Optimal Sovereign Debt Default*

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Abstract

We determine optimal government default policies under commitment for a small open economy with domestic production risk. Contracting frictions make it optimal for the government to issue debt that specifies a non-contingent repayment in explicit terms. Implicitly, however, the government may find it optimal to promise only partial repayment in some contingencies, making sovereign default an equilibrium outcome under commitment. While default events give rise to deadweight costs within our contracting framework, default can remain desirable from an ex-ante welfare perspective: it allows for improved international diversification of domestic output and consumption risk, relative to a situation where risk sharing occurs exclusively via adjustments in the international wealth position. In a quantitative analysis with empirically plausible levels of default costs, we find that default is optimal only in response to disaster-like shocks to domestic output, or when a small adverse shock pushes international debt levels sufficiently close to the country's borrowing limits. Optimal default policies increase welfare significantly compared to a situation where default is ruled out by assumption, even when default costs are sizable.

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1 Introduction

When is it optimal for a sovereign to default on its outstanding debt? We analyze this hotly debated question in a quantitative equilibrium framework in

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which a country can internationally borrow and invest to smooth out shocks to domestic income. Importantly, our analysis assumes the perspective of a benevolent and fully committed social planner that seeks to maximize the ex-ante welfare of society, i.e., we determine the country's fully optimal default policy. In our setting, a committed social planner will find it optimal to occasionally default on outstanding debt and we show that how this significantly increases welfare. Compared to a situation where default is ruled out by assumption, as often occurs in economic models assuming commitment, our quantitative analysis suggests that welfare increases by up to two percentage point each period in consumption equivalent terms.

The fact that sizable welfare gains can arise from sovereign default may appear surprising, given that policy discussions and the academic literature typically emphasize the inefficiencies associated with sovereign default events.[1] Popular discussions, for example, tend to focus on the potential ex-post costs associated with a sovereign default, say, the adverse consequences for the functioning of the banking sector or the economy as a whole. While certainly relevant, we show that sovereign default can remain optimal, even if ex-post costs of an empirically plausible magnitude arise. Similarly, the academic literature, tends to emphasize that the ability to default in the future, limits the ability to issue debt ex-ante, which in turn limits the country's ability to smooth out adverse shocks to domestic income by increased borrowing. Instead, in our setting with fully optimal default policy, the ability to default significantly relaxes the country's borrowing limits, compared to a situation where default is ruled out by assumption. The possibility of a sovereign default thereby allows for increased international risk sharing.

The present analysis emphasizes that sovereign default fulfills a useful economic function, even in a setting with a fully committed government. A default engineers a resource transfer from lenders to the sovereign debtor in times when resources are scarce on the sovereign's side. The option to default thus provides insurance against adverse economic developments in domestic income. And in the absence of other suitable (or less costly) insurance instruments, a fully committed government will find it optimal to make use of the possibility to default. Given this insurance role, default tends to be optimal following negative shocks to domestic output, in line with the observed empirical default patterns. Furthermore, sovereign default can be optimal even if the country has sufficient resources to be able to repay the outstanding debt. The symptoms of a fully optimal default are thus difficult to distinguish from those associated with a strategic default induced by a 'willingness-to-pay' problem (i.e., a commitment problem); or in other words: the fact that a country does not repay its debt, although it would have sufficient resources to do so, is not sufficient to conclude that the default event is inefficient from the viewpoint of ex-ante welfare.

In our setting, sovereign default is optimal under commitment because

[1] An exception within the academic realm is Grossmann and Van Huyck (1988), as discussed in more detail below.

government bond markets are incomplete, so that international bond markets do not provide any explicit insurance against domestic income shocks. The incompleteness of government bond markets thereby emerges endogenously from the presence of contracting frictions. These frictions make it optimal for the government (but not necessarily for private agents) to issue debt contracts that promise a repayment amount that is not contingent on future events. This is in line with empirical evidence, which shows that existing government debt consist predominantly of non-contingent debt instruments.[2] The contracting framework we introduce represents an advance over earlier work studying optimal government policy under commitment and incomplete markets, which simply assumes that government bond markets are incomplete (e.g., Ayiagari et al. (2002), Angeletos (2002), Sims (2001), or Adam (2011)). In the present setting, the incompleteness arises endogenously from the optimal policy problem. And unlike in these earlier contributions, we do not impose that default is ruled out by assumption. Instead, we derive the optimal repayment decisions from the government's optimization problem. Indeed, by allowing for the possibility of partial repayment and by treating repayment as a (continuous) decision variable in a setting with full commitment, we show that the assumption of full repayment can be inconsistent with optimal government behavior once government bond markets are incomplete. As we also show, however, the assumption of full repayment may provide a reasonable approximation to the fully optimal repayment policy in a setting featuring business cycle sized shocks only.

Non-contingent government debt contracts are optimal in our model because we consider a situation where explicit legal contracting is costly. This makes it optimal for the government to shift any desired state-contingency of the repayment profile into the implicit component of the government debt contract. As a result, the government will occasionally find it optimal to pay back less than the explicitly or legally stated repayment amount, and we refer to such situations as 'default events'. In line with the concerns put forward in policy discussions, such default events are costly and give rise to socially wasteful 'default costs'. In our contracting framework, these costs arise naturally from the assumption that explicit legal contracting is costly. Specifically, in our setting a defaulting government is exposed to the threat of being sued in the future by lenders for fulfillment of the explicit contract terms, i.e., for repayment of the full amount specified in the explicit contract.[3]

[2] Most sovereign debt is non-contingent in nominal terms only, and could be made contingent by adjusting the price level, a point emphasized by Chari, Kehoe and Christiano (1991). As shown in Schmitt-Grohe and Uribe (2004), however, such price level adjustments are suboptimal in the presence of even modest nominal rigidities. Morevoer, for countries that are members of a monetary union, non-contingent nominal debt is effectively non-contingent in real terms, since the country cannot control the price level.

[3] As documented in Panizza, Sturzenegger and Zettelmeyer (2009), legal changes in a range of countries in the late 1970's and early 1980's eliminated the legal principle of 'sovereign immunity' when it comes to sovereign borrowing. Specifically, in the U.S. and the U.K. private parties can sue foreign governments in courts, if the complaint relates to a

To prevent this from happening, the government needs to reach an explicit legal settlement with the lender, so as to protect it from such actions. Given that explicit legal contracting is costly, this settlement process gives rise to 'default costs'. Since these default costs have to be paid only in the event when a default contingency is reached, the government will always find it optimal to pay these costs ex-post, rather than to pay for sure ex-ante by specifying an explicitly contingent contract.

