

# CLIMATE TIPPING AND ECONOMIC GROWTH



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# How to model catastrophes?



- Chance that a discontinuous change in damages or carbon cycle takes place. This can be abrupt as with shifts in monsoonal systems. But loss of ice sheets resulting in higher sea levels have slow onsets and can take millennium or more to have its full effect (Greenland 7m and Western Antarctica 3m, say) and may already be occurring.
- 9 big catastrophes are imminent, not all at same time (Lenton and Ciscar, CC, Nature).
- Collapse of the Atlantic thermohaline circulation is fairly imminent and might occur at relatively low levels of global warming. This affects regions differently, but we capture this with a negative TFP shock.
- We look at TFP calamity and also at  $K$ ,  $P$  and climate sensitivity calamities. Expected time of calamity falls with global warming.

## Probabilities of Various Tipping Points from Expert Elicitation

Possible Tipping Points	Duration before effect is fully realized (in years)	Additional Warming by 2100		
		0.5-1.5 C	1.5-3.0C	3-5 C
Reorganization of Atlantic Meridional Overturning Circulation	about 100	0-18%	6-39%	18-67%
Greenland Ice Sheet collapse	at least 300	8-39%	33-73%	67-96%
West Antarctic Ice Sheet collapse	at least 300	5-41%	10-63%	33-88%
Dieback of Amazon rainforest	about 50	2-46%	14-84%	41-94%
Strengthening of El Niño-Southern Oscillation	about 100	1-13%	6-32%	19-49%
Dieback of boreal forests	about 50	13-43%	20-81%	34-91%
Shift in Indian Summer Monsoon	about 1	Not formally assessed		
Release of methane from melting permafrost	Less than 100	Not formally assessed.		

# Previous work



- Gollier (2012): Markov 2-regime switching model to capture exogenous risk of big drop in GDP growth  $\Rightarrow$  much higher *SCC*.
- Threat of doomsday scenario: Bommier et al. (2013).
- Regime shifts with *uncertain* arrival of catastrophe:
  - Partial equilibrium: Tsur & Zemel (1996), Karp & Tsur (2011), Naevdal (2006), Polasky, de Zeeuw & Wagener (2011).
  - General equilibrium: Lemoine and Traeger (2014) use Ramsey model to understand effect of release of permafrost as instantaneous doubling of *ECS* and of learning and multiple catastrophes. Cai, Judd and Lontzeck (2015, NCC) similar and focus on shock to damage function and numerical challenge.

# Messages and aims



- Chance of catastrophe can lead to much higher *SCC* without a very low discount rate provided hazard rises sharply with temperature  $\Rightarrow$  to avert risk.
- There is also a social benefit of capital (*SBC*) which gives a rationale for precautionary capital accumulation  $\Rightarrow$  to be better prepared.
- Calibrate a global IAM with Ramsey growth with both catastrophic and marginal climate damages.
- Show role of convexity of the hazard function.
- Show effect of more intergenerational inequality aversion and thus more risk aversion on *SCC* and *SBC*: i.e., on carbon tax and capital subsidy.

# Ramsey growth with pending climate disaster



- Simplest growth model:
  - Concave time-separable utility function.
  - Concave and CRTS production function.
  - Factors of production: capital  $K$ , labour, fossil fuel and renewables - imperfect substitutes.
  - Fossil fuel  $E$  is abundant at cost  $d$ .
  - Supply of renewable  $R$  is infinitely elastic at cost  $c$ .
- Very simple carbon cycle: nothing stays up forever in the atmosphere, constant decay rate.
- Hazard of catastrophic drop in  $TFP$  is  $H(P)$  and is modelled with Poisson process with  $H' \geq 0$

# Climate disaster and Ramsey growth

$$\max_{C,E,R} E_0 \left[ \int_0^{\infty} e^{-\rho t} U(C(t)) dt \right] \text{ subject to}$$

$$\dot{K}(t) = \tilde{A}F(K(t), E(t), R(t)) - dE(t) - cR(t) - C(t) - \delta K(t),$$
$$\forall t \geq 0, \quad K(0) = K_0,$$

$$\dot{P}(t) = \psi E(t) - \gamma P(t), \quad \forall t \geq 0, \quad P(0) = P_0,$$

$$\tilde{A}(t) = A, \quad 0 \leq t < T, \quad \tilde{A}(t) = (1 - \pi)A, \quad \forall t \geq T, \quad 0 < \pi < 1,$$

$$\Pr[T < t] = 1 - \exp \left( - \int_0^t H(P(s)) ds \right), \quad \forall t \geq 0.$$

# Backward induction: before disaster



- For time being, damages only result from calamities.
- Solve post-catastrophe problem as standard Ramsey problem to give post-calamity value function:  $V^A(K, \pi)$ .
- Solve before-catastrophe problem from the HJB:

$$\rho V^B(K, P) = \text{Max}_{C, E, R} \left\{ U(C) + H(P) \left[ V^A(K, \pi) - V^B(K, P) \right] + \right. \\ \left. V_K^B(K, P) \left[ AF(K, E, R) - dE - cR - C - \delta K \right] + V_P^B(K, P) (\psi E - \gamma P) \right\}$$

with optimality conditions

$$U'(C^B) = V_K^B(K, P), \quad AF_E(K, E, R) = d + \tau, \quad \tau \equiv -\psi V_P^B(K, P) / V_K^B(K, P) > 0, \\ AF_R(K, E, R) = c, \quad AF_K(K, E, R) - \delta \equiv r.$$



