Great Moderation and Great Leverage: Financial trade and asset prices when investors disagree about risk

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Abstract

Whether the "Great Moderation" in macroeconomic volatility observed since the mid-1980s was an outlier of "good luck" or an expression of more structural change was debated in the late 1990s and early 2000s during times of rising leverage and asset prices. We show how disagreement about economic volatility gives rise to perceived gains from trade that can increase leverage and asset prices. This is because investors with dispersed posteriors value more highly the convexity of profits implied by leveraged asset purchases, whose returns increase with high payoffs but are insensitive to a deterioration of bad realisations that lead to bankruptcy of investors. We analyse a simple general equilibrium economy where investors temporarily disagree on the dispersion of payoffs as they update their posteriors at different speeds in reaction to a long-lived fall in observed volatility. Despite constant and common expected payoffs and risk-neutrality, the economy experiences a temporary rise in leverage, price volatility and average asset prices.

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1 Introduction

By the mid-1990s, both investors and academic economists had recognised the fall in macroeconomic volatility that many developed economies had experienced about a decade earlier. Disagreement persisted until the post-2007 financial crisis, however, about the origins and implications of this "Great Moderation" in volatility of GDP and other macroeconomic aggregates. Over the same time period, financial markets in many of the same countries saw both a strong rise in asset prices, and a significant increase in the leverage of households and financial firms particularly in the latter part of the period. This paper looks at the behaviour of asset prices and investor leverage during a period when an observed fall in the dispersion of past output realisations leads to temporary disagreement about the outlook for aggregate risk in the near future. We show how risk-neutral investors with a relatively dispersed posterior distribution of payoffs purchase assets through leveraged investments from those with more concentrated posteriors even though both groups share the same fundamental valuation of (unleveraged) assets. This is because the possibility of bankruptcy by leveraged investors leads to perceived gains from trade from disagreement about risk: limited liability increases the value of leveraged investments to agents with dispersed posteriors, who gain from the upside but go bankrupt when payoffs are low. Investors with concentrated posteriors perceive the risk of bankruptcy to be lower, implying gains from trade that would remain unrealised in the absence of leverage.

We show first theoretically how disagreement about the dispersion of future asset payoffs can lead to an increase in asset trade and leverage as well as a rise in asset prices in the general equilibrium of an economy where investors share the same belief about mean asset payoffs but have different views about their volatility. In a simplified, dynamic version of the model, prices fluctuate in a range strictly above the common fundamental value of the asset. This is because leveraged investors occasionally go out of business but the remaining investors are happy to pay a speculative premium in anticipation of higher prices once leveraged investors re-enter the market in the future. Finally, we use a quantitative version of this simple model to analyse a scenario that captures the experience of the US economy during the Great Moderation period. When some investors update their posteriors about the dispersion of aggregate payoffs less quickly than others, temporary disagreement about volatility leads to a period of increased leverage and price volatility as well as a moderate boom in average asset prices.

The literature on the consequences of heteregenous investor beliefs has focused on disagreement about mean payoffs.[1] Thus, Miller (1977)'s seminal article shows how asset prices rise when investors disagree about future mean payoffs and the absence of short-selling makes the marginal investor become more optimistic. Harrison and Kreps (1978) analyse a dynamic version of the model and show that even in periods where optimists are not in the market, pessimists are happy to pay a speculative premium above their fundamental asset valuation in anticipation of rising prices when optimists return to the market in the future. Geanakoplos (2001) introduces leverage into this framework, whereby investors can issue debt collateralised by the assets they want to buy in order to increase the amount they can invest. This allows optimists to increase their asset purchase and thus makes the marginal investor more optimistic, increasing prices further. At the same time it leads to fluctuations in asset prices in response to changes in investor balance sheets. Simsek (2013) uses a similar model with two groups of investors to show how leveraged investment dampens the effect of belief disagreements on prices when optimism is concentrated on the downside, i.e. when optimists have relatively positive views on the distribution of relatively bad realisations of shocks. If optimists are particularly positive about the upside potential of the asset, however, leveraged investments amplify the effect of belief disagreements.

Interestingly, Miller (1977)'s original article associates higher payoff risk with stronger disagreement about mean payoffs. Neither his article, nor the literature that it preceeds, however, has analysed disagreement in beliefs about risk per se.[2] Although the latter trivially leaves prices unaffected in Miller (1977)'s original framework with risk-neutral investors and no leverage, we show how leverage introduces a convexity in the profit function that makes expected profits rise with the dispersion of asset payoffs. In other words, investors with a relatively more dispersed posterior feature both Simsek (2013)'s upside optimism and downside pessimism, leading to perceived gains from trade and a boom in equilibrium asset prices.

The effect of the Great Moderation on asset prices has been studied previously in environments with Bayesian learning about the volatility of payoffs under homogeneous priors (Lettau et al. (2008), Broer and Kero (2011)). Even when investors share the same prior mean and variance of output growth, however, differences in the "strength" of their priors,

[1]See (Xiong, 2013) for a survey.

[2]In an early reaction to (Miller, 1977), (Jarrow, 1980) has pointed out the importance of the variance-covariance structure of asset returns for the effect of short-selling constraints on asset prices. His focus is very different to the one in this paper, however.

or in the speed at which their posterior reacts to new information, will lead to temporary heterogeneity in posterior beliefs when moments of observed data exhibit a sudden change such as the fall in GDP volatility during the Great Moderation. Particularly, when some agents adjust their posteriors more quickly than others, a permanent fall in volatility leads to a gradual fall in the average of volatility estimates across investors, but also to a temporary rise in their dispersion as posteriors of “quick updaters” react more strongly initially, and those of “latecomers” eventually catch up and both converge to the new data moment. In line with this intuition, both the academic literature and investment research during the second half of the 1990s discussed a variety of potential mechanisms behind the observed fall in volatility. Specifically, some participants saw the period of low volatility as a mere outlier of “good luck” in an unchanged stochastic environment, while others interpreted it as the result of fundamental changes in the economy such as central bank independence, logistical innovation, globalisation, financial innovation, etc.

We see this debate, generally, as an expression of increased heterogeneity in posterior beliefs about the structure of the economy. More specifically, in this paper, we operationalise this heterogeneity as an increase in the dispersion of beliefs about the volatility in future aggregate output and investment payoffs. Due to the convexity of profits from leveraged investment, which are constant and zero below a bankruptcy threshold but increase linearly with asset payoffs above it, this belief dispersion leads to gains from trade as investors with more volatile posteriors perceive both more upside risk and downside risk, but the latter does not affect their expected profits. We see our contribution in drawing attention to this previously overlooked mechanism and formalising it in a simple general equilibrium economy. Specifically, we analyse an economy where risk-neutral investors agree on the expected value of asset payoffs, but differ in their belief about payoff dispersion. In this environment, there are no gains from trading the asset on simple spot markets without leverage. However, investors with more dispersed posteriors can be seen as “upside optimists” and “downside pessimists”, in Simsek (2013)’s language. The result that leveraged investments amplify upside optimism but dampen downside optimism then implies that disagreement about second moments of payoffs necessarily leads to scope for asset trade and a rise in prices in a static economy. Specifically, when the perceived dispersion of asset payoffs takes values on a continuum, there are two cutoff values such that high-dispersion agents, who perceive the upside potential to be large, invest in leveraged assets, while low dispersion investors, who perceive downside risk to be small, buy collateralised loans. Agents with intermediate dispersion do not invest in any assets. Equilibrium asset prices lie strictly above their fundamental value and rise with increasing divergence in investor beliefs about payoff risk. We then present a simplification of the model with only two investor types, equivalent to a general equilibrium version of the static environment in Simsek (2013), that allows us to analyse the choice of leverage levels and the asset price fluctuations in a dynamic version of this economy. There, general equilibrium prices fluctuate in a range strictly above the fundamental value of the asset, equal to the expectation of discounted payoffs, on which optimists and pessimists agree. Thus, despite constant and common fundamental valuations of the asset, disagreement about the second moments introduces both price volatility, as low payoff realisations make leveraged investors go bankrupt and temporarilly leave the market, and a boom in prices. That equilibrium prices exceed the fundamental value of the asset even when only investors with concentrated posteriors are in the market is due to a speculative premium, as in Harrison and Kreps (1978), that makes them pay a higher price today in anticipation of rising prices upon return of the natural buyers in the future. Note that the convexity of payoffs that drives most results is also behind the potential for risk-seeking behaviour in financial markets with limited liability (as in, for example, Murdock et al. (2000)). The environment of this paper, however, lacks any potential for moral hazard as actions are perfectly observed and the buyers of collateralised debt agree with leveraged investors on the functional relationship between exogenous asset payoffs and bankruptcy. The only source of disagreement, and the origin of perceived gains from trade between the two, is the second moment of the distribution of asset payoffs.