Combining data from Klingen et al. (2004) and Cruces and Trebesch (2011), which covers 21 emerging market economies over a period 30 years, allows us to estimate a lower bound for the default costs. Based on this evidence, default costs appear to be sizable, with the baseline estimate showing that default costs amount to at least 7.5% of the defaulted sum. We use this estimate of a lower bound as an input in our quantitative analysis.

To quantitatively assess the role of sovereign debt default as a vehicle for international risk shifting in a setting with a committed government, we consider a small open economy that is subject to domestic productivity shocks. The government can internationally borrow by issuing own bonds with an arbitrary explicit and implicit repayment profile, where the issuance of bonds is subject to the contracting frictions outlined above. The government can also accumulate international reserves by investing in (riskless) bonds issued by foreign lenders. Shocks to the productivity of the domestic capital stock affect domestic income and the incentives for further investment. The government can smooth the consumption implications of such shocks, either by adjusting borrowing and lending in international capital markets, or by making repayment on debt contingent, i.e., by defaulting. The paper determines which of the previous two channels the government should optimally rely on to smooth domestic consumption, and specifically with the question: when is it optimal to (partially) default on government debt in a setting with a fully committed government?

As a benchmark, we first consider the (empirically implausible) case where a default event does not give rise to default costs. Such a setting has previously been analyzed in section II in Grossman and Van Huyck (1988) within an endowment economy with iid income shocks. We extend their results to a production economy with a more general shock process. Specifically, we show analytically how in the absence of default costs, the trade-off between self-insurance and default is fully resolved in favor of default. Debt default then occurs very frequently, generally for all but the best productivity realization, and the optimal amount of default tends to decrease with the aggregate productivity level. As in Grossman and Van Huyck (1988), optimal default decisions implement the first best consumption allocation, i.e., completely stabilize domestic consumption.

commercial activity, amongst which courts regularly count the issuance of sovereign bonds. We implicitly assume that lenders cannot commit not to sue the government in the future. This appears plausible, given that secondary markets allows initial buyers of government debt to sell the debt instruments to other agents.

We then quantitatively evaluate the empirically more relevant case with positive default costs. We find that plausible levels for the default costs make it generally optimal for the government *not* to default following business cycle sized shocks. Only when the country's net foreign debt position approaches its maximum sustainable level (as defined by the marginally binding natural borrowing limit), does sovereign default become optimal following an adverse business cycle shock. With empirically plausible default costs, the optimal default policy thus depends only on whether or not the country is close to its maximally sustainable net foreign debt position. As we show, the ability to default nevertheless significantly relaxes the country' borrowing limits. For our baseline calibration, the maximum sustainable debt to GDP ratio then significantly increases from 100% in the case where default is ruled out by assumption to a level of about 135% of GDP.

Given that reasonably sized default costs largely eliminate sovereign default in response to business cycle sized shocks, we introduce economic 'disaster' risk into the aggregate productivity process, following Barro and Jin (2011). Default then reemerges as part of optimal government policy, following the occurrence of a disaster shock. This is the case even for sizable default costs and even when the country's net foreign asset position is far from its maximally sustainable level. It continues to be optimal, however, not to default following business cycle sized shocks to aggregate productivity, as long as the net foreign debt position is not too close to its maximal level.

Finally, we investigate the utility consequences of using the government default option as a way to insure domestic consumption by comparing the outcome to a situation where the government is *assumed* to repay debt unconditionally. In the latter setting, adjustments in international wealth is the only channel available to smooth domestic consumption. The consumption equivalent welfare gain associated with allowing for default is often in the order of around one percentage point of consumption each period, even when there are sizable costs associated with a government debt default.

In related work, Sims (2001) discusses fiscal insurance in the context of whether or not Mexico should dollarize its economy. Considering a setting where the government is assumed to issue only non-contingent nominal debt that is assumed to be repaid always, he shows how giving up the domestic currency allows for less insurance, as it deprives the government of the possibility to use price adjustments to alter the real value of outstanding debt. The present paper considers a model with real bonds that are optimally non-contingent and allows for outright government debt default. Our setting could be reinterpreted as one where bonds are effectively non-contingent in nominal terms, but where the country has delegated the control of the price level to a monetary authority that pursues price stability, say by dollarizing or by joining a monetary union. As we then show, in such a setting the default option still provides the country with a possible and quantitatively relevant insurance mechanism.

Angeletos (2002) explores fiscal insurance in a closed economy setting with exogenously incomplete government bond markets, assuming also full repayment of debt. He shows how a government can use the maturity structure of domestic government bonds to insure against domestic shocks, by exploiting the fact that bond yields of different maturities react differently to shocks. This channel is unavailable in our small open economy setting, since the international yield curve does not react to domestic events.

The present paper is structured as follows. Section 2 introduces the economic model, including the contracting framework. It derives the optimal government debt contract and the optimal policy problem that the government solves. It also determines an equivalent formulation of the optimal policy problem that facilitates numerical solution of optimal policies. Section 3 derives an analytical result for the case without default costs. Section 4 evaluates the effects of introducing default costs in a setting with business cycle sized shocks. It also provides a lower bound estimate for the default costs. In section 5 we then introduce economic disaster shocks and discuss their quantitative implications for optimal default policies. Section 6 considers the welfare implications of using the default option and section 7 discusses an extension of the model with long maturity bonds. A conclusion briefly summarizes. Technical material is contained in a series of appendices.

2 The Model

This section introduces a small open economy with domestic production risk and contracting frictions. It derives the from of the optimal government debt contract and the government's optimal policy problem.

2.1 Private Sector: Households and Firms

The household side of the domestic economy is described by a representative consumer with utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

where $\beta \in (0,1)$ denotes the discount factor and $u(c)$ the period utility function. The latter is assumed to be twice continuously differentiable, increasing in c and strictly concave, for all values of $c > \overline{c}$ where $\overline{c} \geq 0$ denotes the subsistence level for consumption. We shall assume that $u(c) = -\infty$ for all $c \leq \overline{c}$ and that Inada conditions hold, i.e., $\lim_{c \to \overline{c}^+} u'(c) = +\infty$ and $\lim_{c \to \infty} u'(c) = 0$.

The production side of the economy is described by a representative firm which produces consumption goods using the production function

$$y_t = z_t k_{t-1}^{\alpha},$$

where y_t denotes output in period t, k_{t-1} the capital stock from the previous period, $\alpha \in (0,1)$ the capital share, and $z_t > 0$ an exogenous stochastic productivity disturbance. Productivity shocks assume values from some finite set $Z = \left\{z^1, ..., z^N\right\}$ with $N \in \mathbb{N}$. The transition probabilities for productivity across periods are described by some measure $\pi(z'|z)$ for $z', z \in Z$. Firms are owned by households and must decide on the capital stock one period in advance, i.e., before future productivity is known. For simplicity we assume that capital depreciates fully after one period.