# Precautionary saving and curbing risk of calamity



- The Euler equation has a precautionary return  $\theta$  or social benefit of capital (*SBC*):

$$\dot{C} = \sigma(r + \theta - \rho)C \quad \text{with} \quad r = Y_K^B(K, d + \tau, c, A)$$

$$\theta = H(P) \left[ \frac{V_K^A(K, \pi)}{U'(C)} - 1 \right] = H(P) \left[ \left( \frac{C^B}{C^A} \right)^{1/\sigma} - 1 \right] > 0.$$

- The *SCC* is:

$$\tau(t) = \int_t^\infty \psi H'(P(s)) \frac{V^B(s) - V^A(s)}{U'(C^B(s))} \exp\left(-\int_t^s [r(s') + \theta(s') + \gamma + H(P(s'))] ds'\right) ds$$

$$= \left\{ \int_t^\infty \psi H'(P(s)) [V^B(s) - V^A(s)] \exp\left(-\int_t^s [\rho + \gamma + H(P(s'))] ds'\right) ds \right\} / U'(C^B(t)).$$

# Interpretation

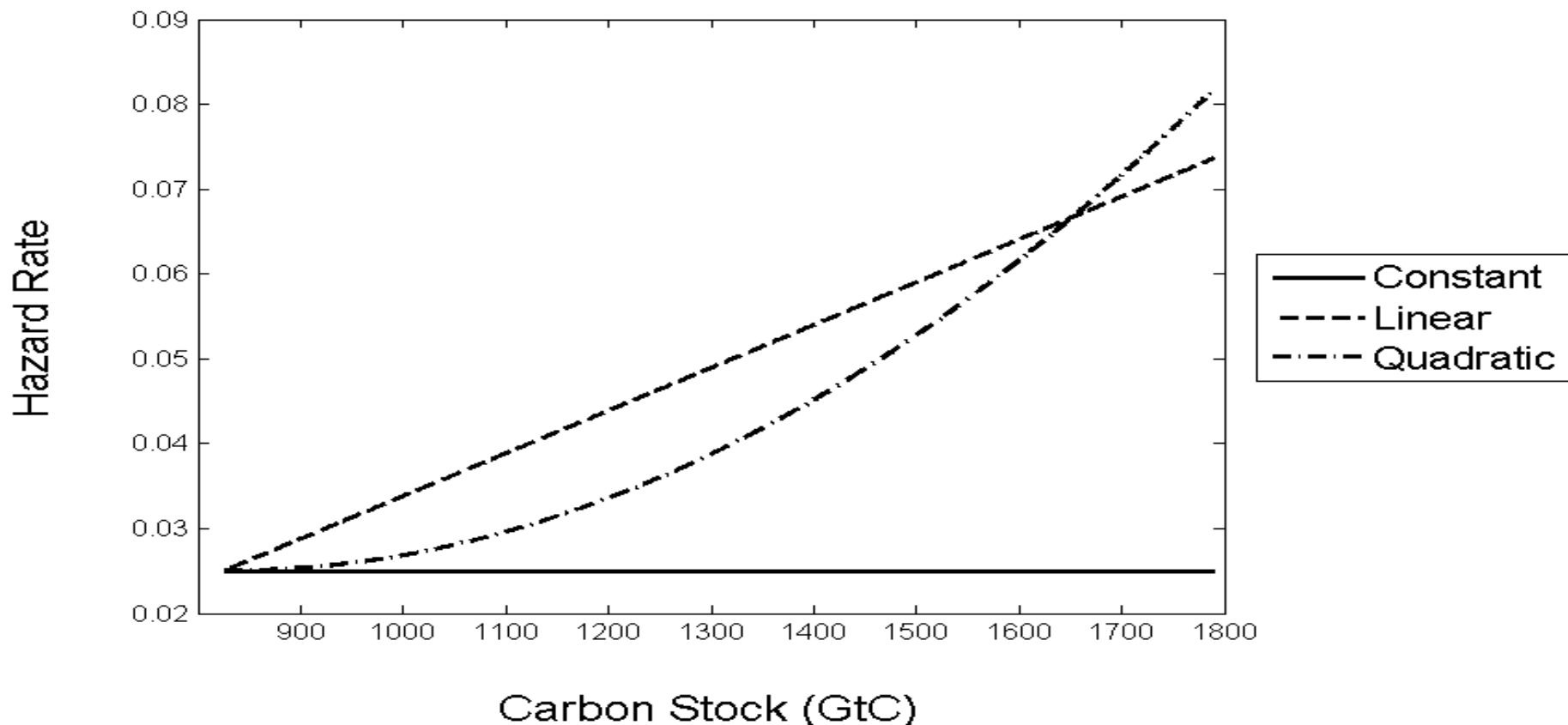


- ‘Doomsday’ scenario has  $V^A = 0$ , so the discount rate is *increased*  $\Rightarrow$  frantic consumption and less investment. Mr. Bean!
- But if world goes on after disaster, precaution is needed. Since consumption will fall after disaster,  $SBC > 0$  and the discount rate is *reduced*. This calls for precautionary capital accumulation (if necessary internalized via a capital subsidy)
- The  $SBC$  is bigger if the hazard and size of the disaster are bigger.
- And if intergenerational inequality aversion ( $CRIIA$ ) or relative prudence ( $1+CRIIA$ ) is bigger.

