Booms and Busts in Asset Prices*

Klaus Adam
Mannheim University and CEPR

Albert Marcet
London School of Economics and CEPR

January 2012 - preliminary

Abstract

We show how low-frequency boom and bust cycles in asset prices can emerge from Bayesian learning by investors. Investors rationally maximize infinite horizon utility but hold subjective priors about the asset return process that we allow to differ infinitesimally from the rational expectations prior. Bayesian updating of return beliefs then gives rise to self-reinforcing return optimism that results in an asset price boom. The boom endogenously comes to an end because return optimism causes investors to make optimistic plans about future consumption. The latter reduces the demand for assets that allow to intertemporally transfer resources. Once returns fall short of expectations, investors revise return expectations downward and set in motion a self-reinforcing price bust. In line with available survey data, the learning model predicts return optimism to comove positively with market valuation. In addition, the learning model replicates the low frequency behavior of the U.S. price dividend ratio over the period 1926-2006.

JEL Class. No.: G12, D84

1 Motivation

Following the recent boom and bust cycles in a number of asset markets around the globe there exists renewed interest in understanding better the factors contributing to the emergence of such drastic asset price movements. This paper adds to this task by constructing a very simple asset pricing model in which learning by investors gives rise to endogenously driven low-frequency waves of optimism and pessimism which are associated with sustained asset price booms

*Thanks go to Fernando Alvarez, Chryssi Giannitsarou, Ricardo Reis, Mike Woodford, conference participants at the Banque de France and Chicago Fed Conference on Asset Price Bubbles, and seminar participants at Columbia University for helpful comments and suggestions. Klaus Adam thanks the Bank of Japan for the hospitality offered during early stages of this project. All errors remain ours.

and busts. These phenomena occur although all investors behave individually rational, i.e., maximize expected infinite horizon utility under a consistent set of beliefs which are updated using Bayesian learning.

The model we construct is close to a standard Lucas (1978) asset pricing model but considers investors that possess only limited knowledge about the equilibrium behavior of asset returns. Specifically, we allow investors to hold subjective prior beliefs about the return process that differ slightly from those entertained by agents in the rational expectations equilibrium (REE). We then show that agents' attempts to improve their knowledge can give rise to self-reinforcing asset price dynamics that generate large deviations from RE prices, even if agents' prior beliefs are arbitrarily close to RE priors. Importantly, these deviations take the form of low-frequency boom and bust cycles in asset prices.

While investors may hold subjective prior beliefs about returns, investors are 'internally rational' in the sense of Adam and Marcet (2009). Specifically, all investors make contingent plans to maximize infinite horizon utility and hold complete and consistent set of probability beliefs about payoff relevant variables. The decision theoretic microfoundations underlying our learning model distinguishes it from much of the earlier learning literature and has the advantage that the present model can serve also to answer important normative questions, although addressing these is beyond the scope of this paper.

Imperfect information about the return process has strong implications for asset prices because agents then use past return realizations to learn about the stochastic process governing returns. Such learning from past observations tends to generate momentum in asset price behavior to the extent that agents become more optimistic (pessimistic) about the return process whenever they are positively (negatively) surprised by realized returns. This is so because increased optimism (pessimism) increases (decreases) investors' asset demand, if the intertemporal elasticity of substitution is larger than unity. Increased (decreased) asset demand in turn leads to further price increases (decreases), thereby reinforcing the initial tendency of increased optimism. As a result, asset prices changes tend to display low frequency momentum, which gives rise to sustained price increases and decreases.

After a sequence of sustained changes countervailing forces come into play that dampen the price momentum, eventually halt it and lead to a reversal. Consider a situation where increased return optimism has given rise to an asset price boom. Investors' return optimism induces them to also make optimistic plans about future consumption. This causes the marginal rate of substitution to fall, thereby reduces agents' demand for assets that allow to transfer resources into the future. As a result, price increases eventually come to an end. At this point, however, agents' return beliefs turn out to be too optimistic relative to the actual return data because large part of returns in the past has been fueled by increases in investor optimism. The subsequent downward revision in beliefs induce negative price momentum and may even cause prices to undershoot their fundamental value substantially and for prolonged periods of time. The effect of future consumption plans then eventually works in reverse and halts this downward momentum.

We show how our simple learning model is able to replicate the low frequency behavior of the price dividend ratio in the United States over the period 1926-2006 and is consistent with survey evidence on investors' return expectations that is available for the internet boom and bust period between 1998 and 2003. Specifically, the learning model is consistent with the empirical evidence that investors' return expectations correlate positively with market valuation (the price dividend ratio) over this period, i.e., that investors' return expectations were highest at the peak of the internet boom period in early 2000. As we explain in the next section, the rational expectations hypothesis counterfactually predicts this correlation to be negative. To the best of our knowledge, the present paper presents the first microfounded asset pricing model that is consistent with the observed survey data.

The learning model we present offers a mechanism for generating asset price booms and busts that is complementary to leading explanations in the rational expectations literature, e.g. Campbell and Cochrane (1999) or Bansal and Yaron (2004). In these latter models, asset price fluctuations are the results of time-variation in risk-aversion or stochastic discount factors and therefore fully efficient. In the present model, the low frequency fluctuations in asset prices are not the result of low frequency components in the stochastic discount factor - agents in our model have standard time separable utility functions - but are due to self-reinforcing endogenous dynamics of investor optimism and pessimism. This suggests that some of the low frequency fluctuations in asset prices that can be observed in the data might be inefficient in the sense of not being the result of changes in fundamentals. Since our model is fairly stylized, e.g., does not take into account important changes in the tax code over time, we do not attempt to decompose to what extent the empirically observed fluctuations are efficient or inefficient.

In a related paper Adam, Marcet, and Nicolini (2009) show that a simple asset pricing model with learning can explain the behavior of second moments of asset prices and that such a model can quantitatively replicate a large number of otherwise puzzling asset price phenomena within a very parsimonious setup. This earlier model, however, could not address the issue of boom and bust behavior in asset prices because the assumed exogeneity of the stochastic discount factor implied that asset price booms would often not come to an end, which required imposing an exogenous upper bound on agents' beliefs (a so-called projection facility).[1] While the earlier model's ability to match second moments of asset prices turned out to be very robust to the precise value chosen for the upper bound, such a model is clearly not suited to address the issue of asset price booms and busts.

Unlike in our earlier work, the present paper considers a model with risk aversion and endogenous discount factors, where stock holding plans and consumption plans interact, so that booms endogenously come to a halt due to the discount factor effects described above. This feature gives rise to a number of

[1] Booms could also end endogenously, but more often than not the exogenous bound triggered the end of an asset price boom.

technical difficulties. First, to determine the stochastic discount factor, one has to solve for agents' optimal state contingent consumption plans which requires solving a non-trivial non-linear optimization problem. Second, since agents are learning, their Bayesian posterior becomes a state variable in their optimization problem. Despite these features, we are able to derive a closed form solution for the equilibrium asset price under learning in the limiting case of vanishing uncertainty, which allows us to illustrate most of our findings analytically. The paper also outlines a numerical solution strategy for the general case with non-vanishing uncertainty.

The present paper also extends the analysis in Adam and Marcet (2009) which considers a risk-neutral asset pricing model and spells out the decision theoretic foundations when agents hold subjective priors about the price process. This paper considers a setting with non-linear utility and provides Bayesian microfoundations for constant gain learning mechanisms, as well as for the information lag in the agents' updating equations. Moreover, none of our earlier contributions dealt with asset price boom and bust cycles, with matching the historical time series of the US PD ratio, or with survey expectations of stock market returns.

Models of learning have been used before to explain some aspects of asset price behavior. Timmermann (1993, 1996), Brennan and Xia (2001), Cogley and Sargent (2008) and Veronesi (2003) consider Bayesian learning to explain various aspects of stock prices. These authors consider agents who learn about the dividend process and set the asset price equal to the discounted expected sum of dividends. This approach is less able to explain asset price volatility: while agents' beliefs about the dividend process influence market prices, agents' beliefs remain unaffected by market outcomes because agents learn only about an exogenous driving process. Agents in our setting are learning about the behavior of market determined variables (asset returns). Other related papers by Bullard and Duffy (2001) and Brock and Hommes (1998) show that learning dynamics can converge to complicated attractors, if the RE equilibrium is unstable under learning dynamics.[2] Branch and Evans (2006) study a model where agents' algorithm to form expectations switches depending on which of the available forecast models is performing best. Also related is Cárceles-Poveda and Giannitsarou (2007) who assume that agents know the mean stock price and learn only about deviations from the mean; they find that the presence of learning does then not significantly alter the behavior of asset prices.[3]

The paper is structured as follows. The next section presents evidence on boom and bust cycles in stock markets. It also discusses survey evidence on investors' return expectations and critically discusses to what extend the rational expectations hypothesis is consistent with the available evidence. Section 3 presents the asset pricing model and section 4 determines for benchmark purposes its Rational Expectations Equilibrium. Section 5 explains how we relax agents' prior beliefs about return expectations and derives the resulting Bayesian

[2] Stability under learning dynamics is defined in Marcet and Sargent (1989).

[3] Cecchetti, Lam, and Mark (2000) determine the misspecification in beliefs about future consumption growth required to match the equity premium and other moments of asset prices.

updating equations. After defining the market equilibrium condition in section 6 we derive a closed form solution for the equilibrium asset price in section 7 for the case with vanishing uncertainty. Section 8 then illustrates the boom and bust episodes to which the learning model gives rise. Section 9 illustrates the model's ability to replicate the low frequency variation of the US price dividend ratio. Section 10 briefly discusses a numerical solution approach for the general case with non-vanishing risk.

2 Stock Market Booms and Busts: Data and Interpretation

This section discusses the empirical evidence on stock market boom and bust behavior and the implication of such behavior for stock market returns. We then discuss to what extend actual stock market return behavior is reflect in investors' expectations as measured by survey evidence.

2.1 Stock Market Prices and Returns

Perhaps not surprisingly, many stock markets historically experienced substantial and sustained price increases that were followed by sustained and long lasting price reversals. Figures 1 - 3 illustrate this behavior for the United States, the Euro Area (using synthetic data before its creation) and Japan, respectively, since the mid 1970's.[4] The figures depict the quarterly price dividend (PD) ratio as well as their HP trend which eliminates high frequency variation in price dividend movements.

In the United States the PD ratio increased more than threefold in the 1990's and then dropped by more than 30% from its peak level after the turn of the century. The Euro Area experienced two cycles over the considered period, with the first starting in the early 1980's and coming to an end around 1990 and the second coinciding with the one in the United States. In both European cycles the PD ratio roughly doubled during the boom and later on approximately reverted to pre-boom levels. Japan also experienced large stock price fluctuations. The PD ratio increased more than four-fold from the mid 1980's until the end of the decade, and subsequently collapsed to one half its peak value. Japan also experienced a second sizable but less persistent increase and reversal around the turn of the century, in line with the experience in Europe and the U.S. at the time.