2.2 The Government

The government seeks to maximize the utility of the representative domestic household (1) and is fully committed to its plans. It can insure against domestic consumption risk by investing in international bonds, i.e., to build a buffer stock of wealth, and issue own bonds so as to borrow internationally. Unless otherwise stated, international bonds are assumed to be risk free and the interest rate r on these bonds satisfies $\frac{1}{1+r} = \beta$.

The government can issue bonds with an arbitrary contingent payment profile. However, as we show below, the presence of contracting frictions and the special nature of government debt contracts make it optimal to issue non-contingent bonds only. The next sections describe the contracting frictions, derive the optimal government debt contract and the present the resulting optimal policy problem.

2.2.1 Government Debt Contracts

Consider a government that can issue arbitrary debt contracts. A contract consist of an implicit and an explicit contract component. The explicit component is written down in the form of a legal text, while the implicit component involves only a common understanding about the nature of the contract between the contracting parties, but no explicit formalization.

Government debt contracts are special contracts because they are held by a large number of investors, unlike contracts involving private parties only. As a result, there typically exists widespread knowledge in society about the implicit contract components of government debt. For this reason, we assume that the implicit component of government debt contracts commonly understood by all economic actors, as is the case for the explicit components. Nevertheless, there exists a crucial difference between these two components. While the common understanding about the explicit contract component exists independent of time, say because agents can always go back and read the written obligations of the contract, the common understanding about the implicit contract is assumed to evaporate with time, and we shall assume that this process is set in motion after the maturity date of the debt contract. This may occur, for example, because circumstances and motives of economic actors change with time, so that agents have difficulties recalling (or agreeing on) the implicit agreements that have been part of government debt contracts that have been signed a long time ago. As a result, the implicit contract components of government debt contract can be verified in court over the lifetime of the contract, but it becomes increasingly difficult to do so after the maturity date of the debt contract.

The fact that - due to the large number of actors involved - the implicit contract component of government debt can be verified in court makes government debt contracts special. Implicit components of private contracts, for example, are often private information available to the contracting parties only, thus cannot be verified in court, not even over the lifetime of the contract. The optimal form of private contracts will therefore generally differ from the optimal form of government debt contracts.

We now describe the explicit and implicit contract components in more detail. For simplicity, we consider a zero coupon bond with a fixed maturity date and omit time subscripts in this and the next section.

The explicit component of a government debt contract is written down in the form of a legal text. In its most general form, the legal text consists of a description of the contingencies z^n and of the repayment obligation l^n for each contingency $n \in \{1, ..., N\}$. Whether or not an explicit contract obligation l^n has been fulfilled can be verified in court even a long time after the maturity date of the contract.

While the explicit contract allows for an arbitrary contingent repayment profile, we assume that the inclusion of contingencies into the legal contract is costly. Such costs arise, for example, because specifying a contingent contract involves the input of lawyers who charge fees for describing contingencies and for writing more complicated contracts.[4] The presence such costs provides an incentive to shift a desired contingent repayment profile into the implicit part of the contract. This is so because the inclusion of contingencies into the implicit contract is not (directly) associated with any legal costs, as no explicit contract has to be formulated.

The implicit contract component credibly specifies - at the time when the contract is issued - a state contingent default profile[5]

$$\Delta = (\delta^1, ..., \delta^N) \in [0, 1]^N$$

The implicit default profile Δ specifies for each possible contingency the share of the legal payment obligation that is *not* fulfilled by the government. An example for such an implicit contract component is 'Repayment will be zero, if the world financial system collapses'; for this contingency we have $\delta = 1$.

The actual repayment at maturity of the government debt contract is thus jointly implied by explicit and implicit contract components and given by

$$l^n(1 - \delta^n)$$

for each contingency $n \in \{1, ..., N\}$. If a contingency arises for which $\delta^n > 0$, the countries pays back less than the legally or explicitly specified amount l^n and we shall say that 'the country is in default'.

Over the lifetime of the contract, there exists a shared and common understanding between the government, investors, and courts about the explicit and

[4] We describe the cost structure in detail below.

[5] The fact that we restrict Δ to the unit cube is not essential for the results that follow. It allows, however, for an easier interpretation of the implicit contract component in terms of 'default'.

implicit payment profiles associated with any government debt contract. All agents are thus fully aware of the possibility that a default can occur, so that government bonds will carry a default premium whenever implicit contract components specify default events. Implicit contract components, however, are forgotten over time. To keep the analysis tractable, we shall assume that all economic actors become oblivious about the implicit contract component in the first period after the debt contract matures. This assumption is not essential for the result that follows and could be relaxed, e.g., memory about the implicit component might be lost only with a certain probability in each period after the bond matures.

Given this assumption, consider a contingency z^n where the government happens to be in default ($\delta^n > 0$). At the time the default occurs, there still exists a shared understanding about the implicit contract components among all economic actors. In the future, however, actors find it difficult to recall (or agree on) the implicit contract components. In the absence of recallable implicit contract components, future court decisions will be based on a comparison of the explicit contract obligations with the actual actions (payments) that occurred. This provides an incentive for lenders to sue the government in the next period for fulfillment of the explicit contract.[6] Anticipating such behavior, the government will engage - at the time the default occurs - in a negotiation process with the lender, with the objective to reach an explicit legal settlement that protects it from being sued in the future.[7] The settlement agreement makes explicit that the debt contract has been fulfilled, even if the actual payment fell short of the legally specified amount. The threat of going to court to obtain such an explicit settlement via a court ruling in the present period will induce the lender to agree to such an agreement. Reaching the explicit legal agreement is, however, costly, as it again requires the input of lawyers who formulate the agreement and charge fees accordingly.

In the setting just described, the government can achieve a contingent repayment by either including the contingency into the explicit contract or by making it part of the implicit contract. The legal costs associated with explicitly describing contingencies in the explicit contract provide an incentive to write non-contingent explicit contracts. Shifting contingencies into the implicit contract, however, is also costly, as it gives rise to a costly ex-post settlement stage following a default. The latter provides an incentive to write legal contracts that avoid default and that incorporate contingencies into the explicit contract component. We investigate this trade-off further in the next section.

[6] This assumes that lenders cannot commit no to sue the government in the future. This assumption appears reasonable, given the existence of secondary markets on which government debt can be traded.

[7] If we allowed for $\delta^n > 1$ in the implicit contract, then the government would have an incentive to sue the lenders for overpayment in future periods, providing the lender with an incentive to engage in an explicit settlement.

2.2.2 The Optimal Government Debt Contract and Default Costs

We now describe the contracting and settlement costs in greater detail and derive the explicit and implicit components of an optimal government debt contract.

