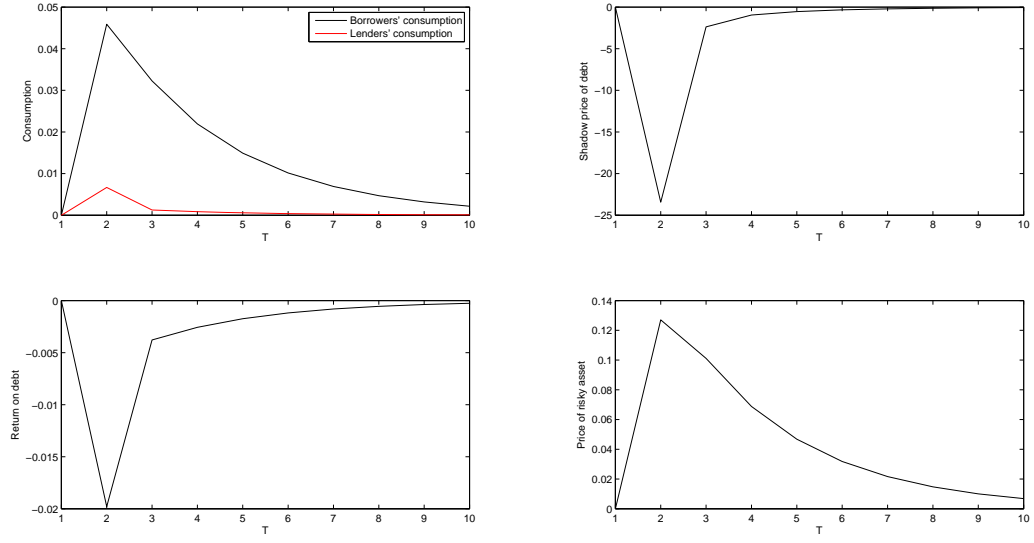
29

Figure 2: **Impulse responses of selected variables to one time shock to the borrowers' income.**
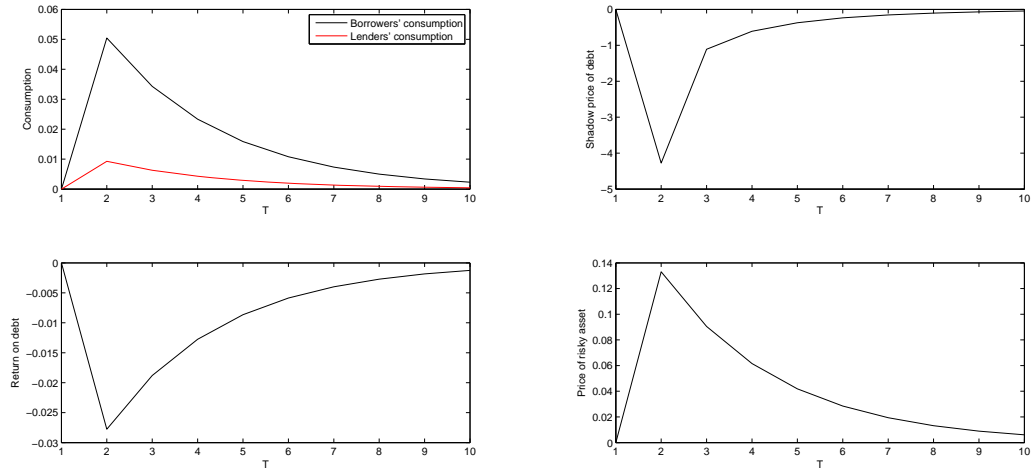

Figure 3: **Impulse responses of selected variables to one time shock to the lenders' income.**

in line with the build-up of leverage observed in expansionary phases. Another notable feature of the impulse response is the asymmetric response of borrowers and lenders consumption. The consumption response of borrowers is always more amplified even when it is lenders to experience the positive income shock. This is due to several reasons. First, borrowers are relatively more impatient and their marginal propensity to consume is higher at the kink. Second, under any shock they experience also an increase in wealth. As the asset price increases, the value of collateral increases too and this raises borrowers' wealth. Moreover as the debt rate goes down the cost of leverage decreases. Borrowers can lever up more and invest more in the risky assets, which in turn increases its value even further. Interestingly this asymmetric response is compatible with previous literature (see Mankiw and Zeldes[48] and Marcet and Singleton[49]), where it has been observed that the consumption by stockholders (borrowers in our case) seems to be much more highly correlated with stock returns than the consumption by non-stockholders.

4.3 Policy functions

Another important aspect in examining the quantitative performance of the model relates to its ability to generate the leverage cycle. The emergence of the leverage cycle can be assessed by examining the policy functions for a sequence of shock simulations and given the change in regimes.

The build up of leverage materializes on average when the borrowing constraint is slack and there positive shocks bring the borrower to the right of the reference point. As the constraint becomes binding de-leverage takes place to the extent that borrowers' precautionary saving is not too large. Korinek and Mendoza [42], Bianchi and Mendoza[11] as well as others show that deleveraging occurs in a model in which debt is constraint by the current value of collateral: as the constraint becomes binding the motives for inter-temporal substitutions prevail over the precautionary savings motives, hence in face of contractionary shocks households act with sharp deleveraging. However when debt is constrained by the future value of collateral it is likely that the precautionary saving motives prevail over inter-temporal substitution. For the model to provide a good and realistic laboratory, episodes of de-leveraging, hence crises, shall materialize. We therefore verify whether our model is capable of generating deleveraging given the shocks realizations and the regime switches. We do so by plotting the policy function of debt with respect to past debt
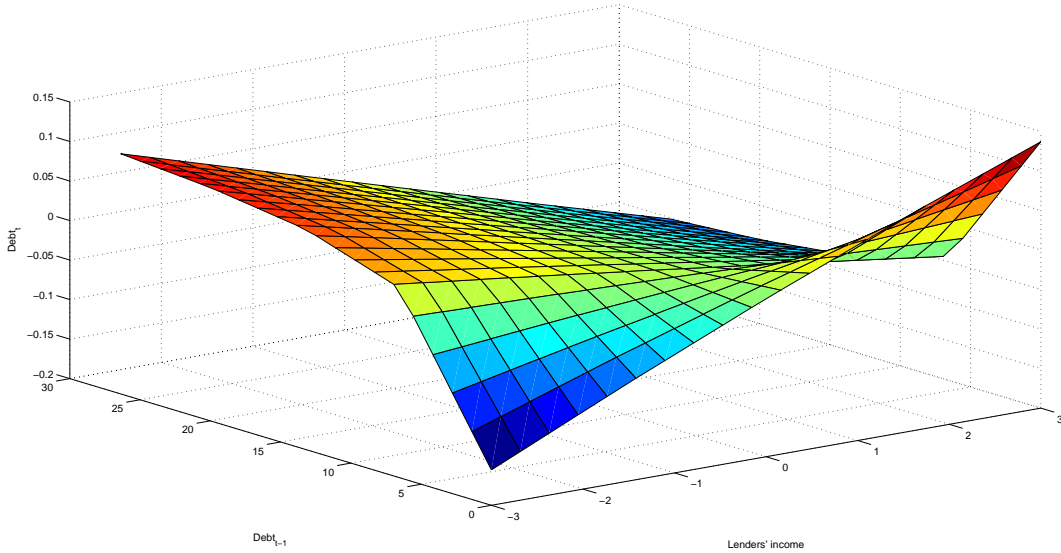
Figure 4: Policy function of debt with respect to past debt and exogenous changes in debt supply.