2 Asset prices in a continuum economy with disagreement about payoff risk

2.1 The General Environment

We study an economy that exists for two periods $t \in \{0, 1\}$. There is a continuum of agents of unit-mass indexed by i with $i \in I = [0, 1]$. The distribution of agents on I is defined by measure $m : \mathbb{I} \longrightarrow [0, 1]$, where $\mathbb{I}$ is the Borel-algebra of I and m has no mass-points. Denote as g the density function induced by m, and by G the cumulative density function of g with $G(0) = 0$,$G(1) = 1$.

In period 0, agents of type i receive an endowment n_i of the unique perishable consumption

good and $\overline{a}_i$ units of a risky asset (a "tree") that pays a stochastic amount $s \in S = [s_{min}, s_{max}]$, $s_{min} > 0$ in period 1. All agents are assumed to be risk-neutral, so they maximise the present discounted sum of expected consumption in both periods i.e. $U_i = c_i + \frac{1}{R}E(c_i')$, where E_i is the mathematical expectation of agent i, c_i (resp. c_i') denotes consumption in period 0 (resp. 1) and $\frac{1}{R}$ is the discount factor.
We assume that types differ in their beliefs about the distribution of random payoffs s, summarised by distribution functions $f_i : S \longrightarrow R^+$. We assume that all agents expect payoffs to be the same on average, but that any type $i : i > j$ believes them to be less tightly distributed than type j. In other words, f_j second-order stochastically dominates f_i whenever $i > j$, or formally:

$$\mathbf{A1} : E_i(s) = E_j(s) \equiv E_s,\ f_j \succ^2 f_i \Leftrightarrow j < i$$

where $\succ^2$ denotes second-order stochastic dominance. Thus i is an index of belief dispersion.

2.2 Asset markets

Agents trade in 2 asset markets: In $t = 0$, agent i purchases $a_i - \overline{a}_i$ units of the physical asset in exchange of $p(a_i - \overline{a}_i)$ units of the consumption good. In addition, agents can borrow by pledging part of their future income. However, agents cannot commit to future payments, and therefore have to collateralise their borrowing. We assume that agents only trade the simplest form of these contracts, namely a debt contract. Debt contracts are characterised by a fixed promised face value. The absence of commitment means that agents transfer to their creditor the face value of the loan or the payoff of the assets that serve as collateral, whatever is smaller. We normalise contracts to be secured by 1 unit of the asset as collateral. Thus collateralised loan contracts have unit-payoffs equal to $\min\{s, \overline{s}\}$, where $\overline{s}$ is the promised face value. In $t = 0$, agents trade these contracts at competitive price $q(\overline{s})$. In the following we assume that this price function is Borel measurable.

2.3 Type i's problem

For a given exogenous $\overline{s}$, the budget constraints of agent i in $t = 0$ and $t = 1$ respectively are:

$$c_i + pa_i + qb_i \leq n_i + p\overline{a}_i, \tag{1}$$

$$c_i^{'} \leq a_i s + min\{s, \bar{s}\} b_i \tag{2}$$

where a_i and b_i represent agent i's total holdings of risky assets, including the initial endowment, and of collateralised loans respectively. Given that borrowing is subject to a collateral constraint, agent i's positions of collateralized loans sold must satisfy the following condition:

$$b_i \geq -a_{i.} \tag{3}$$

Each unit of collateralized loan sold must the secured by at least one unit of the risky asset that agent i possesses and can be used as collateral. Agent i maximizes his expected utility subject to the budget and collateral constraint. Formally his optimization problem is:

$$\max_{c_i, c_i', a_i \geq 0, b_i > -a_i} \quad c_i + \frac{1}{R} E(c_i') \tag{4}$$

subject to

(1), (2) and (3)

2.4 Optimal Behaviour at given prices

2.4.1 Profits

Collateralized Loan Contracts: Buyers of collateralised loans with facevalue $\bar{s}$ pay a sum q to their counterparty today, for a promise whose expected value is $E_i[\min\{s, \bar{s}\}]$. For a quantity of loans b_i, expected discounted profits are

$$\Pi_i^l = b_i [\frac{E_i[min\{s, \bar{s}\}]}{R} - q]$$

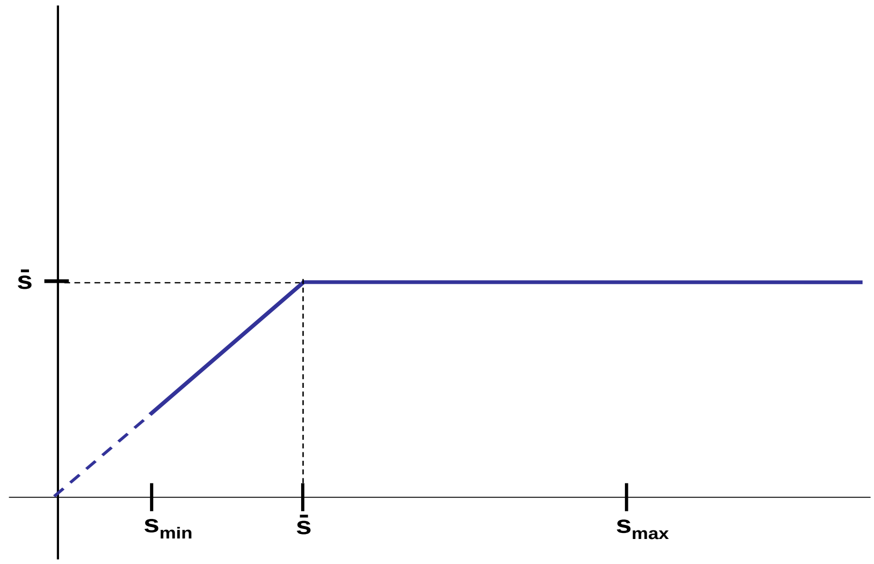
Leveraged Risky Assets: Other than buying assets outright using consumption goods as payment, agents can engage in leveraged asset purchases by using the assets as collateral for collateralised loans. Then, for a given $\bar{s}$, the expected profits from buying a_i units of risky assets partly financed through a collateralised loan of equal size are

$$\Pi_i^a \quad = \quad a_i R_i^a \equiv [\frac{[E_i(s) - E_i(\min(s, \bar{s}))]}{R} - (p - q)] \tag{5}$$

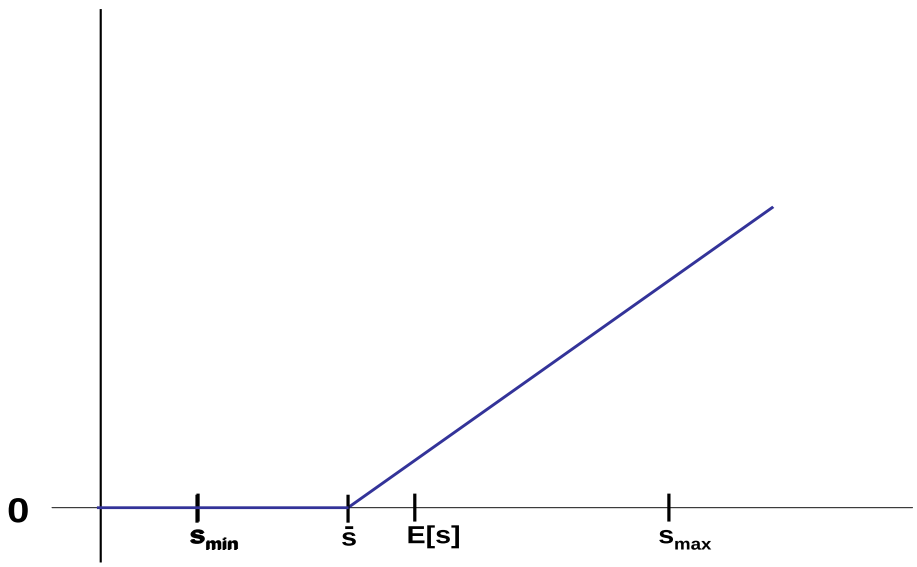
where R_i^a is the return on leveraged asset investment.

Figure 1

Unit-Profits from collateralised loans



Unit-Profits from leveraged asset purchases



The figure plots the profits from collateralised loans (upper panel) that are concave in s, and those from leverage asset purchases (lower panel), which are convex in s.

Figure 1 illustrates how gross unit-profits in period 1 change as a funtion of the asset payoff s. The definition of profits implies that returns on collateralised loans are convex in s, while those on leveraged asset purchases are concave in s. Given the second order stochastic dominance relationship of beliefs, this immediately implies that tight-prior agents (low i) have higher expected profits from investing in collateralised loans than those with dispersed priors (high i). The inverse is true for profits from leveraged asset purchases. We thus have

$$i > j \Rightarrow \Pi_i^l \leq \Pi_j^l \ \forall \ \overline{s} \in (s_{\min}, s_{\max}), \forall p, q, R$$
$$i > j \Rightarrow \Pi_i^a \geq \Pi_j^a \ \forall \ \overline{s} \in (s_{\min}, s_{\max}), \forall p, q, R$$

2.5 General Equilibrium

In this section we look at the equilibrium of an economy with exogenous face value $\overline{s} \leq E_s$. For this, we normalise the asset supply to 1 and assume $\overline{a}_i = 1 \forall i$.