The previous evidence shows that at low frequencies the PD ratio in major stock markets displays substantial momentum, i.e., there are periods in which increases in the PD ratio tend to be followed by further increases, as well as periods in which decreases tend to be followed by additional decreases. This behavior of the PD ratio can be observed in all three stock markets.

[4] The data sources are described in appendix A.

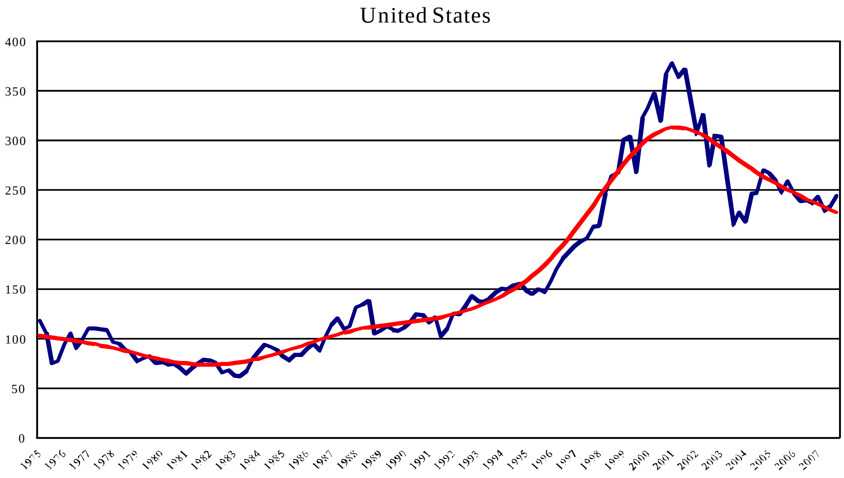

Figure 1: Quarterly PD Ratio and HP Trend

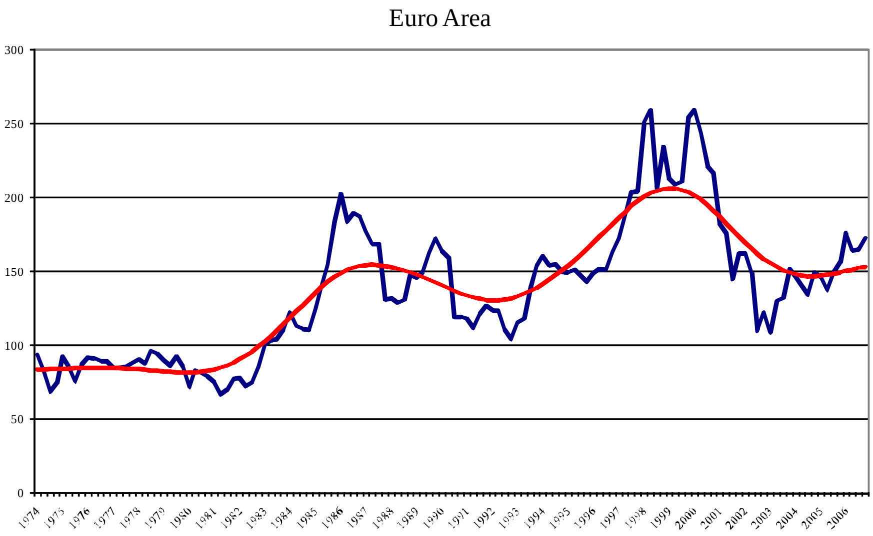

Figure 2: Quarterly PD Ratio and HP Trend

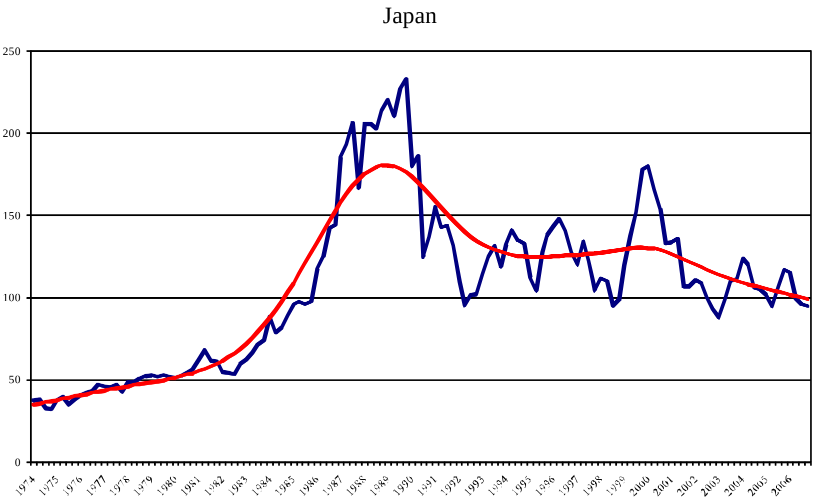

Figure 3: Quarterly PD Ratio and HP Trend

The persistence in the change of the PD ratio documented above implies that stock market returns themselves display persistent low frequency variation over time. This follows from the following simple considerations. Define the asset return R_{t+1} between period t and $t+1$ as

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}$$

where P denotes the stock price and D dividends and use the approximation

$$R_{t+1} = \frac{\frac{P_{t+1}}{D_{t+1}} + 1}{\frac{P_t}{D_t}} \frac{D_{t+1}}{D_t} \approx \frac{\frac{P_{t+1}}{D_{t+1}}}{\frac{P_t}{D_t}} \frac{D_{t+1}}{D_t}$$

which is valid for sufficiently large PD ratios. The previous expression reveals that persistent increases (decreases) in the PD ratio imply persistently high (low) average stock returns, provided dividend growth is uncorrelated or at least not negatively correlated with the changes in the PD ratio, as is actually the case in the data. We can thus summarize the previous discussion as follows:

Observation 1: *Changes in the PD ratio display persistence and average stock market returns display persistent time variation.*

As has been observed before, the PD ratio also has a tendency to mean revert, i.e., sustained increases in the PD ratio - asset price booms - are often partially reversed during subsequent asset price busts. Such behavior took place in all three stock markets around the turn of the millennium, for example. The mean reverting behavior of the PD ratio suggests that future holding period returns are negatively associated with the level of the PD ratio.[5] Specifically, at times where the PD ratio is high, future excess returns are below average. This is illustrated in Table 1 below, which reports the regression coefficient c_1, the standard deviation of the coefficient estimate in brackets, and the R^2 value of the following regression

$$X_{t,t+k} = c_0 + c_1 \frac{P_t}{D_t}$$

where $X_{t,t+k}$ denotes the excess returns of stocks over bonds from period t to $t+k$ and P_t/D_t the price dividend ratio in period t. The table shows that a high PD ratio is associated with below average excess returns in all markets, i.e., c_1 is negative, and that the R^2 of the regression is increasing with the prediction horizon.[6] This empirical relationship is confirmed, for example, in Campbell (2003) for a number of additional stock markets and time periods.

We summarize the previous findings as follows:

Observation 2: *The PD ratio is mean reverting and a high (low) PD ratio predicts future stock market returns to be below (above) average.*

Quarters	EMU (1984-2006)		U.S. (1974-2006)		Japan (1974-2006)	
k	c_1	R^2	c_1	R^2	c_1	R^2
4	-0.20 (0.06)	0.06	-0.0426 (0.02)	0.06	-0.20 (0.035)	0.21
8	-0.16 (0.06)	0.11	-0.0422 (0.01)	0.10	-0.21 (0.02)	0.44
12	-0.16 (0.02)	0.24	-0.0432 (0.01)	0.16	-0.19 (0.01)	0.54

Table 1: Excess Return Predictability

The rational expectations asset pricing literature has offered a consistent explanation for the observed momentum and mean reverting behavior of the PD ratio (and of returns) by considering asset pricing models in which investors' stochastic discount factor is varying over time. If the stochastic discount factor displays persistent changes and slow moving and mean reverting drifts, e.g., as in Campbell and Cochrane (1999), then asset price valuations display corresponding persistent changes and drifts, consistent with Observations 1 and 2 above. The next section assesses to what extent the return expectations implied by the REH actually receive support in the data.

[5] As before, this assumes that dividend growth is uncorrelated or at least not negatively correlated with the changes in the PD ratio.

[6] Due to the difficulties associated with defining the risk free rate, the sample period for the European Monetary Union had to be shortened to start in 1984.

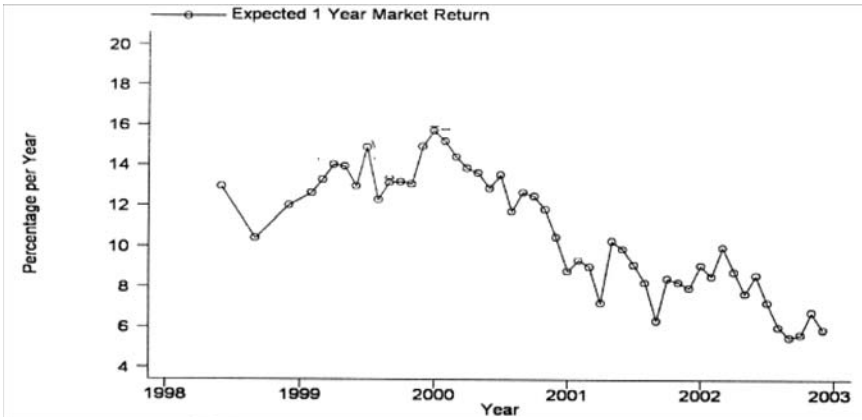

Figure 4: Average 1 year ahead stock market return expectations of US investors, UBS/Gallup Survey Data.

2.2 Expected Stock Market Returns

Observation 2 above implies that agents whose return expectations are rational should expect future stock market returns to be low whenever the PD ratio is high. Observation 2 and the REH thus suggest that at the beginning of the year 2000 when the new economy stock market boom reached its peak, investors have been aware that the expected future returns on their investments would be exceptionally low.[7] Arguably, this is hard to believe on a priori grounds and we document below that this implication of the REH is inconsistent with available survey evidence on expected stock market returns: rather than being pessimistic, investors appear to have been particularly optimistic about returns when the stock price was highest.

Figure 4 which is taken from Vissing-Jorgensen (2003) illustrates this fact. The figure depicts the time series of the average one year ahead stock market return expectations of a representative sample of 1000 U.S. investors from 1998 until the end of 2002. The data is taken from the UBS Gallup Survey and to qualify a household must own at least 10.000 US$ in financial assets. The survey data show that investors' return expectations are rather high in 1999, peak at the beginning of 2000, and gradually come down in the following years. The peak in expected returns thus coincides with the peak of the Nasdaq market, suggesting that market return expectations fail to be negatively associated with market prices, unlike predicted under the REH. Instead, there seems to exist a positive correlation.