and to lenders' income shocks. We focus on this since past debt provides the extent to which the economy is close to the binding region, while the shock to lenders' income represents an exogenous change in debt supply. Figure 4 show the pattern of debt as a function of past debt and shocks to lenders' income.

Deleveraging occurs in our model, either when past debt is high (even for lenders' positive income shocks) or when past debt is low and lenders' income tightens, which implies a tightening of the debt supply. The pattern is rather intuitive and shows that for the benchmark parametrization, as well as under several robustness checks, the model can generate both a build up of leverage and a de-leveraging. Importantly there is no univocal cyclicality of debt with respect to the fundamental shocks (dividend and income) since the debt dynamic properties also depend upon the regime. If there are negative wealth shocks but the borrower is still on the right of the reference point debt build up might continue. Equally if there is a positive shock but the debt constraint is tight and the borrower is on the left of the reference point de-leverage might continue. This complex picture helps to rationalize evidence that debt cyclical properties often depends upon the type of borrower

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and upon the size of his wealth[22]

4.4   Simulated series and model-data moments comparison

We perform random simulations of the model with three goals in mind. First, we show and describe the pattern of simulated variables for the baseline calibration also in comparison to a reference economy in which both agents have the same CRRA utility. This discussion allows us to get further insights on the transmission mechanism of the model and on the role of loss aversion for debt and asset price dynamic. Second, we compute and compare several model- based moments with the data equivalent (for the US and the UK) to verify the ability of the model to match several stylized facts. Third, to quantify the links highlighted in the analytical part we compute the correlations between the distance in the lender/borrower stochastic discount factor and, alternatively, the change in asset prices and debt.

Figure 5 and 6 show the pattern of simulated variables in response to income and dividend shocks and for the baseline calibration. Specifically figure 5 shows debt, the shadow price of debt, the borrowers' stochastic discount factor and the lenders' and borrowers' consumption functions, while figure 6 shows the dynamic of the debt rate and the rate of return on the risky asset. The dynamic of the baseline economy (legend LA) is compared to a reference economy in which both lenders and borrowers exhibit equal standard CRRA preferences (legend CRRA). The comparison gives further insights on the role of loss-aversion in contributing to the dynamic of excessive leverage and risk-taking. At last, notice that the grey shaded areas indicate simulations for which the constraint is binding (and for which the shadow price of debt is positive as shown in panel 2 of figure 5), the opposite is true for white areas.

By examining the path of variables the following considerations emerge. First, the shadow price of λ_t is much more volatile in the economy featuring loss averse borrowers and differences in preferences between borrowers and lenders. This is due to two reasons. First, the λ_t is a function of borrowers' stochastic discount factor (third panel Figure 5) and the latter is much more volatile under loss aversion, due to the non-linear effects induced by the kink in the utility function and by the time-dependence of the reference point. Second, the shadow price of debt is

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[22]See for instance Covas and Den Haan [15] for evidence on corporate debt.

determined endogenously by the distance between lenders and borrowers stochastic discount factors, which is smaller in the reference economy (since both agents have a CRRA utility, albeit different discount factors), while it fluctuates significantly in the LA economy. The larger fluctuations of the λ_t in the LA economy, implies that debt is also much more volatile. As for the pattern of consumption two considerations emerge. First, expected consumption is higher and more volatile for borrowers. Recall that borrowers are relatively more impatient, more prone to risk and hold the risky asset. Second, consumption is more volatile in the loss-averse economy. This is due again to the non-linearities associated with the kink and the time-dependence in the reference point. At last, stock market returns appear more volatile in the loss-averse economy (this will confirmed in the computation of the unconditional moments shown in Table 1 below). Asset returns are function of the borrowers' stochastic discount factor and of the shadow price of debt. Both are more volatile in the LA economy. One last relevant consideration relates to the pattern of the debt returns, which are generally smaller and less volatile when the constraint is binding. In this case indeed the validation of the enforcement constraint provides an insurance device for the lenders, who then require lower and more compressed compensations.

Next we examine the model statistics. Tables 1 presents a set of model statistics which we compare to the data to assess the models' empirical validity. Table 3 examines correlations that speak on the link between the distance in the lender/borrowers stochastic discount factor on the one side and debt and asset prices on the other.

Table 1 below shows model based (second column for the benchmark calibration) and data-equivalent for the expected value and the volatility of the return on debt and of the rate on the risky asset, the expected value and volatility of the equity premium, the volatility of debt and of the equity premium, and the correlation between risky returns and consumption growth over the entire sample of simulations. Table 2 shows the model-based moments for alternative calibrations to test robustness.