# Illustrative calibration of hazard function



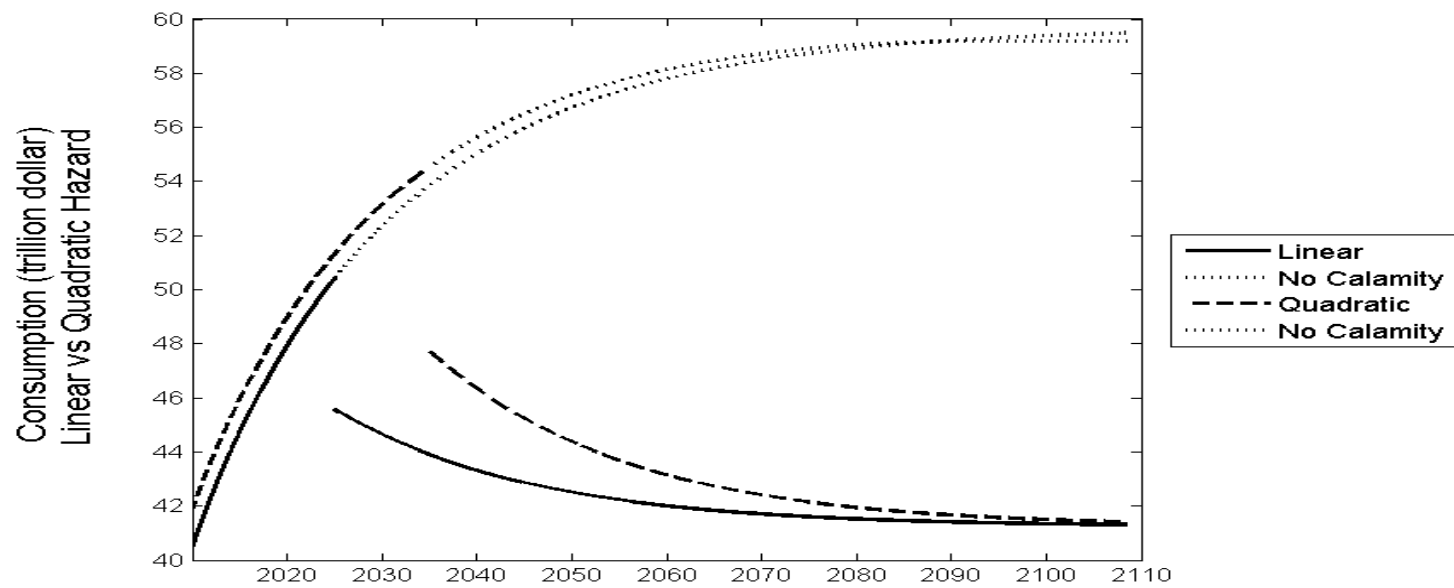
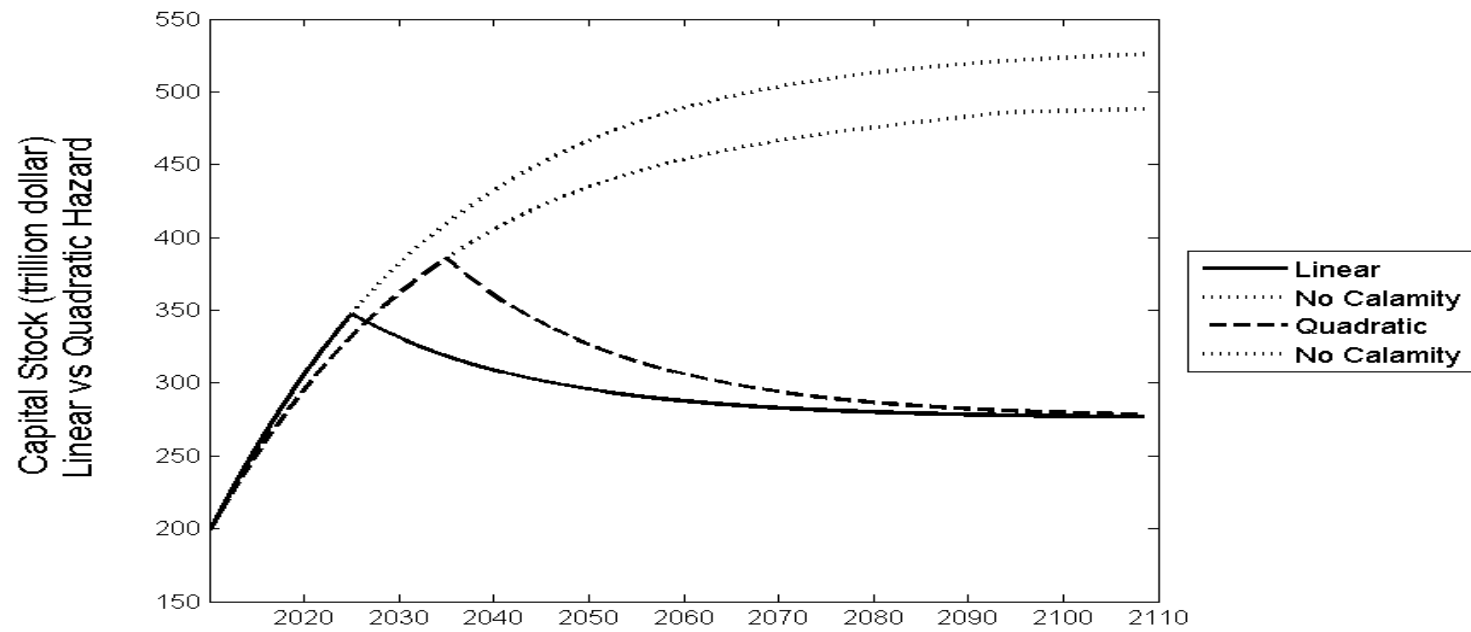
- Use  $H(826) = 0.025$  and  $H(1252) = 0.067$
- So doubling carbon stock (rise in temperature with 3 degrees) brings forward expected time of calamity from 40 to 15 years.

