Inference Based on SVARs Identified with Sign and Zero Restrictions: Theory and Applications *

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* The views expressed here are the authors' and not necessarily those of the Federal Reserve Bank of Atlanta or the Board of Governors of the Federal Reserve System.

Introduction

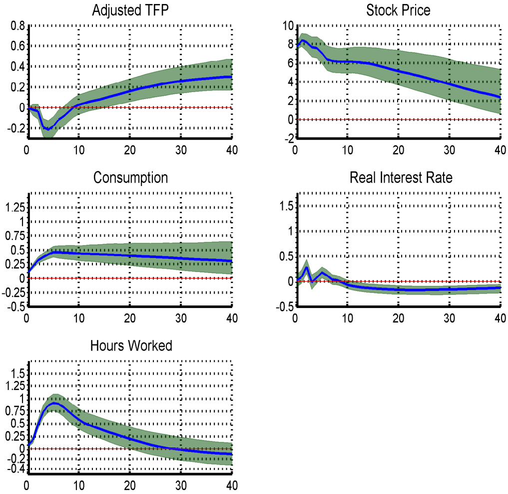
- SVARs useful because explicitly identify shocks.
- Keep the model free of additional restrictive assumptions.
- Sign restrictions appealing: minimal identifying restrictions.
- Zero restrictions added: sign restrictions were too minimal.
- Large literature using SVAR with sign and zero restrictions.
- Baumeister and Benati (2010), Beaudry et al. (2011), Enders et al. (2011), Moench (2010), Mountford and Uhlig (2009).

Beaudry et al.'s (2011) Identification Strategy

	Identification I	Identification II	Identification III
Adjusted TFP	0	0	0
Stock Price	+	+	+
Consumption		+	+
Real Interest Rate			+
Hours			
Investment			
Output			

Table : Beaudry et al. (2011)

Being Confident About The Wrong Thing

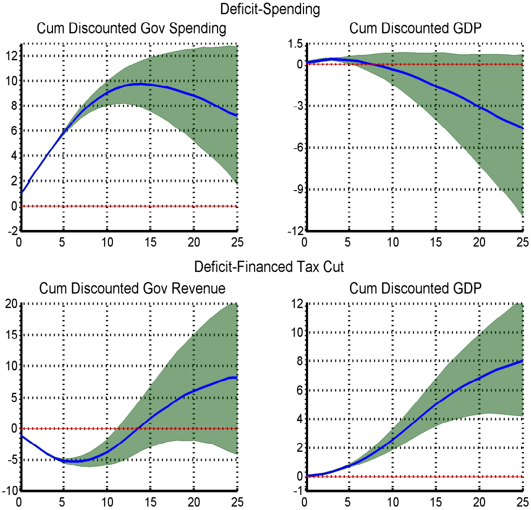


Mountford and Uhlig's (2009) Identification Strategy

	Shocks			
	Business Cycle	Monetary Policy	Gov Revenue	Gov Spending
GDP	+			
Gov Speding				+
Gov Revenue	+		+	
Interest Rate		+		
Adjusted Reserves		−		
Consumer Price Index		−		
GDP Deflator		−		
Consumption	+			
Investment	+			
Real Wage				

Table : Mountford and Uhlig (2009)

Being Confident About The Wrong Thing



The model

- Consider the SVAR

$$\mathbf{y}_t'\mathbf{A}_0 = \sum_{\ell=1}^{p} \mathbf{y}_{t-\ell}'\mathbf{A}_\ell + \mathbf{c} + \varepsilon_t' \quad \text{for } 1 \leq t \leq T.$$

- $\mathbf{y}_t$ is a $n \times 1$ vector of endogenous variables.
- ε_t is a $n \times 1$ vector of exogenous structural shocks.
- $\mathbf{A}_\ell$ is a $n \times n$ matrix of parameters for $0 \leq \ell \leq p$.
- $\mathbf{c}$ is a $1 \times n$ vector of parameters.
- p is the lag length, and T is the sample size.
- $\varepsilon_t \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$.

The model

- The SVAR can be written as

$$\mathbf{y}_t'\mathbf{A}_0 = \mathbf{x}_t'\mathbf{A}_+ + \varepsilon_t' \quad \text{for } 1 \le t \le T$$

$$\mathbf{A}_+' = \left[\begin{array}{cccc} \mathbf{A}_1' & \cdots & \mathbf{A}_p' & \mathbf{c}' \end{array}\right], \mathbf{x}_t' = \left[\begin{array}{cccc} \mathbf{y}_{t-1}' & \cdots & \mathbf{y}_{t-p}' & 1 \end{array}\right].$$

- The reduced-form representation of the SVAR model is

$$\mathbf{y}_t' = \mathbf{x}_t'\mathbf{B} + \mathbf{u}_t' \quad \text{for } 1 \le t \le T.$$

- $\mathbf{B} = \mathbf{A}_+\mathbf{A}_0^{-1}$, $\mathbf{u}_t' = \varepsilon_t'\mathbf{A}_0^{-1}$, and $\mathbb{E}\left[\mathbf{u}_t\mathbf{u}_t'\right] = \mathbf{\Sigma} = \left(\mathbf{A}_0\mathbf{A}_0'\right)^{-1}$.
- $\mathbf{B}$ and $\mathbf{\Sigma}$ are called the reduced-form parameters while $\mathbf{A}_0$ and $\mathbf{A}_+$ are called structural parameters.

Finite horizon IRFs

- The impulse response of the i-th variable to the j-th structural shock at finite horizon h corresponds to the element in row i and column j of the following matrix:

$$\mathbf{L}_h\left(\mathbf{A}_0, \mathbf{A}_+\right) = \left(\mathbf{A}_0^{-1}\mathbf{J}'\mathbf{F}^h\mathbf{J}\right)'$$

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{A}_1\mathbf{A}_0^{-1} & \mathbf{I}_n & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{p-1}\mathbf{A}_0^{-1} & \mathbf{0} & \cdots & \mathbf{I}_n \\ \mathbf{A}_p\mathbf{A}_0^{-1} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, \text{ and } \mathbf{J} = \begin{bmatrix} \mathbf{I}_n \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}.$$

Infinite horizon IRFs

- The impulse response of the i-th variable to the j-th structural shock at infinite horizon (sometimes called long-run impulse responses) corresponds to the element in row i and column j of the following matrix:

$$\mathbf{L}_{\infty}\left(\mathbf{A}_0, \mathbf{A}_+\right) = \left(\mathbf{A}_0' - \sum_{\ell=1}^{p} \mathbf{A}_\ell'\right)^{-1}.$$

Sign Restrictions

- We borrow from Rubio-Ramirez et. al. (2010).
- The function $f(\mathbf{A}_0, \mathbf{A}_+)$ selects the IRFs. For example,

$$f(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} \mathbf{L}_0(\mathbf{A}_0, \mathbf{A}_+) \\ \mathbf{L}_\infty(\mathbf{A}_0, \mathbf{A}_+) \end{bmatrix}.$$

- Let $\mathbf{S}_j$ for $1 \leq j \leq n$ be a matrix of rank s_j.
- These parameters satisfy the sign restrictions if and only if

$$\mathbf{S}_j f(\mathbf{A}_0, \mathbf{A}_+) \mathbf{e}_j > \mathbf{0} \text{ for } 1 \leq j \leq n$$

where $\mathbf{e}_j$ is the j-th column of the identity matrix $\mathbf{I}_n$.

Discussion

- If $\mathbf{Q}$ is an orthogonal matrix, then $(\mathbf{A}_0, \mathbf{A}_+)$ and $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ are observationally equivalent.
- A SVAR with sign restrictions is only set identified. Why?
- For any $(\mathbf{A}_0, \mathbf{A}_+)$ that satisfy the sign restrictions, $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ will also satisfy the sign restrictions if orthogonal matrix $\mathbf{Q}$ is close to the identity.
- The set of structural parameters satisfying the sign restrictions is of positive measure in the set of all structural parameters.

Algorithm 1

1. Draw $(\mathbf{A}_0, \mathbf{A}_+)$ from the unrestricted posterior.
2. Keep the draw if the sign restrictions are satisfied.
3. Return to Step 1 until the required number of posterior draws satisfying the sign restrictions have been obtained.