It is likely that figure 4 understates the positive correlation between time variation in investors' return expectations and asset prices. This is so because

[7] Paying a high price for the asset is still rational, if agents' discount factors are exceptionally high so that the returns that agents require for holding risky stocks are even lower than the returns they expect ex ante. This point goes back to Fama and French (1988).

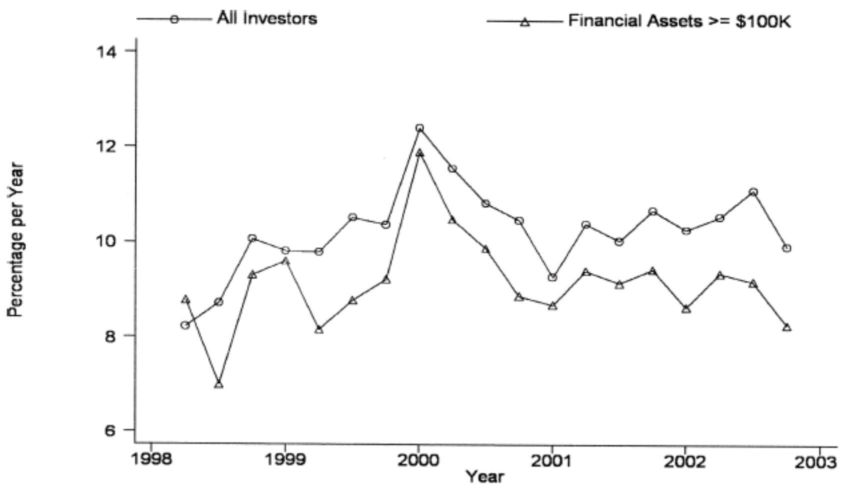

Figure 5: Cross-sectional standard deviation of return expectations.

by averaging the return expectations of investors one implicitly assumes that all investors matter equally for equilibrium asset prices. It appears reasonable to assume, however, that asset prices are influenced more heavily by richer investors, and - to the extent that short-sale constraints are effective - by the expectations of the most optimistic investors. Figure 5, which is taken from Vissing-Jorgensen (2003), displays the cross sectional standard deviation of return expectations. It shows that the cross sectional dispersion comoves positively with the level of return expectations. This is the case for all investors in the survey as well as for those investors holding financial wealth of more than 100.000 US$. This shows that the expected returns of more optimistic investors are even more positively associated with market prices than the average expected returns depicted in figure 4.

The conclusion that can be reached at this point is that there appears to be a positive correlation between asset valuation and investors' return expectations, unlike predicted by Observation 2 and the REH. We summarize this as follows:

Observation 3: High asset prices appear to be associated with overly optimistic return expectations.

Observation 3 is inconsistent with Observation 2 and the REH and suggests that time varying discount factor models in combination with the REH do not offer a complete description of asset price boom and bust movements. Indeed, observation 3 suggests that a potentially important factor contributing to exceptionally high levels of asset prices are what appears ex-post as overly optimistic return expectations. This is consistent with the evidence provided in

Bacchetta et al. (2009) who document for a wide range of asset markets that the same variables that predict excess returns also predict the expectational errors of investors. Specifically, Bacchetta et al. (2009) show that a high PD ratio in stock markets predict that agents are too optimistic about future returns. The asset pricing model with learning that we construct in the next section is consistent with this finding. In particular, it is able to replicate Observations 1 to 3 and thus improves upon RE based explanations which are unable to match Observations 2 and 3 at the same time.

3 The Asset Pricing Model

We consider a simple endowment economy populated by a unit mass of infinitely lived agents trading one unit of a stock in a competitive stock market. Each period the stock yields D_t units of the unique perishable consumption good.

The Investment Problem. Investor $i \in [0,1]$ solves the following infinite horizon maximization problem:

$$\max_{\left\{C_t^i \geq 0, S_t^i \in \left[0,\overline{S}\right]\right\}_{t=0}^{\infty}} E_0^{\mathcal{P}^i} \left[\sum_{t=0}^{\infty} \delta^t \frac{\left(C_t^i\right)^{1-\gamma}}{1-\gamma} \right] \tag{1}$$

$$s.t.$$

$$S_t^i P_t + C_t = S_{t-1}^i \left(P_t + D_t\right) \text{ for all } t \geq 0 \tag{2}$$

$$S_{-1}^i = 1 \text{ given}$$

where C^i denotes consumption, S^i the agent's stockholdings, D dividends, P the (ex-dividend) price of the stock and $\mathcal{P}^i$ the agent's subjective probability measure, which may or may not satisfy the rational expectations hypothesis. Details of $\mathcal{P}^i$ will be specified below. Problem (1) specifies that the agent can not go short on assets, i.e., it imposes the constraint $S^i \geq 0$. This short sale constraint is a consequence of the constraint $C_t \geq 0$ because covering any short position would eventually require negative consumption. We also impose some arbitrarily large but finite upper bound on stock holding $S^i \leq \overline{S} \in (1, \infty)$, which limits the long positions an investor can take. This constraint is introduced for technical reasons only - it insures compactness of the decision space - and $\overline{S}$ is assumed to be sufficiently large so that it does not bind in equilibrium

We assume that the intertemporal elasticity of substitution satisfies

$$\gamma^{-1} > 1$$

The interpretation of γ as a parameter governing intertemporal substitution rather than agents' risk aversion is justified because we will largely eliminate risk considerations from the model later on. The assumption of a more than unitary substitution elasticity then insures that the substitution effect of intertemporal relative price changes dominates the income effect, which turns out to be crucial for the results that follow.[8]

Substituting the constraint into the objective delivers the following alternative description of the investment problem:

$$\max_{\{S_t \in [0,\overline{S}]\}_{t=0}^{\infty}} E_0^{\mathcal{P}^i} \left[\sum_{t=0}^{\infty} \delta^t \frac{\left(S_{t-1}^i \left(P_t + D_t\right) - S_t^i P_t\right)^{1-\gamma}}{1-\gamma} \right] \tag{3}$$
$$s.t.$$
$$S_{t-1}^i \text{ given}$$

Note that we have dropped the constraint $C_t^i \geq 0$. Since marginal utility of consumption increases without bound as at $C_t \to 0$ interior solutions are nevertheless guaranteed.

The Underlying Probability Space. We now construct the underlying probability space. Agents hold a consistent but potentially less-than-fully-rational set of beliefs about all variables that are beyond their control. In the present setup this comprises beliefs about dividends and competitive market prices and potentially beliefs about unknown parameters governing the price and dividend processes. Let Ω denote the space of possible realizations for infinite sequences of dividends and prices. A typical element $\omega \in \Omega$ is then given by $\omega = \{P_t, D_t\}_{t=0}^{\infty}$. As usual, Ω^t denotes the set of price and dividend histories from period zero up to period t and ω^t its typical elements. The agent's plans will be contingent on the history of prices and dividends ω^t, i.e., the agent chooses

$$S_t^i : \Omega^t \to [0, \overline{S}] \tag{4}$$

The corresponding state-contingent consumption process is determined by (4) and the budget constraint (2). The underlying probability space is then given by $(\Omega, \mathcal{B}, \mathcal{P}^i)$ with $\mathcal{B}$ denoting the corresponding σ-Algebra of Borel subsets of Ω, and $\mathcal{P}^i$ a probability measure over $(\Omega, \mathcal{B})$. We make the following assumption

Assumption 1: For all t and all ω^t with $P_t < \infty$ and $D_t < \infty$, the probability measure $\mathcal{P}^i$ satisfies:

$$E_t^{\mathcal{P}^i} \left[\sum_{j=0}^{\infty} \delta^j \frac{\left(P_{t+j} + D_{t+j}\right)^{1-\gamma}}{1-\gamma} \right] < \infty \tag{5}$$

Condition (5) requires that price and dividend beliefs are not 'too optimistic'. Overly optimistic beliefs may pose a problem because they can give rise to a situation where subjective expected utility is infinite, so that problem (3) does not have a well defined solution. Condition (5) is a sufficient condition insuring

[8] Bansal and Yaron (2004) equally require intertemporal elasticity of substitution to be larger than one.

that the maximum achievable utility is finite whenever the current price and dividend are finite.[9]

Existence of Optimal Plans. Since $\gamma^{-1} > 1$ the flow utility is positive each period and thus bounded below. Assumption 1 insures that the objective is bounded above as long as current price and dividend are finite so that the objective function is continuous in these cases. Since the choice set is compact and non-empty in S, a maximum for problem (3) exists provided the current price and dividend are finite.

Sufficiency of First Order Conditions and Uniqueness of Optimal Plans. Provided the current price and dividend are finite the first order conditions are then necessary and sufficient for achieving a maximum because the objective (3) is strictly concave in S_t and because the choice set is convex in S_t. Moreover, strict concavity implies that the optimal policy is unique, so that the optimal stock holding policy is described by a function rather than by a correspondence.

The previous results justify working with the first order conditions of problem (3). Defining the asset return

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} \tag{6}$$

the first order conditions of problem (3) characterizing optimal investment behavior can be written as

$$C_t^{-\gamma} < \delta E_t^{\mathcal{P}^i}\left[(C_{t+1})^{-\gamma} R_{t+1}\right] \qquad \text{and } S_t = \overline{S} \tag{7a}$$

$$C_t^{-\gamma} = \delta E_t^{\mathcal{P}^i}\left[(C_{t+1})^{-\gamma} R_{t+1}\right] \qquad \text{and } S_t \in \left[0, \overline{S}\right] \tag{7b}$$

$$C_t^{-\gamma} > \delta E_t^{\mathcal{P}^i}\left[(C_{t+1})^{-\gamma} R_{t+1}\right] \qquad \text{and } S_t = 0 \tag{7c}$$

Clearly, inequality (7c) will (a.s.) never bind in the optimum. Selling all assets is suboptimal because it implies that consumption in subsequent periods is zero so that marginal utility of consumption is infinite. Likewise by choosing $\overline{S}$ sufficiently large, the upper inequality will not bind. This allows us to focus

