Hours and Participation with Job Heterogeneity*

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Abstract

We consider a competitive equilibrium matching model where technological progress is embodied in new jobs and hence the economy features job heterogeneity. Workers decide on whether to participate in the labor market and on how many hours to work when assigned to a job. This endogenously generates inequality in wages and in labor supply. When the pace of technological progress accelerates, differences in job technologies in the new balanced growth path widen. As a result, the balance of income and substitution effects on labor supply is asymmetric across jobs and it becomes optimal to work longer hours in the top jobs and work less hours in the worst ones. With a fixed cost of labor supply this implies that the participation rate falls as workers work less often in order to avoid the worst jobs, and they supply longer hours on average when employed. This model can explain the simultaneous fall in labor force participation and the increase in working hours experienced by US male workers since the mid 70s in a context of raising wage inequality. In addition, it can explain the differences across education groups. Our model predicts assortative matching, and hence, less educated individuals have access to the worst jobs in the economy, those that worsen the most with the increase in the speed of embodied technical change. Hence, for these workers the returns on market work fall disproportionately and they reduce labor supply in both margins.

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1 Introduction

Heterogeneity in available jobs is an important feature of the labor market: as argued by Akerlof (1981), good jobs —like dam sites— are scarce resources. A natural explanation for this friction is that new technologies come embedded into new capital goods. For instance, when a new microprocessor is invented and makes it into mass production only the new computers come embedded with it, while the existing ones keep working with their older processors. It would be expensive and wasteful to substitute all computers in the economy by the new ones. Heterogeneity in jobs requires, additionally, that machines cannot be lumped together into production such that each worker uses an average fraction of all machines. Instead, job heterogeneity arises in an environment characterized by an assignment friction: every worker can operate only one machine, and the economy has to solve the matching between different machines and different workers.

This non-convexity generates an interesting allocation problem between the number of jobs used in the economy and the time spent working in each of them. To study this problem, we write a neoclassical growth model with vintage capital, where a social planner has to match ex-ante identical workers to heterogeneous machines, and it also has to choose the hours worked in each machine and the number of workers to use. Our first contribution is to show opposite movements of the intensive and extensive margins of labor supply when there is an increase in the dispersion of job qualities. This is important because of the increase in the speed of capital embodied technical change observed in the US after the 1970's — see Greenwood and Yorokoglu (1997), Greenwood, Hercowitz, and Krusell (1997) and Violante (2002). The intuition is as follows. Since jobs of different qualities co-exist in the economy, the aggregate participation rate determines the quality of the worst technologies in use. When the dispersion of jobs increases, the quality of the marginal machine worsens compared to the frontier, and it is not worth to pay the fixed utility costs of operating it. Then, in order to prevent consumption from falling the economy uses more intensively the good jobs, whose productivity falls less compared to the frontier. The assignment friction is essential for this result because it creates asymmetric income and substitution effects across different jobs. Absent job heterogeneity hours per worker could not increase because the change in the distribution of machine qualities would exert income and substitution effects of the same size, which cancel out under balanced growth path preferences.

We next extend the model to allow for heterogeneous workers. In a world without labor choices, the classical work of Becker (1973) tells us that complementarity between machine quality and worker productivity ensures assortative matching. However, endogenous labor supply makes the matching problem between jobs and workers non-trivial and specific conditions for an assortative equilibrium are needed. Our second contribution is to show that an equilibrium with assortative matching requires that human capital differences between worker types be large enough compared to consumption differences. The reason for this is that, while higher human capital is relatively more attractive to better jobs, the higher consumption of more skilled workers discourages their work effort due to the standard income effect. Hence good jobs could find profitable to hire for some periods less productive workers who would work more intensively. Consumption differences between worker types are an equilibrium outcome themselves. In terms of model fundamentals, the condition for assortative matching will be satisfied if skill differences between workers are large enough compared to job heterogeneity. This is because large job heterogeneity would give more skilled workers a consumption advantage that does nor reflect higher productivity. Whenever this condition is not satisfied, an assortative equilibrium requires some redistribution in the economy. That is to say, it requires that low skilled workers enjoy a sufficiently high amount of non-labor income such that they are not willing to supply the extra hours that could lead them to better machines.

To analyze the quantitative relevance of the mechanisms described in the paper, we bring our model to the data and make it face the simultaneous fall in employment rates and increase in hours per male worker during the last 40 years in a context of raising wage inequality. Our model can potentially explain the opposite movements in the extensive and intensive margins of the labor supply as the efficient outcome of an economy that experiences an increase in the speed of embodied technical change. In particular, we paramaterize the model to account for the differences in employment rates, hours per worker, labor income and consumption across education types in the 1970's. The recovered model parameters imply that 70 percent of the ratio of hourly wages between college graduates and high school dropouts was due to differences in worker skills, while 30 due to the heterogeneity of jobs assigned to these types of workers. As shown by Sattinger (1975), Jovanovic (1998) or Violante (2002) dispersion of job qualities may be an important determinant of dispersion of wages above and beyond dispersion of abilities of workers. Then, we consider two exogenous changes in the labor market over this period. First, there has been an increase in the speed of embodied technical change, as documented by Greenwood and Yorokoglu (1997), Greenwood, Hercowitz, and Krusell (1997) and Violante (2002). Second, the supply of workers of different education groups has changed dramatically, with the share of college workers increasing from 15% to 27% and the share of high school dropouts falling from 43% to 14%. When we feed in these two elements into the model, we can explain the observed fall of 8% in the employment rate observed in the U.S. The model also predicts an increase in 1.3 weekly hours , which is a big share of the observed 1.5. Furthermore, the slope by education predicted by the model is also consistent with the data: the model predicts higher increases in hours per worker for the more educated and lower falls in participation. Finally, the model predicts one third of the observed increase in the wage premium between college workers and high school dropouts. The return to skill increases as a side effect of the matching friction: the underlying differences in skill get amplified by an increase in the dispersion of machines quality. In this sense, our model explains the different trends in labor supply by different education groups, and the rise of labor income inequality between them, as the result of a particular form of capital skill complementarity emphasized by Katz and Murphy (1992) and Krusell, Ohanian, Rios-Rull, and Violante (2000).

The remaining of the article is organized as follows. In Section 2 we study an economy with identical workers and the optimal allocation of working time between machines, and the optimal number of machines operated. We also show how these allocations change with an increase in the speed of embodied technical change and what is needed for the opposite movements in the extensive and intensive margins. Then, in Section 3 we show how to decentralize this economy. Section 4 introduces worker heterogeneity and studies the conditions needed for an assortative equilibrium to exist. In Section 5 we evaluate the model ability to account for the labor market trends of the last 40 years.

2 Job heterogeneity and labor supply

The economy is populated by a continuum of measure 1 of identical and infinitely-lived workers. Time is continuous. At every instant in time t, a measure $m < \infty$ of new machines of quality e^{qt} become available. Without loss of generality we normalize $m = 1$. We relax this assumption in Section 4. The constant $q > 0$ determines the speed of embodied technical change.

We assume that every worker is matched with only one machine and a machine can not produce with more than one worker at a time. This is the key friction of our economy, which arises because workers and machines are indivisible. A machine of quality k when matched with a worker who supplies n hours of work produces an output level given by the homogenous of degree one function $f(k, n)$. We restrict this production function to be Cobb-Douglas with capital share α in order to have a balanced growth path with constant participation rate and constant hours per worker.

2.1 Machine ages and qualities

Machines are in excess supply because the number of workers is fixed and new machines become continuously available. This means that there is a critical age τ^* such that all machines older than τ^* are scrapped. The distribution of ages is uniform on the support $\tau \in [0, \tau^*]$. Let p denote the aggregate *participation rate*, i.e. the fraction of workers that participate to the labour market. Since every worker is paired to a machine, p is also the measure of machines operated.[1] Hence, the density of machines of any age is given by $\frac{1}{p}$ and the fraction of machines in operation with age smaller than or equal to $\bar{\tau}$ is given by,

$$\Pr(\tau \leq \bar{\tau}) = \int_0^{\bar{\tau}} \frac{1}{p} \, ds = \frac{1}{p}\bar{\tau}$$

By definition, no machine older than τ^* is in operation. Clearly τ^* solves $\Pr(\tau \leq \tau^*) = 1$, which immediately implies that

$$\tau^* = p.$$

It is easy to map machine ages into machine qualities. The quality $\tilde{k}_t^\tau$ of a machine of age τ at time t is given by $e^{q(t-\tau)}$. Hence, the quality $\tilde{k}_t^*$ of the worst machine in operation at time t can be expressed as

$$\tilde{k}_t^* = e^{q(t-\tau^*)} \tag{1}$$

and the ratio of qualities between the best and the worst machines in operation is given by e^{qp}.

Finally, it will be convenient to define the detrended machine quality as,

$$k^\tau \equiv \tilde{k}_t^\tau e^{-qt} = e^{-q\tau}$$

such that $k^* = e^{-qp}$ and if we call k^0 the detrended quality of the best machine, we have $k^0 = 1$.

Lemma 1 *The distribution of detrended qualities of operating machines has support* $[e^{-qp}, 1]$ *and it is log-uniform with density* $g(k) = \frac{1}{qp}\frac{1}{k}$.

[1] For simplicity, we discuss the model with a constant participation rate p and constant age τ^* of the marginal machine. This does not constraint the solution of the model as we are going to focus on the balanced growth path equilibrium, in which aggregate variables grow at a constant rate and aggregate ratios are constant over time. See Section 3.5 for an exact definition.

2.2 The social planner problem

We want to study the trade-off between using an extra machine from the scrap pool or using the available machines more intensively. To characterize the efficient allocation we set up a large household that cares equally for the utility of all its members and chooses every period the number of household members to send to work, how much each of them works and how much each of them consumes. Since all household members are identical and we concentrate in preferences separable between consumption and leisure, the planner will choose the same consumption for each of them. Also because of the homogeneity of workers, the exact matching with machines is irrelevant and hence indeterminate. With balanced growth preferences and production functions, there will be nothing dynamic about this problem. Then, we can write the detrended planner problem in each period as follows:

$$\max_{c,p,n^{\tau}} \left\{ \log c - \int_0^p v\left(n^{\tau}\right) d\tau \right\} \qquad \text{s.t.} \quad c \leq \int_0^p f\left(k^{\tau}, n^{\tau}\right) d\tau$$

where we have indexed workers by the age τ of the machine with which they are paired. Note that c refers to detrended consumption. The individual disutility of working is given by,

$$v\left(n\right) = \begin{cases} \lambda_0 + \lambda_1 \frac{n^{1+\eta}}{1+\eta} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{2}$$

where $\lambda_0 > 0$ is a fixed cost of going to work while $\lambda_1 > 0$ gives the cost of the variable component. The parameter η regulates the Frisch elasticity.