We set the costs of writing an non-contingent legal contract to zero. A non-contingent explicit contract specifies that the repayment equals $\bar{l} > 0$ for all contingencies. Given the cost structure considered below, we can - without loss of generality - normalize $\bar{l} = 1$.[8] The costs of writing an explicit contingent contract are assumed to take the form of a proportional legal fee $\lambda \geq 0$ that is charged against the value of the contingent agreement. Similarly, we assume that the costs of reaching an explicit settlement agreement following a default event also takes the form of a proportional legal fee $\lambda \geq 0$ that is charged against the value of the ex-post settlement. This is in line with the casual empirical observation that lawyers typically charge fees that are proportional to the value of the agreements they formulate. Specifically, legally incorporating a payment $l^n \leq 1$ for some contingency z^n in the explicit contract, involves the costs

$$\lambda (1 - l^n)$$

per contract issued, where $1 - l^n$ denotes the value of the deviation from the baseline payment of 1 that occurs in contingency z^n.[9] Similarly, the costs of an ex-post legal settlement in case of a default event are given by

$$\lambda l^n \delta^n$$

per contract, where $l^n \delta^n$ denotes the value of the settlement agreement, i.e., the defaulted amount on each contract. For simplicity, we assume that the same proportional fee λ applies to the ex-post settlement as applies to writing an explicit contingent contract ex-ante. While the legal fees associated with writing a legal contract are assumed to be born by the government, the settlement fees may be shared between the lender and the borrower, with the lender paying $\lambda^l \geq 0$ and the borrower $\lambda^b \geq 0$, where $\lambda^l + \lambda^b = \lambda$. In the remaining part of the paper we will refer to these proportional legal settlement costs λ also as 'default costs'.

Consider the situation where the government wishes to implement a contingent payment $p(z) \leq 1$ for some contingency $z \in Z$. Specifying the contingency as part of the legal contract involves the contract writing costs

$$\lambda (1 - p(z))$$

per contract and no ex-post settlement costs in case the contingency arises in the future. Alternatively, not specifying the contingent payment as part of

[8] This is without loss of generality, because the normalization can be undone by issuing $\bar{l}$ units of the normalized bonds. All costs are invariant to this operation.

[9] The fact that we allow only for downward deviations from the benchmark payment $\bar{l}$ amounts to normalizing $\bar{l}$ to $\bar{l} = \max_{n \in N} l^n$. This is without loss of generality as long as upward deviations are equally costly to incorporate as downward deviations, i.e., the costs of the writing the contract cannot be reduced by choosing a different normalization.

the legal contract, gives rise to expected default costs of[10]

$$\Pr(z|z_0)\lambda\,(1-p(z))$$

where z_0 is the contingency prevailing at the time when the contract is issued. Since $\Pr(z|z_0) \leq 1$ and since default costs are born at a later stage, i.e., when the contract matures, the government will always strictly prefer to issue a non-contingent explicit contract and shift contingencies into the implicit contract profile. This feature arises because implicit contract components are

2.2.3 The Government's Optimal Policy Problem

We now consider the government's optimal policy problem. Using the result from the previous section we can assume - without loss of generality - that all government bonds are non-contingent in their explicit component and 'promise' to repay one unit of consumption at maturity. To simplify the exposition, we start by considering zero coupon bonds with a maturity of one period only. Allowing for a richer maturity structure in international bonds would make no difference for the analysis, as foreign interest rates are independent of domestic conditions, so that the government cannot use the maturity structure of foreign bonds to insure against domestic productivity shocks. The effects of introducing domestic bonds with longer maturity will be discussed separately in section 7.

Let $G_t^L \geq 0$ denote the government's holdings of international bonds in period t. These bonds constitute a 'long position', will mature in period $t+1$ and repay G_t^L units of consumption at maturity. Let $G_t^S \geq 0$ denote the bonds issued by the government in period t. These bonds represent a 'short position' and promise - as part of their explicit contract - to repay G_t^S units of consumption in period $t+1$. The government can use adjustments in the long and short positions to insure domestic consumption against domestic productivity shocks.[11]

Total repayment on domestic bonds maturing in period $t+1$ when productivity is equal to z_{t+1} is then given by

$$G_t^S \cdot (1-(1-\lambda)\delta_t^{I(z_{t+1})}) \tag{2}$$

where $I(z_{t+1})$ denotes the index of the productivity shock, i.e., $I(z_{t+1}) = n$ if and only if $z_{t+1} = z^n$, and δ_t the implicit state-contingent default decision, which is (credibly) determined in period t. For simplicity, we assume here that all legal costs associated with reaching the explicit settlement agreement following a default are born by the borrower. Appendix A.1 shows that the

[10] The expected settlement cost for the lender enter here the borrower's optimization reasoning, because the borrower has to compensate the lender for the settlement costs born by the lender.

[11] In the present setting it is actually optimal that the government borrows internationally on behalf of private agents. This allows to economize on contracting costs, as - unlike with private debt contracts - the implicit components of government debt contracts can be verified in court over the lifetime of the contract.

same real allocations are feasible in a setting in which settlement cost are born by the lender instead.

The specification in equation (2) is similar to the specifications in Zame (1993) and Dubey, Geanakoplos and Shubik (2005) who previously introduced proportional default costs for private contracts within a general equilibrium model within an exogenous set of assets. Default costs in our setting represent a resource cost, while the general equilibrium literature models default cost as a direct utility cost, which enters separably into the borrower's utility function. While it is difficult to interpret legal costs as a direct utility cost, we conjecture that imposing a direct utility cost instead of a resource cost would deliver very similar optimal default implications.

We can now define the amount of resources available to the domestic government at the beginning of the period, i.e., before issuing new debt and making investment decisions on international bonds, but after (partial) repayment of maturing bonds.[12] We refer to these resources as beginning-of-period wealth and define them as

$$w_t \equiv z_t k_{t-1}^{\alpha} + G_{t-1}^L - G_{t-1}^S \cdot (1 - (1-\lambda)\delta_{t-1}^{I(z_t)})$$

Beginning-of-period wealth will serve as a useful state variable when computing optimal government policies later on. The government can raise additional resources in period t by issuing own government bonds. It can then use the resulting funds to invest in international riskless bonds, to invest in the domestic capital stock, and to finance domestic consumption. The economy's budget constraint is thus given by

$$c_t + k_t + \frac{1}{1+r} G_t^L = w_t + \frac{1}{1+R(z_t, \Delta_t)} G_t^S$$

where $\frac{1}{1+r}$ denotes the price of the risk-free international bond and $\frac{1}{1+R(z_t,\Delta_t)}$ the price of the domestic bond. The real interest rate $R(z_t, \Delta_t)$ of the domestic bond depends on the (implicit) default profile Δ_t chosen by the government and on the current productivity state, as the latter affects the likelihood of entering different states tomorrow. Due to the small open economy assumption, the government takes the pricing function $R(\cdot,\cdot)$ as given in its optimization problem. Assuming risk-neutral international lenders, no-arbitrage implies that the pricing function for domestic bonds is given by

$$\frac{1}{1+R(z_t, \Delta_t)} = \frac{1}{1+r} \sum_{n=1}^{N} (1 - \delta_t^n) \cdot \pi(z^n | z_t) \qquad (3)$$

so that the expected return on the domestic bond is equal to the return on the riskless international bond.