A Useful Theorem

- Step 1 is the challenge. We develop an algorithm to do that.
- The algorithm requires draw of orthogonal matrix $\mathbf{Q}$ from the uniform distribution with respect to the Haar measure on $O(n)$.

Theorem 1

Let $\mathbf{X}$ be an $n \times n$ random matrix with each element having an independent standard normal distribution. Let $\mathbf{X} = \mathbf{QR}$ be the QR decomposition of $\mathbf{X}$. The random matrix $\mathbf{Q}$ has the uniform distribution with respect to the Haar measure on $O(n)$.

Discussion

- Let h be any continuously differentiable mapping such that

$$h(\mathbf{X})' h(\mathbf{X}) = \mathbf{X}.$$

- For example, $h(\mathbf{X})$ could be the Cholesky decomposition of $\mathbf{X}$.

- Define a function $\hat{h}$ from the set of reduced-form parameters and orthogonal matrices into the set of structural parameters

$$\hat{h}(\mathbf{B}, \mathbf{\Sigma}, \mathbf{Q}) = (h(\mathbf{\Sigma})^{-1}\mathbf{Q}, \mathbf{B}h(\mathbf{\Sigma})^{-1}\mathbf{Q}).$$

- We show how to use $\hat{h}$ to draw from the unrestricted posterior.

Discussion

- Step 1 : Draw from the reduced-form posterior + the Cholesky decomposition + orthogonal matrix $\mathbf{Q}$ to obtain a draw from the unrestricted posterior. How and why?

Theorem 2

Let $\pi(\mathbf{B}, \mathbf{\Sigma})$ be a prior density on the reduced-form parameters. If $(\mathbf{B}, \mathbf{\Sigma})$ is a draw from the reduced-form posterior and $\mathbf{Q}$ is a draw from the uniform distribution with respect to the Haar measure on $O(n)$, then $\hat{h}(\mathbf{B}, \mathbf{\Sigma}, \mathbf{Q})$ is a draw from the unrestricted posterior with respect to the prior $\hat{\pi}(\mathbf{A}_0, \mathbf{A}_+) = \pi(\mathbf{B}, \mathbf{\Sigma}) \left| det\left(\hat{h}'(\mathbf{B}, \mathbf{\Sigma}, \mathbf{Q})\right) \right|^{-1}$.

- Theorem 2 uses Theorem 1 to obtain orthogonal matrix $\mathbf{Q}$.

Algorithm 1 + Theorem 2 = Algorithm 2

1. Draw $(\mathbf{B}, \mathbf{\Sigma})$ posterior of the reduced-form parameters.
2. Use Theorem 1 to draw an orthogonal matrix $\mathbf{Q}$.
3. By Theorem 2, $\hat{h}(\mathbf{B}, \mathbf{\Sigma}, \mathbf{Q}) = (h(\mathbf{\Sigma})^{-1}\mathbf{Q}, \mathbf{B}h(\mathbf{\Sigma})^{-1}\mathbf{Q})$ is a draw from the unrestricted posterior.
4. Keep the draw if $\mathbf{S}_j f\left(h(\mathbf{\Sigma})^{-1}\mathbf{Q}, \mathbf{B}h(\mathbf{\Sigma})^{-1}\mathbf{Q}\right)\mathbf{e}_j > \mathbf{0}$ are satisfied for $1 \leq j \leq n$.
5. Return to Step 1 until the required number of draws satisfying the sign restrictions has been obtained.

Zero Restrictions

- Let $\mathbf{Z}_j$ for $1 \leq j \leq n$ be a matrix of rank z_j.
- Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any value of structural parameters. These parameters satisfy the zero restrictions if and only if

$$\mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right) \mathbf{e}_j = \mathbf{0} \text{ for } 1 \leq j \leq n.$$

- We can no longer apply Algorithm 2. Still set identified but...
- The set of structural parameters satisfying the zero restrictions is of measure zero in the set of all structural parameters.
- Nevertheless, the set of structural parameters satisfying both the zero and sign restrictions is of positive measure in the set of structural parameters satisfying the zero restrictions.

Algorithm 3

1. Draw $(\mathbf{A}_0, \mathbf{A}_+)$ from the posterior satisfying zero restrictions.
2. Keep the draw if the sign restrictions are satisfied.
3. Return to Step 1 until the required number of posterior draws satisfying both the sign and zero restrictions have been obtained.

Discussion

- Step 1 is the challenge again. New algorithm.
- Zero restrictions on the IRFs

$$\mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{e}_j = \mathbf{0} \text{ for } 1 \leq j \leq n$$

  are nonlinear restrictions on $(\mathbf{A}_0, \mathbf{A}_+)$.

- But f has the property that $f(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q}) = f(\mathbf{A}_0, \mathbf{A}_+)\mathbf{Q}$.
- Key: we show that zero restrictions on the IRFs can be converted into linear restrictions on orthogonal matrix $\mathbf{Q}$.

From Nonlinear to Linear Restrictions

- For any orthogonal matrix $\mathbf{Q}$ and for all $1 \leq j \leq n$

$$\mathbf{Z}_j f\left(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q}\right)\mathbf{e}_j = \mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{Q}\mathbf{e}_j = \mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{q}_j$$

- The zero restrictions will hold if and only if

$$\mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right)\mathbf{q}_j = \mathbf{0} \text{ for } 1 \leq j \leq n.$$

- Zero restrictions are linear restrictions on columns of $\mathbf{Q}$.

A Second Useful Theorem

- The algorithm requires draw of orthogonal matrix $\mathbf{Q}$ from the uniform distribution with respect to the Haar measure on $O(n)$ conditional on $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ satisfying the zero restrictions.
- First, we need to find a orthogonal matrix $\mathbf{Q}$ such that $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ satisfies the zero restrictions. How?

Theorem 3

Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any value of structural parameters. The structural parameters $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$, where $\mathbf{Q}$ is orthogonal, satisfy the zero restrictions if and only if $\| \mathbf{q}_j \| = 1$ and $\mathbf{R}_j(\mathbf{A}_0, \mathbf{A}_+)\mathbf{q}_j = \mathbf{0}$, for $1 \leq j \leq n$, where

$$\mathbf{R}_j(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} \mathbf{Z}_j f(\mathbf{A}_0, \mathbf{A}_+) \\ \mathbf{Q}'_{j-1} \end{bmatrix}.$$

If rank of $\mathbf{Z}_j \leq n - j$, then a orthogonal matrix $\mathbf{Q}$ exists.

A Third Useful Theorem

- Second, we need to show that an orthogonal matrix $\mathbf{Q}$ from Theorem 3 is a draw from the uniform distribution with respect to the Haar measure on $O(n)$ conditional on $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ satisfying the zero restrictions.

Theorem 4

Let $(\mathbf{A}_0, \mathbf{A}_+)$ be any value of the structural parameters. Let $j = 1$ and an orthogonal matrix $\mathbf{Q}$ be obtained as follows

1. *Let $\mathbf{N}_j$ be the basis for the null space of $\mathbf{R}_j(\mathbf{A}_0, \mathbf{A}_+)$.*
2. *Draw $\mathbf{x}_j$ from the standard normal distribution of $\mathbb{R}^n$.*
3. *Let $\mathbf{q}_j = \mathbf{N}_j \left(\mathbf{N}_j'\mathbf{x}_j / \parallel \mathbf{N}_j'\mathbf{x}_j \parallel\right)$.*
4. *If $j = n$ stop, otherwise let $j = j + 1$ and move to Step 1.*

The orthogonal matrix $\mathbf{Q}$ has the uniform distribution with respect to the Haar measure on $O(n)$ conditional on $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ satisfying the zero restrictions.

Discussion

- Step 1: Draw from the reduced-form posterior + the Cholesky decomposition + orthogonal matrix $\mathbf{Q}$ conditional on $(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q})$ satisfying the zero restrictions to obtain a draw from the posterior satisfying zero restrictions. How and why?

Theorem 5

Let $\pi(\mathbf{B}, \mathbf{\Sigma})$ be a prior density on the reduced-form parameters. If $(\mathbf{B}, \mathbf{\Sigma})$ is a draw from the reduced-form posterior and $\mathbf{Q}$ is a draw obtained using Theorem 4 for $(\mathbf{A}_0, \mathbf{A}_+) = \left(h(\mathbf{\Sigma})^{-1}, \mathbf{B}h(\mathbf{\Sigma})^{-1}\right)$, then $\hat{h}(\mathbf{B}, \mathbf{\Sigma}, \mathbf{Q})$ is a draw from the posterior satisfying zero restrictions.