Let μ be the lagrange multiplier of the aggregate resource constraint. Then, the first order conditions for an optimum are quite standard:

$$\frac{1}{c} = \mu \tag{3}$$

$$v'\left(n^{\tau}\right) = \mu f_2\left(k^{\tau}, n^{\tau}\right) \tag{4}$$

$$v\left(n^{*}\right) = \mu f\left(k^{*}, n^{*}\right) \tag{5}$$

Equation (3) states that the value of an extra unit of income is equal to the marginal utility of consumption. Equation (4) determines the work effort in every machine, which we can express as:

$$n^{\tau} = \phi\left(k^{\tau}, \mu\right)$$

Since the utility cost of supplying hours of work is the same in all machines but the marginal product of each hour is higher in newer machines ($f_{12} > 0$), this condition states that work effort will be increasing in machine quality, so $\phi_1 > 0$. At the same time, work effort will be increasing in the marginal utility of consumption μ, so $\phi_2 > 0$. Finally, equation (5) determines the number of workers engaged in production by equating the disutility —in terms of forgone leisure— of sending the marginal worker to work and the value of the output produced in the marginal machine. The marginal utility of consumption μ lowers the quality of the marginal machine and hence it raises the number of workers p. The intuition is that when income is more valuable, it is worth operating worse machines that produce less output.

2.3 An increase in the speed of embodied technical change

We want to understand the changes in allocations when we compare different balanced growth path economies with different q. Proposition 5 below states the main results. But before that, it is worth analyzing three intermediate results.

Lemma 2 *The ratio of hours worked between machines of any two vintages increases with q.*

Lemma 2 states that when every newly produced vintage results in a larger improvement compared to the previous one, in steady state the quality ratio between any two vintages increases and so does the cross-sectional variance of machine qualities. Hence, q not only drives the speed of embodied technical change but also the cross-sectional dispersion of available technologies.

Lemma 3 *Hours work n^0 in the newest vintage of machines depend on the marginal utility of consumption μ, but do not depend on the speed of technical change q.*

Lemma 3 comes directly from the normalization of the technology of each vintage.

Lemma 4 *Hours work n^* in the marginal machine depend neither in the speed of technical change q nor in the marginal utility of consumption μ.*

The intuition for Lemma 4 is quite simple. Hours worked in each machine increase with the marginal product of labor, whereas the participation rate increases with total output produced in the marginal machine. The quality of the marginal machine, and hence the speed of technical change q, does not affect hours worked in the marginal machine because under the Cobb-Douglas production function both the marginal product of labor and total output increase with capital at the same rate. The marginal utility of consumption raises equally the value of the output produced in all machines. This increases hours worked in all machines and it also lowers the quality of the marginal machine. Both effects cancel out because the marginal utility of consumption affects both margins equally.

With these results in place, we can state our main proposition

Proposition 5 *An increase in q generates:*

(a) A fall in consumption c

(b) An increase in the quality gap e^{qp} between the top and marginal machines.

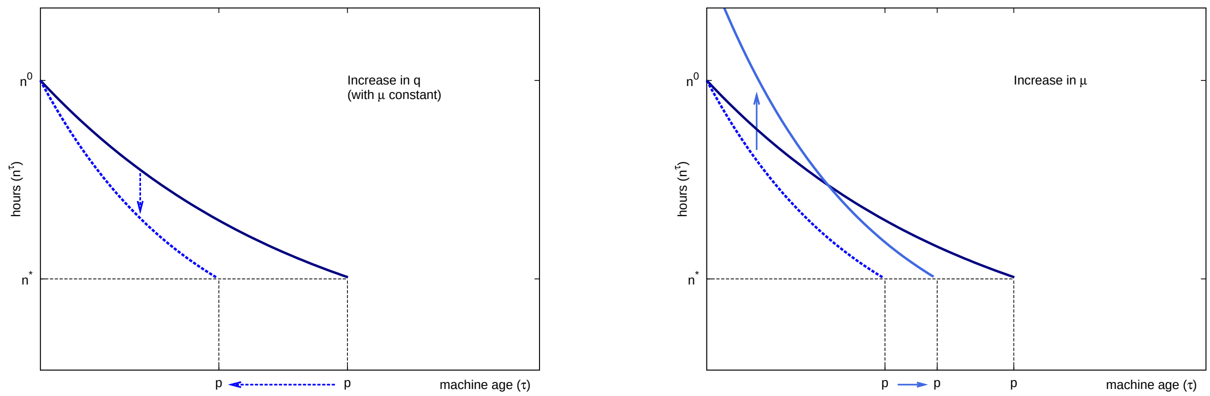
(c) A fall in the fraction of workers p

(d) An increase in hours per worker n

The intuition for part (a) is as follows. When q increases, capital quality relative to the frontier, $k^{\tau} = e^{-q\tau}$, falls in every machine. According to the aggregate resource constraint, if p and n^{τ} do not increase, detrended aggregate output and consumption must fall. The only way output and consumption could go up would be by a large enough increases in the number p of machines used or by a large enough increase in hours worked per machine n^{τ}. But neither of them can happen simultaneously with an increase in consumption (and a fall in its marginal utility μ). The hours equation (4) shows that when q increases, if consumption c goes up, then n^{τ} can only fall. Likewise the participation equation (5) shows that when q increases, if consumption c goes up, participation must fall. The result (b) comes directly from the fall in consumption. With the increase in the value of income μ the participation equation (5) shows that the quality of the marginal machine in use should go down.

Finally, results (c) and (d) come from lemma 3. Notice that the increase in q decreases hours and participation through the substitution effect. At the same time, through the fall in consumption, the increase in q produces an income effect that would lead to increases hours and participation. The trade-off between the extensive and intensive margins comes from the fact that the substitution effect is increasing with the age of machines and it is maximum at the marginal machine, whereas the income effect is the same in all machines. Hence, the fall in quality at the marginal machine is much larger than the fall in consumption (which comes from the average fall in quality through all machines), and then the participation equation (5) leads to a fall in participation and the scrapping of the worse machines. Instead, hours per machine increase in the newest machines because the fall in quality is much lower than the fall in consumption. Therefore, the income effect of the consumption fall generates an increase in hours per worker.

Figure 1: Hours worked per machine age



The solid line in the left panel represents hours worked in machines of different ages. An increase in q while holding consumption constant lowers hours in all machines, more so at older ones. Participation p also falls. The right panel shows the increase in hours worked due to a fall in consumption. The fall in consumption also increases participation.

Figure 1 gives a graphical representation of the economy and the change in q. The solid line in the left panel represents hours worked in each machine vintage. Hours are maximum at $\tau = 0$ and fall exponentially with age until the lowest hours at the oldest machine in use, which is of age p. An increase in q while holding consumption constant generates a fall in hours in all machines except in the newest vintage. Hours in the marginal machines also remain constant because the marginal machine becomes younger as the economy scraps previously used machines. The right panel shows the movement of hours once we let consumption fall. The whole schedule moves upwards: overall, hours in the newest machines increase and hours in the oldest ones fall. Overall, hours per worker fall.

2.4 The model without the assignment friction

In this Section we show that the assignment friction is essential to generate opposite movements in the extensive and intensive margins of labor supply upon an acceleration of embodied technical change. To see this, we look at two different cases where the assignment friction is absent.

Let's assume that production can be described by a representative firm that employs all machines as perfect substitutes and combines them with the labor services of all workers. For consistency with the balanced growth path, the production function $f(K, L)$ is Cobb-

Douglas with capital share α. The aggregate stock of capital is given by,

$$K = \int_0^\infty k^\tau d\tau = \int_0^\infty e^{-q\tau} d\tau = \frac{1}{q}$$

where we are using the fact that without the assignment friction there is no reason to scrap old machines and hence all of them are used. Then, all workers are aggregated into labor:

$$L = np$$

Notice that there is no difference in production between adding an extra hour or adding an extra worker. Finally, the aggregate resource constraint is given by,

$$c \leq f(K, L) = f\left(\frac{1}{q}, np\right)$$

With the same objective function as in Section 2.2 we obtain the FOC

$$\begin{aligned} p\, f_2\left(\frac{1}{q}, np\right)\frac{1}{c} &= v'(n) \\ n\, f_2\left(\frac{1}{q}, np\right)\frac{1}{c} &= v(n) \end{aligned}$$

With a Cobb-Douglas production function, hours n and number of workers p do not depend on q and hence changes in q produce no change in labor supply. To see this, substitute consumption by the resource constraint:

$$\begin{aligned} \frac{1}{n}(1-\alpha) &= v'(n) \\ \frac{1}{p}(1-\alpha) &= v(n) \end{aligned}$$

where $(1-\alpha)$ is the labor share parameter in the Cobb-Douglas production function. The reason for this result is standard. An increase in q changes the average quality of aggregate capital. This exerts income and substitution effects on labor supply. The former through the increase in the total output and the latter through the increase in the marginal product of labor. These two effects cancel out under log preferences.

A less extreme case would be to consider that not all machines ever produced are in use. Instead, we keep our assumption that one needs as many machines as workers, and hence the oldest machines are scrapped. We still keep away from the assignment friction and hence we think of a representative firm lumping together all machines and all workers.