2.5.1 Equilibrium Definition

A general equilibrium given $\overline{s}$ is a set of prices (p, q) and allocations $(c_i, c_i', a_i, b_i)_{i\in[0,1]}$, such that agent $i \in [0, 1]$ behaves optimally given p, q and $\overline{s}$, the demand for assets equals the fixed supply,

$$\int_{i\in I} a_i = 1 \tag{6}$$

and the collateralized loan maket clears,

$$\int_{i\in I} b_i = 0.$$

2.5.2 Uniqueness of equilibrium and asset price bubbles

Note that for any $p < \underline{p} \doteq \frac{E_0}{R}$ all agents would like to buy risky assets, which cannot be an equilibrium. Similarly, for any given $q > \overline{q} \doteq \frac{E_0(min\{s,\overline{s}\})}{R}$, no agent is willing to buy collaterlised loans, but all agents who hold assets make a strict profit by using them as collateral for the issuance of collateralised loans. These equilibria, however, leave perceived gains from trade unexploited as there is a price $q < \overline{q}$ such that some agents would like to sell collateralised loans at q and some others would like to buy them.

Proposition 1: Uniqueness of a bubble equilibrium

There is a unique equilibrium with trade in assets of riskyness $\overline{s}$. This equilibrium has the following properties:

- $\frac{E_s - E_1[min(s,\overline{s})]}{R} + q \doteq \overline{p} \geq p > \underline{p} \doteq \frac{Es}{R}$
- $q < \overline{q} \doteq \frac{E_0(min\{s,\overline{s}\})}{R}$
- There are cutoff values $i_q < i_p$ such that all agents with $i > i_p$ invest their whole endowment $n + p\overline{a}$ in leveraged asset purchases, while all agents with $i < i_q$ invest their whole endowment in collateralised loans. Agents with $i : i_q \leq i \leq i_p$ sell their asset endowment and consume the proceeds together with their endowment of consumption goods.

Proof:

Take any $q < \overline{q}$ and define

$$i_q : q = \frac{E_{i=i_q}(min\{s, \overline{s}\})}{R} \tag{7}$$

Take any $p \geq \underline{p}$ and define

$$i_p : \frac{E_s - E_{i_p}(min\{s, \overline{s}\})}{p - q} = R. \tag{8}$$

Note that for $p = \frac{E_s}{R}$, $i_q = i_p$ and for $p > \frac{E_s}{R}$, $i_q < i_p$.

Note that for any $p \geq \underline{p}$, all agents weakly prefer to sell their assets and consume the proceeds over holding them outright (i.e. without leverage). Since $E_i[min\{s, \overline{s}\}]$ is decreasing in i, agents with $i > i_p$ ($i < i_q$) expect to make strictly positive profits from leveraged asset (collateralised loan) purchases. So all agents with $i > i_p$ ($i < i_q$) sell their assets and invest the proceeds, together with their consumption endowments, in leveraged assets (collateralised loans). Moreover, since for $p = \frac{E_s}{R}$ $i_p = i_q$, and for $p > \frac{E_s}{R}$ any agent with $i : i_q < i < i_p$ strictly prefers selling her assets and consuming, it has to be that $i > i_p$ agents buy all assets, while $i < i_q$ agents buy all collateralised loans, both of which have supply equal to 1. The market clearing condition for leveraged assets thus becomes

$$\int_{i_p}^{1} a_i g(i) = \int_{i_p}^{1} \frac{n + p}{p - q} g(i) = \overline{a} = 1 \tag{9}$$

$$\Rightarrow \qquad (n + p)(1 - G(i_p)) = (p - q)\overline{a}. \tag{10}$$

where the first equality substitutes for a_i from the budget constraint for $i > i_p$ agents with $c_i = 0$, $\overline{a} = 1$ and $b_i = -a_i$. Note that this immediately puts an upper bound $\overline{p}(q)$ on the asset price at the level where even agents of type $i = 1$, whose beliefs are most dispersed and who thus expect to make the highest profit from leveraged asset purchases, do not want to buy assets

$$\overline{p}(q) \leq \frac{E_s - E_1[min(s, \overline{s})]}{R} - q. \tag{11}$$

The market clearing condition for collateralised loans can be written as

$$\int_0^{i_q} b_i g(i) = \int_0^{i_q} \frac{n+p}{q} g(i) = 1 \tag{12}$$

$$\Rightarrow \quad (n+p)G(i_q) = q. \tag{13}$$

where the first equality substitutes for b_i from the budget constraint for $i < i_q$ agents with $a_i = 0$, $\overline{a} = 1$ and $c_i = 0$. Note that, since i_q is decreasing in q, so are $G(i_q)$ and the left-hand side of (13), which thus provides a unique mapping from the asset price p into a market clearing price $q^\star$, thus defining $i_q^\star$. We can substitute the price in (10) to get

$$p(1 - \int_0^{i_q} 1g(i) - \int_{i_p}^{1} 1g(i)) = \int_0^{i_q} ng(i) + \int_{i_p}^{1} ng(i)) \tag{14}$$

$$\Rightarrow \quad p = n(\frac{1}{G(i_p) - G(i_q)} - 1). \tag{15}$$

Clearly, this equation has no finite solution for $i_q = i_p$. Hence $i_q < i_p$ in equilibrium and thus $p > \frac{E_s}{R}$. Note that, without loss of generality, we have assumed a tie-breaking rule for agents with $i = i_q$ and $i = i_p$ both of mass zero.

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Interpretation

Proposition 1 shows how the convexity of payoffs due to leverage allows to exploit perceived gains from trade arising from heterogenous beliefs about payoff dispersion. Investors who perceive risk to be high (low) expect to make strictly positive profits and invest all their funds in leveraged assets (collateralised loans). Market clearing for consumption goods requires that there be a "middle" interval (i_q, i_p) of agents who consume in the first period. For this to be the case, asset prices must exceed their fundamental value $\underline{p}$.

Figure 2 illustrates the equilibrium. Types $i \leq i_q$ invest the value of their whole endowment (equal to $n+p$) in collateralised loans, with a total demand equal to $b = G_1 \frac{n+p}{q}$ for $G_1 = \int_0^{i_q} dG(i)$. Similarly, the demand for consumption goods, by agents with $i : i_1 \leq i < i_p$ equals $c = G_2(n+p)$ for $G_2 = \int_{i_q}^{i_p} dG(i)$. And finally, total asset demand, by agents with $i > i_p$ equals $a = G_3 \frac{n+p}{p-q}$ for $G_3 = \int_{i_p}^{1} dG(i)$.

2.5.3 Comparative statics: increased belief dispersion and asset prices

This section looks at the effect of increasing belief dispersion on asset prices. For this I define an increase in belief dispersion as a perturbation of the distribution of agents $dG(i)$ that reallocates mass from the middle interval $[i_q, i_p]$ to both extremes $[0, i_q], [i_p, 1]$. In other words, I look at a pair of exogenous small changes $dG_1, dG_3 > 0, dG_2 = -(dG_1 + dG_3) < 0$. The following proposition shows how market-clearing prices rise in response to this marginal increase in belief dispersion.

Proposition 2: Increased belief dispersion raises prices
A small increse in belief dispersion $dG_1, dG_3 > 0$ raises prices of both collateralised loans and assets.