In the last columns we show data statistics for the same variables for the U.S. and the U.K.. The data sample for asset returns and prices goes from 1960 for the U.S. and from 1970 for the U.K. up to 2016. We compute statistics for asset returns and prices over two sub-samples: 1970-2016 (for the U.S.) or 1970-2016 (for the U.K.) and 1980-2016 for both countries. The first sample allows us
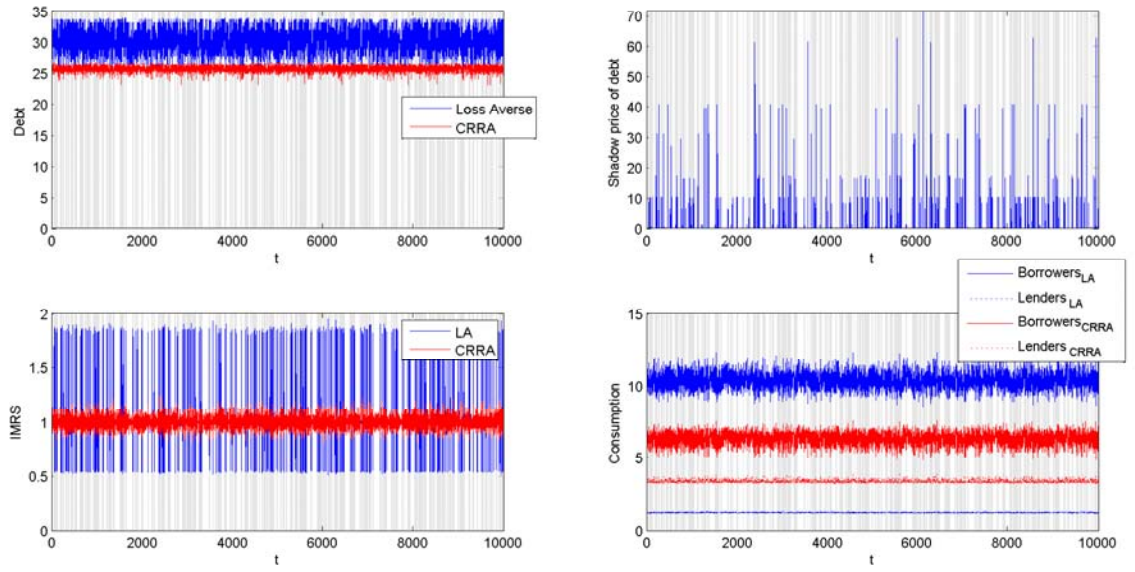
Figure 5: Simulated series in response to income and dividend shocks for debt, shadow price of debt, borrowers' stochastic discount factor and lenders/borrowers' consumption. Comparing the economy with loss averse borrowers (legend LA), versus the economy with CRRA borrowers (legend CRRA).
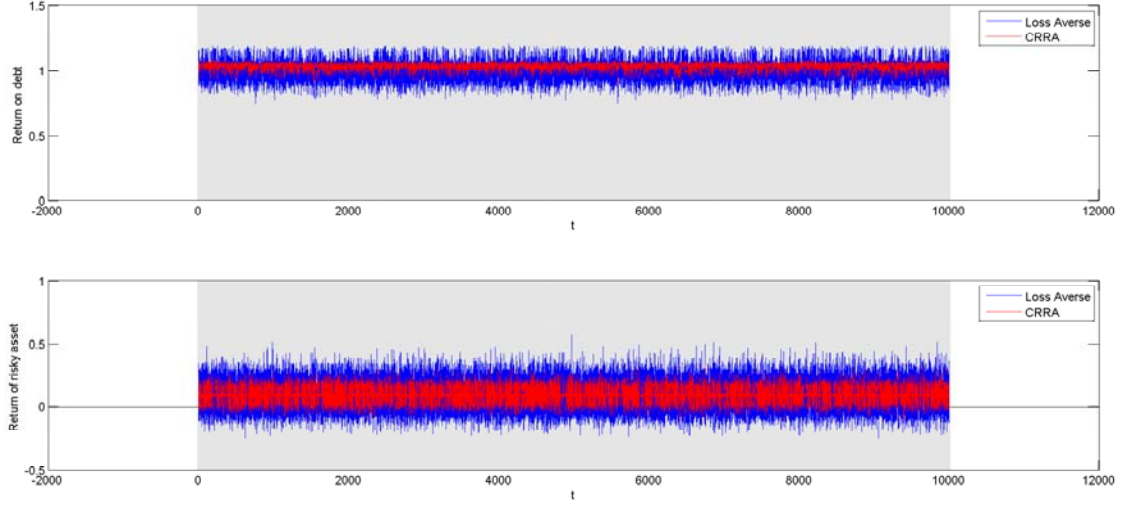
Figure 6: Simulated series in response to income and dividend shocks for debt or return on debt and the return on risky asset. Comparing the economy in which borrowers are loss averse (LA legend) to the one in which they have CRRA utilities (CRRA legend).

to examine long-run trends, while the second is well suited to capture the recent decades' expansion in leverage and asset prices. We collected data for stock market indices with dividends (S&P500 for the U.S. and the Moody's Seasoned Aaa Corporate Bond Yield for the U.K.) and for the return on collateralized debt, which we proxy either with corporate bond rates (a good proxy for market rates of traded debt) or with money market rate (a good anchor for liquid debt contracts). Finally, data for debt quantities are taken from the OECD statistics and are available for both countries only from 1996. Appendix G provides a detailed description of the data and their sources.

Several considerations emerge. First, the model-based statistics of the unrestricted sample match very well the data-equivalent. The expected value of the return on the risky asset ranges in the data between 7% and 9% depending on the sample period and on the country and is higher in the second sample period, namely the period of rapid growth in leverage and asset prices. The model expected value of the return on the risky asset is 7.651% which is well in the range for the observed values. The model also generates a high volatility of the risky asset returns, albeit smaller than the one observed in the data. The model expected value and volatility of the rate of return

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on debt falls in the interval observed in the data for the returns on corporate bonds and on money markets. Furthermore, the model does very well in matching the equity premium and its volatility. Both in the model and in the data the returns on the risky assets are the ones determining the largest part of the equity premium. Several other properties of the equity premium are matched. For instance we find pro-cyclical returns on the risky asset: correlations between returns on the risky assets and consumption are somehow higher in the model, although notice that in the data values for this correlations raise at around 0.70 if one would focus on a sample period starting in 2007. Overall the model does remarkably well in matching several cyclical properties of asset returns, equity premia and debt, along side with generally low returns on debt and the emergence of the leverage cycle. The elements that contribute to this performance are a combination of preference-based and institutional-finance based.