In this case, aggregate capital would be given by:

$$K = \int_0^p k^\tau d\tau = \int_0^p e^{-q\tau} d\tau = \frac{1}{q}\left[1 - e^{-qp}\right]$$

and aggregate labor L would be as above. As before, p and n affect aggregate labor equally. However, they now affect aggregate capital differently: the extensive margin adds machines (albeit the worst ones) and hence it increases the capital stock, whereas the intensive margin does not. Finally, the aggregate resource constraint is given by,

$$c \leq f(K, L) = f\left(\frac{1}{q}\left[1 - e^{-qp}\right], np\right)$$

and the new FOC:

$$\begin{aligned} p\, f_2\left(\frac{1}{q}\left[1 - e^{-qp}\right], np\right)\frac{1}{c} &= v'(n) \\ e^{-qp} f_1\left(\frac{1}{q}\left[1 - e^{-qp}\right], np\right)\frac{1}{c} + n\, f_2\left(\frac{1}{q}\left[1 - e^{-qp}\right], np\right)\frac{1}{c} &= v(n) \end{aligned}$$

Now the first order condition for participation takes into account that by adding an extra worker we will be using an extra machine and hence increasing the capital stock by an amount given by the quality of the marginal machine. Substituting again by the aggregate constraint we get:

$$\frac{1}{n}(1 - \alpha) = v'(n) \tag{6}$$

$$\alpha\frac{k^*}{K} + \frac{1}{p}(1 - \alpha) = v(n) \tag{7}$$

Equation (6) determines hours per worker and it is independent of q for the same reasons as above. Equation (7) determines participation. It shows that the return to the extra worker depends on the ratio of machine qualities between the marginal machine and the aggregate capital stock. This ratio is given by,

$$\frac{k^*}{K} = \frac{q}{e^{qp} - 1}$$

an expression that falls with q as argued in the proof of part (c) of Proposition 5. Hence, an increase in q decreases the value of an extra worker because it comes with a worse machine, and hence it reduces participation.

3 Decentralization

A decentralized economy requires machines and workers to meet and take decisions with the matching between them and the price for labor determined in equilibrium. We have just seen that the optimal allocation gives the same consumption to all workers but requires different work effort from them. In order to decentralize this solution we need some way to insure workers. We opt to do it through saving and borrowing by infinitely-lived workers that trade labor services in a spot market.

3.1 Firms

At any point in time t, a firm with a machine of age τ with quality $\tilde{k}_t^\tau$ is paired with a worker who supplies n_t^τ hours of work to produce output according to a Cobb-Douglas production function f with capital share α. The firm has to pay the worker a market compensation that can depend on hours worked. The optimal demand for hours is then chosen by solving:

$$\tilde{\pi}_t\left(\tilde{k}_t^\tau\right) = \max_{n_t^\tau}\left\{f\left(\tilde{k}_t^\tau, n_t^\tau\right) - \tilde{w}_t(n_t^\tau)\right\}$$

where $\tilde{\pi}_t$ denotes firm profits and $\tilde{w}_t(n_t^\tau)$ is the compensation for a worker that supplies n_t^τ units of labor. The first order condition is given by

$$e^{\alpha q t} f_2\left(k^\tau, n_t^\tau\right) = \tilde{w}_t'\left(n_t^\tau\right) \tag{8}$$

This optimality condition defines a labor demand function,

$$n_t^\tau = \tilde{\phi}_t\left(k^\tau\right)$$

which establishes that the amount of hours worked in every machine type depends on the production function and on the wage function.

Recall that machines are in excess supply and that there is a critical level of capital quality $\tilde{k}_t^*$ such that all machines of smaller quality are scrapped. Free entry must yield zero profits to operating this machine and hence the wage paid to the worker in the worst machine must satisfy:

$$\tilde{w}_t\left(n_t^*\right) = e^{\alpha q t} f\left(k^*, n_t^*\right) \tag{9}$$

where n_t^* are the efficiency units of labor supplied by a worker matched to the worst machine in operation, which since they satisfy equation (8), can be written as $n_t^* = \tilde{\phi}_t\left(k^*\right)$.

3.2 Matching

The equilibrium of the economy has to specify how workers and machines are matched together. No party should have incentive to deviate from the equilibrium matching and market should clear. To model the matching process we define two objects. Let $p_{t,i}$ denote the time t probability that worker i participates in the labor market and p the aggregate fraction of workers participating. Of course, in equilibrium:

$$\int_{[0,1]} p_{t,i} di = p \tag{10}$$

Also let $\varphi_{t,i}(k)$ denote the probability that, conditional on participating in the labor market, worker i is matched with a machine of de-trended quality smaller than or equal to k. Clearly, in equilibrium $\varphi_{t,i}$ must be zero for any $k < k^*$ and it has to satisfy the condition that all machines of quality $k \geq k^*$ are in use, which implies that

$$\int_{[0,1]} p_{t,i}\, \varphi_{t,i}(k)\, di = p \int_{k^*}^{k} g(s)\, ds \qquad \forall k \geq k^* \tag{11}$$

where $g(s)$ is the density function of machine qualities described in Lemma 1. The right hand side of equation (11) gives the number of (non-scrapped) machines of quality equal to or less than k. The left hand side gives the expected number of workers assigned to machines of type equal to or less than k. Since there is a continuum of workers the expectation is equal to the average.

We model the matching process as stochastic for convenience. Since workers are infinitely lived and there are no borrowing constraints, this is without loss of generality due to a law of large numbers.[2] In the continuous time framework, one can always interpret $p_{t,i}$ as the fraction of time within a time interval that the worker chooses to work, and $\varphi_{t,i}(k)$ as the fraction of time within a time interval in which the worker is allocated to machines of quality k or below.

3.3 Workers

Individuals maximize the present discounted value of their utility:

$$\max_{\tilde{c}_{t,i}, n_{t,i}, p_{t,i}} \int_0^\infty e^{-\rho t} \left[\log \tilde{c}_{t,i} - v(n_{t,i})\right] dt$$

[2] We could have instead defined the matching process as a collection of deterministic functions specifying the machine assigned to each worker from time 0 to ∞.

where $\rho > 0$ is the subjective time discount rate, subject to the sequence of budget constraints:

$$\dot{\tilde{b}}_{t,i} = \tilde{w}_{t,i}\left(n_{t,i}\right) - \tilde{c}_{t,i} + r_t \tilde{b}_{t,i} \tag{12}$$

where $\tilde{b}_{t,i}$ are assets, $\tilde{w}_t\left(n_{t,i}\right)$ denotes labour income when supplying $n_{t,i}$ working hours in the market, and r_t is the interest rate. When the worker is not participating in the labor market $n_{t,i}$ and $\tilde{w}_t\left(n_{t,i}\right)$ are both equal to zero. All workers start with the same wealth b_0 and they can choose how much to consume and save every period as well as how many hours of work they supply in the market. There are no liquidity constraints and the consumption good is the numeraire.

By solving the worker's problem we obtain the standard Euler equation for the consumption path,

$$\frac{\dot{\tilde{c}}_{t,i}}{\tilde{c}_{t,i}} = r - \rho \tag{13}$$

where $\dot{\tilde{c}}_{t,i}$ denotes the time derivatives, and the intratemporal condition for labor supply,

$$v'\left(n_{t,i}\right) = \frac{1}{\tilde{c}_{t,i}} \tilde{w}_t'\left(n_{t,i}\right). \tag{14}$$

Moreover it has to be the case that

$$v\left(n_{t,i}\right) \leq \frac{1}{\tilde{c}_{t,i}} \tilde{w}_t\left(n_{t,i}\right), \tag{15}$$

for workers to choose $p_{t,i} > 0$. Whenever the inequality is strict $p_{t,i} = 1$, and whenever it holds as equality the worker is indifferent at time t about the participation probability.

3.4 Financial markets

Firms are owned by workers. In particular, workers own shares $s_{t,i}$ of the diversified portfolio of firms, which entails the payment of aggregate firm profits,

$$\Pi_t = p \int_{k^*}^{1} \tilde{\pi}_t\left(e^{qt} s\right) g\left(s\right) ds$$

Let $\tilde{\mathbf{p}}_t$ denote the price of equity shares at time t. The amount of wealth $\tilde{b}_{t,i}$ of worker i at time t is given by,

$$\tilde{b}_{t,i} = \tilde{\mathbf{p}}_t s_{t,i}$$

Since there are no borrowing constraints in place $s_{t,i}$ can be negative, that is, short selling is allowed. Of course, in equilibrium

$$\int_{[0,1]} \tilde{b}_{t,i} di = \tilde{\mathbf{p}}_t \tag{16}$$

Since all workers start up with the same financial wealth b_0 it means they all start with the same share of firm ownership s_0. The interest rate in the budget constraint (12) is given by the dividend flow and the capital gains,

$$r_t = \frac{\dot{\tilde{\mathbf{p}}}_t}{\tilde{\mathbf{p}}_t} + \frac{\Pi_t}{\tilde{\mathbf{p}}_t} \tag{17}$$

and integrating this expression, the equity price $\tilde{\mathbf{p}}_t$ is equal to the discounted flow of profits:

$$\tilde{\mathbf{p}}_t = \int_t^{\infty} e^{-\int_0^s (r_u - r_t) du} \Pi_s ds \tag{18}$$

3.5 Balanced growth path equilibrium

We focus the analysis on the balanced growth path equilibrium. All time periods are identical and we shall see that in steady state output, consumption, assets and the price of equity shares grow at the constant rate $x = \alpha q$, hours and the participation rate p are constant and the interest rate is given by the modified golden rule, $r = \rho + x$. To characterize the steady state we consider the de-trended variables $c \equiv e^{-xt}\tilde{c}_t$, $b \equiv e^{-xt}\tilde{b}_t$ and $\mathbf{p} \equiv e^{-xt}\tilde{\mathbf{p}}_t$, and the de-trended functions $w(n) \equiv e^{-xt}\tilde{w}_t(n_t)$, $\phi(k) \equiv \tilde{\phi}_t(k)$, and $\pi(k^\tau) \equiv e^{-xt}\tilde{\pi}\left(\tilde{k}_t^\tau\right)$.

Definition 1 *A balanced growth path equilibrium for this economy is characterized by a price of equity shares* $\mathbf{p}$, *a wage function* $w(n)$, *individual participation probabilities* $p_{t,i}$, *assignment functions* $\varphi_{t,i}(k)$, *an aggregate participation rate* p, *an interest rate* r *and individual consumption, saving and working plans and firm labor demands* $\phi(k)$ *such that*

(a) Workers solve their optimization problem, that is, equations (12), (13), (14), and (15) are satisfied.

(b) Firms solve their optimization problem, that is, equation (8) is satisfied,

(c) The free entry condition (9) in production is satisfied,

(d) The labor market clears, that is, equations (10) and (11) hold,

(e) The capital market clears, that is, equation (16) holds,

(f) Aggregate consumption and output grow at the same constant rate,

(g) Individual consumption is identical across workers,

(h) By Walras law, the goods market clears; that is, aggregate output is equal to aggregate consumption.