[12] Below we do not distinguish between the government budget and the household budget, instead consider the economy wide resources that are available. This implicitly assumes that the government can costlessly transfer resources between these two budgets, e.g., via lump sum taxes.

We are now in a position to formulate the government's optimal policy problem (Ramsey allocation problem):

$$\max_{\left\{G_t^L \geq 0, G_t^S \geq 0, \Delta_t \in [0,1]^N, k_t \geq 0, c_t \geq \bar{c}\right\}} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{4a}$$

$$s.t.: c_t = w_t - k_t + \frac{G_t^S}{1 + R(z_t, \Delta_t)} - \frac{G_t^L}{1 + r} \tag{4b}$$

$$w_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z \tag{4c}$$

$$w_0, z_0 : given$$

We have added the natural borrowing limits (4c) so as to prevent explosive debt dynamics (Ponzi schemes). In our numerical application we choose state-contingent values for the natural borrowing limits (NBLs) so that these constraints are just marginally binding. This is required because for beginning-of-period wealth levels below these marginally binding NLBs there exist no policies that are consistent with non-explosive debt dynamics along all contingencies, as we prove in appendix A.6. Such nonexistence generates problems for our numerical solution approach. The marginally binding NBLs that we impose represent the laxest constraints on beginning-of-period wealth levels that are consistent with existence of policies that imply non-explosive debt dynamics. Appendix A.6 shows how one can compute the marginally binding NBLs and proves that they are unique. Naturally, we assume that the initial condition satisfies $w_0 \geq NBL(z_0)$.

While intuitive, the formulation of the optimization problem (4) has a number of unattractive features. First, the price of the domestic government bond depends on the chosen default profile, so that constraint (4b) fails to be linear in the government's choice variables. It is thus unclear whether problem (4) is concave, which prevents us from working with first order conditions. Second, the inequality constraints for G_t^L, G_t^S and especially those for Δ_t are difficult to handle computationally, as they will be occasionally binding.[13] Moreover, the optimal default policies Δ_t turn out to be discontinuous. For these reasons, we derive in the next section an equivalent formulation of the problem that can be shown to be concave, that features fewer occasionally binding inequality constraints, and gives rise to continuous optimal policy functions.

2.2.4 Equivalent Formulation of the Government Problem

We now formulate an alternative optimal policy problem with a different asset market structure than in problem (4) and thereafter show that it is equivalent to the original problem (4).

Specifically, we consider a setting with N Arrow securities and a single riskless bond in which the country can go either long or short. The vector of

[13] The fact that marginal utility increases without bound as $c_t \to \bar{c}$ and that marginal productivity of capital increases without bound as $k_t \to 0$ will insure interior solutions for these two choice variables, allowing to ignore the inequality constraints for these variables when computing numerical solutions.

Arrow security holdings is denoted by $a \in \mathbb{R}^N$ and the n-th Arrow security pays one unit of output tomorrow if productivity state z^n materializes. The associated price vector is denoted by $p \in \mathbb{R}^N$. Given the risk-neutrality of international lenders, the price of the n-th Arrow security in period t is

$$p_t(z^n) = \frac{1}{1+r}\pi(z^n|z_t). \tag{5}$$

Letting b denote the country's holdings of riskless bonds, beginning-of-period wealth for this asset structure is then given by

$$\widetilde{w}_t \equiv z_t \widetilde{k}_{t-1}^{\alpha} + b_{t-1} + (1-\lambda)a_{t-1}(z_t) \tag{6}$$

where $a_{t-1}(z_t)$ denotes the amount of Arrow securities purchased for state z_t, $\widetilde{k}_{t-1}$ capital invested in the previous period, and $\lambda \geq 0$ is the parameter capturing potential default costs in the original problem (4).

Consider the following alternative optimization problem:

$$\max_{\{b_t, a_t \geq 0, \widetilde{k}_t \geq 0, \widetilde{c}_t \geq \bar{c}\}} \quad E_0 \sum_{t=0}^{\infty} \beta^t u(\widetilde{c}_t) \tag{7a}$$

$$s.t. \ \ \forall t: \widetilde{c}_t = \widetilde{w}_t - \widetilde{k}_t - \frac{1}{1+r} b_t - p_t \cdot a_t \tag{7b}$$

$$\widetilde{w}_{t+1} \geq NBL(z_{t+1}) \quad \forall z_{t+1} \in Z$$

$$\widetilde{w}_0 = w_0, z_0 \quad \textit{given.}$$

Problem (7) has the same concave objective function as problem (4), but the constraint (7b) is now linear in the choices, so that first order conditions (FOCs) are necessary and sufficient. The FOCs can be found in appendix A.3. Furthermore, problem (7) reveals that the optimization problem has a recursive structure with the state in period t being described by the vector $(z_t, \widetilde{w}_t)$, allowing us to express optimal policy functions as a function of these two state variables only. Finally, the relevant inequality constraints are given by $a_t \geq 0$ and the marginally binding natural borrowing limits.[14]

As the following proposition shows, the two different asset market structures allow to implement the same set of consumption paths:

Proposition 1 *If a consumption path $\{c_t\}_{t=0}^{\infty}$ is feasible in problem (4), then is it also feasible in problem (7), and vice versa.*

The proof can be found in appendix A.2. It also shows how the financial market choices $\{b_t, a_t\}$ supporting a consumption allocation in problem (7) can be translated into financial market choices $\{G_t^L, G_t^S, \Delta_t\}$ supporting the

[14] As before, the Inada conditions on utility and the fact that marginal productivity of capital increases without bound as $k_t \to 0$ will insure interior solutions for c_t and k_t, allowing to ignore the inequality constraints for these variables when computing numerical solutions.

same consumption allocation in the original problem (4). As is shown in the appendix, the relationship between these choices is given by

$$\begin{aligned} b_{t-1} &= G^L_{t-1} - G^S_{t-1}, \\ a_{t-1} &= G^S_{t-1}\Delta_{t-1} \end{aligned}$$

This shows that b in problem (7) has an interpretation as the net foreign asset position in problem (4), while a in problem (7) can be interpreted as the state contingent default on outstanding own bonds. We will make use of this interpretation in the latter part of the paper. The previous equations allow us to solve the simpler problem (7), but to interpret the solution in terms of the financial market choices for the original problem (4).