Algorithm 3 + Theorem 5 = Algorithm 4

1. Draw $(\mathbf{B}, \mathbf{\Sigma})$ from the posterior distribution of the reduced form parameters.

2. Use Theorem 4, applied to $(\mathbf{A}_0, \mathbf{A}_+) = \left(h(\mathbf{\Sigma})^{-1}, \mathbf{B}h(\mathbf{\Sigma})^{-1}\right)$, to draw an orthogonal matrix $\mathbf{Q}$ such that $(h(\mathbf{\Sigma})^{-1}\mathbf{Q}, \mathbf{B}h(\mathbf{\Sigma})^{-1}\mathbf{Q})$ satisfies the zero restrictions.

3. By Theorem 5, $\hat{h}(\mathbf{B}, \mathbf{\Sigma}, \mathbf{Q}) = (h(\mathbf{\Sigma})^{-1}\mathbf{Q}, \mathbf{B}h(\mathbf{\Sigma})^{-1}\mathbf{Q})$ is a draw from the posterior satisfying zero restrictions.

4. Keep the draw if $\mathbf{S}_j f\left(h(\mathbf{\Sigma})^{-1}\mathbf{Q}, \mathbf{B}h(\mathbf{\Sigma})^{-1}\mathbf{Q}\right)\mathbf{e}_j > \mathbf{0}$ is satisfied for $1 \leq j \leq n$.

5. Return to Step 1 until the required number of posterior draws satisfying both the sign and zero restrictions have been obtained.

Remark

- The $\mathbf{Q}$ such that $(h(\boldsymbol{\Sigma})^{-1}\mathbf{Q}, \mathbf{B}h(\boldsymbol{\Sigma})^{-1}\mathbf{Q})$ satisfies the zero restrictions is not unique.
- In our Algorithm depends on draw of $\mathbf{x}_j$ for $1 \leq j \leq n$.
- Normalization of $\mathbf{q_j}$ needed.
- We use the function $f(\mathbf{A}_0, \mathbf{A}_+)$ to stack the IRFs, but our methodology works as long as the following conditions hold

  Condition 1. Admissible, i.e. $f(\mathbf{A}_0\mathbf{Q}, \mathbf{A}_+\mathbf{Q}) = f(\mathbf{A}_0, \mathbf{A}_+)\,\mathbf{Q}$.

  Condition 2. Regular, i.e. it is continuously differentiable.

  Condition 3. Strongly regular, i.e. it is dense.

Example

MU (2009) Methodology

$$\bar{\mathbf{q}}_j^* = \text{argmin}_{\bar{\mathbf{q}}_j \in S} \quad \Psi\left(\bar{\mathbf{q}}_j\right)$$

subject to

$$\mathbf{Z}_j f\left(\mathbf{A}_0, \mathbf{A}_+\right) \bar{\mathbf{q}}_j = \mathbf{0} \text{ and } \bar{\mathbf{Q}}_{j-1}^{*'} \bar{\mathbf{q}}_j = 0$$

where

$$\Psi\left(\bar{\mathbf{q}}_j\right) = \sum_{i \in I_+} \sum_{h=0}^{H_{i,+}} g\left(-\frac{\mathbf{e}_i' \mathbf{L}_h\left(\mathbf{A}_0, \mathbf{A}_+\right) \bar{\mathbf{q}}_j}{\sigma_i}\right) + \sum_{i \in I_-} \sum_{h=0}^{H_{i,-}} g\left(\frac{\mathbf{e}_i' \mathbf{L}_h\left(\mathbf{A}_0, \mathbf{A}_+\right) \bar{\mathbf{q}}_j}{\sigma_i}\right),$$

$g\left(\omega\right) = 100\omega$ if $\omega \geq 0$ and $g\left(\omega\right) = \omega$ if $\omega \leq 0$, σ_i is the standard error of variable i, and $\bar{\mathbf{Q}}_{j-1}^* = \begin{bmatrix} \bar{\mathbf{q}}_1^* & \cdots & \bar{\mathbf{q}}_{j-1}^* \end{bmatrix}$ for $1 \leq j \leq n$, and $S = \{\bar{\mathbf{q}}_j \in \mathbb{R}^n : \| \bar{\mathbf{q}}_j \| = 1\}$. Applied to $\left(\mathbf{A}_0, \mathbf{A}_+\right) = \left(\mathbf{T}^{-1}, \mathbf{B}\mathbf{T}^{-1}\right)$.

Three Issues

- The Approach may not be really agnostic.
- The Approach obtains only one $\mathbf{Q}$ from the support of conditional uniform.
- Sign restrictions may not hold.
- We will show this with two examples:
  - Beaudry et. al (2011).
  - Mountford and Uhlig (2009).

Is the Penalty Function Approach Agnostic?

- Consider a SVAR with n variables.
- We are interested in identifying the j-th structural shock.
- The restriction on the IRFs are
  - Sign restriction at horizon zero on the second variable.
  - Zero restriction at horizon zero on the first variable.

$$\mathbf{e}_1'\mathbf{L}_0\left(\mathbf{T}^{-1},\mathbf{B}\mathbf{T}^{-1}\right)\bar{\mathbf{q}}_j=\mathbf{t}_{1,1}\bar{\mathbf{q}}_{1,j}=0 \text{ and}$$

$$g\left(-\frac{\mathbf{e}_2'\mathbf{L}_0\left(\mathbf{T}^{-1},\mathbf{B}\mathbf{T}^{-1}\right)\bar{\mathbf{q}}_j}{\sigma_2}\right)\propto-\frac{\mathbf{t}_{2,1}\bar{\mathbf{q}}_{1,j}+\mathbf{t}_{2,2}\bar{\mathbf{q}}_{2,j}}{\sigma_2}=-\frac{\mathbf{t}_{2,2}\bar{\mathbf{q}}_{2,j}}{\sigma_2}.$$

- Hence, $\bar{\mathbf{q}}_j^*=\left[\begin{array}{ccccc}0 & 1 & 0 & \cdots & 0\end{array}\right]'$ and

$$\mathbf{e}_i'\mathbf{L}_0\left(\mathbf{T}^{-1},\mathbf{B}\mathbf{T}^{-1}\right)\bar{\mathbf{q}}_j^*=\mathbf{t}_{2,i} \text{ for } i>2.$$

- If $\mathbf{t}_{2,i}>0$ the penalty function approach imposes an additional positive sign restriction on the i-th variable.
- The penalty function approach imposes sign restrictions on the responses of variables seemingly unrestricted.

Application to Optimism Shocks

- We borrow from Beaudry et al. (2011).
- Are optimism shocks relevant macroeconomic dynamics?
- Sign and zero restrictions + penalty function approach.
- We compare it with our approach.
- Conclusion,
  1. the method is not agnostic and affects the results significantly.
  2. the method is also much slower.

Identification strategy

	Identification I	Identification II	Identification III
Adjusted TFP	0	0	0
Stock Price	+	+	+
Consumption		+	+
Real Interest Rate			+
Hours			
Investment			
Output			

Table : Beaudry et al. (2011)

Sign and zero restrictions

- Sign restrictions

Identification I

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Identification II

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

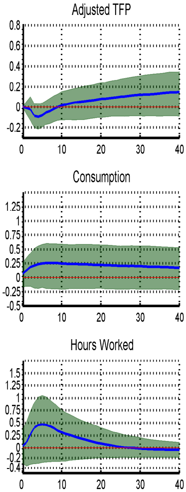
Identification III

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

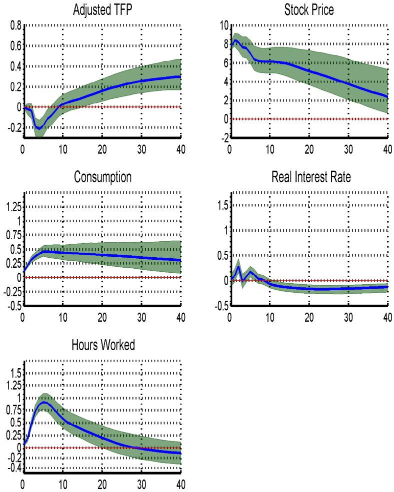
- Zero restriction (Identification I, II, and III)

$$\mathbf{Z}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Identification I