Notice that to characterize the balanced growth path equilibrium we need to characterize simultaneously the wage function and the assignment function. We conjecture the following wage function,

$$w(n) = \begin{cases} a_0 + a_1 \frac{n^{1+\eta}}{1+\eta} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{19}$$

and an assignment such that workers are allocated to different machines during their working life with the constraint that all workers obtain the same permanent income. In equilibrium workers matched with better machines should work longer hours. Since all workers are identical, this can be sustained in equilibrium only if workers obtain the same permanent income. To understand why this has to be the case, argue by contradiction and consider an assignment that puts worker i more often than worker j to work with a good machine. Then worker i will have greater labor income and greater consumption that will disincentive the supply of hours of worker i relative to worker j through the income effect. But then the firm that hires worker i would make greater profits by hiring worker j instead, because he would be willing to supply the same amount of hours as worker i would do but at a cheaper price. As we will see below, the conjectured wage function makes workers indifferent about how many hours to work at any given point in time and hence workers matched to better machines are ready to supply more hours of work. There are many different assignments that can satisfy our conjecture, all yielding the same consumption path and the same lifetime utility for all workers. Without loss of generality, we will focus on symmetric equilibria.[3]

Definition 2 *A symmetric balanced growth path equilibrium is a balanced growth path equilibrium in which, at every time period t*

[3]An alternative way of decentralizing the allocations of the first best would be to allow for complete markets, that is to say, to allow workers to write contracts contingent on the outcome of the assignment and participation lotteries. This would make that at any given period of time all workers had the same level of wealth. In our formulation workers will differ in their level of wealth at any period of time because of their different assignment trajectories so far.

(a) The participation probability $p_{t,i}$ is the same for all individuals,

(b) The assignment function $\varphi_{t,i}(k)$ is the same for all individuals

Note that if the participation probabilities and the assignment functions have to be the same for all workers, then equations (10) and (11) imply

$$p_{t,i} = p \qquad \text{and} \qquad \varphi_{t,i}(k) = \varphi(k) = \int_{k^*}^{k} g(s)\, ds \quad \forall k \geq k^*$$

which makes clear that the individual participation probabilities and the individual assignment functions must be independent of time.

3.6 Equilibrium properties

To understand the nature of the equilibrium, let's have a look at the worker's problem. The Euler equation (13) and the balanced growth path condition determine the interest rate as the modified golden rule,

$$r = \rho + x \tag{20}$$

Integrating the Euler equation (13) and using (20) gives us the standard condition for the consumption path:

$$\tilde{c}_{t,i} = c_{0,i} e^{xt} \quad \Rightarrow \quad c_{t,i} = c_{0,i} \tag{21}$$

Now, we can replace the disutility of work (2), the wage function (19) and the optimal consumption path (21) into the condition for optimal labor supply (14) to obtain,

$$\lambda_1 = \frac{a_1}{c_{0,i}} \tag{22}$$

This equation tells that, at any given point in time, all workers are indifferent about the amount of hours they work because the value of an extra unit of time spent in leisure and the value of an extra unit of time of work grow at the same rate as the amount of working time increases. When paired to a good machine a worker experiences a high utility cost of giving up scarce time for leisure, but he is paid accordingly to compensate for these extra hours. Hence, work effort in a given period is undetermined as in Prescott, Rogerson, and Wallenius (2006). However, this does not mean that lifetime work effort is undetermined: given a market price a_1, equation (22) determines $c_{0,i}$ and this puts a constraint in lifetime work effort through the permanent income (see equation (52) below). Notice also that this condition is identical for all individuals, which implies that all individuals consume the same amount,

$$c_{t,i} = c_{0,i} = c_0 \tag{23}$$

We focus the analysis on the case where the participation rate is positive but strictly less than one, $p \in (0,1)$. This implies that (15) holds as an equality, which after using (19) and (22), yields

$$\lambda_0 = \frac{a_0}{c_0} \tag{24}$$

which says that the fixed utility cost of entering the labour market is equal to the utility gain of participating to the labour market and supplying zero units of labour. Again, at any given point in time the worker is indifferent between going to work or not.

Combining equations (22) with (24) we obtain that the two wage parameters are linked in equilibrium:

$$a_0 = \frac{\lambda_0}{\lambda_1} a_1 \tag{IE}$$

Finally, note that since $c_{0,i}$ is the same across individuals (see equation 22) so is the present value of income. In addition, given the functional form of the equilibrium wages (19), so is the present value of the discounted sum of disutility of working. Hence, all workers get the same utility regardless of the equilibrium working history that they are allocated to.

3.7 Equivalence

Finally, we need to show that the allocations generated by this equilibrium correspond to the ones in Section 2.

Proposition 6 *The allocations of the conjectured balanced growth path equilibrium in Definition 1 also solve the large household problem.*

3.8 The effects of an increase in q

Due to the equivalence result, the changes in allocations in the decentralized economy will be as in the large household problem. Since detrended consumption falls, equations (22) and (24) say that so will the wage parameters a_0 and a_1. It remains to be seen what happens with labor income inequality.

Let's call $\mathbf{LI}_t$ the ratio between the labor income in the top machine and in the marginal machine at time t,

$$\mathbf{LI}_t = \frac{\tilde{w}_t\left(n_t^0\right)}{\tilde{w}_t\left(n_t^*\right)}$$

This is a measure of income inequality, and it is very easy to see that it increases with q. The basic ideas is that, as seen in Proposition 5, an increase in q increases the quality gap between the top and bottom machines, which commands and increase in the dispersion of hours worked and hence of compensation.

Proposition 7 *When q increases labor income inequality as measured by **LI** increases.*

4 Heterogeneity in workers and the matching problem

We now analyze the model where workers differ in their skills and possibly in their non labor income. We assume that there are N types of workers with skill level $h_i > h_{i+1}$. We normalize h_1 to one. The mass of type i workers is $z_i \in (0,1)$ so that $\sum_{i=1}^{N} z_i = 1$. We assume that workers with human capital h_i supply efficiency units of labour according to

$$e = h_i^{1-\theta} n^{\theta}, \quad i = 1, 2, \ldots, N.$$

This specification allows the existence of a steady state with constant growth. To allow for differences in non-labor income we assume that workers of different types may differ in their initial wealth $b_{0,i}$. Finally, for quantitative reasons, we add two new elements to the model. First, following Hornstein, Krusell, and Violante (2007), we allow for machine qualities to depreciate over time at the rate δ, so that the detrended quality of a machine of age τ is given by $k^{\tau} = e^{-(q+\delta)\tau}$. Second, the measure m of new machines entering the economy every period will be different from one, so that the density of available machines of any age is given by $1/m$.

4.1 The conjectured BGP equilibrium

We conjecture an equilibrium where:

(a) The assignment of workers to machines is such that there is assortative matching: the best machines are given to the workers with skill h_1, then the best machines left are assigned to workers of skill group h_2, and so forth.

(b) Within the same skill type, the allocation of machines to workers requires a balanced rotation of workers between machines such that all workers of the same type obtain the same permanent income.

(c) workers of different skills are offered different wage schedules given by

$$w\left(h_i, n\right) = \begin{cases} a_{0i} + a_{1i}\frac{n^{1+\eta}}{1+\eta}, & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \tag{25}$$

where a_{0i} and a_{1i} are shorthands for $a_0\left(h_i\right)$ and $a_1\left(h_i\right)$.

Without loss of generality, we will focus on the symmetric equilibrium such that all workers of a given type face the same participation probability and the same assignment function. Hence, i will denote worker type, not individual. We will see in Section 4.6 that this equilibrium configuration requires that the skill differences between worker types are large enough compared to their consumption differences. This will happen whenever (a) the skill differences between worker types are large enough compared to the differences between machine qualities, or (b) there is some redistribution in the economy, that is to say, the share of consumption not coming from labor income is higher for low skilled workers.

To describe the assignment with more precision we introduce some more notation. Let p_i denote the participation probability of workers of type i. Then the number of machines assigned to workers of type i is given by $p_i z_i$. The maximal duration of a machine operated by workers of type 1 will be given by

$$\tau_1^* = \frac{p_1 z_1}{m}$$

while workers of type i will operate machines with age in the interval $\left[\tau_{i-1}^*, \tau_i^*\right]$ with

$$\tau_i^* = \frac{\sum_{j=0}^{i} p_j z_j}{m} = \tau_{i-1}^* + \frac{p_i z_i}{m} \tag{26}$$

where we define $\tau_0^* = 0$ and $p_0 z_0 = 0$. Let's define k_i^* as the quality of the worst machine assigned to workers of type i. It is easy to prove that

Lemma 8 *For type i workers, the distribution of detrended qualities of operating machines has support $\left[k_i^*, k_{i-1}^*\right] = \left[e^{-(q+\delta)\tau_i^*}, e^{-(q+\delta)\tau_{i-1}^*}\right]$ and it is log-uniform with density $g_i\left(k\right) = \frac{m}{(q+\delta)p_i z_i}\frac{1}{k}$*

Finally, let's characterize the equilibrium matching function. Let's define $\varphi_i\left(k\right)$ as the *cdf* that determines, for a worker of type i, the probability of being matched with a machine of de-trended quality k or less conditional on being selected to work. This *cdf* has to be zero for $k < k_i^*$ and $k > k_{i-1}^*$. Moreover it has to satisfy that all machines of quality $k_{i-1}^* > k \geq k_i^*$ are in use by workers of skill type i. Hence,

$$\varphi_i(k) = \int_{k_i^*}^{k} g_i(s)\, ds \qquad \forall k \in [k_i^*, k_{i-1}^*] \tag{27}$$

4.2 Firms

A firm with capital k paired with a worker with human capital h_i will choose its demand of hours by solving,

$$\pi(k, h_i) = \max_n \left\{ k^\alpha \left(h_i^{1-\theta} n^\theta \right)^{1-\alpha} - w(h_i, n) \right\}$$