3 Zero Default Costs: Analytical Results

In the absence of default costs, our setting reduces essentially to that analyzed in section II in Grossman and Van Huyck (1988), who consider an endowment economy with iid income risk. Our setting is slightly more general, as we analyze a production economy and allow for serially correlated productivity shocks. In the absence of default costs, the solution to problem (7) can be determined analytically. The following proposition summarizes our main finding:[15]

Proposition 2 *Without default costs ($\lambda = 0$) the solution to problem (7) involves constant consumption equal to*

$$c = (1-\beta)(\Pi(z_0) + \widetilde{w}_0) \tag{8}$$

where $\Pi(\cdot)$ denotes the maximized expected profits from future production, defined as

$$\Pi(z_t) \equiv E_t\left[\sum_{j=0}^{\infty} \beta^j \left(-k^*(z_{t+j}) + \beta z_{t+j+1}\left(k^*(z_{t+j})\right)^{\alpha}\right)\right]$$

with

$$k^*(z_t) = (\alpha\beta E(z_{t+1}|z_t))^{\frac{1}{1-\alpha}} \tag{9}$$

denoting the profit maximizing capital level. For any period t, the optimal default level satisfies

$$a_t(z_t) \propto -\left(\Pi(z_t) + z_t\left(k^*(z_{t-1})\right)^{\alpha}\right) \tag{10}$$

As in Grossman and Van Huyck (1988), it is optimal - in the absence of default costs - to fully smooth consumption. The optimal commitment policy thereby involves frequent default: equation (10) reveals that default must

[15] The proof of proposition 2 can be found in appendix A.4.

occur for virtually all productivity realizations.[16] Such default insures the country against two risk components: first, against (adverse) news regarding the expected profitability of future investments, as captured by $\Pi(z_t)$; second, against low output due to a low realization of current productivity, as captured by $z_t\,(k^*(z_{t-1}))^\alpha$. If expected future profits commove positively with current productivity, e.g. if z_t is a persistent process, or in the special case with iid productivity shocks, where expected future profits are independent of current productivity, it follows from equation (10) that the optimal default levels are inversely related to the current level of productivity. Default is then optimal whenever z_t falls short of its highest possible value and the optimal size of default is increasing in the amount by which productivity falls short of its maximal level.

The previous proposition is also of interest, because it shows that the assumption of full repayment of government debt is suboptimal in a setting with a fully committed government, whenever government bond market are incomplete and promise a non-contingent repayment in explicit terms. Due to the continuity of the optimal solution, this will remain optimal for small but positive default costs. The optimal policies for positive default costs will be explored in detail in the next section.

4 Optimal Default Policies with Default Costs

While being a useful reference point, the setting with zero default costs analyzed in the previous section appears empirically implausible. For this reason this section considers the effects of introducing positive default costs. Such costs decrease the attractiveness of the default option, and as is clear from equation (6), it becomes optimal to rely exclusively on self-insurance via international wealth adjustments (i.e., to set $a_t \equiv 0$), if the dead weight costs from default become sufficiently high, e.g., if $\lambda \geq 1$. The resolution of the trade-off between insurance via default and via international reserve adjustment thus shifts from an exclusive reliance on default for $\lambda = 0$ to an exclusive reliance on international reserve adjustment for $\lambda \geq 1$. The assumption of full repayment of non-contingent debt under commitment entertained in earlier incomplete market models can thus be interpreted as a situation where default costs are sufficiently high, so as to make repayment in all states optimal.

To quantitatively evaluate how the trade-off between default and self-insurance is resolved for empirically relevant levels of default cost, we estimate a lower bound for the size of default costs from financial market data and information on default events. Our model-based estimation approach is presented in the next section. Calibrate the remaining model parameters

[16] Default is not required for states z_t achieving the maximal value for $\Pi(z_t)+z_t(k^*(z_{t-1}))^\alpha$ across all $z_t \in Z$. For such states default can be set equal to zero, otherwise default levels are strictly positive. This default pattern is, however, not the only one implementing full consumption stabilization in a setting with zero default/contracting costs ($\lambda = 0$). For $\lambda = 0$ it is equally optimal to write a fully contingent contract and to never default. Yet, for vanishingly small but positive default costs, frequent default will be optimal even in the limit, while the no-default policy with explicitly contingent contracts will remain suboptimal.

we then quantitatively determine the resulting optimal default policies in a setting with business cycle sized shocks to aggregate productivity.

4.1 Default Costs: An Estimated Lower Bound

While default or contracting costs are notoriously difficult to estimate, we can exploit some restrictions from our structural model to obtain a lower bound for the costs associated with a default event. The basic idea underlying our estimation approach is to exploit the fact that default costs that accrue to the lender (but not those that accrue to the borrower) can be estimated from financial market prices and information on default events. This is possible because the borrower has to compensate the lender ex-ante for the default costs arising on the lender's side, so that these costs are reflected in financial market prices.

To pursue this idea, we now consider a slightly more general setting where the lender also bears some default costs ($\lambda^l > 0$). Appendix A.1 shows that such a setting allows to support all allocations that can be supported in a setting where default costs are born exclusively by the borrower. The only difference arising in this setting is that the bond pricing equation now depends on λ^l. Consider a one-period bond issued in period t with a non-contingent explicit profile ($l(z) = 1$ for all $z \in Z$) and with an implicit default profile $\delta(z)$. A risk-neutral foreign lender who can earn the gross return $1 + r$ on alternative investments will invest in this bond if it offers the same expected return net of default costs, i.e., if

$$1 + r = \frac{1 - (1 + \lambda^l) \sum_{z' \in Z} \delta(z')\Pi(z'|z_t)}{\frac{1}{1+R(z_t,\delta)}} \tag{11}$$

where $1/(1 + R(z_t, \delta))$ denotes the issue price of the bond and the numerator on the right-hand side of the expression captures the expected repayments on the bond, net of the lender's settlement costs λ^l.

The ex-post return on the government bond is defined as

$$1 + epr_t = \frac{1 - \sum_{z' \in Z} \delta(z')\Pi(z'|z_t)}{\frac{1}{1+R(z_t,\delta)}}$$

which is an object that can be measured from financial market and default information. Using this definition and applying the unconditional expectations operator to equation (11) to integrate over the stationary distribution of the z_t we obtain

$$\lambda^l = \frac{E\left[\frac{1}{1+R(z_t,\delta)} \sum_{z' \in Z} \delta(z')\Pi(z'|z_t)\right]}{E\left[epr_t - r\right]} \tag{12}$$

We can obtain an estimate for the average excess return, which shows up in the denominator of the previous equation, from Klingen, Weder, and Zettelmeyer (2004), who consider 21 emerging market economies over the period 1970-2000. Using data from table 3 in Klingen, Weder, and Zettelmeyer (2004), the average excess return varies between -0.2% and +0.5% for publicly guaranteed debt, depending on the method used.[17,18] For our estimation we will use the average value for the estimated ex-post excess return, i.e., $E[epr_t - r] = 0.15\%$.