As noted also in Cochrane[13] the ability to match contemporaneously the long run equity premia and asset returns and their cyclical properties is related to the agents' attitude toward losses. As agents become very afraid of bad times, they tend to shy away from risky investment and tend to over-react to the possibility of future bad events. In our model the loss averse reference dependent utility achieves this outcome. Most asset price models however are silent about the joint dynamic of leverage and risky assets, while our model can capture both very well. The institutional-finance elements related to the agency problems behind the collateral constraint and its interaction with the risk-attitudes contribute to the explain this joint dynamic. Fluctuations in the price of collateral, its warrant value, contribute to fluctuations in the price of the risky asset. However this effect would not necessarily induce deleveraging in bad times, excess volatility of debt and counter-cyclicality of the equity premium. The channel that in our model helps to assemble the full puzzle is given by the dependence of the shadow price of debt upon the stochastic discount factor. As agents are hit by negative shocks it is more likely that the constraint binds (either because a fall in the lenders' income debt supply or because a fall in borrowers' income reduces his desire to invest). In this case the shadow price of debt becomes positive and is determined by borrowers'/lenders' discount factors. Consider again risk-neutral lenders. As the borrower is debt constrained, its precautionary saving attitude expands, pushing consumption to the left of the reference level. Here losses resonate more than gains, inducing the borrower to fire sales the risky

Table 1: **Comparison model-based versus empirical moments of selected variables. All values are in percentage terms.**

Statistics and Mnemonics	Model	U.S.		U.K.	
		1960-2016	1980-2016	1970-2016	1980-2016
Exp ret risky asset, R_t^s	7.651	7.05	9.35	7.17	8.23
Vol risky asset, R_t^s	9.300	16.19	15.57	19.34	15.55
Exp ret risk-free, R_t^f	2.627	$3.43^*, 1.42^{**}$	$4.30^*, 1.70^{**}$	$1.22^*, 1.43^{**}$	$2.40^*, 2.62^{**}$
Vol risk-free, R_t^f	7.573	$2.36^*, 2.41^{**}$	$2.22^*, 2.71^{**}$	$4.25^*, 4.37^{**}$	$2.85^*, 3.04^{**}$
Equity premium	5.024	$3.62^*, 5.63^{**}$	$5.05^*, 7.65^{**}$	$5.95^*, 5.77^{**}$	$5.83^*, 5.62^{**}$
Vol. equity premium	14.167	$15.65^*, 15.90^{**}$	$16.49^*, 16.30^{**}$	$19.59^*, 19.68^{**}$	$15.12^*, 15.19^{**}$
Vol of debt, B_t^*	6.711	5.28^{***}	5.28^{***}	4.06^{***}	4.06^{***}
$\text{Corr}(R_t^s, \frac{C_t}{C_{t-1}})$	0.71458	0.41	0.47	0.22	0.29

*Based on corporate bond rate
Based on money market rate;*Debt data are available only from 1996.

assets, which per se induces a sharp raise in equity premia, and to de-leverage.

The set of model statistics has been tested under alternative parameter specifications (see Table 2) for the loan to value ratio and for the value of risk-aversion, γ. In the first column of Table 2 we show results under a higher value for the loan to value ratio. This value might in fact change over time and across countries. Results remain generally robust under this new parametrization. At last, in column 2 of the same table we show the results for the a lower value of γ. Results are confirmed again showing that the equity premium and asset price properties can be matched in this model even with very low values of the risk aversion parameter.

So far we have assessed the empirical validity of the model. We shall now quantify the link, discussed in the analytical part, between leverage growth and asset price growth on the one side and the agents' risk-attitudes on the other. As borrowers become more risk-tolerant, hence more risk-takers, with respect to lenders, they lever up more and invest more in risky assets. Hence an increase in the distance between lenders'/borrowers' stochastic discount factors, namely their price of risk, due for instance to increased borrowers' risk-tolerance shall induce an positive leverage and asset price growth. This link has been discussed extensively in the analytical part. Table 3 below

Table 2: **Model-based moments under alternative parametrizations. All values are in percentage terms.**

Statistics and Mnemonics	Model	Model
	$\phi = 0.65$	$\gamma = 2.5$
Exp ret risky asset, R_t^s	7.583	7.698
Vol risky asset, R_t^s	8.931	9.114
Exp ret risk-free, R_t^f	2.712	2.778
Vol risk-free, R_t^f	7.807	7.611
Equity premium	4.871	4.920
Vol. equity premium	14.076	13.993
Vol of debt, B_t^*	6.762	6.6675
$\text{Corr}(R_t^s, \frac{C_t}{C_{t-1}})$	0.72582	0.71806

Table 3: **Model correlations computed for the full sample of simulated series with occasionally binding constraints.**

Model correlations	$\gamma = 3$	$\gamma = 2.5$	$\gamma = 5$
$Corr(\Delta p_t, \Delta m_t)$	0.54634	0.53707	0.56927
$Corr(\Delta B_t, \Delta m_t)$	0.29241	0.30820	0.27989
$Corr(C_t^b, R_t^s)$	0.52000	0.50550	0.50689
$Corr(C_t^l, R_t^s)$	0.50000	0.49114	0.49336

reports (for the baseline calibration) the correlation between leverage growth and the (absolute) distance in lenders/borrowers stochastic discount factor, labelled Δm, and between asset price growth and Δm. Both are positive. To check robustness checks we show results also for a different value of the risk-aversion parameter.

At last, our model generates a slightly higher correlation of consumption to stock returns for borrowers than for lenders (see Table 3). The first are indeed the actual stockholders in the model. This fact is consistent with evidence and results reported in Mankiw and Zeldes[48] and Marcet and Singleton[49]. Notice that lenders in the model are also exposed to the risk of changes in collateral value, since the latter provides the guarantee in case of contract renegotiation. Therefore their consumption is also correlated to risky asset returns. In fact in our model the difference in the correlations of consumption with asset returns are not too large between the two agents. This

is so since we do not include other institutional features that differentiate the portfolio allocation of two households with respect to various risky asset classes.

5   Conclusions

The rapid growth in leverage and asset prices which preceded the 2007 financial crisis has been fuelled by a combination of collateralized debt contracts and diminishing risk-tolerance of borrowers at large gambles, an aspect which is the main driver of risk-taking attitudes. More generally endogenous fluctuations in the tightness of the borrowing limit (the shadow price of debt) can be well explained by differences in lenders'/borrowers' valuations of debt. As borrowers become risk-insensitive at large gambles, their desire to invest in risky asset (risk-taking behavior) increases their demand for debt. They are therefore willing to transfer large compensation premia to entice lenders into offering funds, something which in turn affects the leverage/de-leverage cycle.

We capture those mechanisms through a model with two main features. First, there is a difference in risk-sensitivity between lenders and borrowers, with the latter becoming risk-insensitive (hence risk-takers) at large gambles. Second, borrowers face occasionally binding collateral constraints. Analytically we discuss the role of divergent risk-attitudes for the endogeneity of the collateral constraint and for the dynamic of assets prices. We also assess the quantitative properties of our model by implementing a numerical algorithm based on policy function iteration augmented with Markov-switching regimes. Results show that the model does remarkably well in accounting jointly for the long-run and the cyclical properties of asset prices and returns, equity premia and debt. This confirms that the combination of behavioral and institutional-finance elements, as well as their interaction, can not only account for the emergence of crises, but also explain long-run and cyclical dynamics of financial markets.