After rearranging, this gives the following demand function for hours:

$$n = \phi(k, h_i) = \left[\frac{(1-\alpha)\,\theta k^\alpha h_i^{(1-\alpha)(1-\theta)}}{a_{1i}} \right]^{\frac{A}{1+\eta}} \tag{28}$$

with

$$A = \frac{(1+\eta)}{1 - (1-\alpha)\,\theta + \eta} > 1.$$

Note that the optimal demand of hours is increasing with the worker's skills h_i and decreasing with the worker's cost of compensating hours, a_{1i}. A given firm with capital k will demand more hours from a higher skilled-worker if the former compensates the latter. That is to say,

$$\phi(k, h_i) > \phi(k, h_{i+1}) \quad \Leftrightarrow \quad \left(\frac{h_i}{h_{i+1}} \right)^{(1-\alpha)(1-\theta)} > \frac{a_{1i}}{a_{1i+1}} \tag{29}$$

Of course, if the relation (29) holds, a firm with capital k would produce more output when hiring a higher skilled worker. However, a necessary condition for output to be increasing in worker quality is weaker, and it states that only efficiency units of labor need to be increasing with worker type. Hence,

$$y(k, h_i) > y(k, h_{i+1}) \quad \Leftrightarrow \quad \left(\frac{h_i}{h_{i+1}} \right)^{(1-\alpha)(1-\theta)} > \left(\frac{a_{1i}}{a_{1i+1}} \right)^{\frac{A-1}{A}} \tag{30}$$

4.3 Free entry

In equilibrium we must have that

$$\pi\left(k_i^*, h_i\right) = \pi\left(k_i^*, h_{i+1}\right), \ \forall i \geq 1 \tag{31}$$

and that

$$\pi\left(h_N, k_N^*\right) = 0 \tag{32}$$

The first condition says that at the critical technological gap τ_i^* a firm should be indifferent between hiring a type i worker or type $i+1$. The second condition says that at the critical technological gap τ_N^* a firm should make zero profits. This last is really a free entry condition that arises because in the model there is an excess supply of machines relative to workers.

4.4 Aggregates

Let Y_i denote the average output produced by workers of type $i = 1, 2, \ldots, N$ at any given point in time, i.e.

$$Y_i = p_i \int_{k_i^*}^{k_{i-1}^*} y(s, h_i) g_i\left(s\right) ds \qquad i = 1, 2, \ldots, N$$

Notice that this quantity is not multiplied by z_i. So Y_i denotes average output per worker of type i and hence aggregate output Y in the economy is given by

$$Y = \sum_{i=1}^{N} z_i Y_i$$

Since optimal output $y\left(k, h_i\right)$ is equal to

$$y\left(k, h_i\right) = \left[\frac{\left(1-\alpha\right)\theta}{a_{1i}}\right]^{A-1} h_i^{(1-\alpha)(1-\theta)A} k^{\alpha A} \tag{33}$$

one obtains a closed form expression for Y_i:

$$Y_i = p_i \left[\frac{\left(1-\alpha\right)\theta}{a_{1i}}\right]^{A-1} h_i^{(1-\alpha)(1-\theta)A} E_i\left[k^{\alpha A}\right] \tag{34}$$

where

$$
\begin{aligned}
E_i\left[k^{\alpha A}\right] &= \int_{k_i^*}^{k_{i-1}^*} s^{\alpha A} g_i(s)\, ds = \frac{m}{\alpha A(q+\delta)\, p_i z_i}\left(k_{i-1}^{*\,\alpha A} - k_i^{*\,\alpha A}\right) \\
&= \frac{m}{\alpha A(q+\delta)\, p_i z_i}\left(1 - e^{-\alpha A(q+\delta)\frac{p_i z_i}{m}}\right) e^{-\alpha A(q+\delta)\tau_{i-1}^*}
\end{aligned}
$$

denotes the output part that comes from the machine qualities assigned to workers of type i.

Likewise, let Π_i denote average firm profits generated by workers of type i and hence aggregate profits are equal to

$$
\Pi = \sum_{i=1}^{N} z_i \Pi_i \tag{35}
$$

Since firm profits are equal to

$$
\pi(k, h_i) = \frac{1}{A} y(k, h_i) - a_{0i} \tag{36}
$$

we have that

$$
\Pi_i = p_i \int_{k_i^*}^{k_{i-1}^*} \pi(s, h_i) g_i(s) = \frac{1}{A} Y_i - a_{0i} p_i, \qquad i = 1, 2, \ldots, N \tag{37}
$$

Finally, let L_i denote the average labor income obtained by workers of type i. Then, given (25) and (28) we can write:

$$
L_i = p_i \int_{k_i^*}^{k_{i-1}^*} w(h_i, \phi(s, h_i))\, g(s)\, ds = a_{0i} p_i + \frac{(1-\alpha)\theta}{(1+\eta)} Y_i \qquad i = 1, 2, \ldots, N \tag{38}
$$

Notice that (38) together with (37) immediately imply that $L_i + \Pi_i = Y_i$.

4.5 Workers

Now, let's turn to the problem of a given household i. Following derivations analogous to the one-type model of Section 3, the intensive and extensive margin first order conditions are given by:

$$
\lambda_1 = \frac{a_{1i}}{c_{0,i}} \qquad i = 1, 2, \ldots, N \tag{39}
$$

$$
\lambda_0 = \frac{a_{0i}}{c_{0,i}} \qquad i = 1, 2, \ldots, N \tag{40}
$$

The first equation states that all workers are indifferent about the amount of hours they work at a given point in time. The second equation says that the utility gains of participating to the labour market and supplying zero units of labour compensate the worker for the fixed cost of entering the labour market. We have a strict equality because we focus the analysis on the case where the participation rate for any type of workers is positive but strictly less than one, $p_i \in (0, 1)$. By combining (39) with (40) we obtain that

$$a_{0i} = \frac{\lambda_0}{\lambda_1} a_{1i} \qquad i = 1, 2, \ldots, N \tag{IEH}$$

We can think of equation (39) determining consumption for workers of type i given the wage parameter a_{1i}. As in the one type model, we can write consumption of type i workers as being equal to permanent income,

$$c_i = \rho \left[b_{0i} + \int_0^\infty e^{-\rho t} \left(p_i \int_{k_i^*}^{k_{i-1}^*} w\left(h_i, \phi\left(s, h_i\right)\right) d\varphi_i\left(s\right) \right) dt \right]$$

and using the matching function (27)

$$c_i = \rho b_{0i} + p_i \int_{k_i^*}^{k_{i-1}^*} w\left(\phi\left(s, h_i\right), h_i\right) g\left(s\right) ds$$

Note that the second term in the right hand side tells us that the present value of labor income for workers of type i is equal to the cross-sectional average of labor income generated by workers of this same type. In particular, this second term is equal to L_i in equation (38). Let's denote by μ_i, $i = 1, 2, \ldots, N$, the share of aggregate profits appropriated by a worker of type i. Of course it will have to be the case that

$$\sum_{i=1}^{N} z_i \mu_i = 1 \tag{41}$$

Then equation (17) and the balanced growth path conditions imply that $\rho b_{0,i} = \mu_i \Pi$. Hence, we can write,

$$c_i = \mu_i \Pi + L_i \qquad i = 1, 2, \ldots, N$$

This tells us that consumption for workers of group i is equal to their labor income plus their share of aggregate profits.

4.6 Verifying the equilibrium

To prove that the conjectured assignment is indeed an equilibrium we have to show that firms with capital of high quality are satisfied with hiring top workers and that they do not have incentives to deviate and hire a low skilled worker. That could happen if low skilled workers, because their consumption is lower, were ready to work long hours for small wages in such a way that this more than compensated their lower skills.

Proposition 9 below states that if the human capital ratio between skill types is large enough compared to their consumption ratio, then firms never have incentives to hire workers less qualified than the ones assigned to them in equilibrium because the disincentive effect of higher consumption of highly skilled workers is not large enough to undo their productive advantage.

Proposition 9 *$\pi(k, h_i) \geq \pi(k, h_{i+1})$ for $k \in [k_i^*, k_{i-1}^*]$ if and only if the following condition holds*

$$\left(\frac{h_i}{h_{i+1}}\right)^{(1-\alpha)(1-\theta)} > \left(\frac{c_i}{c_{i+1}}\right)^{\frac{A-1}{A}} \tag{42}$$

Note that this condition is equal to equation (30) —which guarantees that a firm produces more output with a higher skilled worker— after replacing a_{1i} by its equilibrium value in (39).

This condition is useful because it allows us to verify whether the conjectured assignment is indeed an equilibrium. However, since the consumption ratio is an equilibrium outcome itself, the condition is not very useful to gain intuition. To understand it better, let's look at a case without participation decision, $\lambda_0 = 0$ and *without redistribution.* We describe an allocation as *without redistribution* when the consumption ratio between worker types is equal to their ratio of permanent labor incomes. In our model this situation will arise if (a) profits are not given back to workers, $\mu_i = 0$, or (b) profits generated in machines allocated to workers of type i are given back to workers of the same type i, $\mu_i = \Pi_i/\Pi$. In this case we can express the consumption ratio as:

$$\frac{c_i}{c_{i+1}} = \left(\frac{h_i}{h_{i+1}}\right)^{(1-\alpha)(1-\theta)} \left(\frac{E_i\left[k^{\alpha A}\right]}{E_{i+1}\left[k^{\alpha A}\right]}\right)^{1/A}$$

Hence, better workers enjoy more consumption due to both their higher skills and the better machines they are allocated to. Now, equation (53) becomes,

$$\left(\frac{h_i}{h_{i+1}}\right)^{(1-\alpha)(1-\theta)} > \left(\frac{E_i\left[k^{\alpha A}\right]}{E_{i+1}\left[k^{\alpha A}\right]}\right)^{\frac{A-1}{A}}$$

This tells us that an equilibrium with assortative matching requires the skill advantage of workers of type i to be large enough to compensate the higher income that workers of type i obtain due to being allocated to better machines. If the dispersion of machine qualities tends to zero, assortative matching arises due to the standard complementarity between skill types and machine quality: higher skilled workers would always be paired with better machines because they work the same amount of hours —income and substitution effects due to h cancel out— and their skill advantage ensures that they supply more efficiency units of labor. Instead, if the dispersion of machine qualities is large compared to the differences in worker skills, then highly skilled workers suffer a negative income effect in working hours due to the rents of good machines that is not offset by their higher skills. In this situation for an assortative matching to be an equilibrium the economy requires some *redistribution*, that is to say, it requires that the consumption ratio between types be smaller than their labor income ratio:

$$\frac{c_i}{c_{i+1}} = \zeta \frac{L_i}{L_{i+1}} \quad \text{with } \zeta < 1$$

which in the model is achieved by setting $\mu_i > \Pi_i/\Pi$ for less skilled workers.