We can also obtain an estimate of the numerator on the r.h.s. of equation (12). Specifically, to a first order approximation, we have

$$E\left[\frac{1}{1+R(z_t,\delta)}\sum_{z'\in Z}\delta(z')\Pi(z'|z_t)\right] \approx E\left[\frac{1}{1+R(z_t,\delta)}\right]E\left[\sum_{z'\in Z}\delta(z')\Pi(z'|z_t)\right]$$

where the last term equals (again to a first order approximation) the average default probability times the average default size, conditional on a default occurring. Based on the data compiled in Cruces and Trebesch (2011), who kindly provided us with the required information, we observe for the 21 countries considered in Klingen et al. (2004) and for the period 1970-2000 a total of 58 default events, so that the average yearly default probability equals 8.9%. The average haircut conditional on a default was 25%, so that these figures imply that

$$E\left[\sum_{z'\in Z}\delta(z')\Pi(z'|z_t)\right] \approx 8.9\% \cdot 25\% = 2.225\%$$

The average ex-ante interest rate $R(z_t,\delta)$ can be computed by adding to the average ex-post return of 8.8% reported in table 3 in Klingen, Weder, and Zettelmeyer (2004) for publicly guaranteed debt, the average loss due to default, which equals 2.225%. This provides us with an estimate for the default costs accruing to lenders, which equals

$$\lambda^l \approx 7.5\%$$

Given that this estimate provides a lower bound for the total default costs $\lambda = \lambda^l + \lambda^b$ with $\lambda^b \geq 0$, we will consider in our quantitative analysis default cost levels above this value.

[17] As suggested in Klingen, Weder, and Zettelmeyer (2004), we use the return on a 3 year US government debt instrument as the safe asset, since it approximately has the same maturity as the considered emerging market debt.

[18] The fact that ex-post excess returns on risky sovereign debt are relatively small or sometimes even negative is confirmed by data provided in Eichengreen and Portes (1986) who compute ex-post excess returns using interwar data.The negative ex-post excess returns likely arise due to the presence of sampling uncertainty: the high volatility of the nominal exchange rate makes it difficult to estimate the mean ex-post excess returns.

4.2 Model Calibration

We now calibrate the remaining model parameters. We begin with the quarterly productivity process. A standard parameterization is given by a first order autoregressive process with quarterly persistence of 0.9 and a standard deviation of 0.5% for the quarterly innovation.[19] Since we use a yearly model, we annualize these values by choosing an annual persistence of technology equal to $(0.9)^4$ and use an annual standard deviation of the innovation of 1%. We then use Tauchen's (1986) procedure to discretize the shock process into a process with a high and a low productivity state. Normalizing average productivity to one, the resulting high productivity state is $z^h = 1.0133$ and the low productivity state $z^2 = 0.9868$**.** The procedure also yields a transition matrix for the states, given by

$$\pi = \begin{pmatrix} 0.8077 \, 0.1923 \\ 0.1923 \, 0.8077 \end{pmatrix}. \tag{13}$$

We set the capital share parameter in the production function to $\alpha = 0.34$. The annual discount factor is $\beta = 0.97$ and we consider households with a flow utility function given by

$$u(c) = \frac{(c - \bar{c})^{1-\sigma}}{1 - \sigma}$$

where $\bar{c} \geq 0$ denotes the subsistence level of consumption and σ parameterizes risk aversion. We choose $\sigma = 2$ and calibrate the subsistence level of consumption $\bar{c}$ such that in an economy where the government is forced to repay debt always, the marginally binding natural borrowing limit implies that the net foreign asset position of the country cannot fall below -100% of average GDP in any productivity state.[20] We thereby seek to capture the fact that in industrialized countries, which are the countries that only rarely default, the net foreign asset position is rarely below -100% of GDP, see figure 10 in Lane and Milesi-Ferretti (2007). Moreover, three out of the five industrialized countries approaching this boundary in the year 2004 later on faced fiscal solvency problems (Greece, Portugal and Iceland). In the light of this evidence it appears plausible to consider a calibration implying that countries cannot sustain higher external debt levels without running the risk of a sovereign default.

[19] The quantitative results reported below are not very sensitive to the precise numbers used. A similar calibration is employed in Adam (2011).

[20] Appendix A.6 explains how one can compute the marginally binding NBL for each productivity state. Average GDP is defined as the average output level associated with efficient investment, i.e., when $k_t = k^*(z_t)$ each period, and where we average over the ergodic distribution of the z process. For our parameterization this yields an average output level of 0.5661. Furthermore, the net foreign asset position of the country is independent of government policy at the marginally binding NBL, instead exlusivley determined by the desire to prevent debt from exploding, so that this measure can be used to calibrate the model. The resulting level for subsistence consumption is $\bar{c} = 0.3542$.