[9] This follows from:

$$\begin{aligned}
\max_{\left\{S_{t+j}^i \in [0,\overline{S}]\right\}_{j=0}^{\infty}} \quad & E_t^{\mathcal{P}^i}\left[\sum_{j=0}^{\infty} \delta^j \frac{\left(S_{t+j-1}^i\left(P_{t+j} + D_{t+j}\right) - S_{t+j}^i P_{t+j}\right)^{1-\gamma}}{1-\gamma}\right] \\
\leq \quad & E_t^{\mathcal{P}^i}\left[\sum_{j=0}^{\infty} \delta^j \frac{\left(\overline{S}\left(P_{t+j} + D_{t+j}\right)\right)^{1-\gamma}}{1-\gamma}\right] \\
= \quad & \frac{\overline{S}^{1-\gamma}}{1-\gamma} E_t^{\mathcal{P}^i}\left[\sum_{t=0}^{\infty} \delta^j \left(P_{t+j} + D_{t+j}\right)^{1-\gamma}\right]
\end{aligned}$$

on equation (7b). Using the budget constraint and the definition (6) future consumption can be expressed as

$$\begin{aligned} C_{t+1} &= (S_t - S_{t+1})\, P_{t+1} + S_t D_{t+1} \\ &= (S_t - S_{t+1})\,(P_t R_{t+1} - D_{t+1}) + S_t D_{t+1} \\ &= (S_t - S_{t+1})\, P_t R_{t+1} + S_{t+1} D_{t+1} \end{aligned}$$

so that the FOC (7b) can alternatively be written as

$$\begin{aligned} &\left((S_{t-1} - S_t)\,\frac{P_t}{D_t} + S_{t-1}\right)^{-\gamma} \\ =\;& \delta E_t^{\mathcal{P}^i}\left[\left((S_t - S_{t+1})\,\frac{P_t}{D_t} R_{t+1} + S_{t+1}\frac{D_{t+1}}{D_t}\right)^{-\gamma} R_{t+1}\right] \end{aligned} \tag{8}$$

The previous equation illustrates that evaluating the first order conditions requires that agents formulate beliefs about dividend growth and asset returns one period ahead. Agents' current economic situation is thereby described by the stocks that they purchased in the previous period (S_{t-1}), the current price dividend ratio (P_t/D_t) at which they can trade the asset and by their beliefs $\mathcal{P}^i|\omega^t$. The solution to FOC (8) is a stock demand function $S(S_{t-1}, P_t/D_t, \omega^t)$ that specifies how much assets to demand as a function of previous stock holdings, the current price dividend ratio and the beliefs about the future, which are potentially a function of the entire history ω^t.

4 Rational Expectations (RE) Equilibrium

This section specifies a dividend process and determines the resulting equilibrium outcome when agents' beliefs $\mathcal{P}^i$ are rational. The standard assumption in the literature is to assume that dividends evolve according to

$$\ln D_t = \ln D_{t-1} + \ln \beta^D + \ln \varepsilon_t^D \tag{9}$$

with

$$\ln \varepsilon_t^D \sim N(-\frac{\sigma_D^2}{2}, \sigma_D^2) \tag{10}$$

so that $\beta^D > 0$ denotes dividend growth and ε_t^D is a shock to dividend growth with mean 1.

Appendix B shows the following results. When agents know (9) and hold rational price expectations, then the FOC (8) implies a constant price dividend ratio which is given by

$$PD^{RE} = \frac{\delta\beta^{RE}}{1 - \delta\beta^{RE}} \tag{11}$$

$$\beta^{RE} = \left(\beta^D\right)^{1-\gamma} e^{-\gamma(1-\gamma)\frac{\sigma_D^2}{2}} \tag{12}$$

The equilibrium price under rational expectations thus follows

$$\ln P_t = \ln P_{t-1} + \ln \beta^D + \varepsilon_t^D \tag{13}$$

and stock returns under RE are given by:

$$\ln R_t = \ln \overline{R} + \ln \varepsilon_t^D \tag{14}$$

with

$$\overline{R} = \frac{\delta^{-1}\left(\beta^D\right)^{\gamma}}{e^{-\gamma(1-\gamma)\frac{\sigma_D^2}{2}}}$$

The mean stock return in the Rational Expectations Equilibrium is thus constant over time. For the case with vanishing risk ($\sigma_\varepsilon^2 \to 0$) that we consider below, the previous solution simplifies to the perfect foresight outcome

$$\begin{aligned}
\ln D_t &= \ln D_{t-1} + \ln \beta^D && (15)\\
\ln R_t &= \ln\left(\delta^{-1}\left(\beta^D\right)^{\gamma}\right) && (16)\\
\frac{P_t}{D_t} &= \frac{\delta\left(\beta^D\right)^{1-\gamma}}{1-\delta\left(\beta^D\right)^{1-\gamma}}
\end{aligned}$$

5 Learning about Return Behavior

We now relax the Rational Expectations Hypothesis and endow agents with a model of the asset return process that is slightly more general than the behavior of returns (14) emerging in the rational expectations equilibrium. Specifically, we consider agents who doubt that the mean asset return is constant over time and instead believe that mean returns may drift over time. This is in line with the empirical observations made in section 2. In the special case with vanishing risk, which we will consider later on, this generalized return process converges to the perfect foresight rational expectations outcome (16). Agents' prior beliefs thus converge to the RE priors in this limiting case, so that the deviations from the RE beliefs become arbitrarily small.

To emphasize the importance of learning about returns rather than learning about dividend behavior, which was the focus of much of the earlier literature on learning in asset markets, e.g., Timmermann (1993, 1996), we continue to assume that agents know the dividend process (9), i.e., hold rational dividend expectations.

Generalized Return Beliefs. For simplicity we consider a situation where all agents i hold the same beliefs. We thus drop the superscript i from agents' probability measure. In the REE the asset return process is composed of a constant mean and an unpredictable component, see equation (14). In the data, however, returns display persistent time variation, as discussed in section

2. To capture this feature we now suppose that agents entertain the following generalized model for asset returns

$$\ln R_t = \ln \overline{R}_t + \ln \varepsilon_t \tag{17}$$

where ε_t is a transitory component and $\overline{R}_t$ denotes a persistent time varying return component which follows the process

$$\ln \overline{R}_t = \ln \overline{R}_{t-1} + \ln \nu \tag{18}$$

The disturbances are given by

$$\begin{pmatrix} \ln \varepsilon_t \\ \ln \nu_t \end{pmatrix} \sim iiN\left(\begin{pmatrix} -\frac{\sigma_\varepsilon^2}{2} \\ -\frac{\sigma_v^2}{2} \end{pmatrix}, \begin{pmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_\nu^2 \end{pmatrix} \right) \tag{19}$$

and are assumed independent of the dividend innovations ε^D. The specification (17) implies that return innovations are unpredictable but that expected returns vary over time in a persistent way. Specifically, there exist periods in which expected returns are high ($\overline{R}_t > \overline{R}$) and periods with low returns ($\overline{R}_t < \overline{R}$). We assume that the agents' prior beliefs about the persistent component are given by

$$\ln \overline{R}_0 \sim N(\ln m_0, \sigma_0^2) \tag{20}$$

and that these are independent of $\varepsilon_t, \varepsilon_t^D$ and ν_t for all t. Equations (17)-(20) together with knowledge of the dividend process (9) jointly specify agents' probability beliefs $\mathcal{P}^i$.[10]

Learning about Returns. The agent can observe the asset return R_t but can not directly tell which part of the observed return is due to the persistent component $\overline{R}_t$ and which part due to the transitory element ε_t. Instead, agents formulate beliefs about the persistent return component $\ln \overline{R}_t$ using standard (Bayesian) filtering techniques. Assuming that agents know σ_ε^2 and σ_ν^2, Bayesian updating of beliefs implies that (e.g. Theorem 3.1 in West and Harrison (1997))

$$\ln \overline{R}_t | \omega^t \sim N(\ln m_t, \sigma^2)$$

with

$$\begin{aligned} \ln m_t &= \ln m_{t-1} + g\left(\ln R_t + \frac{\sigma_\varepsilon^2 + \sigma_v^2}{2} - \ln m_{t-1} \right) && (21) \\ \sigma^2 &= \sigma_0^2 = \frac{-\sigma_\nu^2 + \sqrt{\left(\sigma_\nu^2\right)^2 + 4\sigma_\nu^2 \sigma_\varepsilon^2}}{2} && (22) \\ g &= \frac{\sigma_0^2}{\sigma_\varepsilon^2} && (23) \end{aligned}$$

[10]The price pricess implied by $\mathcal{P}^i$ follows recursively from equation (6).

where we have chosen the prior uncertainty σ_0^2 about the unobserved state $\overline{R}_0$ to be equal to its steady state value. Agents' beliefs are thus summarized by a single state variable (m_t) which evolves recursively according to equation (21).

Appendix C proves the following result:

Proposition 1 *The beliefs $\mathcal{P}$ defined by equations (9)-(10) and (17)-(20) satisfy assumption 1.*

The previous proposition implies that maximum achievable utility is finite, provided $P_t < \infty$ and $D_t < \infty$, so that optimal plans exist and can be characterized using the first order conditions of the investment problem.

How this nest RE beliefs. The belief specification (9)-(10) and (17)-(20) nests RE beliefs in the special case with vanishing risk. Specifically, consider the limiting case without uncertainty where $\left(\sigma_\varepsilon^2, \sigma_\nu^2, \sigma_{\varepsilon^D}^2\right) \to 0$. If at $t = 0$ agents initial belief about the persistent return component is centered at the perfect foresight outcome (16), as we assume from now on, i.e., if

$$\ln m_0 = \ln\left(\delta^{-1}\left(\beta^D\right)^\gamma\right)$$

then agents' prior probability mass about returns increasingly concentrates at the perfect foresight outcome $m_t = \delta^{-1}\left(\beta^D\right)^\gamma$ for all t, as noise vanishes. This is the case because prior uncertainty then vanishes ($\sigma_0^2 \to 0$) as well as the variance of the return innovations σ_ε^2 and σ_ν^2. Since agents' dividend expectations are rational, agents prior beliefs $\mathcal{P}$ then approach the perfect foresight outcome (15)-(16). The limiting Kalman gain parameter g is thereby determined by equations (22) and (23) which implicitly define $\frac{\sigma_\nu^2}{\sigma_\varepsilon^2} \to \frac{g^2}{4-2g}$. Since g will generally not converge to zero as noise vanishes, one can study learning dynamics even in the limiting case with vanishing risk.

6 Market Equilibrium

The belief specification introduced in the previous section implies that agents' beliefs about returns are summarized by a single state variable, namely the mean belief about the permanent return component $\ln m_t$. The asset demand function solving the first order condition (8) therefore takes the form

$$S(S_{t-1}, \frac{P_t}{D_t}, \ln m_t) \tag{24}$$

Normalizing total asset supply to one and imposing market clearing in all periods implies that the equilibrium price dividend ratio and beliefs in period t must solve

$$1 = S(1, P_t/D_t, \ln m_t) \tag{25}$$

The current beliefs $\ln m_t$ and the current price dividend ratio P_t/D_t are determined simultaneously via equations (21) and (25). Generally, there may thus exist multiple market clearing pairs for the PD ratio and agents' beliefs. This potential for multiplicity arises from the complementarity between realized returns and expected future returns. Intuitively, a higher PD ratio also implies higher asset returns and thus higher expected future returns via equation (21). Higher expected future returns may then induce agents to be willing to buy the asset at a higher price. While this multiplicity may be a potentially interesting avenue to explain asset price booms and busts, we wish to abstract from such simultaneities between beliefs and outcomes, as this would require us to select between multiple market clearing prices.