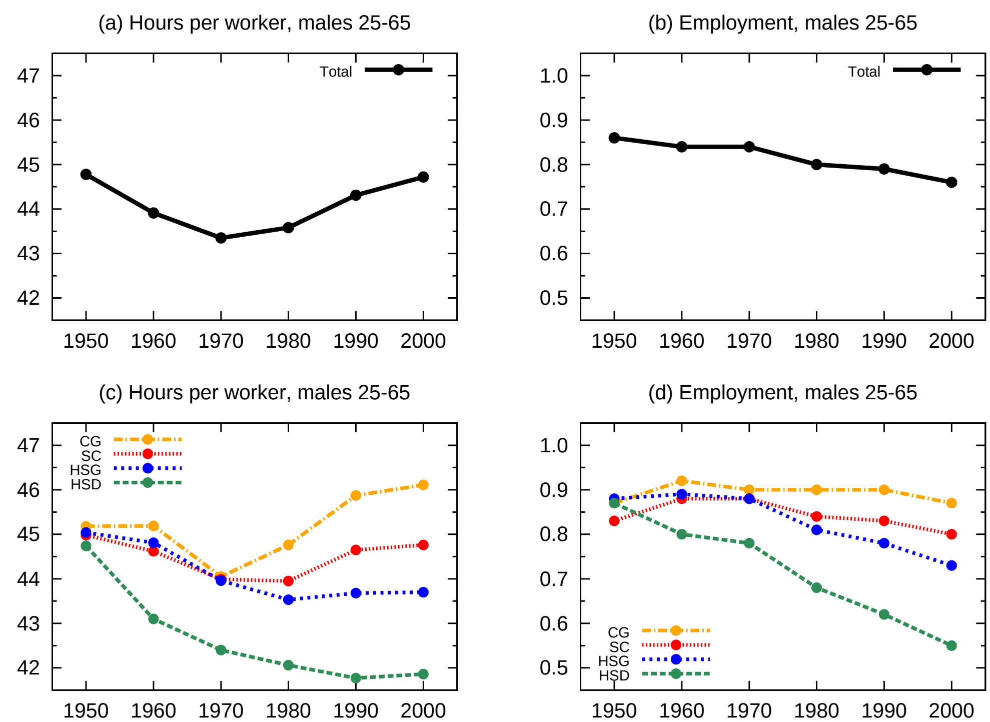
5 A quantitative exercise

There is much evidence that the pace of investment specific technological progress has sped up since the 1970's, see Greenwood and Yorokoglu (1997), Greenwood, Hercowitz, and Krusell (1997) and Violante (2002). At the same time, there have been important changes in the labor market. First, the aggregate participation rate for males has gone down while the hours per employed worker have gone up, see Juhn (1992), Aaronson, Fallick, Figura, Pingle, and Wascher (2006) and Michelacci and Pijoan-Mas (2012). We also document these patterns by use of data from the US Census.[4] We restrict the analysis to male workers aged between 25 and 64 years old. In Panels (a) and (b) in Figure 2 we report the series for hours per employed worker and the employment rate, respectively. Second, this process of intensive-extensive margin substitution masks important differences by education groups, see Juhn and Potter (2006) and Michelacci and Pijoan-Mas (2012). In Panels (c) and (d) in Figure 2 we report the hours per worker and the employment rates by education group. We clearly see that more educated workers increased hours the most and decreased participation the least; instead less educated workers see their

[4]We use the 1 percent sample of the decennial Census, as provided by the Integrated Public Use Microdata Series (IPUMS) at the University of Minnesota (www.ipums.org).

hours increase less or even fall while the fall in the employment rate is larger. And third, all these changes happened in a context of increasing wage inequality, both between and within education types, see for instance Katz and Autor (1999) or Heathcote, Perri, and Violante (2010).

Figure 2: Hours and Participation



To analyze the quantitative relevance of the mechanisms described in the previous sections, we use our model to quantify the importance of the acceleration of the speed of investment specific technological progress in accounting for these outcomes. Within our model, this corresponds to an increase in q. We also explore the role played by the change of the relative supply of workers by education type, as the share of college workers increased from 15% to 27% and the share of high school dropouts fell from 43% to 14% (see Table 1).

5.1 Calibration in 1970

We solve the model with 4 types, corresponding to different education groups: college graduates, workers with some college education but no college degree, high school graduates and high school drop outs. With four types we have a total of 18 independent parameters. We are going to set 8 of them directly and for the other 10 we will need to compute statistics within the model in equilibrium. In Table 2 there is a summary of parameter values and calibration targets.

Table 1: Population share by education groups

	1970	2000	Δ_{00-70}
College graduates	15	27	+12
Some college	11	28	+17
High school graduates	31	31	0
High school dropouts	43	14	-29

Note. All statistics are computed over population aged 25-65 from the 1970 U.S. Census.

5.1.1 Parameters set directly

We choose an annual discount rate ρ of 4% and a curvature parameter for the disutility of hours η of 2. These values are more or less standard. We set the depreciation rate δ equal to 6%.[5] Following Greenwood, Hercowitz, and Krusell (1997) we map the rate of growth of capital-embodied technical change, q in our model, to the rate of fall of the quality adjusted price of capital. Hornstein, Krusell, and Violante (2007) document that the quality adjusted price of capital fell at an average rate of 2% before the 70's and 4.5% in the late 90's. The value for m is chosen to match the average age of private fixed assets in the mid 60's of 11.5 years, as reported by the Bureau of Economic Analysis.[6] Note that the age of the oldest machine is given by p/m and the distribution of ages is uniform, hence the average age of machines in the economy is given by $\frac{p}{2m}$. Since p will be a calibration target (see below), m can be chosen directly.[7] The shares z_i of workers of each type are taken from the U.S. Census in 1970 corresponding to males aged 25-65.

5.1.2 Parameters set in equilibrium

We want the model to deliver in equilibrium a series of properties from the data. In particular, we want the model to reproduce the average participation rate of the economy,

[5] We take this value from the estimate of Nadiri and Prucha (1996)

[6] See Table 2.10 at http://www.bea.gov/national/FA2004/

[7] In any case, the value of m in the model is irrelevant, or in other words, machine ages are irrelevant. What matters is the spread of machine qualities, not the spread of ages. If we change m and hence average machine age, we can pick new values of α, θ and h_i such that the economy is unchanged. In particular, if we keep constant α/m, $(1-\alpha)\,\theta$ and $h_i^{(1-\alpha)(1-\theta)}$ all the relevant statistics of the model economy remain unchanged.

Table 2: Parameter values and calibration targets, 1970

Model parameter		Calibration target	
symbol	**value**	**Statistic**	**value**
preferences			
ρ	0.04	–	
η	2	–	
λ_0	0.65	average employment to population ratio	0.84
λ_1	6.88	average hours per employed person	43.3
technology			
δ	0.06	–	
q	0.02	rate of fall of price of investment goods	0.02
m	0.03652	average age of fixed assets (in years)	11.5
α	0.45	capital share	0.33
θ	0.64	income ratio between group 4 and 1	0.54
population			
z_2	0.11	population share of group 2	0.11
z_3	0.31	population share of group 3	0.31
z_4	0.43	population share of group 4	0.43
h_2	0.83	consumption for group 2 relative to group 1	0.84
h_3	0.75	consumption for group 3 relative to group 1	0.76
h_4	0.64	consumption for group 4 relative to group 1	0.68
μ_2	0.92	participation rate for group 2	0.88
μ_3	0.90	participation rate for group 3	0.88
μ_4	1.09	participation rate for group 4	0.78

Note. Group 1 refers to college graduates, group 2 refers to high school graduates with some college education, group 3 refers to high school graduates and group 4 to high school dropouts. All statistics are computed over population aged 25-65. Population shares, employment rates, hours per worker and income differences from the 1970 U.S. Census. Consumption from 1980 CEX.

the average hours per worker, the aggregate labor share and then differences in hours, participation, labor income and consumption between types of workers. Our empirical strategy is to choose as many moments from data as parameters we need to set. Hence, we will choose a subset of these moments and use the other ones as over-identifying restrictions to assess the model.

We choose λ_0 and λ_1 to match average participation and average hours per worker. We measure both quantities in our Census sample for 1970, looking at males aged 25-65. We obtain an employment ratio of 0.84 and 43.3 weekly hours per employed worker. We choose α to match the labor share of gdp, which we set to the standard value of 2/3.

Now, we have to determine h_i for three types, μ_i for three types and θ. We choose to match the participation rate for every education type, the consumption level of every education type relative to the consumption level of the highest type, and the labor income gap between the best and worst types. The problem with our calibration strategy is that we do not observe differences in consumption in 1970. The first year that we can use is 1980 with the CEX. For differences in labor income we have two options. The first option is to use the Census data for 1970. This option has the problem that the sampling and year are different from the ones we use for consumption. The second option is to use the CEX for 1980. This option makes the consumption and income data consistent but pays the cost that we impose into the model the labor income inequality of 1980 instead of 1970. We take the first choice.

5.1.3 Assessment

Columns 1 and 2 in Table 3 present the data for employment rates and hours worked in 1970 and how they changed in 2000. These numbers are the same used in Figure 3. In Column 3 we report the employment rates and hours per worker predicted by our model in 1970. The average employment rate, the average hours per worker and the employment rates by education groups are calibration targets and hence are matched perfectly. The hours per worker by education group are not targeted at all. We observe that the model rightly predicts that better educated workers work more hours. However, the model over predicts the difference in hours by education group: for instance, the difference in weekly hours between college workers and high school dropouts is 1.8 hours in the data and 7.4 hours in the model. Columns 1 and 2 in Table 4 report the data for the labor income gaps and consumption gaps between education types. Again, the consumption gaps are matched by design and so is the labor income gap between high school dropouts and college workers. However, the labor income gaps between the other education types are not targeted and the model reproduces them quite nicely.