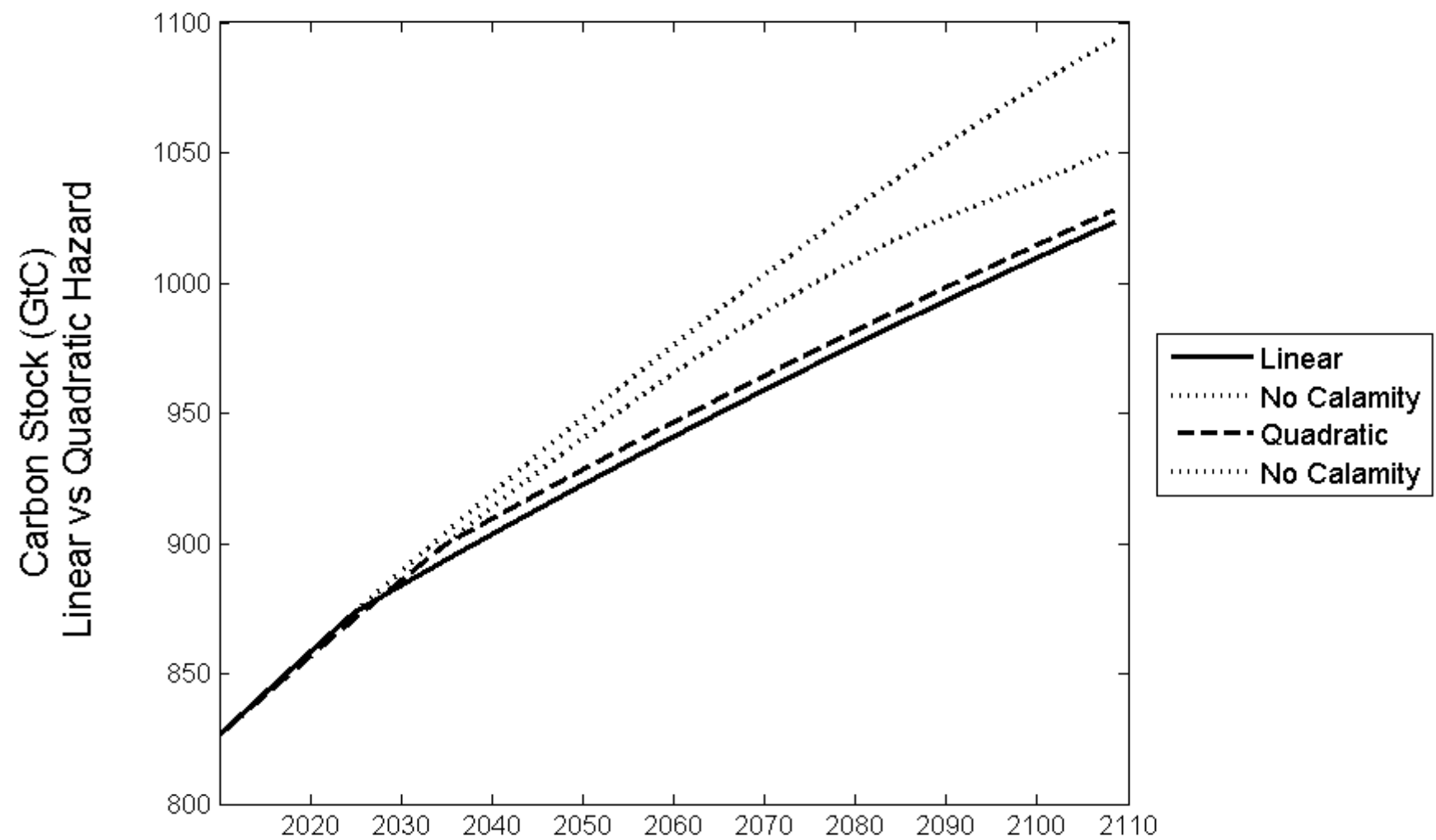


# After-disaster, naïve and before-disaster steady states



	After disaster	Naïve solution	Constant hazard $h = 0.25$	Linear hazard	Quadratic hazard	EIS = 0.8
Capital stock (T \$)	276	392	472	530	486	436
Consumption (T \$)	41.3	58.6	59.4	59.6	59.2	58.9
Fossil fuel use (GtC/year)	7.3	10.4	11.0	9.7	7.7	7.7
Renewable use (million GBTU/year)	8.2	11.7	12.4	12.7	12.2	11.8
Carbon stock (GtC)	1218	1731	1838	1623	1281	1279
Precautionary return (%/year)	0	0	0.76	1.24	0.99	0.57
SCC (\$/tCO <sub>2</sub> )	0	0	0	22.4	56.9	51.0





# Precautionary capital can be negative if hazard function is very convex



- Steady-state pre-disaster  $K$  is bigger than naive  $K$  iff:

$$\left( \frac{\delta + \rho}{\delta + \rho - \theta^{B^*}} \right)^{1-\beta} > \left( \frac{d + \tau^{B^*}}{d} \right)^{\beta\omega} .$$

- This is always so if hazard constant and  $SCC$  zero.
- With convex enough hazard function effects of  $SCC$  can outweigh effect of  $SBC$ , so inequality need not hold.
- With quartic  $SCC$  is very high and  $SBC$  very low. The high carbon tax averts disaster so much that there is less need for precautionary capital accumulation. Put differently, precautionary capital is bad as it induces more fossil fuel use, more global warming and a relatively big increase in hazard of climate disaster. So avoid Green Paradox.

# Role of intergenerational inequality aversion



- Much debate is about discount rate but  $CRIIA = 1/\sigma$  is at least as important.
- Higher  $\sigma$  or lower  $CRRA$  and  $CRIIA$  has two effects:
  - Lower  $CRRA$ , so lower  $SBC$ , less precautionary saving and thus less fossil fuel demand and emissions. Need lower carbon tax
  - Lower  $CRIIA$  so more prepared to sacrifice consumption and have a higher carbon tax.
- With  $\sigma = 0.8$  and linear hazard first effect dominates: lower  $SCB$  and lower  $SCC$  so less capital before disaster and less sacrificing of consumption.