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6 Appendix A. Enforcement constraint

Debt contracts emerge as result of bargaining agreements between lenders and borrowers. The agency problems are mitigated by imposing a borrower's enforcement constraint that prevents rational default. As it is common in the standard debt contract literature (see Gale and Hellwig[23] and Townsend[65]) we assign all the bargaining power to the borrower. This implies that the lender receives only the threat value of collateral (liquidation value in case of default). Given this all that remains to characterize is the functional form of the enforcement constraint on the borrower, which, as we show below, once manipulated results in a collateral constraint (see also Jermann and Quadrini[36] for a similar characterization).

Typical of debt contracts is a double-ignorance (see Holmstrom [34]) such that neither the lender nor the borrower observe the liquidation value of collateral before a default event. Consider a debt contract that runs over several periods. Once the contract starts the borrower must service the debt in every period and repay the full amount at the final date. To enforce repayment the loan contract shall be written to accommodate an enforcement constraint, for which the value from servicing the debt in every period and from repaying the final stock is higher than the value of defaulting, (hence appropriating current asset returns but loosing the collateral).

To incorporate the double-ignorance we assume that at the time in which the contract is stipulated both the lender and the borrower are uncertain about the future value of collateral, hence they assign a probability ϕ that the value can be recovered through a market sale and a probability $(1 - \phi)$ that the recovery value will be zero. Let's denote by $\mathbb{E}_t \left\{ m_{t,t+1}^b V_{t+1}^b \right\}$ the expected future discounted value of the period $t + 1$ borrowers' value function in case he services the debt in every period. In case the borrower would decide not to service the debt in the current period, he would be able to retain $R_{t+1}^f B_t^b$, however collateral would be liquidated. Hence the borrower expects a loss in value of $\phi S_t^b \mathbb{E}_t \left\{ p_{t+1} \right\}$. After that the borrower can enter a new debt contract and enjoy the same discounted value function from that point onward[23]. In this context the enforcement constraint (or no-rational default constraint) reads as follows:

$$\mathbb{E}_t \left\{ m_{t,t+1}^b V_{t+1}^b \right\} \geq R_{t+1}^f B_t^b + \mathbb{E}_t \left\{ m_{t,t+1}^b V_{t+1}^b \right\} - \phi S_t^b \mathbb{E}_t \left\{ p_{t+1} \right\} \tag{33}$$

---

[23] We are assuming no exclusion from the market. That does not impair the constraint, as adding the punishment of the market exclusion would only make more likely that the constraint is satisfied.

By re-arranging equation 33 we obtain our collateral constraint:

$$R_{t+1}^f B_t^b \leq \phi S_t^b \mathbb{E}_t \{p_{t+1}\} \tag{34}$$

7 Appendix B. Derivations of the gain-loss utility IMRS

Given the utility:

$$U^i(C_t^i, X_t) = \begin{cases} \Lambda \frac{|\frac{(C_t^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma}|^{1-\theta}}{1-\theta}, & \text{if } C_t^i < X_t, \\ \frac{\left(\frac{(C_t^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma}\right)^{1-\theta}}{1-\theta}, & \text{if } C_t^i \geq X_t, \end{cases} \tag{35}$$

It follows that:

$$IMRS_{t,t+s} = \rho^s \frac{U_C(C_{t+1}^i, X_{t+1})}{U_C(C_t^i, X_t)} = \begin{cases} \Lambda \frac{|\frac{(C_{t+s}^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_{t+s})^{(1-\gamma)}}{1-\gamma}|^{-\theta}(C_{t+s}^i)^{-\gamma}}{|\frac{(C_{t+s}^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_{t+s})^{(1-\gamma)}}{1-\gamma}|^{-\theta}(C_t^i)^{-\gamma}} & \text{if } C_t^i < X_t, \\ \Lambda \frac{(\frac{(C_{t+s}^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_{t+s})^{(1-\gamma)}}{1-\gamma})^{-\theta}(C_{t+s}^i)^{-\gamma}}{(\frac{(C_{t+s}^i)^{(1-\gamma)}}{1-\gamma} - \frac{(X_{t+s})^{(1-\gamma)}}{1-\gamma})^{-\theta}(C_t^i)^{-\gamma}} & \text{if } C_t^i \geq X_t, \end{cases} \tag{36}$$

Recall that $G_{t+s}^i = \frac{C_{t+s}^i}{C_t^i}, \hat{g}_{t+s}^i = \log(G_{t+s}^i)$ hence $G_{t+s}^i = \exp(\hat{g}_{t+s}^i)$. Working with logarithms and recalling that $\hat{y}_{t+s} = \log\left(\frac{C_{t+s}^i}{X_{t+s}}\right)$ we obtain the expression for the $IMRS_{t,t+s}$ as given by equations 18,19, 20.

8 Appendix C. Borrowers' stochastic discount factor

Following Tallarini[64] and Yogo[66] we assume that consumption growth, $\hat{g}_t^b$, follows a normal distribution $N(\mu, \sigma^2)$ at any date t. Imposing the random structure already on consumption rather than on income allows us to obtain analytically tractable expressions for the computation of the stochastic discount factor. Such approximation does not affect the main channels of our model compared to the situation in which the ultimate source of randomness lies in income or wealth and is consistent with empirical evidence (Lettau and Ludvigson[45]) showing that consumption and wealth are cointegrated, albeit the volatility of consumption falls short that of wealth.