We can use the calibrated model to measure how much of the college premium is due to direct skill advantage and how much due to the matching process allocating better workers to better machines. To do so, we solve our model with a very large m, which gives makes *de facto* all machines identical. We find that 70% of the hourly wage ratio between college workers and high school dropouts remains. Hence we conclude that, in 1970, 70% of the college premium was due to intrinsic differences in human capital, while the remaining 30% was due to access to better machines.

5.2 Calibration in 2000

We consider two different changes, separately and together. First, we increase q from 2% to 4.5% as discussed above. Second, we replace the population shares μ_i used in 1970 by the ones corresponding to 2000 (see Table 1)

5.3 Results

Table 3: Labor supply

	Data		Model			
Statistic	1970	Δ_{00-70}	1970	Δq	Δz_i	both
Participation rate	0.84	-0.08	0.84	-0.08	0.00	-0.08
College graduates	0.90	-0.03	0.90	-0.06	-0.05	-0.12
Some college	0.88	-0.08	0.88	-0.07	-0.03	-0.09
High school graduates	0.88	-0.15	0.88	-0.07	-0.01	-0.09
High school dropouts	0.78	-0.23	0.78	-0.09	-0.04	-0.14
Hours per worker	43.3	+1.5	43.3	+1.2	+0.1	+1.3
College graduates	44.1	+2.5	47.4	+2.0	0.0	+2.0
Some college	44.0	+1.0	46.7	+1.8	-2.2	-0.8
High school graduates	44.0	-0.2	44.2	+1.3	-3.3	-2.7
High school dropouts	42.3	-0.5	40.0	+0.4	-1.8	-1.7

In Column 4 of Table 3 we report the model predicted results when we increase q as in the 2000's. This increase in q allows us to characterize the observed acceleration in the pace of investment specific technological progress. The model predicts a fall in the participation rate of 8 percentage points as in the data. Likewise, the model predicts and increase of 1.2 weekly hours, just a bit short of the increase of 1.5 hours observed in the data. The intensive-extensive margin tradeoff that we analyzed in the previous sections is quantitatively consistent with the experience in the US labor market during the last decades. Furthermore, the slope by education predicted by the model is also consistent with the data. The model predicts higher increases in hours per worker for the more educated and lower falls in participation. In Column 4 of Table 4 we report the implications for labor income and consumption inequality. We see that the increase in q generates a small increase in labor income inequality, with the gap between high school dropouts and college workers falling from 0.54 to 0.49, which is one third of the fall observed in the data. This is a side effect of the matching friction: small differences in skill get amplified by an increase in the dispersion of machines quality. Finally, consumption inequality, increases very little, with the gap between high school dropouts and college workers falling from 0.68 to 0.66. This is because low skilled workers receives a substantial amount of non labor income in the form of transfers. This suggests that there is substantial redistribution in the US economy. According to Budría, Díaz-Giménez, Quadrini, and Ríos-Rull (2002) 25% of total income of low educated workers comes from transfers (data from Survey of Consumer Finances).

Table 4: Labor income and consumption

	Data		Model			
Statistic	1980	2000	1980	Δq	Δz_i	both
Average labor income						
College graduates	1.00	1.00	1.00	1.00	1.00	1.00
Some college	0.79	0.66	0.81	0.79	0.82	0.79
High school graduates	0.69	0.54	0.71	0.68	0.73	0.70
High school dropouts	0.54	0.39	0.54	0.49	0.56	0.50
Average consumption						
College graduates	1.00	1.00	1.00	1.00	1.00	1.00
Some college	0.84	0.77	0.84	0.83	0.85	0.84
High school graduates	0.76	0.68	0.76	0.74	0.78	0.76
High school dropouts	0.68	0.54	0.68	0.66	0.71	0.70

When we change the composition of the labor force, the aggregate participation and hours per worker are hardly affected (see Column 5 in Table 3). In spite of this, the model predicts substantial action by education group. In particular, it predicts falls in the participation and hours per worker in every education group. The reason for this is that all education groups get worse machines on average. The college educated, a minority in 1970, are now a large group so they have to rotate through a larger —and less privileged— pool of machines. By a cascade effect, this makes everybody else's machine relatively worse, and the situation becomes dramatic for the high-school dropouts because they become a minority in 2000, which means that they have to content themselves with the very worst jobs in the economy.

Finally, Column 6 of Tables 3 and 4 report the overall effects combined.

A Theorems and proofs

Lemma 1 *The distribution of detrended qualities of operating machines has support* $[e^{-qp}, 1]$ *and it is log-uniform with density* $g(k) = \frac{1}{qp}\frac{1}{k}$.

Proof: Notice first that the distribution of the age of operating machines is uniform with support $[0, p]$. The detrended quality of a machine of age τ can be expressed as $k^\tau = e^{-q\tau}$. This implies that the age of a machine, τ can be expressed as the ratio of the log of its quality and q: $\tau = -\frac{1}{q}\log k^\tau$. This can be used to express the *cdf* of machine qualities $G(k)$:

$$\begin{aligned} G(k) &\equiv \Pr\left(\tilde{k} \leq k\right) = \Pr\left(\log\tilde{k} \leq \log k\right) = \Pr\left(-\frac{\log\tilde{k}}{q} \geq -\frac{\log k}{q}\right) \\ &= 1 - \Pr\left(\tau \leq -\frac{\log k}{q}\right) = 1 + \frac{\log k}{qp} \end{aligned}$$

and hence, the *pdf* is given by,

$$g(k) = \frac{1}{qp}\frac{1}{k}.$$

We can easily check that the density integrates to one over the support of machine qualities $[e^{-qp}, 1]$:

$$\int_{e^{-qp}}^{1} g(s)\, ds = \int_{e^{-qp}}^{1} \frac{1}{qp}\frac{1}{s}\, ds = \frac{1}{qp}(0 + qp) = 1$$

■

Lemma 3 *Hours work* n^0 *in the newest vintage of machines depend on the marginal utility of consumption* μ*, but do not depend on the speed of technical change* q.

Proof: This is self-evident from the definition of machine quality relative to frontier: $k^0 = 1$.

■

Lemma 2 *The ratio of hours worked between machines of any two vintages increases with* q.

Proof: Let's take two ages τ_a and τ_b. The ratio of hours worked in machines of those ages can be obtained by dividing the hours equation (4):

$$\frac{v'(n^{\tau_a})}{v'(n^{\tau_b})} = \frac{f_2(k^{\tau_a}, n^{\tau_a})}{f_2(k^{\tau_b}, n^{\tau_b})}$$

Using our functional forms and rearranging leads to,

$$\frac{n^{\tau_a}}{n^{\tau_b}} = e^{\frac{\alpha}{\eta+\alpha}q(\tau_b - \tau_a)}$$

■

Lemma 4 *Hours work n^* in the marginal machine depend neither in the speed of technical change q nor in the marginal utility of consumption μ.*

Proof: Take equation (4) and evaluate it for hours n^* at the marginal machine. Divide it by the participation equation (5). This yields a solution for hours in the marginal machine given by,

$$\frac{v'(n^*)}{v(n^*)} = \frac{\mu}{\mu}\frac{f_2(k^*, n^*)}{f(k^*, n^*)} = \frac{1-\alpha}{n^*}$$

■

Proposition 5 *An increase in q generates:*

(a) A fall in consumption c

(b) An increase in the quality gap e^{qp} between the top and marginal machines.

(c) A fall in the fraction of workers p

(d) An increase in hours per worker n

Proof: Our economy is characterized by the three first order conditions and the aggregate resource constraints. Let's rewrite them here as,

$$u'(c) = \mu \tag{43}$$

$$v'(n^\tau) = \mu f_2\left(e^{-q\tau}, n^\tau\right) \tag{44}$$

$$v(n^*) = \mu f\left(e^{-qp}, n^*\right) \tag{45}$$

$$c = \int_0^p f\left(e^{-q\tau}, n^\tau\right) d\tau \tag{46}$$

and let's use (44) to write an implicit function of hours,

$$n^\tau = \phi\left(e^{-q\tau}, \mu\right) \tag{47}$$

where we have already seen that $\phi_1 > 0$ and $\phi_2 > 0$. Now, let's prove each part in turn:

(a) Note that totally differentiating the resource constraint (46) we obtain,

$$\left[1 - u''(c)\int_0^p f_2\phi_2 d\tau\right] dc = f^* dp + \left[\int_0^p (f_1 + f_2\phi_1)\left(-\tau e^{q\tau}\right) d\tau\right] dq$$

and totally differentiating the participation equation (45):

$$dc = \alpha \frac{u'(c)}{u''(c)} [pdq + qdp]$$

Combining these two equations we obtain:

$$\left[1 - u''(c) \int_0^p f_2 \phi_2 d\tau - \frac{u''(c)}{u'(c)}\right] dc = \left[\int_0^p (f_1 + f_2 \phi_1)(-\tau e^{q\tau}) d\tau - \frac{qp}{f^*}\right] dq$$

The term in brackets in the left hand side is positive and the one in the left is negative. Hence, $\frac{dc}{dq} < 0$.

(b) Notice that the participation equation (45) requires that whenever μ goes up (which happens when c falls) then qp has to increase too.