# Gradual damages $A(Temp)$ and the SCC: $\Omega$ and $\Xi$ are $K$ - and $P$ -catastrophes



- Before-disaster SCC has in general 3 components:

$$\begin{aligned}
 \tau(t) = & \underbrace{\frac{-\psi}{U'(C(t))} \int_t^\infty A'(P(s)) F(s) U'(C(s)) e^{-\int_t^s [\rho + \gamma + H(P(s'))] ds} ds}_{\text{conventional Pigouvian social cost of carbon}} \\
 & + \underbrace{\frac{-\psi}{U'(C(t))} \int_t^\infty H(P(s)) V_P^A(K(s) - \Omega, \Delta, P(s) + \Xi) e^{-\int_t^s [\rho + \gamma + H(P(s'))] ds} ds}_{\text{'raising the stakes' effect}} \\
 & + \underbrace{\frac{\psi}{U'(C(t))} \int_t^\infty H'(P(s)) \{V^B(s) - V^A(s)\} e^{-\int_t^s [\rho + \gamma + H(P(s'))] ds} ds}_{\text{'risk averting' effect}}, \quad 0 \leq t < T.
 \end{aligned}$$

Alternative expressions: discounting goods units instead of utils

$$\begin{aligned}
 \tau(t) = & \underbrace{-\psi \int_t^\infty A'(P(s)) F(s) e^{-\int_t^s [r+\theta+\gamma+H(P(s'))] ds} ds}_{\text{conventional Pigouvian social cost of carbon}} \\
 & \underbrace{-\psi \int_t^\infty \frac{H(P(s)) V_P^A(K(s) - \Omega, \Delta, P(s) + \Xi)}{U'(C(s))} e^{-\int_t^s [r+\theta+\gamma+H(P(s'))] ds} ds}_{\text{'raising the stakes' effect}} \\
 & \underbrace{+\psi \int_t^\infty \frac{H'(P(s)) \{V^B(s) - V^A(s)\}}{U'(C(s))} e^{-\int_t^s [r+\theta+\gamma+H(P(s'))] ds} ds}_{\text{'risk averting' effect}}, \quad 0 \leq t < T.
 \end{aligned}$$

Golosov et al. (2014, *Ectra*): first term with  $\sigma = 1$ ,  $\theta = H(P) = 0 \Rightarrow = \frac{0.00238\psi GDP}{\rho + \gamma}$

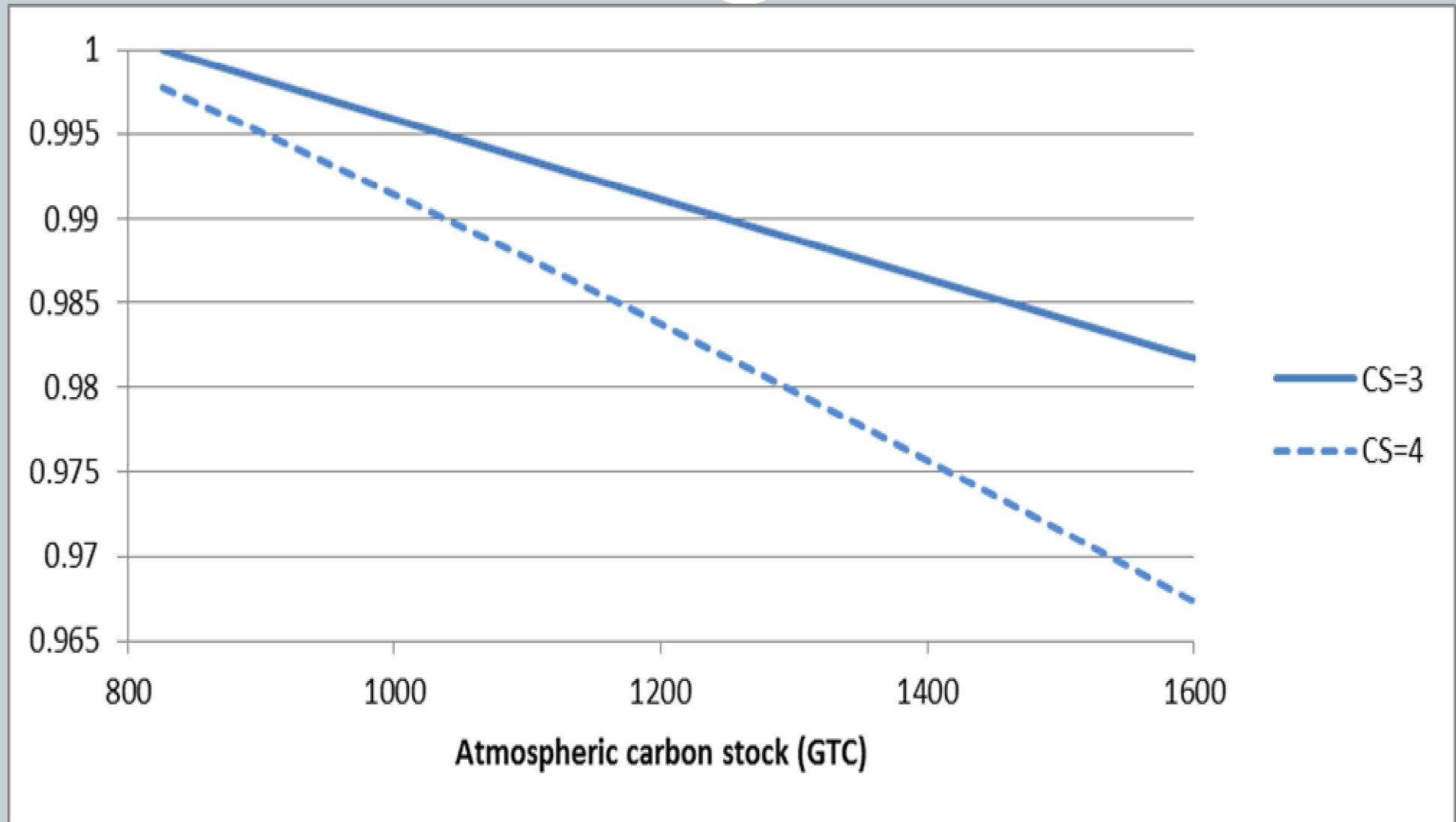
(they also need  $A(P) = e^{-0.00238P}$ , 100% depreciation & Cobb-Douglas production)