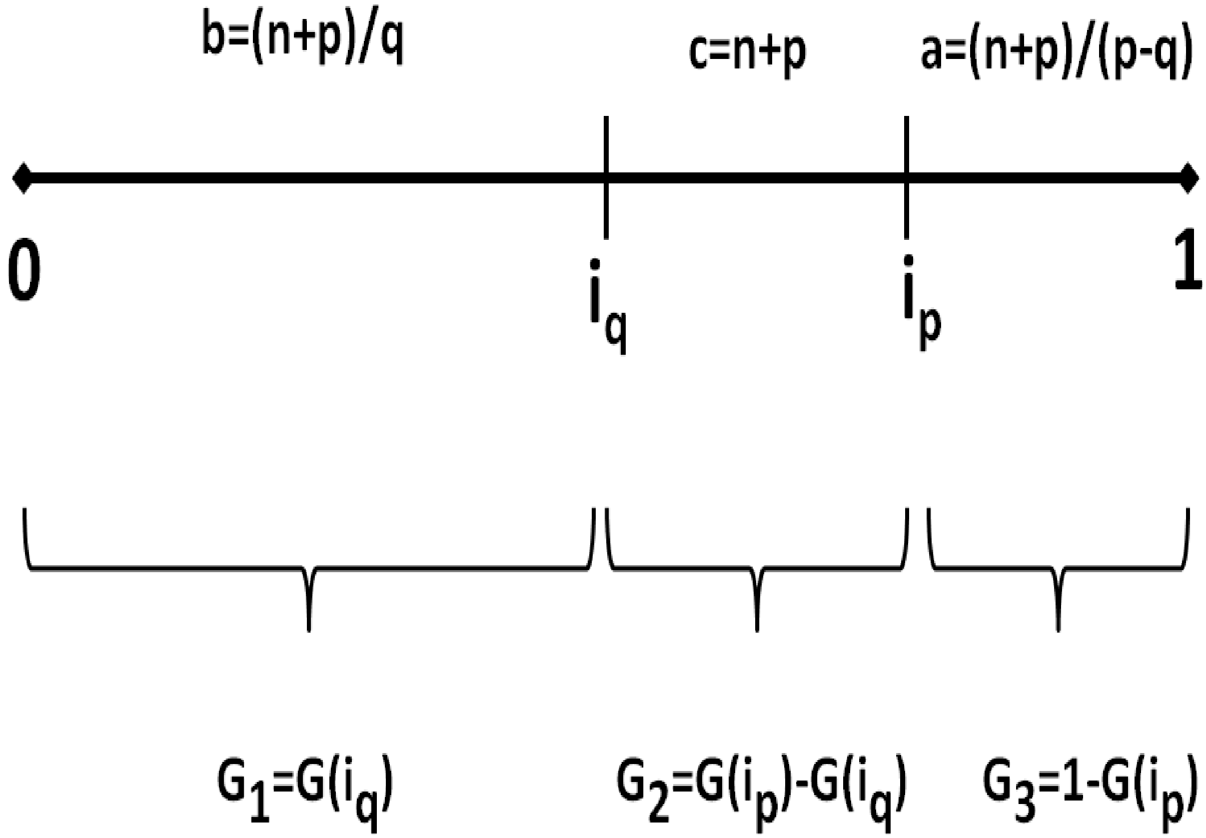
Proof:
Note that at given prices p, q, the change in excess demand for bonds and assets eqals their unit demands multiplied by the change in the mass of agents in the extreme intervals, respectively, $d\widetilde{a} = dG_1 \frac{n+p}{q}$ and $d\widetilde{b} = dG_3 \frac{n+p}{p-q}$. We are thus looking for a pair of price changes dp, dq such that

$$\begin{aligned} db &= -d\widetilde{b} = \frac{n+p}{q}[\frac{\delta G_1}{\delta q} - G_1 \frac{1}{q}]dq \; + \; \frac{1}{q}[\frac{\delta G_1}{\delta p}(n+p) + G_1]dp < 0 \\ da &= -d\widetilde{a} = \frac{n+p}{p-q}[\frac{\delta G_3}{\delta q} + G_3 \frac{1}{p-q}]dq + \frac{1}{p-q}[\frac{\delta G_3}{\delta p}(n+p) + G_3(1 - \frac{n+p}{p-q})]dp < 0 \end{aligned} \tag{16}$$

Note that, from the definition of G_1, G_3 as well as i_p and i_q in (8) and (7), we have $\frac{\delta G_1}{\delta q} = g(i_q)\frac{i_q}{\delta q} < 0$, $\frac{\delta G_3}{\delta q} = -g(i_p)\frac{i_p}{\delta p} < 0$, $\frac{\delta G_1}{\delta p} = 0$ and $\frac{\delta G_3}{\delta p} = -\frac{\delta G_3}{\delta p} > 0$. Use this, and the market-clearing conditions $G_1 \frac{n+p}{q} = G_3 \frac{n+p}{p-q} = 1$,to simplify (16)

$$\begin{aligned} db &= = [\frac{\delta G_1}{G_1 \delta q} - \frac{1}{q}]dq \; + \; \frac{1}{q} G_1 dp \\ da &= = [\frac{\delta G_3}{G3\delta q} + \frac{1}{p-q}]dq + [\frac{\delta G_3}{G_3 \delta p} + (G_3 - \frac{1}{p-q}1))]dp \end{aligned}$$

Figure 2



The figure plots the three intervals on [0,1] that correspond to 1. investors with low belief dispersion ($i \leq i_q$), who have mass G_1 and prefer to buy collateralised loans to consuming or buying assets; 2. investors with medium belief dispersion ($i_q < i \leq i_p$), who have mass G_2 and prefer to consume, rather than invest; 3. and finally investors with high belief dispersion ($i > i_p$), who have mass G_3 and prefer to buy leverage assets.

Denoting the vector of price and quantity changes as $\overline{dp}$ and $\overline{dx}$ respectively, and writing

$$\overline{dx} \quad = \quad A\overline{dp} \Rightarrow \overline{dp} = A^{-1}\overline{dx}$$

we can sign the row i-column j elements of A as $A_{11} < 0$, $A_{12} > 0$, $A_{22} < 0$, $A_{21} > 0$. In other words, the "own-price effects" on asset demand are negative, while the "cross-price effects" (the off-diagonal elements of A) are positive, implying cofactor matrix of A C_A with only negative entries. To conclude the proof, we thus have to show that D_A, the determinant of A, is positive.

$$\begin{aligned} D_A &= [\frac{\delta G_1}{G_1 \delta q} - \frac{1}{q}][\frac{\delta G_3}{G_3 \delta p} + (G_3 - \frac{1}{p-q}1)] \quad - \quad [\frac{\delta G_3}{G3 \delta q} + \frac{1}{p-q}]\frac{1}{q}G_1 \\ &= \frac{1 - G_3 - G_1}{q(p-q)} + \frac{1}{q}\frac{g_3}{G_3}\frac{di_p}{dp}(1 - G_1) + \frac{g_1}{G_1}\frac{di_q}{dq}(-\frac{g_3}{G_3}\frac{di_p}{dp} - \frac{1 - G_3}{p-q}) > 0 \quad (17) \end{aligned}$$

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Interpretation

Proposition 2 shows how we need a rise in both prices to "undo" a rise in excess demand that results from an exogenous increase in belief dispersion. The challenge is three-fold: first, a change in prices changes both the unit demands as well as the size of the intervals G_1 and G_3; second, the unit demands comprise the asset endowment, leading to a positive effect of a rise in asset prices on both quantities; and finally, the cross-price effect of a rise in the price of collateralised loans $dq > 0$ on asset demand is positive, as it makes it cheaper to raise outside funds. The proof exploits market-clearing, and the fact that an equal increase in dp and dq leaves leaveraged asset demand (excluding the endowment effect) unchanged to show that the own price effects dominate the endowent and cross-price effects.

2.5.4 Endogenous choice of $\overline{s}$

So far, we have taken $\overline{s}$, the face value of the loan, as exogenous. This section looks at the optimal choice of $\overline{s}$ subject to an upper bound: $s \leq \overline{s}^{max}$. In other words, we assume that there are some non-modelled features of the economy that put an upper bound to the riskyness of collateralised loans.

The net benefit of a marginal change $d\overline{s}$ to an investor in collateralised asset equals the additional returns on outside funds that increase when selling collateralised loans at a higher price, equal to $R_i^a \frac{\delta q(\bar{s})}{\delta \bar{s}}$, minus the increase in expected payments on the loan, equal to 1 minus the probability of default $(1 - F_i(\bar{s}))$.

$$\frac{d\Pi_i}{d\bar{s}} = \frac{n+p}{p-q}[R_i^a \frac{\delta q(\bar{s})}{\delta \bar{s}} - (1 - F_i(\bar{s}))] \tag{18}$$

Conversely, the net beneft to an agent j from increasing the $\bar{s}$ of the collateralised loan she purchases equals the expected rise in payments $(1 - F_j(\bar{s}))$ minus the loss in profits due to a reduced quantity of loans she can afford at the higher price, equal to $\frac{E_j[min(s,\bar{s})}{q} \frac{\delta q(\bar{s})}{\delta \bar{s}}$

$$\frac{d\Pi_j}{d\bar{s}} = \frac{n+p}{q}[(1 - F_i(\bar{s})) - \frac{E_j[min(s,\bar{s})}{q} \frac{\delta q(\bar{s})}{\delta \bar{s}}] \tag{19}$$

In order to characterise the equilibrium with endogenous leverage choice we make the following additional assumption:

A2: $\bar{s}^{max} \leq s^\star = min_i min_j (s : F_i(s) = F_j(s) \quad j, i \in [0,1]) > s_{min}$

Note that second-order stochastic dominance implies single-crossing of "adjacent" distributions F_i. The assumption ensures that the maximum leverage is smaller than the minimum of all single-crossing points. For example, if we were to restrict our attention to beliefs that are symmetric around E_s, then we would have $s^\star = E_s$ and $\bar{s}^{max} \leq E_s$, which is equivalent to assuming that the bankruptcy probability cannot exceed fifty percent.

Proposition 3: Degenerate choice of $\bar{s}$
Under assumption Ai, only one collateralised loan contract with $\bar{s} = \bar{s}^{max}$ gets traded in equilibrium.