Instead, we slightly modify the information setup for agents. The modification implies that the Bayesian posterior estimate depends on lagged returns only which eliminates the simultaneity problem. Specifically, we generalize the perceived return process (17) by splitting the temporary return innovation $\ln \varepsilon_t$ into two independent subcomponents

$$\ln R_t = \ln \overline{R}_t + \ln \varepsilon_t^1 + \ln \varepsilon_t^2$$

where $\ln \varepsilon_t^1 \sim N(-\frac{\sigma_{\varepsilon,1}^2}{2}, \sigma_{\varepsilon 1}^2)$ and $\ln \varepsilon_t^2 \sim N(-\frac{\sigma_{\varepsilon 2}^2}{2}, \sigma_{\varepsilon 2}^2)$ and $\sigma_\varepsilon^2 = \sigma_{\varepsilon 1}^2 + \sigma_{\varepsilon 2}^2$. We then assume that agents observe the innovations ε^1 with a one period lag, i.e., $\left\{\varepsilon_{t-1}^1, \varepsilon_{t-2}^1, ...\right\}$ is part of agents' time t information set.[11] One possible interpretation of this setup is that agents learn over time something about the temporary return components. The process for the persistent return component $\ln \overline{R}_t$ remains as in equation (18) but now has innovation variance $\sigma_{\widetilde{v}}^2$ instead σ_v^2. Appendix D proves the following result:

Proposition 2 *Consider the limit $\sigma_{\varepsilon 2}^2 \to 0$ and let $\sigma_{\varepsilon 1}^2 = \sigma_\varepsilon^2 - \sigma_{\varepsilon 2}^2$ and $\sigma_{\widetilde{v}}^2 = \sigma_{\varepsilon 2}^2 g^2/(1-g)$. The Bayesian posterior mean of $\ln \overline{R}_t$ using information up to period t is then given by*

$$\ln m_t = \ln m_{t-1} + g\left(\ln R_{t-1} - \ln \varepsilon_{t-1}^1 - \ln m_{t-1}\right) \tag{26}$$

The modified information structure thus implies that only lagged returns R_{t-1} enter the current state estimate. Intuitively, this is so because lagged returns become infinitely more informative relative to current returns as $\sigma_{\varepsilon 2}^2 \to 0$. This eliminates the simultaneity problem. For non-vanishing uncertainty $\sigma_{\varepsilon 2}^2$ the weight of the last observation actually remains positive but would still be lower than that given to the lagged return observation, see equation (39) in appendix D and the subsequent discussion for details. In the case with vanishing noise $(\sigma_\varepsilon^2, \sigma_v^2, \sigma_{\varepsilon^D}^2) \to 0$, which we consider below, equation (26) implies that under the modified information structure beliefs evolve according to

$$\ln m_t = \ln m_{t-1} + g\left(\ln R_{t-1} - \ln m_{t-1}\right) \tag{27}$$

[11] Although agents observe only the lagged values of ε^1, they continue to observe the contemporaneous value of the asset return and the dividend.

which in this case is identical to the original updating equation (21) but with lagged returns now entering instead of current returns.

For simplicity, we continue to parameterize the model using the original information structure and only use the modified updating equation (27) to describe the evolution of beliefs. Proposition 2 indicates the parameterization of the modified structure implied by this approach.

Importantly, agents' beliefs $\ln m_t$ are now predetermined at the time the market clears and the equilibrium price materializes, thereby eliminating any simultaneities. The economy then evolves according to a simple recursive process: given the beliefs $\ln m_t$, equation (25) determines the market clearing price dividend ratio for period t; equation (26) in turn determines how the beliefs are updated using this information. Equation (25) then determines the equilibrium price in the subsequent period, and so on.

7 Solving the Learning Model

The learning model has a closed form solution in the limiting case with vanishing risk, i.e., $(\sigma_\varepsilon^2, \sigma_\nu^2, \sigma_{\varepsilon^D}^2) \to 0$. This limiting case is of interest because the generalized probability measure $\mathcal{P}$ that we specified in section 5 then converges to the perfect foresight RE outcome, so that the belief deviations from the prior beliefs that agents are assumed to entertain in the rational expectations equilibrium become infinitesimally small. A solution approach for the case with non-vanishing risk is discussed in section 10.

The following proposition summarizes the main result of this section. The proof is contained in appendix E.

Proposition 3 *Under vanishing uncertainty, i.e.,* $\left(\sigma_\varepsilon^2, \sigma_\nu^2, \sigma_{\varepsilon^D}^2\right) \to 0$, *the equilibrium price is given by*

$$\frac{P_t}{D_t} + 1 = \sum_{j=0}^{\infty} \left(\left(\delta^{\frac{1}{\gamma}} \right)^j \prod\nolimits_{i=1}^{j} \left(E_t^{\mathcal{P}} R_{t+i} \right)^{\frac{1-\gamma}{\gamma}} \right) \tag{28}$$

The result of the previous proposition holds true independently of the belief specification we assume for agents. For the belief specification from section 5 and with vanishing risk we have that

$$E_t^{\mathcal{P}} R_{t+i} = m_t$$

for all $i \geq 0$. Proposition 3 then implies that the equilibrium price dividend ratio is

$$\frac{P_t}{D_t} = \frac{\left(\delta \left(m_t \right)^{1-\gamma} \right)^{\frac{1}{\gamma}}}{1 - \left(\delta \left(m_t \right)^{1-\gamma} \right)^{\frac{1}{\gamma}}} \tag{29}$$

More optimistic return expectations (higher m_t) thus imply a higher asset price as long as intertemporal substitution elasticity satisfies $\gamma^{-1} > 1$. The learning model thus associates high values of the price dividend ratio with optimistic return expectations, unlike rational expectations models.

The actual asset returns implied by equation (29) are given by

$$\begin{aligned} R_t &= \frac{P_t + D_t}{P_{t-1}} \\ &= \frac{\frac{P_t}{D_t} + 1}{\frac{P_{t-1}}{D_{t-1}}} \frac{D_t}{D_{t-1}} \\ &= \frac{1 - \left(\delta \left(m_{t-1}\right)^{1-\gamma}\right)^{\frac{1}{\gamma}}}{1 - \left(\delta \left(m_t\right)^{1-\gamma}\right)^{\frac{1}{\gamma}}} \frac{1}{\left(\delta \left(m_{t-1}\right)^{1-\gamma}\right)^{\frac{1}{\gamma}}} \frac{D_t}{D_{t-1}} \end{aligned} \tag{30}$$

This equation together with the belief updating equation (27) jointly determines the evolution of returns and beliefs.[12] The implied path for the price dividend ratio under learning follows from equation (29).

Since $\left(\delta \left(m_{t-1}\right)^{1-\gamma}\right)^{\frac{1}{\gamma}}$ will take on values close to one, the behavior of actual asset returns (30) is dominated by the behavior of the first fraction in (30). Specifically, if agents have become more optimistic $m_t > m_{t-1}$ then realized returns will also increase. Since realized returns are used to update beliefs, see equation (27), there will be a tendency for beliefs to increase further, i.e., $m_{t+1} > m_t$.

Suppose, for example, that agents hold beliefs consistent with the perfect foresight RE outcome. The sensitivity of realized returns with respect to the current return expectations m_t is then given by

$$\left.\frac{\partial R_t}{\partial m_t}\right|_{m_t = m_{t-1} = \delta^{-1}\left(\beta^D\right)^{\gamma}} = \frac{1-\gamma}{\gamma} \frac{\delta^{-1}\left(\beta^D\right)^{1+\gamma}}{1 - \delta\left(\beta^D\right)^{1-\gamma}}$$

Taking the approximation $\beta^D = 1$ we have that $\frac{\partial R_t}{\partial m_t} > 1$ whenever $\gamma < \delta$. Since δ is close to one, the learning model displays momentum of returns and return expectations around the RE as long as the intertemporal elasticity of substitution is somewhat larger than one. Specifically, if return expectations increase above (fall below) the RE value, realized returns will also increase (decrease) but stronger than expected returns, so that future return expectations are even higher (lower).

The next section investigates more closely the behavior of the model under learning.

[12] As discussed before, the Kalman gain parameter g is implicitly defined by $\frac{\sigma_\nu^2}{\sigma_\varepsilon^2} \to \frac{g^2}{1-g}$ and depends on the relative variance of the transitory and persistent return shocks in the limit.

8 Asset Price Booms and Busts from Learning

This section illustrates the behavior of the model under learning using the closed form solution derived in the previous section. It shows that the model strongly propagates initial shocks to return expectations and gives rise to low frequency movements in asset prices similar to the ones we observed when discussing the empirical evidence in section 2. We also show that the model can give rise to asymmetric asset price fluctuations, e.g. a protracted asset price boom that is followed by a sharper and faster asset price bust. Conversely, starting from the RE price level a decrease in return expectations sets in motion a relatively sharp asset price bust which tends to be followed by a slow and long-lived recovery of the price dividend ratio.

To illustrate these model properties we use the following baseline parameterization. We set the quarterly discount factor to $\delta = 0.995$ and choose $\beta^D = 1.0035$, which is the value for quarterly US dividend growth used in Adam, Marcet and Nicolini (2009). We then choose $\gamma = 0.8$ and set the gain to $g = 0.014$ so that agents attribute 1,4% of any return observation to the persistent component and 98,6% to the transitory component. We discuss the robustness of our findings to alternative model parameterizations at the end of this section.

Figure 6 depicts the impulse response of the price dividend ratio to a 10 basis points (bp) increase of the quarterly real return expectations above its rational expectations starting value (which lies at 78 bp per quarter).[13] The figure illustrates the strong momentum that is present in the model: following the initial impulse, the PD ratio displays further increases for about 15 quarters. The increase eventually stops and is followed by a much faster decline: the PD ratio falls back to baseline in about 7 quarters, i.e., just about half the time it took to increase. Due to the momentum that is present in returns and return expectations around the RE value for beliefs, the PD ratio actually undershoots its initial value and then slowly returns over time to its baseline value.