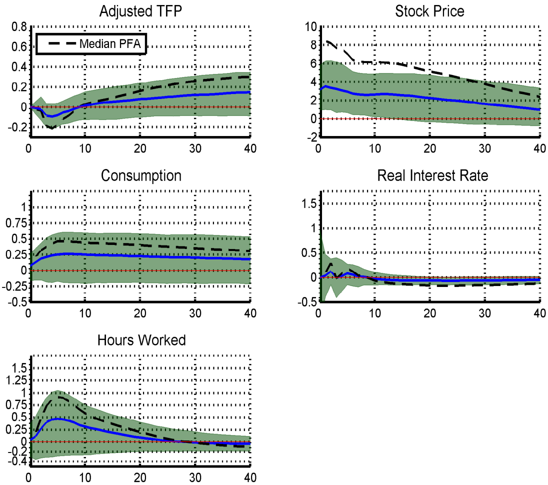


(a) ARRW (2013) methodology



(b) MU (2009) methodology

Identification I



Identification I

Identifications II and III

Seemingly unrestricted variables: Identification I

	MU (2009)			ARRW(2013)		
	Mean	Std dev	$\Pr(\cdot < 0)$	Mean	Std dev	$\Pr(\cdot < 0)$
Consumption	0.1034	0.0260	0.0000	0.0532	0.1914	0.3980
Hours Worked	0.0736	0.0379	0.0250	0.0355	0.2891	0.4490

Replicating Beaudry et al. (2011)

- Did we make a mistake?
- We can replicate Beaudry et al. (2011) using our approach.
- In their Identification I, they have one sign restriction and one zero restriction at horizon 0.
- Hence, we can replicate them using

$$f\left(\mathbf{A}_0, \mathbf{A}_+\right) = \left[\begin{array}{c} \mathbf{L}_0\left(\mathbf{A}_0, \mathbf{A}_+\right) \\ \mathbf{I}_n \end{array}\right].$$

- Where $\mathbf{I}_n$ is related to the constraint on $\mathbf{Q}$.

Sign and zero restrictions

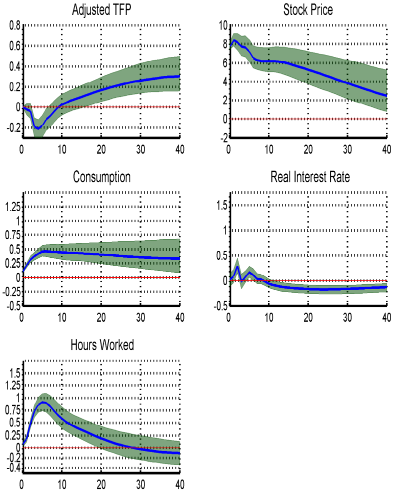
- Sign restriction

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

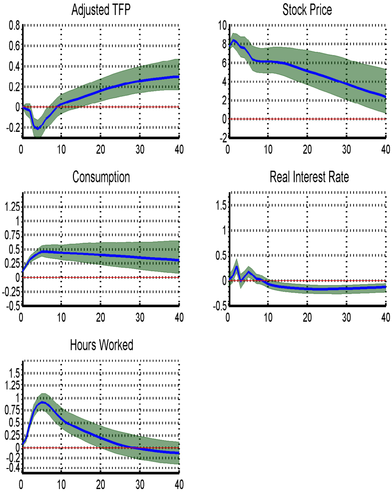
- Zero restrictions

$$\mathbf{Z}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Replicating Beaudry et al. (2011)



(a) ARRW (2013) with MU (2009) Restriction



(b) MU (2009)

Forecast Error Decomposition at Horizon 40

	ARRW (2013)			MU (2009)		
	Identification I	**Identification II**	**Identification III**	**Identification I**	**Identification II**	**Identification III**
Adjusted TFP	0.09 [0.03 , 0.22]	0.12 [0.04 , 0.28]	0.17 [0.06 , 0.33]	0.17 [0.08 , 0.30]	0.22 [0.10 , 0.37]	0.28 [0.14 , 0.43]
Stock Price	0.16 [0.03 , 0.47]	0.26 [0.07 , 0.58]	0.31 [0.09 , 0.62]	0.72 [0.55 , 0.85]	0.71 [0.57 , 0.82]	0.57 [0.42 , 0.72]
Consumption	0.17 [0.02 , 0.49]	0.28 [0.06 , 0.59]	0.40 [0.13 , 0.66]	0.26 [0.13 , 0.43]	0.69 [0.53 , 0.83]	0.76 [0.59 , 0.87]
Real Interest Rate	0.18 [0.07 , 0.39]	0.20 [0.08 , 0.40]	0.23 [0.09 , 0.44]	0.13 [0.07 , 0.22]	0.13 [0.07 , 0.22]	0.35 [0.29 , 0.43]
Hours Worked	0.18 [0.04 , 0.48]	0.27 [0.07 , 0.55]	0.29 [0.07 , 0.57]	0.31 [0.21 , 0.44]	0.62 [0.48 , 0.73]	0.49 [0.34 , 0.64]

Seven-Variable SVAR

Application to Fiscal Policy Shocks

- Mountford and Uhlig (2009).
- They analyze the effects of fiscal policy shocks on key economic variables such as GDP.
- They develop a sign and zero restrictions approach building on Uhlig (2005) penalty function approach.
- Why? They claim they want an agnostic approach.
- We compare it with our approach.
- Conclusion,
    - the method affects the results significantly.

Identification strategy

	Shocks			
	Business Cycle (BC)	Monetary Policy (MP)	Gov't Rev	Gov't Exp
GDP	+			
Gov't Expenditure				+
Gov't Revenue	+		+	
Interest Rate		+		
Adjusted Reserves		−		
Consumer Price Index		−		
GDP Deflator		−		
Consumption	+			
Non-Res Investment	+			
Wage				

Table : Mountford and Uhlig (2009)

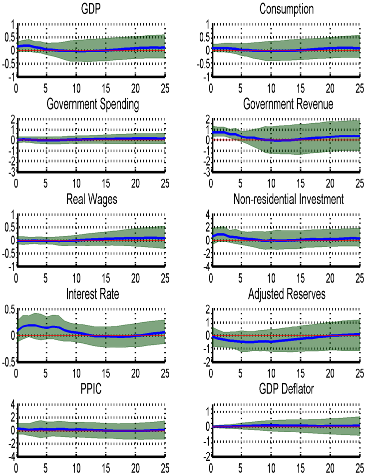
- Each sign restriction is enforced during 4 quarters.
- Gov't rev shock is orthogonal to BC and MP shocks.
- Gov't exp shock is orthogonal to BC and MP shocks.
- BC is orthogonal to MP shock.
- Gov't rev and exp shocks are not required to be orthogonal.

Fiscal Policy Scenarios

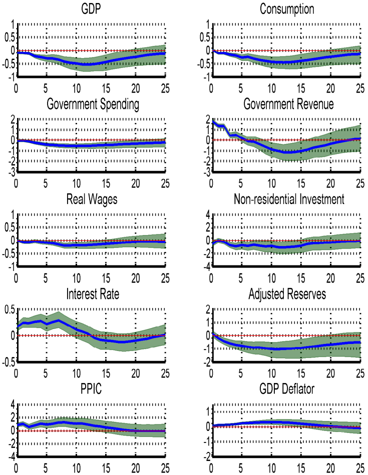
Let gov't rev and gov't exp shocks denote basic fiscal policy shocks. Then,

1. Deficit Spending Policy Scenario
   - It is a sequence of basic fiscal policy shocks where government spending is raised by 1% and government revenue remains unchanged for four quarters following the initial shock.
2. Deficit Tax Cut Policy Scenario
   - It is a sequence of basic fiscal policy shocks where government spending remains unchanged and government revenue falls by 1% for four quarters following the initial shock.
3. Balanced Budget Policy Scenario
   - It is a sequence of basic fiscal policy shocks where government spending is raised by 1% and government revenue is raised by 1.28% for four quarters following the initial shock.