Proof
The choice of $\bar{s}$ depends crucially on the slope of the equilibrium price function $q(\bar{s})$. Note that, for any $\widehat{\bar{s}}$ that gets traded in equilibrium, it has to be that the marginal buyer $i_q(\widehat{\bar{s}})$ weakly prefers $\widehat{\bar{s}}$ to a marginal increase $\widehat{\bar{s}} + d\bar{s}$. Thus

$$\frac{d\Pi_{i_q}(\widehat{\bar{s}})}{d\bar{s}}^+ \leq 0 \Rightarrow \frac{\delta q(\widehat{\bar{s}})}{\delta \widehat{\bar{s}}} \geq (1 - F_{i_q}(\widehat{\bar{s}})) \tag{20}$$

where $\frac{d\Pi_{i_q}(\widehat{\bar{s}})}{d\bar{s}}^+$ denotes the right-hand-side derivative of profits with respect to $\bar{s}$. We can substitute this into (18), to get

$$\frac{d\Pi_i}{\widehat{\bar{s}}}^+ \geq \frac{n+p}{p-q}[R_i^a(1 - F_{i_q}(\widehat{\bar{s}})) - (1 - F_i(\widehat{\bar{s}}))] \geq \frac{n+p}{p-q}[(1 - F_{i_q}(\widehat{\bar{s}})) - (1 - F_{i_p}(\widehat{\bar{s}}))] > 0 \tag{21}$$

where the second-to-last inequality follows from $R_i^a \geq 1 \forall i \geq_p$, and $(1 - F_i(s)) \leq (1 - F_{i_p}(s) \forall i \geq i_p$, and the last inequality follows from $(1 - F_{i_q}(\widehat{\overline{s}})) - (1 - F_{i_p}(\widehat{\overline{s}})) > 0 \forall \widehat{\overline{s}} < s^\star$. So agents only want to issue loans with maximum leverage $\overline{s}^{max}$.

■

Interpretation: The marginal buyer of loan with $\overline{s} = \widehat{\overline{s}}$ has to weakly prefer the collateralised loan of face value $\widehat{\overline{s}}$ to the one with a slightliy higher face value. This puts a lower bound on the slope of the price function $p(\overline{s})$ at that point. Moreover, for $\overline{s} < \overline{s}^\star$, higher i implies a higher bankruptcy-probability, so the additional expected payment on collateralised loans from a small rise $d\overline{s} > 0$ falls with belief dispersion. Issuers of collaterlised loans thus gain from a rise in prices more than they loose from higher expected payments. So they choose the maximum face value and leverage, equal to $\overline{s}^{max}$. Note that the assumption of an upper bound for the face value $\overline{s}$ is crucial here. Without it, issuers of collaterlised loans would potentially choose different $\overline{s}$.

3 Endogenous leverage in an economy with two types

The analysis so far was silent about how the riskyness of loans $\overline{s}$ is determined in equilibrium. Ideally, we would model both the optimal decision of leveraged investors and of buyers of collateralised loans, both of whom choose among a menu of collateral levels $\overline{s} \in [s_{min}, s_{max}]$ at which they can buy or sell collateralised assets. Their optimal choice would depend on individual beliefs f_i, the asset price p and the equilibrium price function for collateralised loans of different risk levels $q(\overline{s})$. Unfortunately, with heterogeneous beliefs about the dispersion of payoffs, there will typically not be a "natural" leverage level as in Geanakoplos (2001)[xxx Check xxx], where the unique equilibrium contract features maximum leverage at zero riskyness ($\overline{s} = s_{min}$). This is because in his model, "optimists' " beliefs first order stochastically dominate those of pessimists. So optimists always perceive a smaller risk $F(\overline{s})$ than pessimists, who thus demand a price of risk that is higher than what optimists are happy to pay. With disagreement about second moments, gains from trade arise precisely because more volatile beliefs imply both higher riskyness of collateralised loans $F_i(\overline{s})$ and higher expected payoffs of leveraged assets. Moreover, different beliefs imply different appetite for leverage and thus different choices of $\overline{s}$, as agents with more dispersed payoff perception value the upside potential contained in high-leverage assets more than agents with low risk perception. But unfortunately, the mapping from price function $q(\overline{s})$ and asset price p to excess demands for assets and collateralised loans (via matching the two continua of buyers and sellers to different optimal leverage levels, not all of whom will be traded in equilibrium), is impossible analytically to the best of our knowledge.

The previous section could be viewed as the characterisation of an economy where leverage is regulated to be at a constant exogenous level. This section, in contrast, briefly looks at an economy where the distribution of types has two mass points $i \in 0, 1$. This assumption allows us, in a first step, to characterise optimal leverage choices, and, in a second step, to look at a dynamic economy without the need to track the wealth distribution across a continuum of investor types. The environment is similar to Simsek (2013) who, however, assumes that beliefs of one group first-order stochastically dominate those of the other. He shows that leverage amplifies the effect of "upside optimism" about high payoff realisations (which increases expected returns of leveraged assets) but disciplines "downside optimism" about low payoffs (since profits are 0 independently of the exact realisation of s below $\overline{s}$). When agents differ in the perceived dispersion of payoffs, however, investors with more dispersed beliefs can be seen as "upside optimists" and "downside pessimists", in Simsek (2013)'s language. The result that leveraged investments amplify upside optimism but dampen downside optimism then implies that disagreement about second moments of payoffs necessarily leads to scope for asset trade and a rise in prices.

3.1 The General Environment

We analyse a simplification of the environment where the distribution g has two mass points at $i \in \{0, 1\}$, corresponding to two groups of agents whose beliefs satisfy $f_0 \succ^2 f_1$, as before. So type 0 agents are the natural buyers of collateralised loans, and type 1 agents the natural investors in leveraged assets, since as before $\Pi_1^l <= \Pi_0^l, \Pi_1^a > \Pi_0^a \; \forall \; \overline{s} \in (s_{\min}, s_{\max}), \forall p, q, R$. In other words, if there is trade in collateralised loans in equilibrium $-b_1 = b_0 > 0$. We make the additinoal assumption that type 0, or "steady", agents are relatively cash-rich, in the sense that they can in principle buy all assets at their fundamental valuation $\frac{E_s}{R}$, or formally

$$A3: \; n_0 \geq \overline{a}\frac{E_s}{R}.$$

We think this is a reasonable assumption because many institutions and agents do not make leveraged investments in risky assets due to reasons not modelled here. It can be viewed as an alternative to Simsek (2013)'s assumption that issuers have all bargaining power in splitting the perceived surplus from collateralised loan issuance. This is because assumption A3 immediately implies that type 0 agents bid up the price of any collateralised loan issued by type 1 agents to their expected discounted value, where they are indifferent between investing and consuming. Importantly, this implies that the price function of collateralised loans is

$$q(\overline{s}) = \frac{E_i[\min\{s, \overline{s}\}]}{R} \tag{22}$$

3.2 Type 1's problem and the choice of $\overline{s}$

In contrast to the exogenous $\overline{s}$ in the previous section, Type 1 now chooses both current consumption and the level of leverage $\overline{s}$ to solve the following problem given p and the price function $q(\overline{s})$

$$\begin{aligned} \max_{c_1, \overline{s}} \quad U_1 &= c_1 + \frac{(n_1 + \overline{a}_1 p - c_1)}{R} \frac{[E_s - E_1(\min\{s, \overline{s}\})]}{p - \frac{E_0\{\min\{s, \overline{s}\}\}}{R}} \\ &= c_1 + \frac{(n_1 + \overline{a}_1 p - c_1)}{R} R_1^a \end{aligned} \tag{23}$$

where again $R_1^a \doteq \frac{[E_s - E_1(\min\{s,\overline{s}\})]}{p - \frac{E_0\{\min\{s,\overline{s}\}\}}{R}}$ is the leveraged return of the asset using a loan with riskiness $\overline{s}$. The first order condition for $\overline{s}$ can be written as

$$\frac{(n_1 + \overline{a}_1 p)}{Rp - E_0\{min\{s, \overline{s}\}\}}[(1 - F_1(\overline{s})) - \frac{R_1^a}{R}(1 - F_0(\overline{s}))] = 0 \tag{24}$$

Proposition 3: Unique interior choice of $\overline{s}$.
If p is such that $\frac{E_s}{R} = \underline{p} > p > \overline{p} \doteq \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R}$ holds for some $\overline{s} \in (s_{min}, s_{max})$, $R_1^a(p, \overline{s})$ has an interior maximum at some $\overline{s}^\star \in (s_{min}, s_{max})$.
Proof: Note that$R_1^a = 0$ at $\overline{s} = s_{max}$. Also, if $p > \frac{E_s}{R}$, $R_1^a(s_{min}) = \underline{R_1^a} < 1$. But if at some $\overline{s}'$ $p < \frac{E_s + E_0(min\{s,\overline{s}'\}) - E_1(min(s,\overline{s}'))}{R}$, then $R_1^a(\overline{s}') > 1$. The statement then follows from continuity of R_1^a.