Figure 7 depicts the impulse response to a 10 bp drop in the return expectations. It shows that the drop leads to a very quick fall in the PD ratio that is followed by a very gradual return over time. The return to the baseline value of the PD ratio is slow because for low values of m_t actual returns (30) react less strongly to changes in beliefs. This is so because the sensitivity of the first fraction in (30), which is the dominant factor determining actual asset returns, is highly non-linear. As $\left(\delta\left(m_t\right)^{1-\gamma}\right)^{\frac{1}{\gamma}}$ falls further below 1, changes in m_t relative to m_{t-1} influence returns less than in the case where $\left(\delta\left(m_t\right)^{1-\gamma}\right)^{\frac{1}{\gamma}}$ is closer to one. This asymmetry also explains why the overall fall in the PD ratio following a drop in return expectations is less pronounced than the increase in the PD ratio following an increase in return expectations: momentum is less

[13] Given a gain value of $g = 0.014$ such an increase would be triggered by the observation of a quarterly real asset return that exceeds its average value by 7.14%. Given the variance of asset returns in the data, this is not an unlikely event.

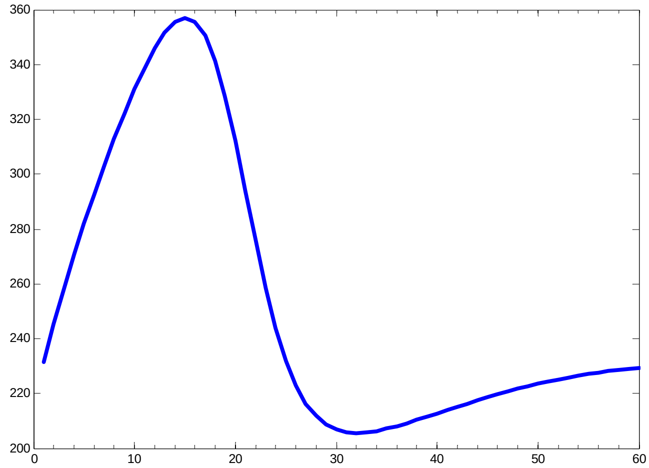

Figure 6: Response of the PD Ratio to a 10bp Increase in Return Expectations (Baseline Parameterization)

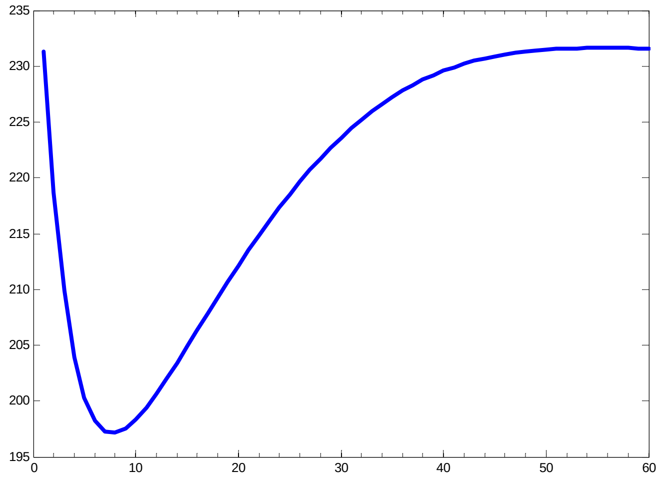

Figure 7: Response of the PD Ratio to a 10bp Decrease in Return Expectations (Baseline Parameterization)

pronounced when return expectations are low.

We now briefly discuss how these findings are affected by different model parameterizations. Increasing the gain above the baseline value will increase the size of the fluctuations. The PD ratio then starts to display very persistent low frequency variation and is persistently oscillating between high and low values. If the gain parameter becomes even larger, then the positive momentum in beliefs becomes eventually so strong as to cause asset prices to increase without bound. Lower gain values reduce the internal propagation of the model. Following an increase in the return expectations, the PD ratio then increases less strongly and reverts back to baseline more slowly, so that the asymmetry is now reversed: the boom then occurs faster than the price bust. The asymmetry following a negative return innovation, however, remains unchanged even for smaller gains. Finally, reducing the intertemporal elasticity of substitution or the discount factor both reduce the model's sensitivity to return expectations and thereby tend to dampen the internal propagation of shocks.

9 Matching the Empirical Behavior of the U.S. PD Ratio

This section evaluates the ability of the learning model to replicate the low frequency behavior of the PD ratio in the data. Since the longest historical time series are available for the U.S., we restrict consideration to the behavior of the U.S. price dividend ratio over the period 1926-2006.

The learning model is described by 3 parameters (δ, γ, g) and the initial return beliefs m_0. Given these values, the sequence of historical returns in the data define a model implied path for the return beliefs m_t according to equation (27). Theses beliefs and the parameter values for δ and γ then define the model implied PD ratio via equation (29). We compare this model implied PD ratio with that observed in the data.

We parameterize the learning model as follows. For the gain we choose $g = 0.014$, which is the same value as has been used in the simulations before. We then set $\delta = 0.988$ and $\gamma = 0.72$, which are chosen informally so as to help matching the empirical behavior of the U.S. PD ratio in the data. The initial value m_0 is chosen so as to match the PD ratio in the data at the start of the sample.

Figure 8 depicts the actual and the model implied quarterly PD ratio over the period 1926:4-2006:1. The model predicted PD ratio replicates the behavior of the PD ratio in the data surprisingly well. It matches all the low frequency variation and also generates the asset price boom at the end of the century. The largest discrepancy actually emerges at the end of the sample period where the model predicts a much stronger fall in the PD ratio than can be observed in the data.

Figure 9 depicts the model-implied return expectations over the period 1998-2003. Comparing figure 9 with the survey expectations of US investors shown

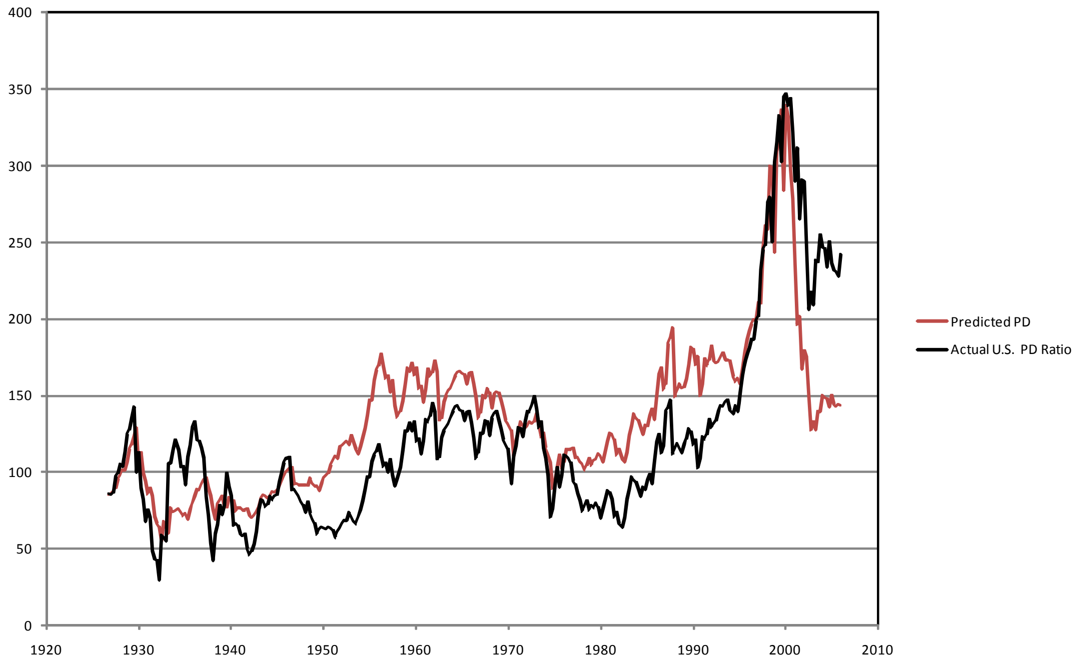

Figure 8: United States: Actual and Model Predicted Price Dividend Ratio.

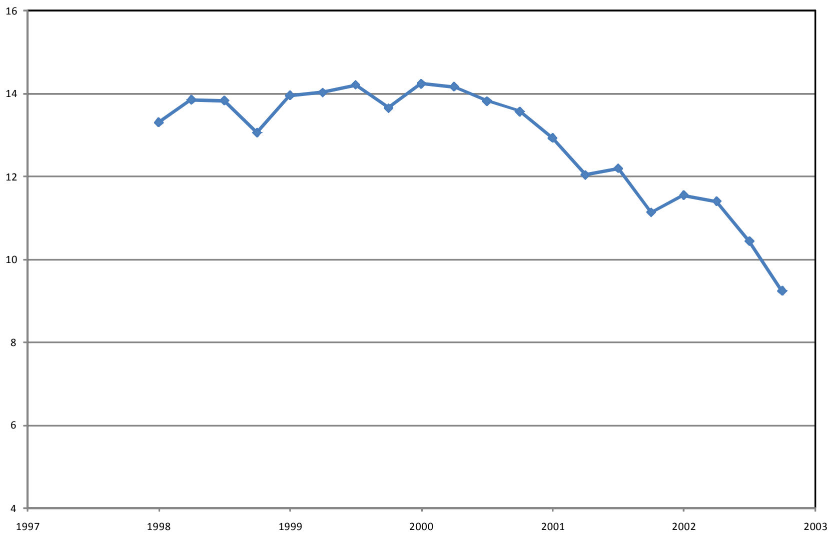

Figure 9: Model Implied Return Expectations

in figure 4 reveals a number of striking similarities. Prior to 2000 the model predicts investors' annual return expectations to be around 14 percent, similar to what US investors actually seem to have expected. Moreover, after the year 2000 return expectations implied by the model significantly fall, as is the case in the data. The drop in the data, however, appears to have been somewhat more pronounced than that implied by the model.

Overall, we conclude that the learning model successfully replicates the low frequency swings in the U.S. PD ratio over the period 1926-2006 and the behavior of return expectations around the turn of the century.

10 General Solution Approach

This section outlines how one can solve the model numerically in the general stochastic case. Due to difficulties associated with Jensen's inequality we were not able to solve the model analytically in the case with non-vanishing risk.