We define $\overset{\wedge}{\kappa}_{t+1} = \hat{x}_{t+1} - \hat{c}_t = -\log(b) + \hat{s}_t^b$. Under this assumption it holds that:

$$\mathbb{E}_t \left[\exp(\hat{g}_{t+1}^b) \mid \hat{g}_{t+1}^b > \overset{\wedge}{\kappa}_{t+1} \right] = \exp \left\{ \mu + \frac{\sigma^2}{2} \right\} \frac{F(-(\overset{\wedge}{\kappa}_{t+1} - \mu - \sigma^2)/\sigma)}{F(-(\overset{\wedge}{\kappa}_{t+1} - \mu)/\sigma)} \tag{37}$$

48

$$\mathbb{E}_t\left[\exp(\hat{g}^b_{t+1}) \mid \hat{g}^b_{t+1} < \hat{\kappa}_{t+1}\right] = \exp\left\{\mu + \frac{\sigma^2}{2}\right\} \frac{F((\hat{\kappa}_{t+1} - \mu - \sigma^2)/\sigma)}{F((\hat{\kappa}_{t+1} - \mu)/\sigma)} \tag{38}$$

where F is the cumulative conditional distribution of the standard normal. The borrowers' *IMRS* given by equation 18 and 19 can be written as:

$$m^b_{t,t+1} = \begin{cases} \dfrac{\Lambda\rho\exp\left\{-\gamma\hat{g}^b_{t+1}\right\}}{k(\hat{y}_t)} & \text{if } \hat{g}^b_{t+1} < \hat{\kappa}_{t+1}, \\[2em] \dfrac{\rho\exp\left\{-\gamma\hat{g}^b_{t+1}\right\}}{k(\hat{y}_t)} & \text{if } \hat{g}^b_{t+1} > \hat{\kappa}_{t+1}, \end{cases} \tag{39}$$

Given the above we can compute the first moment of $m^b_{t,t+1}$ as follows:

$$\begin{aligned} \mathbb{E}_t\left\{m^b_{t,t+1}\right\} &= \frac{\rho}{k(\hat{y}_t)} F(-\frac{(\hat{\kappa}_{t+1} - \mu)}{\sigma})\mathbb{E}_t\left[\exp\left\{-\gamma\hat{g}^b_{t+1}\right\} \mid \hat{g}^b_{t+1} > \hat{\kappa}_{t+1}\right] \times \\ &\quad \times\Lambda F(\frac{(\hat{\kappa}_{t+1} - \mu)}{\sigma})\mathbb{E}_t\left[\exp\left\{-\gamma\hat{g}^b_{t+1}\right\} \mid \hat{g}^b_{t+1} < \hat{\kappa}_{t+1}\right] \end{aligned} \tag{40}$$

Using formulas in 37 and 38 we can re-write 40 as follows:

$$\begin{aligned} \mathbb{E}_t\left\{m^b_{t,t+1}\right\} &= \frac{\rho}{k(\hat{y}_t)}\exp\left\{\gamma\mu + \frac{(\gamma\sigma)^2}{2}\right\} \times \\ &\quad \times\left[1 + (\Lambda - 1)F(\gamma\sigma + \frac{(\hat{\kappa}_{t+1} - \mu)}{\sigma})\right] \end{aligned} \tag{41}$$

9   Appendix D. Derivation of asset prices. Proposition 3.

We start by rearranging borrowers' optimality condition on risky assets as follows:

$$p_t = \mathbb{E}_t\left\{m^b_{t,t+1}(p_{t+1} + d_{t+1})\right\} + \lambda'_t\phi\mathbb{E}_t\left\{p_{t+1}\right\} \tag{42}$$

Next we can rearrange equation 42 as follows:

$$p_t = \mathbb{E}_t\left\{m^b_{t,t+1}d_{t+1}\right\} + \mathbb{E}_t\left\{K_{t,t+1}p_{t+1}\right\} \tag{43}$$

where $K_{t,t+1} = m_{t,t+1}^b + \phi\lambda_t'$. We now iterate forward equation 43:

$$
\begin{aligned}
p_t &= \mathbb{E}_t\left\{m_{t,t+1}^b d_{t+1}\right\} + \mathbb{E}_t\left\{K_{t,t+1}p_{t+1}\right\} = \\
&= \mathbb{E}_t\left\{m_{t,t+1}^b d_{t+1}\right\} + \mathbb{E}_t\left\{K_{t,t+1}\left[m_{t+1,t+2}^b(p_{t+2}+d_{t+2}) + \lambda_{t+1}'\phi\mathbb{E}_t\left\{p_{t+2}\right\}\right]\right\} = \\
&= \mathbb{E}_t\left\{m_{t,t+1}^b d_{t+1} + K_{t,t+1}m_{t+1,t+2}^b d_{t+2}\right\} + \mathbb{E}_t\left\{K_{t,t+1}\left[m_{t+1,t+2}^b + \lambda_{t+1}'\phi\right]p_{t+2}\right\} = \\
&= \mathbb{E}_t\left\{m_{t,t+1}^b d_{t+1} + K_{t,t+1}m_{t+1,t+2}^b d_{t+2}\right\} + \mathbb{E}_t\left\{K_{t,t+1}K_{t+1,t+2}p_{t+2}\right\} = \\
&= \mathbb{E}_t\left\{m_{t,t+1}^b d_{t+1} + K_{t,t+1}m_{t+1,t+2}^b d_{t+2}\right\} + \mathbb{E}_t\left\{K_{t,t+1}K_{t+1,t+2}\left[m_{t+2,t+3}^b d_{t+3} + (m_{t+2,t+3}^b + \lambda_{t+2}'\phi)p_{t+3}\right]\right\} = \\
&= \mathbb{E}_t\left\{m_{t,t+1}^b d_{t+1} + K_{t,t+1}m_{t+1,t+2}^b d_{t+2} + K_{t,t+1}K_{t+1,t+2}m_{t+2,t+3}^b d_{t+3}\right\} + \\
&\quad + \mathbb{E}_t\left\{K_{t,t+1}K_{t+1,t+2}K_{t+2,t+3}p_{t+3}\right\}
\end{aligned}
\tag{44}
$$

After merging terms we obtain:

$$
\begin{aligned}
p_t &= \mathbb{E}_t\left\{m_{t,t+1}^b d_{t+1}\right\} + \mathbb{E}_t\left\{\sum_{i=1}^{T} m_{t+i,t+i+1}^b d_{t+i+1}\prod_{j=1}^{i}K_{t+j-1,t+j}\right\} + \\
&\quad + \left[\prod_{i=0}^{T}K_{t+i,t+i}p_{t+T}\right]
\end{aligned}
\tag{45}
$$