3.3 Equilibrium Characterisation

Definition: A general equilibrium is a set of prices (p, q) and allocations $(c_i, c_i', a_i, b_i)_{i \in \{0,1\}}$, such that agent $i \in \{0, 1\}$ solves the optimization problem (23), the demand for assets equals the fixed supply,

$$a_0 + a_1 = 1$$

and the collateralized loan maket clears,

$$b_1(\overline{s}) + b_0(\overline{s}) = 0 \ \forall \overline{s}.$$

Note that for any $p < \frac{E_s}{R}$ all agents would like to buy risky assets, which cannot be an equilibrium. Equivalently, for any $p > \overline{p} \doteq \frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R}$ both type 0 and type 1 agents would like to sell their risky assets, again contradicting equilibrium. Whenever $p : \overline{p} > p \geq \frac{E_s}{R}$, however, type 1 agents optimally make leveraged asset purchases from type 0 agents. Specifically, since expected returns from leveraged investments exceed their rate of time preference, she prefers not to consume in period 0 in order to invest all her resources in leveraged asset purchases, equal to

$$a_1 = \frac{n_1 + \overline{a}_1 p}{p - \frac{E_0[\min\{s,\overline{s}\}]}{R}}.$$

Given the unit supply of risky assets, $p(\overline{s})$ is determined in general equilibrium by

$$p(\overline{s}) = \frac{E_0[\min\{s,\overline{s}\}]}{R(1-\overline{a}_1)} + \frac{n_1}{(1-\overline{a}_1)} \tag{25}$$

Equation (25) is an inverse demand function that determines the level of loan riskiness $\overline{s}$ as a function of the asset price p.
Finally, when $p = \overline{p}$ type 1 agents are indifferent between consuming in period 0 or investing in leveraged assets.

The exact equilibrium depends on the supply of assets relative to amount of ressources available to type 1 agents for leveraged asset purchases in $t = 0$, which is

$$n_1^{\max}(\overline{s}) = n_1 + \frac{E_0[\min\{s,\overline{s}\}]}{R},$$

$\frac{E_0[\min\{s,\overline{s}\}]}{R}$ is the maximum amount that he can borrow, assuming that he has purchased all the remaining risky assets of the economy, i.e. $a_1 = 1$. Specifically

1. If $n_1 = 0, a_1 = 0$, then $n_1^{\max}(\overline{s}) \leq \frac{E_s}{R}$. Thus type 1's ressources are insufficient to buy all assets. Therefore, the only equilibrium price is the one at which both type 0 and type 1 agents are willing to hold the asset. So $p = \frac{E_s}{R}$ and there is no trade in collateralised loans.

2. If $n_1 > 0$ or $a_1 > 0$ or both, p and $\overline{s}$ are given by the following equations

$$PS^1: \quad \mathbb{C} = [E_s - E_1(\min\{s, \overline{s}\})](1 - F_0) - (1 - F_1)(Rp - E_0[min\{s, \overline{s}\}]) = 0 \tag{26}$$

$$p = max\{\overline{p}, p^\star\} \tag{27}$$

$$PS^2: \quad (1 - \overline{a}_1)p^\star - n_1^{\max}(\overline{s}) = 0 \tag{28}$$

   (a) Particularly, for $\overline{s}(\overline{p})$ the value of $\overline{s}$ that solves (27) when $p = \overline{p}$, if $n_1^{\max}(\overline{s}(\overline{p})) \geq (1 - \overline{a}_1)\overline{p} = (1 - \overline{a}_1)\frac{E_s + E_0(min(s,\overline{s})) - E_1(min\{s,\overline{s}\})}{R}$, type 1's endowment is large enough to buy type 0's assets at the maximum price $\overline{p}$ that ensures her participation. The only equilibrium price is thus $\overline{p}$, at which type 1 agents are happy to consume in period 0 any resources that remain after purchasing all of type 0's assets.

   (b) If $\frac{E_s}{R} < n_1^{\max}(\overline{s}) < (1 - \overline{a}_1)\overline{p}$, we have $p = p^\star < \overline{p}$, which implies $R_1^a > R$ and $c_1 = 0$. Type 1 agents expect a return on leveraged investments larger than R, so invest all their funds to buy type 0's assets.

Result 4: Uniqueness of equilibrium
PS^2 is upward-sloping. PS^1 is downward-sloping.
Proof: The second part of the statement follows from

$$\frac{dp}{d\overline{s}}|_{PS^1} = -\frac{\frac{d\mathbb{C}}{d\overline{s}}}{\frac{d\mathbb{C}}{d\overline{p}}} \tag{29}$$

Concavity of R_1^a at the optimum implies that the numerator is negative. Since $\frac{(C)}{d\overline{p}} < 0, \forall P, \overline{s}$ the result follows.

3.4 Comparative statics

This section looks at the effect of "belief-divergence", in the sense of a further mean-preserving contraction to f_0. For this we assume that the distribution function f_0 is parameterised by a variable v such that

1. f_0 is continuous in v for all s

2. $E_{0,v}(s) = E_s, \forall v$

3. $f_0(v_1)$ second order dominates $f_0(v_2)$ whenever $v_1 > v_2$

4. $F_0(v2, s) - F_0(v1, s)$ is downward sloping in s whenever $v_1 > v_2$ and crosses the zero line once at $s^\star$.

Result 5: Conditions for belief-divergence to increase asset prices
$\frac{\delta p}{\delta v} > 0$ if either $n_1^{\max}(\overline{s}(\overline{p})) \geq (1 - \overline{a}_1)\overline{p}$ or $\frac{\delta PS^1}{\delta \overline{v}} > 0$. The latter is necessarily the case when equilibrium riskyness s is below $s^\star$.
If $n_1 > (1 - \overline{a}_1)\overline{p}$, $F_0 = F_1 = \frac{1}{2}$ from PS_1 and $R_1^a = 1$. So $\overline{s}$ does not change in response to v, but $\overline{p}$ rises.
If $0 < n_1 < (1 - \overline{a}_1)\overline{p}$, we can use PS^1 and PS^2 to get from the implicit function theorem the partial derivative of the price with respect to v as

$$\frac{\delta p}{\delta v} = \frac{\frac{\frac{\delta PS^1}{\delta \overline{s}}}{\frac{\delta PS^2}{\delta \overline{s}}}[\frac{\delta PS^2}{\delta v}] - \frac{\delta PS^1}{\delta v}}{\frac{\delta PS^1}{\delta p} - \frac{\delta PS^2}{\delta p}} \tag{30}$$

We know:

1. $\frac{\delta PS^1}{\delta \overline{s}} < 0$ (from optimality of $\overline{s}$)
2. $\frac{\delta PS^2}{\delta \overline{s}} = -(1 - F_0(\overline{s})) < 0$
3. $\frac{\delta PS^2}{\delta v} = -\frac{\delta E_0(min\{s,\overline{s}\})}{\delta v} < 0$
4. $\frac{\delta PS^1}{\delta p} - \frac{\delta PS^2}{\delta p} = -R(1 - F_1(\overline{s})) - (1 - \overline{a_1}) < 0$
5. $\frac{\delta PS^1}{\delta v} = -[E_s - E_1(min\{s, \overline{s}\})]\frac{\delta F_0}{\delta v} + (1 - F_1)\frac{\delta E_0(min\{s,\overline{s}\})}{\delta v}$

Thus $\frac{\delta p}{\delta v} > 0$ if $\frac{\delta PS^1}{\delta v} > 0$. [To be completed.]

□

Discussion
A mean-preserving contraction in f_0 is equivalent to lenders updating their beliefs to a lower level of risk. This increases the expected payoff from a collateralised loan of given riskyness, and thus increases the amount they are willing to lend to investors for leveraged investment. For investors, this always increases expected profits at a given price and level of riskyness. However, it has an ambiguous effect on marginal profits and thus the optimal value of riskyness $\overline{s}$. Specifically, while a rise in v increases the return at any given riskyness, it can increase or decrease $1 - F_0$, the marginal effect of a change in

$\overline{s}$ on profits at given returns. Only when the marginal benefit of a change in riskyness rises with v ($\frac{\delta PS^1}{\delta v} > 0$), however, is the effect on asset demand, and thus the general equilibrium effect on prices unambiguously positive.

4 A Dynamic model

In an environment like that of the previous section, where assets are claims to a single stochastic payoff in the future, limited liability of leveraged investments naturally implies the convex profits of leveraged investors the are behind most of the results. When assets are claims on a sequence of payoffs, however, investor wealth at any point in time equals the sum of the realised payoff and the price of the asset. The curvature of profits thus depends on the function that links general equilibrium asset prices to current payoffs and other state variables in the economy. This section circumvents this problem by assuming that payoffs have a discrete support, as opposed to the continuous support in the previous section. Specifically, we aim to present the simplest example of such an economy. To introduce mean-preserving spreads, the analysis requires distributions with at least 3 points of strictly positive mass. As it turns out, however, payoff realisations close to the value of promised net payments by leveraged investors to creditors introduce a potential for multiple equilibria: if prices are high these payoffs are consistent with credit repayments as promised and continued leveraged investment that justifies high prices. If prices, on the other hand, are low, leveraged investors cannot pay back their creditors, go bankrupt and leave the market for one period, warranting the fall in prices. While this multiplicity is interesting and potentially also arises in other models of collateralised investments, we choose an environment where it does not arise because payoff realisations are always sufficiently "far away" from the value of promised net payments to be consistent with either bankruptcy or continued investment, but not both. This requires at least 4 support points of payoffs, which is the case we focus on.