# Catastrophic and marginal climate damages



	Naïve solution	20% shock in TFP			10% shock in TFP		
		after shock	linear	quadratic	after shock	linear	quadratic
Capital stock (T \$)	378	271	492	465	323	431	421
Consumption (T \$)	57.1	40.8	58.3	58.2	48.7	57.8	57.8
Carbon stock (GtC)	1502	1107	1287	1161	1303	1425	1320
Temperature (degrees Celsius)	4.00	2.68	3.33	2.88	3.38	3.77	3.44
Precautionary return (%/year)	0	0	1.10	0.90	0	0.57	0.49
SCC (\$/GtCO <sub>2</sub> )	15.4	11.0	54.8	71.2	13.2	29.8	41.5
marginal	15.4	11.0	4.3	5.7	13.2	3.8	4.7
risk averting	0	0	35.0	51.9	0	12.4	24.2
raising stakes	0	0	15.4	13.7	0	13.7	12.5

# Effect of climate sensitivity on damages: CS = 3 approximates DICE damages of Nordhaus



# Carbon and capital catastrophes



	Naïve solution	CS jumps from 3 to 4		20% drop in $P$	20% drop in $K$
		After calamity	Before calamity		
Capital stock (T \$)	379	372	382	381	433
Consumption (T \$)	57.1	56.3	57.3	57.1	57.6
Carbon stock (GtC)	1503	1374	1400	1490	1534
Temperature (degrees Celsius)	4.00	4.82	3.69	3.96	4.09
Precautionary return (%/year)	0	0	0.05	0.03	0.57
SCC (\$/GtCO <sub>2</sub> )	15.5	26.7	26.5	16.9	18.5
Marginal	15.5	26.7	4.1	3.8	3.8
risk averting	0	0	2.2	1.4	2.5
raising stakes	0	0	20.2	11.7	12.2

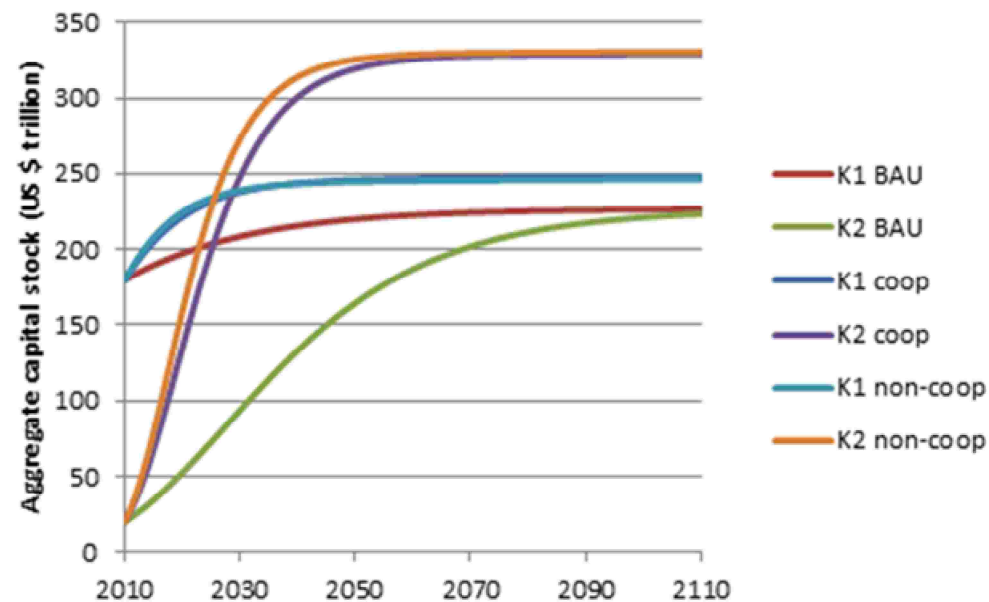
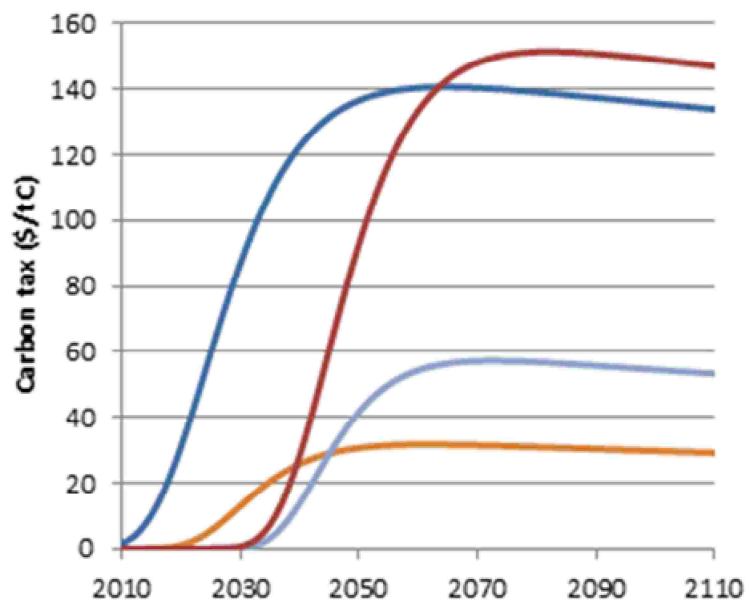
# Conclusions



- Small risks of climate disasters may lead to a much bigger *SCC* even with usual discount rates. Rationale is to avoid risk.
- Also need for precautionary capital accumulation.
- Need estimates of current risks of catastrophe and how these increase with temperature.
- Recoverable shocks such as *P* or *K* calamities are less problematic.
- Catastrophic changes in system dynamics unleashing positive feedback may be much more dangerous than *TFP* calamities.