Unanticipated Gov't Revenue Shock

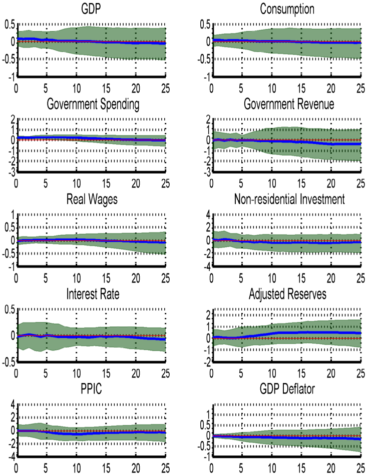


(a) ARRW (2013)

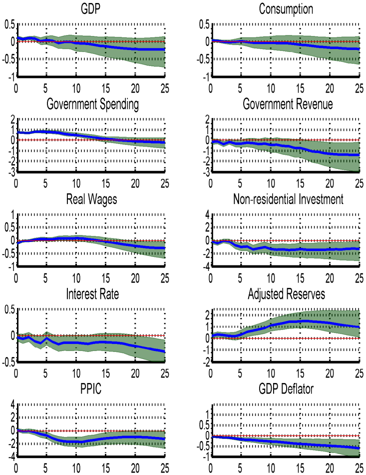


(b) MU (2009)

Unanticipated Gov't Expenditure Shock

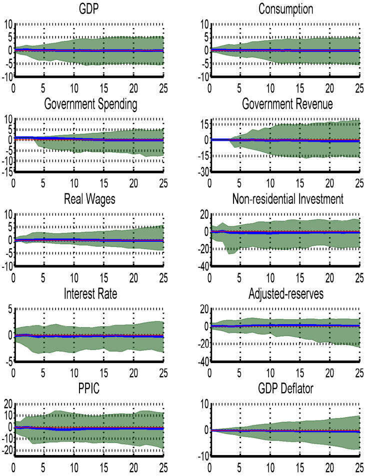


(a) ARRW (2013)

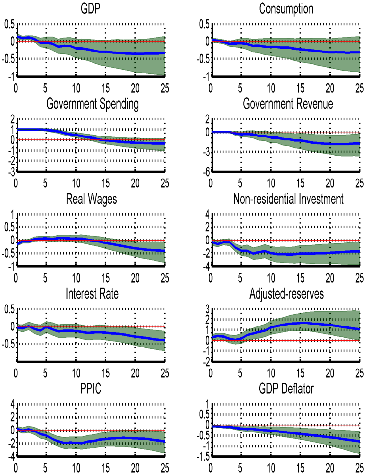


(b) MU (2009)

Deficit Spending Policy Scenario

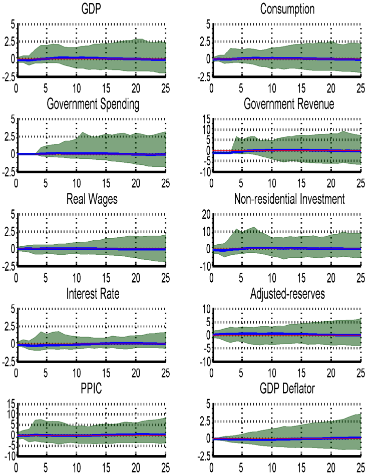


(a) ARRW (2013)

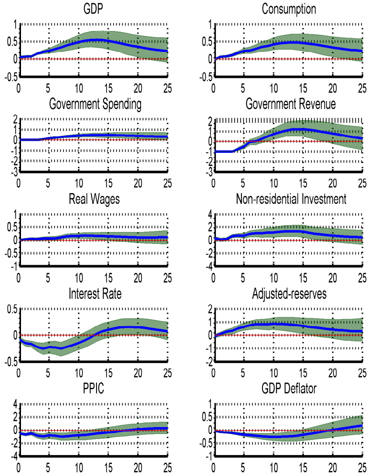


(b) MU (2009)

Deficit Tax Cut Policy Scenario

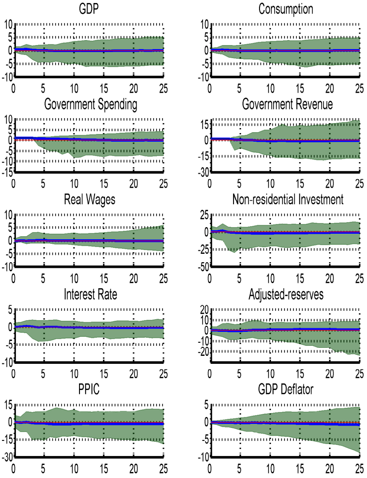


(a) ARRW (2013)

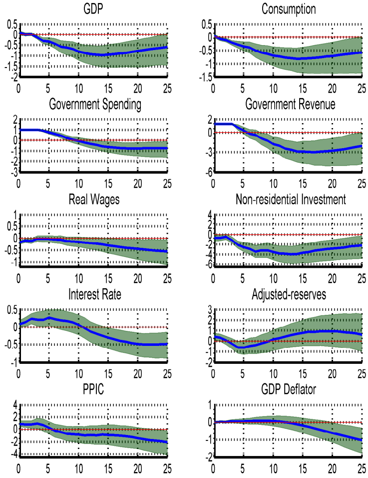


(b) MU (2009)

Balanced Budget Policy Scenario

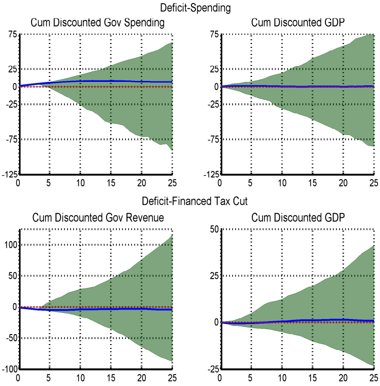


(a) ARRW (2013)

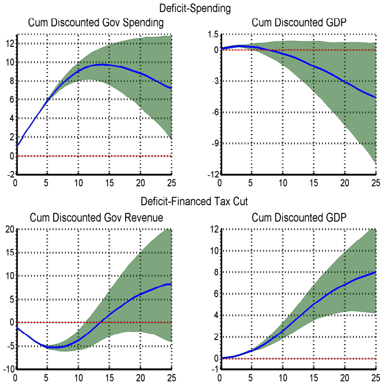


(b) MU (2009)

Cumulative IRFs of Deficit Spending and Deficit Tax Cut Policy Scenarios



(a) ARRW (2013)

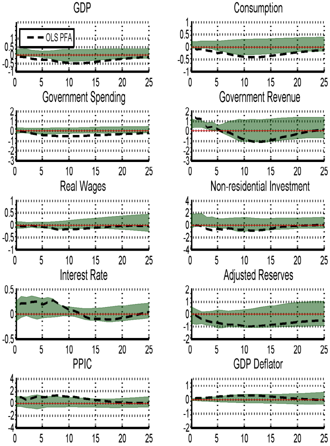


(b) MU (2009)

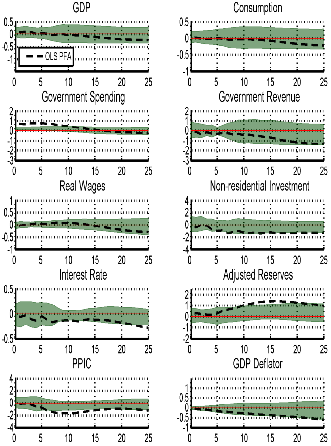
Understanding the Results

- Let the reduced-form parameters be the OLS parameters.
- Find $\mathbf{Q}$ using MU (2009) and plot IRFs.
- From the set of feasible $\mathbf{Q}$s associated with the OLS parameters and plot the 68% confidence bands of IRFs.

Understanding the Results



(a) ARRW (2013)



(b) MU (2009)

Conclusion

- We develop an algorithm for inference based on SVARs identified with sign and zero restrictions.
- Our algorithm draws from the correct posterior distribution conditional on the sign and zero restrictions holding.
- We present two applications and we show the advantages of our methodology relative to existing methods.