4.1 The environment

We look at a version of the previous model where time is infinite $t = 0, 1, 2, ...$ Every period physical assets pay a random amount of the consumption good s_t, $\forall t$ that is independent across periods. Agents of both types are infinitely lived and receive consumption endowment $n_i, \forall t$. So the resources available to agent i at the beginning of the period

equal her investment payoff plus her endowment. Agents maximise the present discounted value of consumption through decisions on consumption and asset purchases every period. As before they trade physical assets and collateralised loans whose face value for next period $\overline{s}_{t+1}$ is agreed on in t. For simplicity I normalise total assets to 1 and assume that type 0 agents own all asset in $t = 0$.
Agents' liability is limited to their total period t assets plus their period t endowment. The budget constraint for leveraged investors is

$$a_{it+1}(p_t - \frac{E_j[min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}]}{R}) + c_{it} \leq max\{n_i + a_{it}(p_t + s_t - \overline{s}), 0\}) \tag{31}$$

For providers of collateralised loans we have

$$a_{jt+1}\frac{E_i[min\{n_j + p_{t+1} + s_{t+1}, \overline{s}\}]}{R} + c_{it} \leq n_i + a_{jt}(min\{n_j + p_t + s_t, \overline{s}\} \tag{32}$$

4.2 3 dimensions of dynamic behaviour

There are three dimensions of dynamic bevaviour:

1. The endogenous evolution of relative wealth
2. Learning about the distribution of s_t
3. Endogenous price fluctuations, with potentially different beliefs about the price process p_t by type 1 and 2 agents

This section focusses on dimensions 1 and 3. We thus abstract from 2. by assuming that belief disagreements are constant, but introduce learning explicitly in the quantitative analysis of the next section.

4.3 Equilibrium definition

1. Sequences of prices and quantities as functions of the state of the economy $(s_t, \overline{s}_t, a_{it}, a_{jt})$ s.t.
2. agents optimise given belief $f_{it} = f_i$
3. markets for consumption and assets clear

Discussion: Note how we specify an equilibrium where agents agree on the price and policy functions of the underlying state of the economy, but disagree on the distribution of exogenous shocks.

4.4 Equilibrium Description and Problem

agents invest in leveraged assets if

$$R_i^a = \frac{E_i[p_{t+1} + s_{t+1} - min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}]}{p_t - \frac{E_j[min\{n_i+p_{t+1}+s_{t+1},\overline{s}\}]}{R}} \tag{33}$$

$$= \frac{E_i[p_{t+1}] + E_s - E[min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}]}{p_t - \frac{E_j[min\{n_i+p_{t+1}+s_{t+1},\overline{s}\}]}{R}} > R \tag{34}$$

Asset market clearing requires $R_i^a \geq R \geq R_j^a$: only 1 type can expect to make strictly positive profits from leveraged investments. However, asset prices are not necessarily weakly convex in s_t, partly because they are bounded below and above respectively by the minimum and maximum of their expected discounted value to either agent with or without leverage. So $E_1[p_{t+1}]$ may be greater or smaller than $E_0[p_{t+1}]$.

4.5 A discrete process for s_t

This section shows an example of an economy where disagreement in beliefs about the volatility of asset payoffs have the following implications:

1. Prices are strictly above the discounted expected value of payoffs.

2. Prices fluctuate over time although the expected payoff is constant.

For this, we make the following assumption:

A3: The perceived distribution of s_t $f_{i,0}$ is symmetric around E_s with mass points at $\{s_1, s_2, s_3, s_4\} = \{E_s - 2n, E_s - n, E_s + n, E_s + 2n\}$. Particularly, $f_{1,0}(s_i) = \frac{1}{4}, i = 1, 2, 3, 4,$ while $f_{0,0}(s_2) = f_{0,0}(s_3) = \frac{1-\epsilon}{2}$ and $f_{0,0}(s_3) = f_{0,0}(s_4) = \frac{\epsilon}{2}$ where ϵ is an infinitesimally small number.

4.5.1 Equilibrium Dynamics

Proposition 1
There is an equilibrium with the following properties:

- $\overline{s}_t = \underline{p} + E_s, \forall t$
- If type 1 invested in $t-1$, she goes bankrupt in the two low-income states where $s_t < E_s$. If bankrupt, $c_1 = a_1 = 0$. In case she did not invest in period $t-1$, or if she did and $s_t > E_s$, she buys all assets and consumes a positive amount $a_1 = 1, c_1 > 0$.
- The price process has only two support points:
  1. If agent 1 is not bankrupt, $p_t(s_t) = \overline{p} = \frac{E_s + \frac{R-\frac{1}{2}}{R}\Delta E[\varphi(s)]}{R-1}$
  2. If agent 1 is bankrupt, $p_t(s_t) = \underline{p} = \frac{E_s + \frac{1}{2}\frac{\Delta E[\varphi(s)]}{R}}{R-1}$

Proof:
The proof is by guess and verify: guess that $p(s_1) = p(s_2) = \underline{p} < E_s < p(s_3) = p(s_4) = \overline{p}, \forall t$ and define $\Delta p = \overline{p} - \underline{p} > 0$. Guess also that agent 1 buys as much leveraged claims as she can and sets $\overline{s} = E_s$. To verify that this is an equilibrium, first calculate the price in periods where agent 1 is bankrupt as

$$\underline{p} = \frac{E_0[p_{t+1}] + E_s}{R} = \frac{E_s + \frac{1}{2}\Delta p}{R-1} \tag{35}$$

The expected return to agent i from a leveraged investment is

$$R_i^a = \frac{E_i[p_{t+1}] + E_s - E_i[min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}]}{p_t - \frac{E_j[min\{n_i+p_{t+1}+s_{t+1},\overline{s}\}]}{R}} \tag{36}$$

Since $E_1[p_{t+1}] = E_0[p_{t+1}]$ and $E_1[min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}] < E_0[min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}]$, $R_1^a > R_0^a$. So whenever agent 1 has funds to invest, she will spend them on leveraged asset puchases in equilibrium. The maximum value of $\overline{p}$ at which she is willing to do so is that which implies $R_1^a = 1$. If at this value agent 1 can afford to buy all assets whenever she has positive resources, this is the only equilibrium price outside bankruptcy periods: outside bankruptcy agent 1 is “too rich” for asset purchases to exhaust her budget even at the maximum price. So asset market clearing requires that she has strictly positive consumption. For this to be an equilibrium, she needs to be indifferent between investing and consuming, implying an equilibrium price consistent with $R_i^a = 1$. To see this, first calculate $\overline{p}$ from $R_1^a = 1$ as

$$\overline{p} = \frac{E_1[p] + E_s + \Delta E[\varphi(s_{t+1})]}{R} \tag{37}$$

$$= \underline{p} + \frac{\Delta E[\varphi(s_{t+1})] - \Delta p}{R-1} \tag{38}$$

$$= \frac{E_s + \frac{R-\frac{1}{2}}{R}\Delta E[\varphi(s_{t+1})]}{R-1} \tag{39}$$

where the last line uses the definition of $\underline{p}$. This also yields

$$\Delta p = \frac{\Delta E[\varphi(s)]}{R} = \frac{\Delta E[\varphi(s)]}{R} = \frac{\frac{1}{4}n}{R} \tag{40}$$

To show that at this price, agent 1 can indeed always buy all assets in states s_3, s_4, calculate the excess ressources (multiplied by R for convenience) without initial asset holdings ($a_{1,t} = 0$) in state s_3, the non-bankruptcy state with lowest income, after buying all assets ($a_{1,t+1} = 1$) at the maximum price

$$Rn + E_0[min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}]) - R\overline{p} \tag{41}$$

$$= Rn + E_0[min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}]) - E_1[p] - E_s - \Delta E[\varphi(s_{t+1})] \tag{42}$$

$$= Rn + E_1[min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}]) - E_1[p] - E_s \tag{43}$$

$$= Rn + \frac{1}{4}[\underline{p} + E_s - n] + \frac{1}{4}[\underline{p} + E_s] + \frac{1}{2}[\underline{p} + E_s] - E_1[p] - E_s \tag{44}$$

$$= Rn - \frac{1}{2}\Delta p - \frac{1}{4}n \tag{45}$$

$$= Rn - (\frac{1}{8R} + \frac{1}{4})n > 0 \tag{46}$$

Since in all other non-bankrupt states agent 1 has funds that are larger, she can always buy all assets at the maximum price. So $\overline{p}$ is the only equilibrium price outside bankruptcy periods.[3]