# Extension: North-South perspective

- Carbon taxes rise in line with GDP; lots of precaution.
- South is poor and is hit more by global warming than North  $\Rightarrow$  taxes carbon later and eventually more.
- Big non-cooperative bias in carbon tax, but not in precautionary return on capital.



# Other extensions



- Adaptation capital (sea walls, storm surge barriers) increases with global warming: trade-off with productive capital.
- Positive feedback in the carbon cycle changes carbon cycle dynamics (e.g., Greenland or West Antarctica ice sheet collapse).
- Multiple tipping points with different hazard functions and impact lags (Cai, Judd, Lontzek). ‘Strange’ cost-benefit analysis (Pindyck).
- Learning about probabilities of tipping points, but also about whether they exist all (cf. ‘email-problem’). How to respond to a tipping point which may never materialize?
- Exhaustibility of fossil fuel: so anticipation of tip  $\Rightarrow$  Green Paradox.
- Second-best issues: Green Paradox can lead to ‘runaway’ global warming if system is tipped due to more rapid depletion of oil, gas and coal in face of a future tightening of climate policy (Winter, 2014).



# Other applications of regime switches



- Probability of a probability of arrival of breakthrough new carbon-free technology in the next few decades (e.g., fusion energy – Tokamak technology?) with hazard rate depending on cumulative R&D efforts.
- Three decades of almost no success on climate policy. Have temperature-dependent hazard of political regime change to one with an aggressive climate policy. Stranded assets.
- Political economy: double externality whammy – international free rider problems & sacrifices now for less global warming many decades or centuries ahead.

# BACKGROUND STUFF



- Calibration details
- After-catastrophe solution of Ramsey growth IAM
- Loglinear approximation of post-catastrophe stable manifold
- State space of pre-catastrophe system
- Multiple tipping points

Pure rate of time preference, $\rho$	0.014
Elasticity of intertemporal substitution, $\sigma$	0.5 (and 0.8)
Share of capital in value added, $\alpha$	0.3
Share of fossil fuel (oil, gas, coal) in value added, $\beta \omega$	0.0626
Share of fossil fuel in total energy, $\omega$	0.9614
Share of energy in value added, $\beta$	0.0651
Share of labour in value added, $1 - \alpha - \beta$	0.6349
Depreciation rate of manmade capital, $\delta$	0.05
Initial level of GDP, $Y_0$	63 trillion US \$
Initial capital stock, $K_0$	200 trillion US \$
Initial fossil fuel use, $E_0$	468.3 million G BTU = 8.3 GtC
Initial renewable use, $R_0$	9.4 million G BTU
Total factor productivity, $A$	11.9762
Cost of fossil fuel, $d$	9 US \$/million BTU = 504 US \$/tC
Cost of renewable, $c$	18 US \$/million BTU
Initial stock of carbon, $P_0$	826 GtC = 388 ppm by vol. CO2
Pre-industrial carbon stock	596.4 GtC = 280 ppm by vol. CO2
Fraction of carbon that stays up in atmosphere, $\psi$	0.5
Eventual climate shock, $\pi$	0.2 (and 0.1)
Equilibrium climate sensitivity	3 (and 4)

# After climate disaster: optimality conditions

Energy demands:

$$\tilde{A}F_E(K, E^A, R^A) = d, \quad \tilde{A}F_R(K, E^A, R^A) = c.$$

Net output function for after calamity:

$$Y^A(K, d, c, \pi) \equiv \underset{E, R}{\text{Max}} \left[ (1 - \pi)AF(K, E, R) - dE - cR \right] - \delta K,$$

$$Y_K^A = (1 - \pi)AF_K - \delta > 0, \quad Y_d^A = -E < 0, \quad Y_c^A = -R < 0,$$

$$Y_\pi^A = -AF < 0.$$

HJB equation for after the calamity has occurred:

$$\rho V^A(K, \pi) = \underset{C}{\text{Max}} \left\{ U(C) + V_K^A(K, \pi) \left[ Y^A(K, d, c, \pi) - C \right] \right\},$$

$$\text{so } U'(C) = V_K^A(K, \pi).$$

# After climate disaster



- Keynes-Ramsey rule:

$$\dot{C}(t) = \sigma \left[ Y_K^A(K(t), d, c, \pi) - \rho \right] C(t), \quad \sigma \equiv -U' / CU'' = EIS = \frac{1}{C R IIA} > 0.$$

- Post-catastrophe value function from HJB equation:

$$V^A(K, \pi) = \frac{U(C^A(K, \pi)) + U'(C^A(K, \pi)) \left[ Y^A(K, d, c, \pi) - C^A(K, \pi) \right]}{\rho}.$$