10 Appendix E. Sharpe Ratio. Proposition 4.

We have to derive the Sharpe ratio, namely the slope of the portfolio frontier, for the borrower. We start from re-arranging the borrower's first order conditions (equations 12 and 13) as follows:

$$
1 = \mathbb{E}_t\left\{m_{t,t+1}^b R_{t+1}^f\right\} + \lambda_t' R_{t+1}^f
\tag{46}
$$

$$
1 = \mathbb{E}_t\left\{m_{t,t+1}^b R_{t+1}^s\right\} + \lambda_t'\phi\mathbb{E}_t\left\{\frac{p_{t+1}}{p_t}\right\}
\tag{47}
$$

where as in the text $\lambda_t' = \frac{\lambda_t}{U_c(C_t^b, X_t)}$ and $R_{t+1}^s = \frac{p_{t+1}+d_{t+1}}{p_t}$. We now subtract 46 from 47. The goal is that of deriving the Hansen and Jagannathan bounds on the excess return between the risky asset and the risk-free asset. We obtain:

$$
0 = \mathbb{E}_t\left\{m_{t,t+1}^b Z_{t+1}\right\} + \lambda_t'\mathbb{E}_t\left\{\Sigma_{t+1}\right\}
\tag{48}
$$

where $Z_{t+1} = (R_{t+1}^s - R_{t+1}^f)$ and $\mathbb{E}_t\left\{\Sigma_{t+1}\right\} = \mathbb{E}_t\left\{\phi\frac{p_{t+1}}{p_t}\right\} - R_{t+1}^f$. Let's assume that the stochastic discount factor takes a general linear functional form as: follows:

$$
m_{t,t+1}^b = \mathbb{E}_t\left\{m_{t,t+1}^b\right\} + \beta_M(Z_{t+1} - \mathbb{E}_t\left\{Z_{t+1}\right\})
\tag{49}
$$

The functional form above is compatible with a log-linear approximation of the Euler equation. If the above is a valid stochastic discount factor it must satisfy equation 48 which once expanded delivers the following expression:

$$
\begin{aligned}
0 &= \mathbb{E}_t \left\{ m^b_{t,t+1} Z_{t+1} \right\} + \lambda'_t \mathbb{E}_t \left\{ \Sigma_{t+1} \right\} = \qquad (50) \\
&= \mathbb{E}_t \left\{ m^b_{t,t+1} \right\} \mathbb{E}_t \left\{ Z_{t+1} \right\} + Cov(m^b_{t,t+1} Z_{t+1}) + \lambda'_t \mathbb{E}_t \left\{ \Sigma_{t+1} \right\} = \\
&= \mathbb{E}_t \left\{ m^b_{t,t+1} \right\} \mathbb{E}_t \left\{ Z_{t+1} \right\} + \mathbb{E}_t \left\{ (m^b_{t,t+1} - \mathbb{E}_t \left\{ m^b_{t,t+1} \right\})(Z_{t+1} - \mathbb{E}_t \left\{ Z_{t+1} \right\}) \right\} + \lambda'_t \mathbb{E}_t \left\{ \Sigma_{t+1} \right\} = \\
&= \mathbb{E}_t \left\{ m^b_{t,t+1} \right\} \mathbb{E}_t \left\{ Z_{t+1} \right\} + \mathbb{E}_t \left\{ (Z_{t+1} - \mathbb{E}_t \left\{ Z_{t+1} \right\})(Z_{t+1} - \mathbb{E}_t \left\{ Z_{t+1} \right\}) \beta_M \right\} + \lambda'_t \mathbb{E}_t \left\{ \Sigma_{t+1} \right\} = \\
&= \mathbb{E}_t \left\{ m^b_{t,t+1} \right\} \mathbb{E}_t \left\{ Z_{t+1} \right\} + \sigma^2_z \beta_M + \lambda'_t \mathbb{E}_t \left\{ \Sigma_{t+1} \right\}
\end{aligned}
$$

Hence inverting 50 we obtain:

$$
\beta_M = -(\sigma^2_z)^{-1} \mathbb{E}_t \left\{ m^b_{t,t+1} \right\} \mathbb{E}_t \left\{ Z_{t+1} \right\} - (\sigma^2_z)^{-1} \lambda'_t \mathbb{E}_t \left\{ \Sigma_{t+1} \right\} \qquad (51)
$$

Given the above we can compute the variance of the stochastic discount factor as follows:

$$
\begin{aligned}
Var(m^b_{t,t+1}) &= Var((Z_{t+1} - \mathbb{E}_t \left\{ Z_{t+1} \right\})' \beta_M) = \qquad (52) \\
&= \beta'_M \sigma^2_z \beta_M = \\
&= \left[-(\sigma^2_z)^{-1} \mathbb{E}_t \left\{ m^b_{t,t+1} \right\} \mathbb{E}_t \left\{ Z_{t+1} \right\} - (\sigma^2_z)^{-1} \lambda'_t \mathbb{E}_t \left\{ \Sigma_{t+1} \right\} \right]' \sigma^2_z \times \\
&\quad \times \left[-(\sigma^2_z)^{-1} \mathbb{E}_t \left\{ m^b_{t,t+1} \right\} \mathbb{E}_t \left\{ Z_{t+1} \right\} - (\sigma^2_z)^{-1} \lambda'_t \mathbb{E}_t \left\{ \Sigma_{t+1} \right\} \right]
\end{aligned}
$$

Once re-arranged the last delivers:

$$
\begin{aligned}
Var(m^b_{t,t+1}) &= -(\sigma^2_z)^{-1} (\mathbb{E}_t \left\{ m^b_{t,t+1} \right\})^2 (\mathbb{E}_t \left\{ Z_{t+1} \right\})^2 + \qquad (53) \\
&\quad + \mathbb{E}_t \left\{ \Sigma_{t+1} \right\} 2 \lambda'_t \mathbb{E}_t \left\{ m^b_{t,t+1} \right\} (\sigma^2_z)^{-1} + (\sigma^2_z)^{-1} (\lambda'_t)^2 (\mathbb{E}_t \left\{ \Sigma_{t+1} \right\})^2
\end{aligned}
$$

After re-arranging 53 we obtain the Sharpe ratio in Proposition 4.

11 Appendix F. Numerical solution

Our simulation method is based on a policy function iteration augmented with a Markov-switching structure. We start with a guess on the consumption decision rule as a policy function of the pre-determined variables, namely the three exogenous shocks, debt and the reference point, and we

iterate until convergence. The policy function shall arise as solution to the system of competitive equilibrium conditions. The latter however take different functional forms depending on the economy's regime, namely on whether the constraint binds or not ($\lambda_t > 0$ or $\lambda_t = 0$) and on whether consumption is above or below the reference level. Therefore in each period we verify if the policy function is such that the constraint binds or not and if consumption is above or below the reference point. Based on that we update expectations. This allows for an endogenous regime switching: in every period agents form expectation on the consumption policy function, which in turn depend upon the state variables. The latter summarize information about the relevant economy's regime. Methodologically the procedure is based on a monotone mapping algorithm that finds the fixed point in decisions rules (Coleman[14]).