To show that $\overline{s} = \underline{p} + E_s$ is an optimal choice for loan riskyness, note that agent 1's total funds in state s_i before paying creditors equal $p(s_i) + s_i + n$. Thus, for $\overline{s} : \underline{p} + E_s > \overline{s} > \underline{p} + E_s - n$ she expects to go bankrupt with probability $\frac{1}{4}$, while agent 0 expects her to go bankrupt with probability 0. Thus, in this range, agent 1 expects to make

[3] In a similar fashion, it is easy to show that the agent 0 always consumes a positive amount after buying collateralised loans worth $\frac{E_0[min\{n_i + p_{t+1} + s_{t+1}, \overline{s}\}]}{R}$.

profits from issuing claims that she perceives as risky, but agent 0 does not. Similarly, for $\overline{s} : \overline{p} + E_s + 3n > \overline{s} > \overline{p} + E_s + 2n$ agent 1 expects to go bankrupt with probability $\frac{3}{4}$, while agent 0 expects her to go bankrupt with certainty. So agent 1 will not be able to increase the funds available for asset purchase by increasing $\overline{s}$ beyond $E_s + 2n$. In other words

$$\frac{dR_1^a}{d\overline{s}}|\underline{p} + E_s - n < \overline{s} < \underline{p} + E_s = R_1^a \times 1 - \frac{3}{4} > 0 \tag{47}$$

$$\frac{dR_1^a}{d\overline{s}}|\underline{p} + E_s \leq \overline{s} \leq \overline{p} + E_s + 2n = R_1^a \times \frac{1}{2} - \frac{1}{2} = 0 \tag{48}$$

$$\frac{dR_1^a}{d\overline{s}}|\overline{p} + E_s + 2n < \overline{s} \leq \overline{p} + E_s + 3n = R_1^a \times 0 - \frac{1}{4} < 0 \tag{49}$$

Since $R_1^a = 1$ in equilibrium outside bankruptcy, agent 1 is indifferent between any $\overline{s} : \underline{p} + E_s \leq \overline{s} < \overline{p} + E_s + 2n$.

Conjecture 1

The equilibrium prices are unique. Multiplicity in consumption follows from that of loan riskyness which can take values $\overline{s} : \underline{p} + E_s \leq \overline{s} < \overline{p} + E_s + 2n$ in equilibrium.

5 The Great Moderation, belief disagreement and asset prices

When investors are risk-neutral, a fall in volatility such as during the Great Moderation should leave their fundamental valuation unchanged. The previous sections showed, however, that with leveraged asset purchases, equilibrium prices nevertheless rise. This section uses the simple dynamic model from the previous section to illustrate the effect of the Great Moderation on asset prices when the convexity-effect of heterogeneous beliefs about second moments is the only channel through which a fall in volatility affects asset prices.

The general environment is the same as in the previous section. Particularly, payoffs are distributed on 4 support points. In order to introduce learning in this environment in the simplest possible way, we make the following assumptions: The Great Moderation takes the form of a contraction in the support of a binomial distribution of payoffs. Thus, before the Great Moderation, payoffs take values in $\{s_1, s_4\}$ only, while during the Great Moderation they fluctuate between s_2 and s_3. Investors' prior before the Great Moderation coincides with the true distribution. Once they observe realisations of payoffs in $\{s_2, s_3\}$ they therefore notice a change in the environment. To update their prior for the payoff distribution in the following periods $\widehat{f}_{it+s}$, investors use a simple statistical updating rule with constant gain ξ_i

$$\widehat{f}_{it+1} = \xi_i \pi_{it-1} + (1 - \xi)\widehat{f}_{it-1} \tag{50}$$

I assume $\xi_0 > \xi_1$ such that $\widehat{f}_{0t} \succeq^2 \widehat{f}_{0t}$.

5.1 Calibration

We normalise both expected payoffs and their standard deviation during the pre-Great Moderation period to 1. In line with the fall in the standard deviation of both US GDP and consumption growth during the Great Moderation to half their previous values, I set the standard deviation of payoffs after the fall in volatility to 1/2. This yields $\{s_1, s_2, s_3, s_4\} = \{0, 0.5, 1.5, 2\}$. Finally, I choose learning parameters $\xi_1 = 0.4\%$ and $\xi_2 = 7\%$, which is consistent with a peak of the asset price boom after 45 quarters, in line with the US experience where the great moderation started in the second half of the 1980s and prices peaked towards the end of the 1990s. Finally I set quarterly interest rates to 1 percent.

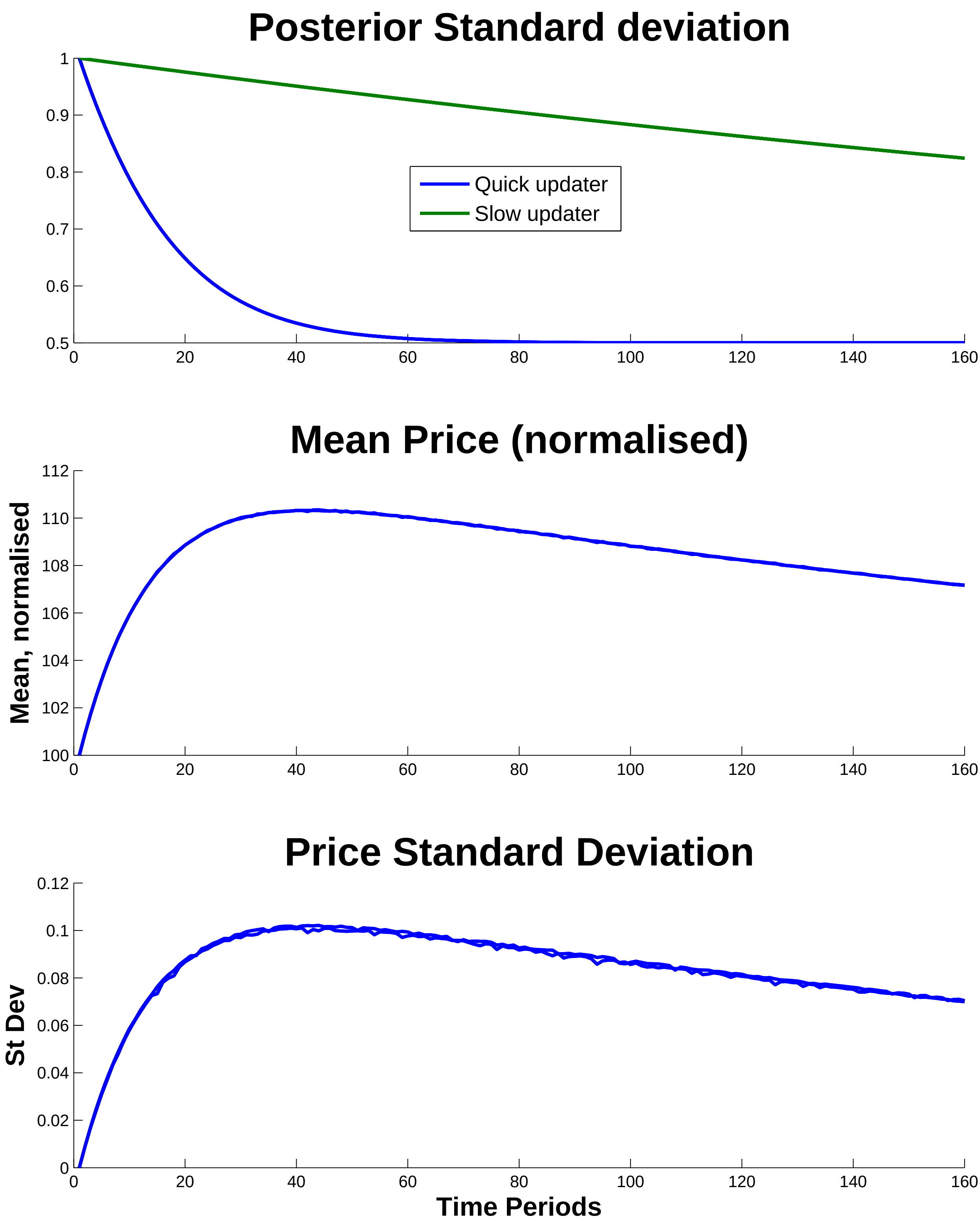
Figure 3 presents the results. The top panel shows how, after the onset of the Great Moderation in $t = 0$, the standard deviations of the posterior payoff distribution first diverge before slowly re-converging. Mean asset prices, calculated across 100 simulations of the economy, peak at a moderate 10.4 percent above their initial value. The standard deviation of prices, calculated across 100 simulated price paths also increases and follows a path very similar to that of mean prices. Importantly, leverage jumps up to values that imply a bankruptcy probability of 50%.

6 Extensions

6.1 Short Contracts

To be added.

Figure 3


Posterior Standard deviation
Quick updater
Slow updater
1
0.9
0.8
0.7
0.6
0.5
0
20
40
60
80
100
120
140
160
Mean Price (normalised)
Mean, normalised
112
110
108
106
104
102
100
Price Standard Deviation
St Dev
0.12
0.1
0.08
0.06
0.04
0.02
0
Time Periods

6.2 Franchise Value

To be added.

7 Conclusion

To be added.

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