(c) We need to work a bit with our functional forms. We can write (47) as

$$n^{\tau} = \phi\left(e^{-q\tau}, \mu\right) = \left(\mu \frac{1-\alpha}{\lambda_1}\right)^{\frac{1}{\eta+\alpha}} e^{-\frac{\alpha q}{\eta+\alpha}\tau} \tag{48}$$

Then let's rewrite the resource constraint (46) as,

$$c = \int_0^p \left(\mu \frac{1-\alpha}{\lambda_1}\right)^{\frac{1-\alpha}{\eta+\alpha}} e^{-\frac{\alpha q(1+\eta)}{\eta+\alpha}\tau} d\tau$$

Solving for the integral and substituting μ by $1/c$ we arrive at,

$$c^{\frac{1+\eta}{\eta+\alpha}} = \left(\frac{1-\alpha}{\lambda_1}\right)^{\frac{1-\alpha}{\eta+\alpha}} \frac{\eta+\alpha}{\alpha q (1+\eta)} \left[1 - e^{-\frac{\alpha(1+\eta)}{\eta+\alpha} qp}\right] \tag{49}$$

Now, we can rewrite the participation equation (45) as,

$$v(n^*) = \frac{1}{c} e^{-\alpha pq} (n^*)^{1-\alpha}$$

which let us write consumption as

$$c = e^{-\alpha pq} \frac{(n^*)^{1-\alpha}}{v(n^*)} \tag{50}$$

substituting back into the resource constraint (49) we obtain

$$\left[\frac{(n^*)^{1-\alpha}}{v(n^*)}\right]^{\frac{1+\eta}{\eta+\alpha}} = \left(\frac{1-\alpha}{\lambda_1}\right)^{\frac{1-\alpha}{\eta+\alpha}} \frac{\eta+\alpha}{\alpha q (1+\eta)} \left[e^{\frac{\alpha(1+\eta)}{\eta+\alpha} qp} - 1\right]$$

The left hand side is a bunch of constants. The right hand side is increasing in p. If we can show that the right hand side is also increasing in q, then it must be the case that $\frac{dp}{dq} < 0$. To see that the right hand side is increasing in q notice that the sign of its derivative with respect to q is the same as the sign of the derivative of the function

$$z(q) = \frac{e^{\gamma_0 q} - 1}{q} \tag{51}$$

where $\gamma_0 = \frac{\alpha(1+\eta)}{\eta+\alpha} p > 0$. The derivative of this function has the same sign as

$$g(q) = \gamma_0 e^{\gamma_0 q} q - e^{\gamma_0 q} + 1.$$

which is positive. To see that $g(q)$ is positive one can notice that $g(0) = 0$ and that $g(q)$ is increasing in q for all $q > 0$:

$$g'(q) = \gamma_0^2 e^{-\gamma_0 q} q$$

(d) We define average hours per worker,

$$n \equiv \frac{1}{p} \int_0^p n^\tau d\tau$$

Substituting n^τ by expression (48) and integrating the resulting equation we can rewrite,

$$n = \left(\frac{1}{c} \frac{1-\alpha}{\lambda_1} \right)^{\frac{1}{\eta+\alpha}} \frac{\eta+\alpha}{\alpha} \frac{1}{qp} \left[1 - e^{-\frac{\alpha}{\eta+\alpha} qp} \right]$$

Substituting out consumption from equation (50) we obtain,

$$n = \left(\frac{v(n^*)}{(n^*)^{1-\alpha}} \frac{1-\alpha}{\lambda_1} \right)^{\frac{1}{\eta+\alpha}} \frac{\eta+\alpha}{\alpha} \frac{1}{qp} \left[e^{\frac{\alpha}{\eta+\alpha} qp} - 1 \right]$$

Hence, average hours per worker n are increasing in the quality gap qp. Sine part (b) of this proposition shows that qp is increasing in q, it must be the case that hours per worker increase with q.

■

Proposition 6 *The allocations of the conjectured balanced growth path equilibrium in Definition 1 also solve the large household problem.*

Proof: To prove this result we just need to show that the BGP equilibrium yields conditions (4), (5) and the aggregate resource constraint of the problem in Section 2.2.

- To show that equation (4) holds, just combine the optimal demand of hours by firms (8) with the optimal supply by workers (14). And note that equations (23) and (22) state that $\tilde{c}_{t,i} = c_0$. Hence,

$$v'(n^\tau) = \frac{1}{c_0} f_2(k^\tau, n^\tau)$$

- To show that equation (5) holds, just combine equations (9) with (15) and also substitute consumption at t to obtain,

$$v(n^*) = \frac{1}{c_0} f(k^*, n^*)$$

- To show that the aggregate resource constraint holds, we integrate forward the period budget constraints (12) to obtain,

$$\int_0^\infty e^{-rt} \tilde{c}_{t,i} dt = b_0 + \int_0^\infty e^{-rt} \tilde{w}_t(n_{t,i}) dt$$

Then given (21) and (20), we obtain

$$c_{0,i} = \rho \left[b_0 + \int_0^\infty e^{-\rho t} w(n_{t,i}) dt \right]$$

which is the standard permanent income condition that determines consumption $c_{0,i}$.

Now, note that the labor income $w_\tau(n_{\tau,i})$ at any period of time is stochastic as it depends on the assignment. However, given a law of large numbers the present value of labor income is equal to the cross-sectional average of labor income, which is not stochastic. To see this note that the present value of labor income can be written as

$$\int_0^\infty e^{-\rho t} w(n_{t,i}) dt = \int_0^\infty e^{-\rho t} \left(p \int_{k^*}^1 w(\phi(s)) d\varphi(s) \right) dt = \frac{p}{\rho} \int_{k^*}^1 w(\phi(s)) g(s) ds$$

where we have imposed the conditions for the symmetric equilibrium.

Initial wealth is given by the price of shares $\mathbf{p}_0$, which according to expression (18) equals the discounted present value of profits,

$$b_0 = \mathbf{p}_0 = \int_0^\infty e^{-rt} \Pi_t dt = \frac{p}{\rho} \int_{k^*}^1 \left[f(s, \phi(s)) - w(\phi(s)) \right] g(s) ds$$

Hence the detrended consumption level at every point in time is given by the detrended sum of output produced in all machines at every point in time,

$$c_0 = p \int_{k^*}^1 f(s, \phi(s)) g(s) ds \tag{52}$$

■

Proposition 7 *When q increases labor income inequality as measured by **LI** increases.*

Proof: We can write labor income in t at a machine of given age τ as $\tilde{w}_t\left(n_t^{\tau}\right) = \tilde{w}_t\left(\tilde{\phi}_t\left(k^{\tau}\right)\right)$. Then,

$$\mathbf{LI}_t = \frac{w\left(\phi\left(1\right)\right)}{w\left(\phi\left(e^{-qp}\right)\right)} = \frac{\frac{\lambda_0}{\lambda_1} + \frac{\phi(1)^{1+\eta}}{1+\eta}}{\frac{\lambda_0}{\lambda_1} + \frac{\phi(e^{-qp})^{1+\eta}}{1+\eta}}$$

where for the last equality we have used the equilibrium wage equation (19) and the equilibrium relationship between a_0 and a_1 in (IE).

Now, Proposition 5 shows that when q increase the quality gap e^{qp} also increases. Hence, the detrended quality e^{-qp} of the marginal machine goes down and **LI** goes up. ■

Lemma 8 *For type i workers, the distribution of detrended qualities of operating machines has support* $\left[k_i^*, k_{i-1}^*\right] = [e^{-(q+\delta)\tau_i^*}, e^{-(q+\delta)\tau_{i-1}^*}]$ *and it is log-uniform with density* $g_i\left(k\right) = \frac{m}{(q+\delta)p_i z_i}\frac{1}{k}$

Proof: The first part of the Lemma follows directly from Lemma 1. To prove the second part notice first that the distribution of the age of machines operated by workers of type i is uniform with support $\left[\tau_{i-1}^*, \tau_i^*\right]$. The detrended quality of a machine of age τ can be expressed as $k^{\tau} = e^{-(q+\delta)\tau}$. This implies that the age of a machine, τ can be expressed as the ratio of the log of its quality and $(q+\delta)$: $\tau = -\log k^{\tau}/\left(q+\delta\right)$. This can be used to express the *cdf* of machine qualities $G_i\left(k\right)$ for type i workers:

$$\begin{aligned} G_i\left(k\right) &\equiv \Pr\left(\tilde{k} \leq k\right) = \Pr\left(\log \tilde{k} \leq \log k\right) = \Pr\left(-\frac{\log \tilde{k}}{q+\delta} \geq -\frac{\log k}{q+\delta}\right) \\ &= 1 - \Pr\left(\tau \leq -\frac{\log k}{q+\delta}\right) = 1 - \int_{\tau_{i-1}^*}^{-\frac{\log k}{q+\delta}} \frac{1}{\tau_i^* - \tau_{i-1}^*} ds \\ &= 1 + \frac{m}{p_i z_i}\left(\frac{\log k}{q+\delta} - \tau_{i-1}^*\right) \end{aligned}$$

and hence, the *pdf* is given by,

$$g_i\left(k\right) = \frac{m}{\left(q+\delta\right)p_i z_i} \cdot \frac{1}{k}.$$

We can easily check that the density integrates to one over the support of machine qualities $[e^{-(q+\delta)\tau_i^*}, e^{-(q+\delta)\tau_{i-1}^*}]$:

$$\int_{e^{-(q+\delta)\tau_i^*}}^{e^{-(q+\delta)\tau_{i-1}^*}} g_i\left(s\right)\, ds = \int_{e^{-(q+\delta)\tau_i^*}}^{e^{-(q+\delta)\tau_{i-1}^*}} \frac{m}{\left(q+\delta\right)p_i z_i}\frac{1}{s}\, ds = \frac{m}{\left(q+\delta\right)p_i z_i}\left[-\left(q+\delta\right)\tau_{i-1}^* + \left(q+\delta\right)\tau_i^*\right] = 1.$$

■

Proposition 9 $\pi\left(k, h_i\right) \geq \pi\left(k, h_{i+1}\right)$ *for* $k \in \left[k_i^*, k_{i-1}^*\right]$ *if and only if the following condition* *holds*

$$\left(\frac{h_i}{h_{i+1}}\right)^{(1-\alpha)(1-\theta)} > \left(\frac{c_i}{c_{i+1}}\right)^{\frac{A-1}{A}} \tag{53}$$

Proof: Note that the free entry conditions (31) state that $\pi(k, h_i) = \pi(k, h_{i+1})$ whenever $k = k_i^*$. For $k > k_i^*$ the inequality of the lemma will be met if and only if as capital quality increase, profits increase more for the firm with the better worker. That is to say, we require,

$$\frac{\partial\pi(k, h_i)}{\partial k} \geq \frac{\partial\pi(k, h_{i+1})}{\partial k}$$

Going to the profit function (36) and the output function (33) we see that the above inequality requires,

$$\frac{h_i^{(1-\alpha)(1-\theta)A}}{a_{1i}^{A-1}} \geq \frac{h_{i+1}^{(1-\alpha)(1-\theta)A}}{a_{1,i+1}^{A-1}}$$

Finally, equation (22) gives an expression for a_{1i} and $a_{1,i+1}$ as a function of consumption that leads to,

$$\frac{c_{i+1}}{c_i} \geq \left(\frac{h_{i+1}}{h_i}\right)^{\left(\frac{1-\theta}{\theta}\right)(1+\eta)}$$

■

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