The solution of the model is an asset demand function of the form (24) solving the first order condition (8) under the perceived the state dynamics

$$
\begin{aligned}
S_t &= S(S_{t-1}, \frac{P_t}{D_t}, \ln m_t) && (31)\\
\frac{P_{t+1}}{D_{t+1}} &= \frac{R_{t+1}P_t - D_{t+1}}{D_{t+1}} \\
&= \frac{\overline{R}_{t+1}\varepsilon_{t+1}}{\beta^D \varepsilon^D_{t+1}}\frac{P_t}{D_t} - 1 \\
&= \frac{\overline{R}_t v_{t+1}\varepsilon_{t+1}}{\beta^D \varepsilon^D_{t+1}}\frac{P_t}{D_t} - 1 \\
&= \frac{m_t \eta_t v_{t+1}\varepsilon_{t+1}}{\beta^D \varepsilon^D_{t+1}}\frac{P_t}{D_t} - 1 && (32)\\
\ln m_{t+1} &= \ln m_t + g\left(\ln R_{t+1} + \frac{\sigma_\varepsilon^2 + \sigma_v^2}{2} - \ln m_t\right) \\
&= \ln m_t + g\left(\ln \overline{R}_{t+1} + \ln \varepsilon_{t+1} + \frac{\sigma_\varepsilon^2 + \sigma_v^2}{2} - \ln m_t\right) \\
&= \ln m_t + g\left(\ln \overline{R}_t + \ln \nu_{t+1} + \ln \varepsilon_{t+1} + \frac{\sigma_\varepsilon^2 + \sigma_v^2}{2} - \ln m_t\right) \\
&= \ln m_t + g\left(\ln m_t + \ln \eta_t + \ln \nu_{t+1} + \ln \varepsilon_{t+1} + \frac{\sigma_\varepsilon^2 + \sigma_v^2}{2} - \ln m_t\right) \\
&= \ln m_t + g\left(\ln \eta_t + \ln \nu_{t+1} + \ln \varepsilon_{t+1} + \frac{\sigma_\varepsilon^2 + \sigma_v^2}{2}\right) && (33)
\end{aligned}
$$

where

$$\ln \eta_t \sim iiN(0, \sigma_0^2)$$

The noise term η_t captures the uncertainty associated with not knowing the true value for β_t. Due to the assumed independence between the prior beliefs about β_0 and ε_t and ν_t and due to the independence of ε_t and ν_t over time, η_t is independent of ε_{t+1} and ν_{t+1}.

Equation (31) says that agents expect tomorrow's stockholdings to be equal to their current demand. Equation (32) determines the future PD ratio. The future PD ratio is thereby uncertain due to future unknown shocks and due to uncertainties associated with the true value of $\overline{R}_t$. Finally, equation (33) specifies that beliefs about $\overline{R}_t$ follow a martingale process.

Standard projection methods allow to determine the asset demand function (24) that solves the functional equation defined by the first order condition (8) and the state transition laws (31)-(33).

The market equilibrium price is then implicitly determined by the market clearing condition

$$1 = S(1, \frac{P_t}{D_t}, \ln m_t)$$

and the beliefs recursively evolve according to (21).

11 Conclusions

This paper constructs a simple asset pricing model in which low frequency boom and bust cycles in asset prices emerge from learning and rational investment behavior by agents that hold priors about return behavior that differ only infinitesimally from the rational expectations priors. Agents attempts to improve their forecasts is shown to give rise to price movements that reinforce the initial beliefs revision and thereby generate waves of optimism and pessimism that are associated with large and persistent swings in asset prices. The model is able to replicate the low frequency behavior of the US PD ratio and is consistent with survey evidence on investors' return expectations around the tech stock boom period.

A Data Sources

Euro Area: We used the Datastream Global Equity Index "TOTMKEM". This index is a value-weighted average of the national equity indexes of the euro area and is available on a quarterly basis from the first quarter of 1973. National indices are value weighted averages of individual stocks, covering a minimum of 75-80% of national market capitalisation. Datastream converts all numbers into US Dollars using the daily exchange rate of the local currency vis-à-vis the US Dollar. We reconverted the series into Euro using an exchange rate series from Global Fin Data (Global Fin code "_EURO_D"), which is available from 1950 to 2007 on a daily basis. This series is constructed merging the European Composite Unit from 1950 to August 1969, the EUA/ECU from September

1969 to 31 December 1998 and the exchange rate of the Euro from January 1999 onwards. Our quarterly series for the exchange rate is simply the average of the daily rates of a considered quarter. The corresponding dividend series has been constructed using Datastream's TOTMKEM dividend yield series (Datastream "data type" DY). The resulting dividend series has been seasonally adjusted by taking averages of dividend payments over the last 4 quarters.

United States: We used the series "TOTMKUS" from Datastream's Global Equity Index and the corresponding dividend yield series. As before, dividends have been seasonally adjusted by taking averages over the last 4 quarters. TOTMKUS is available on quarterly basis from the first quarter of 1973.

Japan: We used "TOTMKJP" from Datastream's Global Equity Index and the corresponding dividend yield series. Dividends have been seasonally adjusted by taking averages over the last 4 quarters. TOTMKJP is available on quarterly basis from the first quarter of 1973.

B PD Ratio under Rational Expectations

Market clearing requires $S_t = 1$ and $C_t = D_t$ for all t. Provided

$$\lim_{j\to\infty} E_t\left[\delta^j\left(\frac{D_{t+j}}{D_t}\right)^{-\gamma}P_{t+j}\right]=0$$

for all ω^t and t, the FOC (8) under RE implies

$$\begin{aligned}
P_t &= \delta E_t\left[\left(\frac{D_{t+1}}{D_t}\right)^{-\gamma}(P_{t+1}+D_{t+1})\right] \\
&= E_t\left[\sum_{j=1}^{\infty}\delta^j\left(\frac{D_{t+j}}{D_t}\right)^{-\gamma}D_{t+j}\right] \\
&= E_t\left[\sum_{j=1}^{\infty}\delta^j\left(\left(\beta^D\right)^j\prod_{i=1}^{j}\varepsilon_{t+i}^D\right)^{-\gamma}D_t\left(\beta^D\right)^j\prod_{i=1}^{j}\varepsilon_{t+i}^D\right] \\
&= D_t\sum_{j=1}^{\infty}\left(\delta\left(\beta^D\right)^{1-\gamma}\right)^j\prod_{i=1}^{j}E_t\left(\varepsilon_{t+i}^D\right)^{1-\gamma} \\
&= D_t\sum_{j=1}^{\infty}\left(\delta\left(\beta^D\right)^{1-\gamma}e^{-\gamma(1-\gamma)\frac{\sigma_D^2}{2}}\right)^j \\
&= \frac{\delta\left(\beta^D\right)^{1-\gamma}e^{-\gamma(1-\gamma)\frac{\sigma_D^2}{2}}}{1-\delta\left(\beta^D\right)^{1-\gamma}e^{-\gamma(1-\gamma)\frac{\sigma_D^2}{2}}}D_t
\end{aligned}$$

as claimed in (11) and (12). Letting PD^{RE} denote the PD ratio in the REE, the REE stock returns are given by

$$\begin{aligned}
\ln R_t &= \ln\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right)\\
&= \ln\left(\frac{PD^{RE}+1}{PD^{RE}}\frac{D_{t+1}}{D_t}\right)\\
&= \ln\left(\frac{PD^{RE}+1}{PD^{RE}}\right)+\ln\beta^D+\ln\varepsilon_t^D\\
&= \ln\left(\frac{\beta^D}{\delta\beta^{RE}}\right)+\ln\varepsilon_t^D\\
&= \ln\left(\frac{\delta^{-1}\left(\beta^D\right)^\gamma}{e^{-\gamma(1-\gamma)\frac{\sigma_D^2}{2}}}\right)+\ln\varepsilon_t^D
\end{aligned}$$

C Proof of Proposition 1

Using the return definition (6) one has

$$E_t^{\mathcal{P}}\left[(P_{t+j}+D_{t+j})^{1-\gamma}\right]=E_t^{\mathcal{P}}\left[(R_{t+j}P_{t+j-1})^{1-\gamma}\right] \tag{34}$$

Moreover,

$$\begin{aligned}
P_{t+j-1} &= R_{t+j-1}P_{t+j-2}-D_{t+j-1}\\
&\leq R_{t+j-1}P_{t+j-2}
\end{aligned}$$

so that

$$R_{t+j}P_{t+j-1}\leq P_t\prod\nolimits_{i=1}^{j}R_{t+i} \tag{35}$$

Equations (34) and (35) then imply

$$\begin{aligned}
&E_t^{\mathcal{P}}\left[(P_{t+j}+D_{t+j})^{1-\gamma}\right]\\
\leq\ & (P_t)^{1-\gamma}E_t^{\mathcal{P}}\left[\left(\prod\nolimits_{i=1}^{j}R_{t+i}\right)^{1-\gamma}\right]\\
=\ & (P_t)^{1-\gamma}E_t^{\mathcal{P}}\left[\left(\prod\nolimits_{i=1}^{j}\overline{R}_{t+i}\varepsilon_{t+i}\right)^{1-\gamma}\right]\\
=\ & (P_t)^{1-\gamma}E_t^{\mathcal{P}}\left[\left(\prod\nolimits_{i=1}^{j}\left(\overline{R}_t\left(\prod\nolimits_{k=1}^{i}v_{t+k}\right)\varepsilon_{t+i}\right)\right)^{1-\gamma}\right]\\
=\ & (P_t)^{1-\gamma}E_t^{\mathcal{P}}\left[\left(\overline{R}_t\right)^{1-\gamma}\right]\left(\prod\nolimits_{i=1}^{j}\left(\left(\prod\nolimits_{k=1}^{i}E_t^{\mathcal{P}}\left[(v_{t+k})^{1-\gamma}\right]\right)E_t^{\mathcal{P}}\left[(\varepsilon_{t+i})^{1-\gamma}\right]\right)\right)
\end{aligned} \tag{36}$$

where the last line uses the fact that $\overline{R}_t, v_{t+k}$ and ε_{t+i} are all mutually independent.

$$
\begin{aligned}
E_t^{\mathcal{P}}\left[\left(v_{t+k}\right)^{1-\gamma}\right] &= E_t^{\mathcal{P}}\left[e^{(1-\gamma)\ln v_{t+k}}\right] \\
&= e^{-(1-\gamma)\frac{\sigma_v^2}{2}+(1-\gamma)^2\frac{\sigma_v^2}{2}} \\
&= e^{-\gamma(1-\gamma)\frac{\sigma_v^2}{2}} \\
&\leq 1
\end{aligned}
$$

Correspondingly

$$
E_t^{\mathcal{P}}\left[\left(\varepsilon_{t+i}\right)^{1-\gamma}\right] \leq 1
$$

The previous two inequalities and equation (36) imply

$$
\begin{aligned}
E_t^{\mathcal{P}}\left[\left(P_{t+j}+D_{t+j}\right)^{1-\gamma}\right] &\leq \left(P_t\right)^{1-\gamma} E_t^{\mathcal{P}}\left[\left(\overline{R}_t\right)^{1-\gamma}\right] \\
&= \left(P_t\right)^{1-\gamma} E_t^{\mathcal{P}}\left[e^{(1-\gamma)\ln \overline{R}_t}\right] \\
&= \left(P_t\right)^{1-\gamma} E_t^{\mathcal{P}}\left[e^{(1-\gamma)m_t+(1-\gamma)\sigma_0^2}\right]
\end{aligned}
$$

so that

$$
E_t^{\mathcal{P}}\left[\sum_{j=0}^{\infty}\delta^j\frac{\left(P_{t+j}+D_{t+j}\right)^{1-\gamma}}{1-\gamma}\right] \leq \frac{1}{1-\gamma}\frac{1}{1-\delta}\left(P_t\right)^{1-\gamma} E_t^{\mathcal{P}}\left[e^{(1-\gamma)m_t+(1-\gamma)\sigma_0^2}\right] < \infty
$$