Example: A draw for the reduced form parameters

- Consider a five variables SVAR with one lag and let

$$\mathbf{B} = \begin{bmatrix} 0.7577 & 0.7060 & 0.8235 & 0.4387 & 0.4898 \\ 0.7431 & 0.0318 & 0.6948 & 0.3816 & 0.4456 \\ 0.3922 & 0.2769 & 0.3171 & 0.7655 & 0.6463 \\ 0.6555 & 0.0462 & 0.9502 & 0.7952 & 0.7094 \\ 0.1712 & 0.0971 & 0.0344 & 0.1869 & 0.7547 \end{bmatrix}$$

and

$$\mathbf{\Sigma} = \begin{bmatrix} 0.0281 & -0.0295 & 0.0029 & 0.0029 & 0.0024 \\ -0.0295 & 3.1850 & 0.0325 & -0.0105 & 0.0315 \\ 0.0029 & 0.0325 & 0.0067 & 0.0054 & 0.0030 \\ 0.0029 & -0.0105 & 0.0054 & 0.1471 & 0.0021 \\ 0.0024 & 0.0315 & 0.0030 & 0.0021 & 0.0140 \end{bmatrix}.$$

- Let the structural parameters be $(\mathbf{A}_0, \mathbf{A}_+) = (\mathbf{T}^{-1}, \mathbf{B}\mathbf{T}^{-1})$

$$\mathbf{A}_0 = \begin{bmatrix} 5.9655 & 0.5911 & -1.4851 & -0.0035 & -0.4591 \\ 0 & 0.5631 & -0.1455 & 0.0321 & -0.0566 \\ 0 & 0 & 12.9098 & -2.2906 & -3.5385 \\ 0 & 0 & 0 & 2.6509 & 0.0072 \\ 0 & 0 & 0 & 0 & 8.9469 \end{bmatrix}$$

and

$$\mathbf{A}_+ = \begin{bmatrix} 4.5201 & 0.8454 & 9.4033 & -0.7034 & 1.0835 \\ 4.4330 & 0.4572 & 7.8615 & -0.5815 & 1.1879 \\ 2.3397 & 0.3878 & 3.4710 & 1.3104 & 4.4701 \\ 3.9104 & 0.4135 & 11.2867 & -0.0694 & 2.6867 \\ 1.0213 & 0.1559 & 0.1757 & 0.4192 & 6.5477 \end{bmatrix}.$$

- Restrictions at horizon zero, two, and infinity

$$\mathbf{L}_0(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} 0.1676 & 0 & 0 & 0 & 0 \\ -0.1760 & 1.7760 & 0 & 0 & 0 \\ 0.0173 & 0.0200 & 0.0775 & 0 & 0 \\ 0.0173 & -0.0042 & 0.0669 & 0.3772 & 0 \\ 0.0143 & 0.0192 & 0.0306 & -0.0003 & 0.1118 \end{bmatrix}$$

$$\mathbf{L}_2(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} 0.1468 & 2.1329 & 0.2138 & 0.5832 & 0.0522 \\ 0.0316 & 1.3934 & 0.0989 & 0.3142 & 0.0241 \\ 0.1447 & 2.2170 & 0.2294 & 0.6235 & 0.0473 \\ 0.1181 & 2.2576 & 0.2302 & 0.6779 & 0.0479 \\ 0.1405 & 2.5858 & 0.2838 & 0.7751 & 0.0952 \end{bmatrix}$$

$$\mathbf{L}_\infty(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} 0.1159 & -0.2625 & -0.0832 & -0.2330 & -0.0145 \\ -0.1149 & 1.3281 & -0.0594 & -0.2142 & -0.0044 \\ -0.0194 & -0.3461 & 0.0057 & -0.1048 & -0.0486 \\ -0.0449 & -0.9519 & 0.0389 & 0.2935 & -0.0268 \\ -0.0999 & -1.6985 & -0.0220 & -0.2832 & 0.2129 \end{bmatrix}$$

The restrictions

- $f(\mathbf{A}_0, \mathbf{A}_+)$ stacks these impulse response functions

$$f(\mathbf{A}_0, \mathbf{A}_+) = \begin{bmatrix} \mathbf{L}_0(\mathbf{A}_0, \mathbf{A}_+) \\ \mathbf{L}_2(\mathbf{A}_0, \mathbf{A}_+) \\ \mathbf{L}_\infty(\mathbf{A}_0, \mathbf{A}_+) \end{bmatrix}$$

- Assume that we want to impose the following restrictions

$$\begin{aligned}
\mathbf{Z}_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{Z}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{S}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{S}_3 &= \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
\mathbf{S}_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

- We do not need to specify $\mathbf{S}_1$, $\mathbf{S}_4$, $\mathbf{Z}_2$, $\mathbf{Z}_3$, and $\mathbf{Z}_5$.

Sign restrictions

- Using Algorithm 2, assume that we draw

$$\mathbf{X} = \begin{bmatrix} 0.9195 & 0.1651 & 0.7871 & 0.0329 & 0.3847 \\ -0.2499 & -0.4216 & -0.4650 & -1.8634 & 1.0269 \\ -0.2079 & 0.2769 & 1.3521 & -0.2368 & -1.3322 \\ 0.9978 & 1.3410 & -0.2697 & 0.0062 & 0.4697 \\ -0.0693 & 1.7345 & 0.8953 & -0.2012 & 0.0055 \end{bmatrix},$$

Then,

$$\mathbf{Q} = \begin{bmatrix} 0.6582 & -0.2495 & 0.5362 & -0.1878 & 0.4263 \\ -0.1789 & -0.1192 & -0.2003 & -0.9551 & 0.0375 \\ -0.1488 & 0.2124 & 0.7167 & -0.1734 & -0.6238 \\ 0.7143 & 0.3059 & -0.3889 & -0.1094 & -0.4827 \\ -0.0496 & 0.8859 & 0.0866 & -0.1021 & 0.4413 \end{bmatrix}.$$

- Given $\mathbf{Q}$ the sign restrictions are satisfied

$$
\begin{aligned}
\mathbf{S}_2 f(\mathbf{A}_0, \mathbf{A}_+)\,\mathbf{q}_2 &= \begin{bmatrix} 0.0190 & 0.0002 \end{bmatrix}' > \mathbf{0} \\
\mathbf{S}_3 f(\mathbf{A}_0, \mathbf{A}_+)\,\mathbf{q}_3 &= 0.4500 > 0 \\
\mathbf{S}_5 f(\mathbf{A}_0, \mathbf{A}_+)\,\mathbf{q}_5 &= 0.1394 > 0.
\end{aligned}
$$

- However, the zero restrictions are not satisfied for such $\mathbf{Q}$

$$
\mathbf{Z}_1 f(\mathbf{A}_0, \mathbf{A}_+)\,\mathbf{q}_1 = \begin{bmatrix} 0.1103 & -0.0037 \end{bmatrix}' \neq \mathbf{0},
$$

and

$$
\mathbf{Z}_4 f(\mathbf{A}_0, \mathbf{A}_+)\,\mathbf{q}_4 = -0.0377 \neq 0.
$$

Sign and Zero Restrictions

- Using Step 2 of Algorithm 4:

1. Let $j = 1$.
2. Find a matrix $\mathbf{N}_{j-1}$ whose columns form an orthonormal basis for the null space of $\mathbf{R}_j(\mathbf{A}_0, \mathbf{A}_+)$

$$\mathbf{N}_0 = \begin{bmatrix} 0 & 0 & 0 \\ -0.9682 & 0 & 0 \\ 0.2502 & 0 & 0 \\ 0 & 1.0000 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}.$$

3. Draw $\mathbf{x}_j$ from the standard normal distribution of $\mathbb{R}^n$,

$$\mathbf{x}_1 = \begin{bmatrix} 0.3409 \\ -0.5418 \\ 1.5292 \\ 0.3320 \\ -0.4429 \end{bmatrix}.$$

4. Let $\mathbf{q}_j = \mathbf{N}_{j-1} \left(\mathbf{N}'_{j-1}\mathbf{x}_j / \parallel \mathbf{N}'_{j-1}\mathbf{x}_j \parallel \right)$,

$$\mathbf{q}_1 = \begin{bmatrix} 0 \\ -0.8265 \\ 0.2135 \\ 0.3124 \\ -0.4168 \end{bmatrix}.$$