The general functional form for the equilibrium conditions describing the economy are as follows:

$$U_C^l\left(C_t^l\right) = \beta \mathbb{E}_t\left\{U_C^l\left(C_{t+1}^l\right)R_{t+1}^f\right\} \tag{54}$$

$$U_C^b(C_t^b, X_t) = \rho \mathbb{E}_t\left\{U_C^b(C_{t+1}^b, X_{t+1})R_{t+1}^f\right\} + \lambda_t R_{t+1}^f \tag{55}$$

$$p_t U_C^b(C_t^b, X_t) = \rho \mathbb{E}_t\left\{U_C^b(C_{t+1}^b, X_{t+1})(p_{t+1} + d_{t+1})\right\} + \lambda_t \phi \mathbb{E}_t\left\{p_{t+1}\right\} \tag{56}$$

$$(1 - \nu)S_t^b = 1 \tag{57}$$

$$\nu B_t^l + (1 - \nu)B_t^b = 0$$

$$\nu C_t^l + (1 - \nu)C_t^b = \nu w_t^l + (1 - \nu)w_t^b + d_t$$

Notice that the lenders and borrowers individual budget constraints are not needed. Indeed by Walras law their linear combination is accounted for by the resource constraint. In each period t the policy function is computed conditional on expectations that in the next period λ_t can be either positive or zero and that $U_C^b(C_{t+s}^b, X_{t+s})$ could be either $\Lambda \frac{|\frac{(C_t^b)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma}|^{1-\theta}}{1-\theta}$ or $\frac{|\frac{(C_t^b)^{(1-\gamma)}}{1-\gamma} - \frac{(X_t)^{(1-\gamma)}}{1-\gamma}|^{1-\theta}}{1-\theta}$. This amounts to 16 combinations for λ_t, λ_{t+1} and $IMRS_t, IMRS_{t+1}$.

In the numerical implementation we simulate the income and dividend processes with 1100 simulations. We drop the first 100 observations. The grid points are chosen to be always on an interval around the previous point which is ± 3 times the standard deviation of each process.

12   Appendix G. Data description

We start to detail data sources, samples and definitions for asset prices and returns. Data are monthly (except UK T-bill rates, which is interpolated linearly from quarterly data). Asset returns for the US are available from the 1960, whole for the UK they are available from 1970. We compute statistics over two sub-samples: 1960-2016 for the US (and 1970-2016 for the UK) and 1980-2016 (for both countries). Data for the U.S. are as follows. For stock market indices we take the S&P500, taken from Bob Shiller database. We choose the stock market index that includes dividends as this is the definition which is closer to our model based. However we have checked and verified that the moments for the dividend free stock index are not very different, albeit smaller as one would expect. The return on debt is proxied with the corporate bond rate (3-Month Treasury Bill: Secondary Market Rate) or with the money market rate (Effective Federal Funds Rate): both are taken from the FRED database.

Data for the UK are as follows. Stock market indices (inclusive of dividends) are from Moody's Seasoned Aaa Corporate Bond Yield, obtained from Morgan Stanley via Datastream. Data for corporate bonds rates (Discount Rate on Short-Term Commercial Paper in the United Kingdom) and data for money market rates (3-month London Interbank Offered Rate, LIBOR, in the United Kingdom) are taken from FRED Database.

Notice that all equity premia are calculated in real terms, deflated by the CPI series. Specifically the premium (in logs) has been calculated as follows:

$$EP_t = RER_t - RRFR_t \tag{58}$$

where RER_t is the real equity premium and $RRFR_t$ is the real return on debt. The latter is calculated as:

$$RRFR_t = RFR_t - \pi_t^c \tag{59}$$

where π_t^c is CPI inflation. The real equity premium (inclusive of dividends) is calculated for the US and the UK as:

$$RER_t^{Div} = \frac{p_t + d_t - p_{t-1}}{p_{t-1}} \tag{60}$$

where p_t is the real share price index (here S&P500 for the US) and d_t are real dividends.

When using data from Moody's Seasoned Aaa Corporate Bond Yield the premium is calculated as follows:

$$RER_t^{Div} = \frac{T_t - T_{t-1}}{T_{t-1}} \tag{61}$$

where T_t is a real total return index (i.e., including dividends) based on MSCI data.

Data on debt are taken from OECD statistics. For the US we take data on debt to non-financial corporation. This definition fits well our model based corporate debt definition. For the UK debt to non-financial corporation is much less important as the manufacturing sector is tiny compared to the financial sector. Hence for this country we take total debt of the economy. Both are available in the OECD statistics as percentage of GDP. We transform them to make them comparable to the model. For debt data are available only from 1996 and at annual series (we interpolate the series to obtain quarterly frequency).

13   Appendix H. Estimation of the income processes

To estimate the income process we use PSID data (see also Heatcote, Perri and Violante[33]). The PSID is a longitudinal study of a sample of the US population. It has been conducted annually since 1968 until 1997 and biannually afterwards. It consists of two independent samples. One of about 2,000 households drawn from the Survey of Economic Opportunity respondents (SEO sample) which represents low-income families and one drawn by the Survey Research Center (SRC sample) that includes about 3,000 households representative of the US population. We only consider the SRC sample since we are interested in a representative sample. From this sample we excluded families that stopped responding to the survey at some point in time or families which reported no income for one or more years. This leaves us with a sample of 375 families.

For those families we construct income as follows. The survey follows both the original families as well as their split-offs. Total income of the family is obtained by adding together the reported taxable income and transfer income from all sources for the family head, the wife and all other earners in the family. Taxable income includes labor income (wages and salaries, bonuses, overtime, tips, commissions, professional practice or trade, and market gardening) and income from other sources. Transfer income includes social security income, unemployment and workers compensation, child support, retirement income as well as other welfare transfers to the head and wife. Total family

income (constructed as per sum above) is weighted by the number of family members and deflated by the CPI.

We divide the entire group of the families into two, borrowers and non-borrowers, by using the Wealth supplement search. Families sometimes switch back and forth from the borrower to the non-borrower status. We defined a borrower as the family that has debt in at least 50% of all the years considered. The fraction of borrowers is slightly above 30%. For each household income series we fit a AR(1) process through an OLS estimation (see Heaton and Lucas [32]). We then obtain the coefficient for the aggregate income process of borrowers and lenders by computing sample averages.

The dividend process is estimated as follows. We use NIPA data from Bureau of Economic Analysis for the years 1960 to 2016. We compute real dividend per capita by deflating for CPI and by dividing for population in a given year (population is taken from U.S. Census Bureau). We also de-trend the series through HP-filter. Also in this case we use the time series of the dividend data to fit an AR(1) through OLS estimation.