# Approximate after-calamity solution

$$C^A(K) \cong C^{A*} \left( \frac{K}{K^{A*}} \right)^\phi, \quad \phi \equiv \left[ \frac{1}{2} \rho + \frac{1}{2} \sqrt{\rho^2 - 4\sigma Y_{KK}^A C^{A*}} \right] \frac{K^{A*}}{C^{A*}} > \rho,$$

where the steady state follows from:

$$Y_K^A(K^*, d, c, \pi) = \rho \quad \Rightarrow \quad K^{A*} = K^A(\bar{\rho}, \bar{d}, \bar{c}, \bar{\pi}),$$

$$C^{A*} = Y^A(K^*(\bar{\rho}, \bar{d}, \bar{c}, \bar{\pi}), d, c, \pi) \equiv C^A(\bar{\rho}, \bar{d}, \bar{c}, \bar{\pi}).$$

Cobb-Douglas production (with  $\alpha$  and  $\beta$  capital & energy shares):

$$\phi = \left[ \frac{1}{2} \rho + \frac{1}{2} \sqrt{\rho^2 + 4\sigma(\rho + \delta) \left( \frac{1 - \alpha - \beta}{1 - \beta} \right) \frac{C^{A*}}{K^{A*}}} \right] \frac{K^{A*}}{C^{A*}} = 0.431.$$

# State space of pre-catastrophe system



- Can reduce two HJBs to 4-dimensional dynamics:

$$\dot{K} = Y^B(K, d + \tau, c, A) - \tau Y_{d+\tau}^B(K, d + \tau, c, A) - C, \quad K(0) = K_0,$$

with  $Y^B(K, d + \tau, c, A) = \text{Max}_{E, R} [AF(K, E, R) - (d + \tau)E - cR] - \delta K,$

$$\dot{P} = -\psi Y_{d+\tau}(K, d + \tau, c, A) - \gamma P, \quad P(0) = P_0,$$

$$\dot{C} = \sigma \left[ Y_K^B(K, d + \tau, c, A) + \Theta(K, P) - \rho \right] C, \quad \theta = \Theta(K, P) \equiv H(P) \left[ \frac{V_K^A(K, \pi)}{U'(C)} - 1 \right],$$

$$\dot{\tau} = \left[ Y_K^B(K, d + \tau, c, A) + \Theta(K, P) + \gamma + H(P) \right] \tau - \frac{\psi H'(P) [V^B(K, P) - V^A(K, \pi)]}{U'(C)}.$$

# Multiple tipping points



- Multiple tipping points corresponding to different types of catastrophe (local versus regional, fast versus slow, damages versus changes in carbon cycle dynamics, etc.) each with different hazard rates.
- With 2 catastrophes must solve for 5 coupled HJB equations. If value to go after 2 shocks is independent of order of shocks, need to solve for 4.
- With 3 (4, 5 or 6) catastrophes and symmetry need to solve for 8 (26, 122 or 722) value functions.
- With 9 catastrophes we need to solve at least  $2 + 9! = 362,882$  coupled HJB equations!



# The SCC with two catastrophes



- Independent disasters, but hazard of the next will go up if first disaster leads to more global warming!

$$\begin{aligned}
 \tau(t) = & \underbrace{\psi \int_t^\infty e^{-\int_t^s [\rho + \gamma + H_1(P(s')) + H_2(P(s'))] ds} ds}_{\text{modified conventional no-shock social cost of carbon}} + \underbrace{\int_t^\infty H_1(P(s)) \tau_1(s) e^{-\int_t^s [\rho + \gamma + H_1(P(s')) + H_2(P(s'))] ds} ds}_{\text{'raising the stakes' effect: 1}} \\
 & \underbrace{\int_t^\infty H_2(P(s)) \tau_2(s) e^{-\int_t^s [\rho + \gamma + H_1(P(s')) + H_2(P(s'))] ds} ds}_{\text{'raising the stakes' effect: 2}} \\
 & + \underbrace{\int_t^\infty H_1'(P(s)) \{V^B(P(s)) - V^1(P(s))\} e^{-\int_t^s [\rho + \gamma + H(P(s'))] ds} ds}_{\text{'risk averting' effect: 1}}, \\
 & + \underbrace{\int_t^\infty H_2'(P(s)) \{V^B(P(s)) - V^2(P(s))\} e^{-\int_t^s [\rho + \gamma + H(P(s'))] ds} ds}_{\text{'risk averting' effect: 2}}, \quad 0 \leq t < T_1 \text{ or } T_2.
 \end{aligned}$$

# SCC after one calamity has occurred



If shock 1 has occurred first:

$$\begin{aligned} \tau_1(t) = & \underbrace{\psi \int_t^\infty e^{-\int_t^s [\rho + \gamma + H_2(P(s'))] ds} ds}_{\text{modified conventional no-shock social cost of carbon}} + \underbrace{\int_t^\infty H_2(P(s)) \tau^A e^{-\int_t^s [\rho + \gamma + H_2(P(s'))] ds} ds}_{\text{'raising the stakes' effect}} \\ & + \underbrace{\int_t^\infty H_2'(P(s)) \{V^1(P(s)) - V^A(P(s))\} e^{-\int_t^s [\rho + \gamma + H_2(P(s'))] ds} ds}_{\text{'risk averting' effect}}, \quad 0 \leq t < T. \end{aligned}$$

If shock 2 has occurred first:

$$\begin{aligned} \tau_2(t) = & \underbrace{\psi \int_t^\infty e^{-\int_t^s [\rho + \gamma + H_1(P(s'))] ds} ds}_{\text{modified conventional no-shock social cost of carbon}} + \underbrace{\int_t^\infty H_1(P(s)) \tau^A e^{-\int_t^s [\rho + \gamma + H_1(P(s'))] ds} ds}_{\text{'raising the stakes' effect}} \\ & + \underbrace{\int_t^\infty H_1'(P(s)) \{V^2(P(s)) - V^A(P(s))\} e^{-\int_t^s [\rho + \gamma + H_1(P(s'))] ds} ds}_{\text{'risk averting' effect}}, \quad 0 \leq t < T. \end{aligned}$$