5. If $j = n$ stop, otherwise let $j = j + 1$ and move to Step 2.

$$\mathbf{N}_1 = \begin{bmatrix} 0.8265 & -0.2135 & -0.3124 & 0.4168 \\ 0.3169 & 0.1765 & 0.2582 & -0.3445 \\ 0.1765 & 0.9544 & -0.0667 & 0.0890 \\ 0.2582 & -0.0667 & 0.9024 & 0.1302 \\ -0.3445 & 0.0890 & 0.1302 & 0.8263 \end{bmatrix},$$

$$\mathbf{N}_2 = \begin{bmatrix} 0.4549 & -0.1767 & 0.8185 \\ 0.4177 & 0.3072 & -0.1995 \\ 0.6276 & -0.1331 & -0.1074 \\ 0.0468 & 0.9254 & 0.1984 \\ -0.4718 & 0.0164 & 0.4893 \end{bmatrix},$$

$$\mathbf{N}_3 = \begin{bmatrix} -0.6323 \\ -0.3924 \\ -0.5271 \\ -0.2887 \\ 0.2917 \end{bmatrix}, \text{ and } \mathbf{N}_4 = \begin{bmatrix} 0.6092 \\ -0.3678 \\ 0.0459 \\ -0.6484 \\ 0.2668 \end{bmatrix}.$$

$$\mathbf{x}_2 = \begin{bmatrix} -0.4423 \\ 2.0019 \\ 0.5116 \\ -0.7100 \\ 1.9563 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0.2203 \\ -0.1524 \\ 0.0247 \\ 0.7181 \\ 1.0279 \end{bmatrix},$$

$$\mathbf{x}_4 = \begin{bmatrix} 0.3112 \\ 1.2880 \\ 0.2050 \\ -0.3948 \\ -1.0959 \end{bmatrix}, \text{ and } \mathbf{x}_5 = \begin{bmatrix} -0.3828 \\ -0.3661 \\ 0.3669 \\ 0.2647 \\ 0.8716 \end{bmatrix}.$$

$$\mathbf{q}_2 = \begin{bmatrix} -0.3033 \\ -0.0908 \\ 0.7289 \\ 0.0664 \\ 0.6034 \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} 0.3704 \\ -0.1394 \\ -0.3783 \\ 0.6279 \\ 0.5532 \end{bmatrix},$$

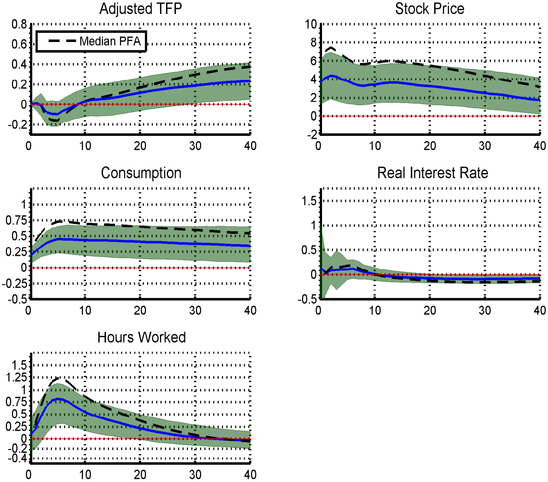
$$\mathbf{q}_4 = \begin{bmatrix} 0.6323 \\ 0.3924 \\ 0.5271 \\ 0.2887 \\ -0.2917 \end{bmatrix}, \text{ and } \mathbf{q}_5 = \begin{bmatrix} -0.6092 \\ 0.3678 \\ -0.0459 \\ 0.6484 \\ -0.2668 \end{bmatrix}.$$

- In this case, the sign restrictions also hold

$$\begin{aligned}
\mathbf{S}_2 f(\mathbf{A}_0, \mathbf{A}_+)\,\mathbf{q}_2 &= \begin{bmatrix} 0.0082 & 0.0008 \end{bmatrix}' > \mathbf{0} \\
\mathbf{S}_3 f(\mathbf{A}_0, \mathbf{A}_+)\,\mathbf{q}_3 &= 0.3127 > 0 \\
\mathbf{S}_5 f(\mathbf{A}_0, \mathbf{A}_+)\,\mathbf{q}_5 &= 0.4235 > 0.
\end{aligned}$$

- This fact depends on the draw of $\mathbf{x}_j$ for $1 \leq j \leq n$.