D Proof of Proposition 2

Consider the modified information structure described in section 6. The posterior mean estimate of $\ln \overline{R}_t$ based on information up to period $t-1$ is then recursively given by

$$
\ln m_{t|t-1} = \ln m_{t-1|t-2} + \widetilde{g}\left(\ln R_{t-1} - \ln \varepsilon_{t-1}^1 + \frac{\sigma_{\varepsilon 2}^2}{2} - \ln m_{t-1|t-1}\right) \tag{37}
$$

and the posterior uncertainty and the Kalman gain by

$$
\begin{aligned}
\sigma_{-1}^2 &= \sigma_{0|-1}^2 = \frac{-\sigma_{\widetilde{\nu}}^2 + \sqrt{\left(\sigma_{\widetilde{\nu}}^2\right)^2 + 4\sigma_{\widetilde{\nu}}^2\sigma_{\varepsilon 2}^2}}{2} \\
\widetilde{g} &= \frac{\sigma_{-1}^2}{\sigma_{\varepsilon 2}^2}
\end{aligned} \tag{38}
$$

Standard updating formulas for normal distributions then imply that the posterior mean based on information up to t can be expressed as

$$\ln m_{t|t} = \ln m_{t|t-1} + \frac{\sigma_{-1}^2}{\sigma_{-1}^2 + \sigma_{\varepsilon 1}^2 + \sigma_{\varepsilon 2}^2 + \sigma_{\widetilde{v}}^2}(\ln R_t + \frac{\sigma_{\varepsilon 1}^2 + \sigma_{\varepsilon 2}^2 + \sigma_{\widetilde{v}}^2}{2} - \ln m_{t|t-1}) \tag{39}$$

Since $\frac{\sigma_{-1}^2}{\sigma_{-1}^2+\sigma_{\varepsilon 1}^2+\sigma_{\varepsilon 2}^2+\sigma_{\widetilde{v}}^2} < \frac{\sigma_{-1}^2}{\sigma_{\varepsilon 2}^2} = \widetilde{g}$, the weight of the observation dated t is reduced relative to the earlier observation dated $t-1$ because it is 'noisier'. Now consider the limit $\sigma_{\varepsilon 2}^2 \to 0$ and along the limit choose $\sigma_{\varepsilon 1}^2 = \sigma_{\varepsilon}^2 - \sigma_{\varepsilon 2}^2$ and $\sigma_{\widetilde{v}}^2 = \frac{g^2}{1-g}\sigma_{\varepsilon 2}^2$. The latter implies that $\widetilde{g} = g$. From equation (38) then follows that $\sigma_{0|-1}^2 \to 0$ and thus $\sigma_{-1}^2 \to 0$. Equation (39) then implies that the weight of the last observation converges to zero so that $\ln m_{t|t} = \ln m_{t|t-1}$ in the limit. Equation (37) can then be written as stated by equation (26) in the main text.

E Proof of Proposition 3

We consider the situation with vanishing noise ($\sigma_D^2 \to 0$,$\sigma_\varepsilon^2 \to 0, \sigma_v^2 \to 0$), so that agents' become increasingly certain about future outcomes. To simplify notation we suppress the expectations operator and use $E_t^{\mathcal{P}} X_{t+j} = X_{t+j}$ so that X_{t+j} now denotes the value that investors expect the variable X to take on in period $t+j$. The first order condition (7b) under vanishing noise can then be written as

$$\begin{aligned} 1 &= \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \delta R_{t+1} \Longleftrightarrow \\ \frac{C_{t+1}}{P_{t+1}+D_{t+1}} &= \delta^{\frac{1}{\gamma}} \left(R_{t+1}\right)^{\frac{1-\gamma}{\gamma}} \frac{C_t}{P_t} \end{aligned} \tag{40}$$

which holds for all t. The budget constraint implies

$$S_{t-1}(P_t + D_t) = C_t + S_t P_t \Longrightarrow S_{t-1} = \frac{C_t}{P_t + D_t} + \frac{P_t}{P_t + D_t} S_t$$

Iterating forward on the latter equation gives

$$S_{t-1} = \frac{C_t}{P_t + D_t} + \frac{P_t}{P_t + D_t}\frac{C_{t+1}}{P_{t+1} + D_{t+1}} + \frac{P_t}{P_t + D_t}\frac{P_{t+1}}{P_{t+1} + D_{t+1}}\frac{C_{t+2}}{P_{t+2} + D_{t+2}} + \ldots$$

Substituting out the fractions that involve future consumption by using repeatedly equation (40) gives

$$\begin{aligned}
S_{t-1} &= \frac{C_t}{P_t+D_t} + \frac{P_t}{P_t+D_t}\delta^{\frac{1}{\gamma}}\left(R_{t+1}\right)^{\frac{1-\gamma}{\gamma}}\frac{C_t}{P_t} \\
&\quad + \frac{P_t}{P_t+D_t}\frac{P_{t+1}}{P_{t+1}+D_{t+1}}\delta^{\frac{1}{\gamma}}\left(R_{t+2}\right)^{\frac{1-\gamma}{\gamma}}\frac{C_{t+1}}{P_{t+1}} + \ldots \\
&= \frac{C_t}{P_t+D_t} + \delta^{\frac{1}{\gamma}}\left(R_{t+1}\right)^{\frac{1-\gamma}{\gamma}}\frac{C_t}{P_t+D_t} \\
&\quad + \frac{P_t}{P_t+D_t}\delta^{\frac{1}{\gamma}}\left(R_{t+2}\right)^{\frac{1-\gamma}{\gamma}}\frac{C_{t+1}}{P_{t+1}+D_{t+1}} + \ldots \\
&= \frac{C_t}{P_t+D_t} + \delta^{\frac{1}{\gamma}}\left(R_{t+1}\right)^{\frac{1-\gamma}{\gamma}}\frac{C_t}{P_t+D_t} \\
&\quad + \frac{P_t}{P_t+D_t}\delta^{\frac{1}{\gamma}}\left(R_{t+2}\right)^{\frac{1-\gamma}{\gamma}}\delta^{\frac{1}{\gamma}}\left(R_{t+1}\right)^{\frac{1-\gamma}{\gamma}}\frac{C_t}{P_t} + \ldots \\
&= \frac{C_t}{P_t+D_t} + \delta^{\frac{1}{\gamma}}\left(R_{t+1}\right)^{\frac{1-\gamma}{\gamma}}\frac{C_t}{P_t+D_t} \\
&\quad + \left(\delta^{\frac{1}{\gamma}}\right)^2\left(R_{t+1}R_{t+2}\right)^{\frac{1-\gamma}{\gamma}}\frac{C_t}{P_t+D_t} + \ldots \\
&= \frac{C_t}{P_t+D_t}\sum_{j=0}^{\infty}\left(\left(\delta^{\frac{1}{\gamma}}\right)^j \prod\nolimits_{i=1}^{j}\left(R_{t+i}\right)^{\frac{1-\gamma}{\gamma}}\right) \qquad (41)
\end{aligned}$$

which is the result stated in the proposition under the convention that $R_{t+i} = E_t^{\mathcal{P}} R_{t+i}$.

References

Adam, K., and A. Marcet (2009): "Internal Rationality and Asset Prices," *CEPR Discussion Paper No. 7498.*

Adam, K., A. Marcet, and J. P. Nicolini (2009): "Learning and Stock Market Volatility," *Mannheim University Mimeo.*

Bacchetta, P., E. Mertens, and E. V. Wincoop (2009): "Predictability in Financial Markets: What Do Survey Expectations Tell Us?," *Journal of International Money and Finance*, 28, 406–426.

Bansal, R., and A. Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59, 1481–1509.

Branch, W., and G. W. Evans (2006): "Asset Return Dynamics and Learning," University of Oregon mimeo.

Brennan, M. J., and Y. Xia (2001): "Stock Price Volatility and Equity Premium," *Journal of Monetary Economics*, 47, 249–283.

Brock, W. A., and C. H. Hommes (1998): "Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model," *Journal of Economic Dynamics and Control*, 22, 1235–1274.

Bullard, J., and J. Duffy (2001): "Learning and Excess Volatility," *Macroeconomic Dynamics*, 5, 272–302.

Campbell, J. Y. (2003): "Consumption-Based Asset Pricing," in *Handbook of Economics and Finance*, ed. by G. M. Constantinides, M. Harris, and R. Stulz, pp. 803–887. Elsevier, Amsterdam.

Campbell, J. Y., and J. H. Cochrane (1999): "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107, 205–251.

Carceles-Poveda, E., and C. Giannitsarou (2007): "Asset Pricing with Adaptive Learning," *Review of Economic Dynamics (forthcoming).*

Cecchetti, S., P.-S. Lam, and N. C. Mark (2000): "Asset Pricing with Distorted Beliefs: Are Equity Returns Too Good to Be True?," *American Economic Review*, 90, 787–805.

Cogley, T., and T. J. Sargent (2008): "The Market Price of Risk and the Equity Premium: A Legacy of the Great Depression?," *Journal of Monetary Economics*, 55, 454–476.

Fama, E. F., and K. R. French (1988): "Dividend Yields and Expected Stock Returns," *Journal of Financial Economics*, 22, 3–25.

Lucas, R. E. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1426–1445.

Marcet, A., and T. J. Sargent (1989): "Convergence of Least Squares Learning Mechanisms in Self Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

Pastor, L., and P. Veronesi (2003): "Stock Valuation and Learning About Profitability," *Journal of Finance*, 58(5), 1749–1789.

Timmermann, A. (1993): "How Learning in Financial Markets Generates Excess Volatility and Predictability in Stock Prices," *Quarterly Journal of Economics*, 108, 1135–1145.

——— (1996): "Excess Volatility and Predictability of Stock Prices in Autoregressive Dividend Models with Learning," *Review of Economic Studies*, 63, 523–557.

Vissing-Jorgensen, A. (2003): "Perspectives on Behavioral Finance: Does "Irrationality" Disappear with Wealth? Evidence from Expectations and Actions," in *2003 Macroeconomics Annual*, Boston. NBER.

West, M., and J. Harrison (1997): *Bayesian Forecasting and Dynamic Models*. Springer, Berlin.