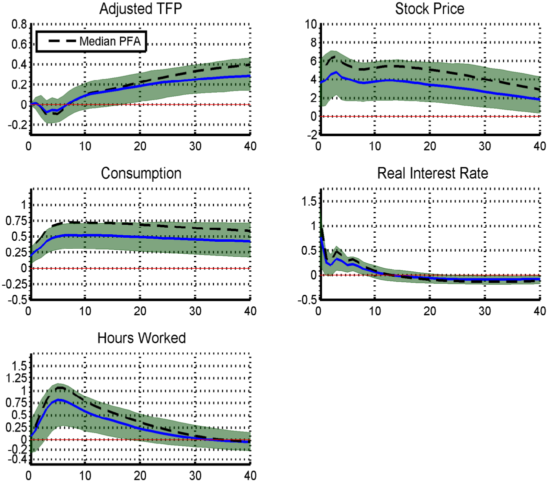
Identification II



Identification II

Go Back

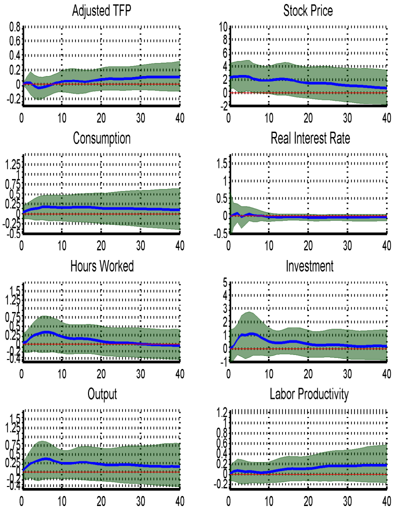
Identification III



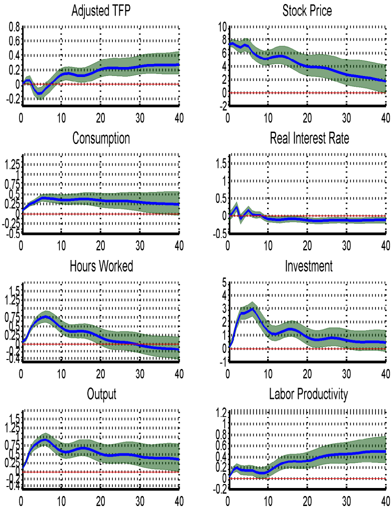
Identification III

Go Back

Identification I

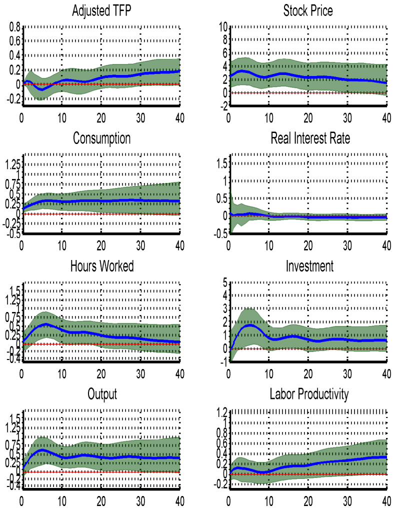


(a) ARRW (2013) methodology

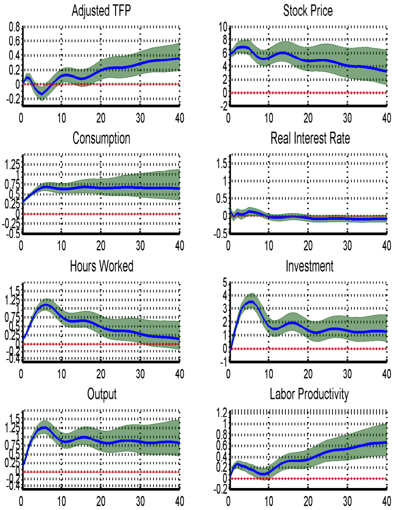


(b) MU (2009) methodology

Identification II

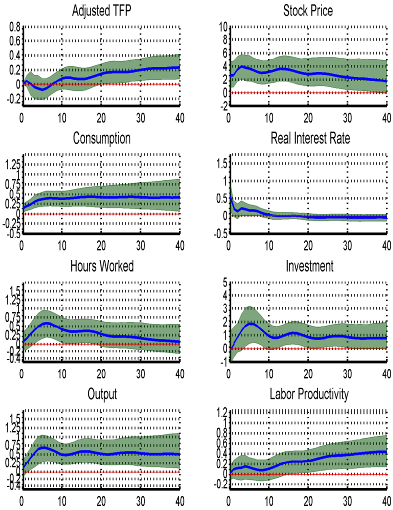


(a) ARRW (2013) methodology

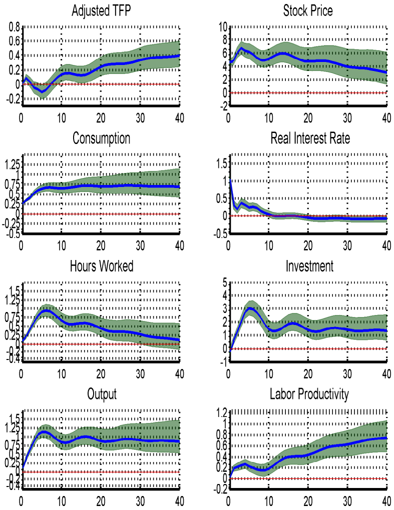


(b) MU (2009) methodology

Identification III

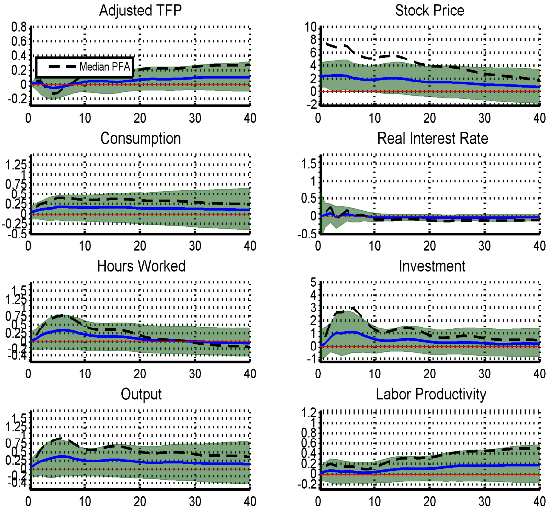


(a) ARRW (2013) methodology



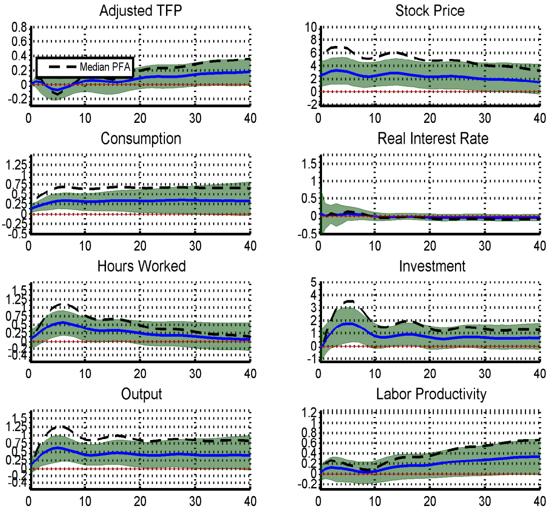
(b) MU (2009) methodology

Identification I



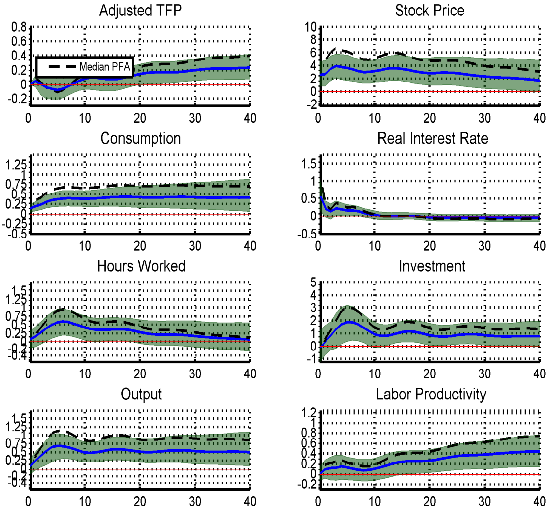
Identification I

Identification II



Identification II

Identification III

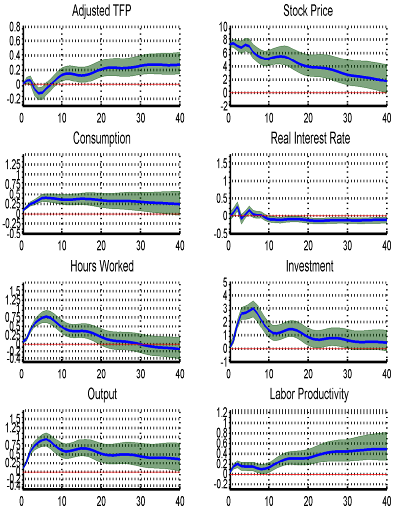


Identification III

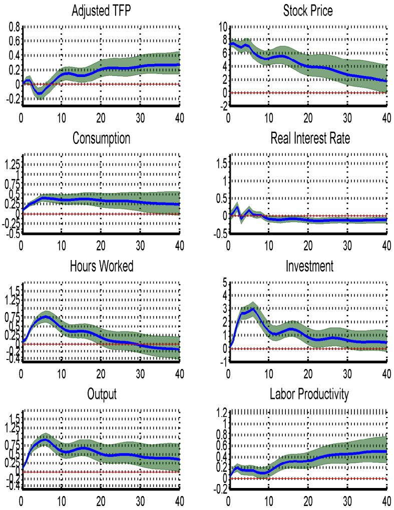
Seemingly unrestricted variables: Identification I

	MU (2009)			ARRW(2013)		
	Mean	Std dev	$\Pr(\cdot < 0)$	Mean	Std dev	$\Pr(\cdot < 0)$
Consumption	0.0793	0.0247	0	0.0259	0.1440	0.4490
Real Interest Rate	-0.0196	0.1174	0.5570	0.0047	0.7299	0.5050
Hours	0.0680	0.0391	0.0350	0.0280	0.2266	0.4830
Investment	0.1387	0.2032	0.2490	-0.0031	1.1442	0.5120
Output	0.0966	0.0333	0.0040	0.0333	0.1917	0.4440
Labor Productivity	0.0286	0.0198	0.0810	0.0054	0.1189	0.4860

Replicating Beaudry et al. (2011)



(a) ARRW (2013) with MU (2009) Restriction



(b) MU (2009)

Forecast Error Decomposition at Horizon 40

	ARRW (2013)			MU (2009)		
	Identification I	**Identification II**	**Identification III**	**Identification I**	**Identification II**	**Identification III**
Adjusted TFP	0.08	0.10	0.13	0.17	0.21	0.29
	[0.03 , 0.21]	[0.03 , 0.23]	[0.05 , 0.28]	[0.08 , 0.30]	[0.08 , 0.38]	[0.13 , 0.46]
Stock Price	0.12	0.17	0.23	0.52	0.62	0.57
	[0.03 , 0.34]	[0.04 , 0.39]	[0.07 , 0.46]	[0.34 , 0.70]	[0.48 , 0.73]	[0.43 , 0.69]
Consumption	0.11	0.14	0.21	0.13	0.55	0.61
	[0.02 , 0.35]	[0.03 , 0.40]	[0.05 , 0.49]	[0.06 , 0.28]	[0.39 , 0.69]	[0.43 , 0.76]
Real Interest Rate	0.13	0.13	0.14	0.10	0.06	0.26
	[0.06 , 0.27]	[0.05 , 0.27]	[0.06 , 0.28]	[0.05 , 0.17]	[0.03 , 0.13]	[0.19 , 0.34]
Hours Worked	0.12	0.14	0.16	0.16	0.38	0.30
	[0.03 , 0.33]	[0.04 , 0.34]	[0.05 , 0.37]	[0.09 , 0.26]	[0.25 , 0.53]	[0.18 , 0.44]
Investment	0.13	0.15	0.18	0.25	0.45	0.39
	[0.05 , 0.30]	[0.05 , 0.35]	[0.07 , 0.38]	[0.16 , 0.37]	[0.34 , 0.57]	[0.26 , 0.52]
Output	0.12	0.16	0.22	0.23	0.59	0.60
	[0.03 , 0.33]	[0.05 , 0.39]	[0.07 , 0.46]	[0.13 , 0.39]	[0.46 , 0.72]	[0.44 , 0.73]
Labor Productivity	0.10	0.12	0.20	0.24	0.37	0.49
	[0.02 , 0.29]	[0.03 , 0.34]	[0.04 , 0.44]	[0.12 , 0.39]	[0.17 , 0.55]	[0.27 , 0.65]

▸ Five-Variable SVAR