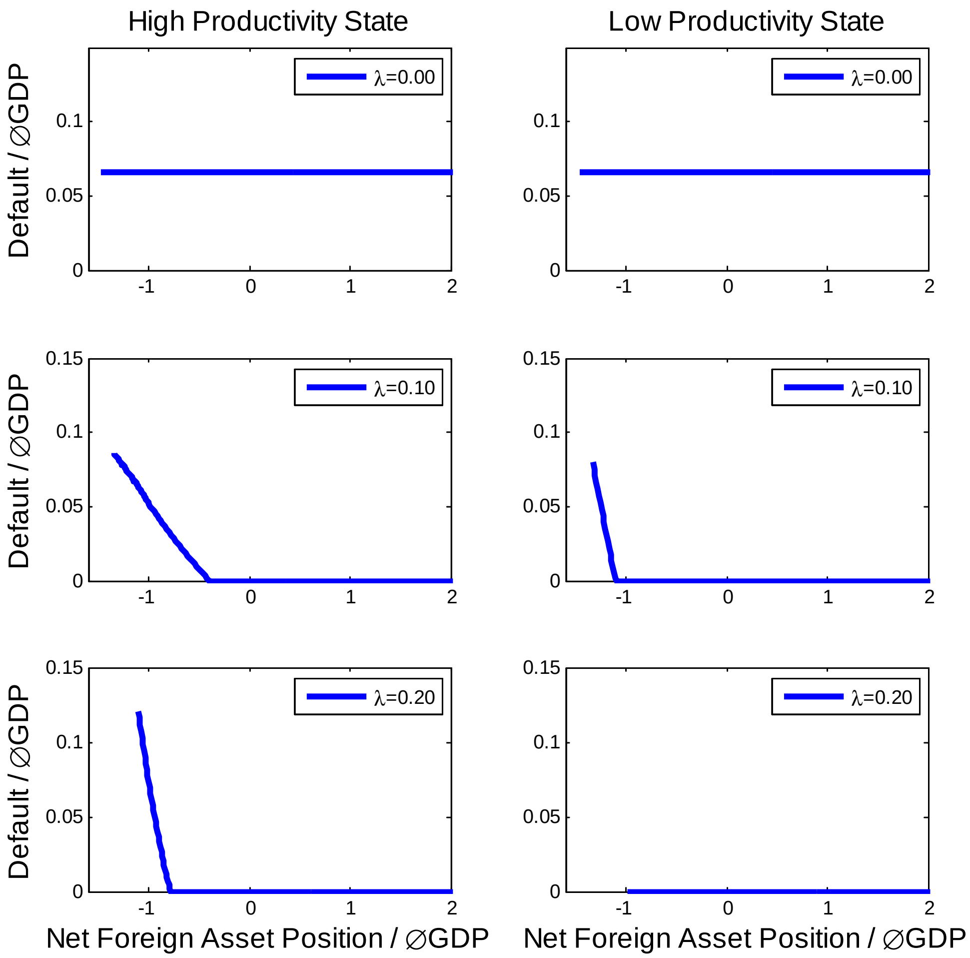

Figure 1: Optimal Default Policies: The Effect of Default Costs

Positive default costs and the small open economy assumption jointly imply that the equilibrium outcomes are non-stationary, unless we choose $1 + r < 1/\beta$ to insure that net foreign assets remain also bounded from above. To insure that the equilibrium process is ergodic, we set the annual international interest rate five basis points below the rate implied by the inverse of the discount factor. Optimal default policies turn out to be very robust to the precise number chosen.[21]

[21] We also experimented with larger gaps of 50 basis points. This leads to no noticable difference for the graphs shown in this paper.

4.3 Evaluating the Effect of Default Costs

This section reports the optimal default policies arising for various levels of the default costs. It shows that allowing for (partial) default significantly relaxes the country's borrowing limits when default costs are low, but that relatively modest levels for these costs eliminate default over large parts of the state space. We illustrate this result by analyzing optimal default policies for three different levels of default costs: we first consider - for benchmark purposes - the case without costs, thereafter default costs of 10% and 20%, which are, respectively, cost levels either mildly or significantly exceeding our lower bound estimate from section 4.1. A value of $\lambda = 10\%$ implies that three quarters of the overall costs are born by the lender.[22] A setting with $\lambda = 20\%$ implies that the majority of the default costs accrues on the borrowers side, which is arguably a more plausible assumption.

The top row in figure 1 reports the optimal default policies for the case without default costs ($\lambda = 0$). Graphs on the left of the figure show the optimal default policy if the current productivity state is high (z^h), and graphs on the right the optimal policy if the current state is low (z^l). Each graph depicts the optimal amount of default chosen for the next period, if the low productivity state will materialize (default is optimally zero if the high state materializes).[23] All graphs in the figure report default amounts on the y-axis as a function of the current (end-of period) net foreign asset position, as indicated on the x-axis.[24] To facilitate interpretation, the default amounts and the net foreign asset positions are both normalized by average GDP. A value of -1 on the x-axis, for example, corresponds to a situation where the government has issued explicit repayment claims equal to 100% of average GDP. Default policies are always depicted up to the maximum negative value of the net foreign asset position, i.e., up to the marginally binding natural borrowing limit (NBL) pertaining to the respective productivity state. Appendix A.5 explains how the optimal policies can be determined numerically.

For the case with zero default costs shown in the first row of figure 1, the optimal amount of default is independent of the country's net foreign asset position and almost independent of the current productivity state.[25] As is clear from proposition 2, these default policies fully insure future consumption against fluctuations in productivity. Of interest is the fact that the marginally binding natural borrowing limits (NBL) falls significantly once the repayment decision is treated as a decision variable: while the model has been calibrated such that the NBL is at most -100% of average GDP in a setting

[22] Recall that our estimate from section 4.1 was $\lambda^l = 7.5\%$, so that $\lambda = 10\%$ implies $\lambda^b = 2.5\%$.

[23] Since there exists a multiplicity of optimal default policies when $\lambda = 0$, the figure reports the poloicies for the limiting outcome $\lambda \to 0$.

[24] As explained in section 2.2.4, the net foreign asset position is given by the optimal value of b chosen for the corresponding period.

[25] From equation (10) follows that the default in the next period does depend on the current state because the optimal investment $k^*(z_t)$ depends on the current productivity, but this effect is quantitatively small.

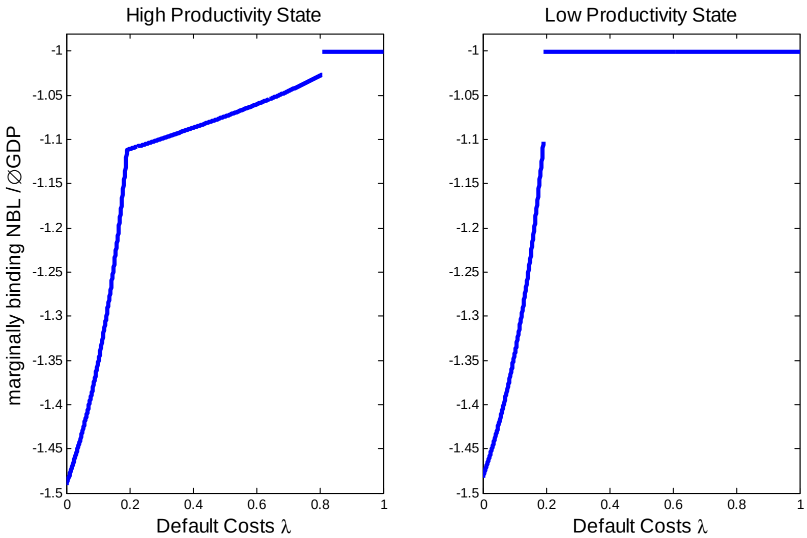

Figure 2: Default Costs and the Marginally Binding Natural Borrowing Limits

where debt has to be repaid always, figure 1 shows that the borrowing limits fall to approximately -150% of GDP, once one considers optimal repayment decisions.

As similar, albeit quantitatively smaller effect can be observed for higher levels of default costs, see the second and third row in the figure, which depict the outcomes for the case where the default cost λ equals 10% and 20% respectively. Importantly, however, for these (still relatively modest) levels of the default costs, default ceases to be optimal over a wide range of net foreign asset positions. Specifically, for $\lambda = 20\%$ it is optimal to default only if the net foreign asset position approaches the borrowing limit, otherwise insurance is exclusively provided by adjusting the international wealth position. Comparing the graphs on the right and left of figure 1 reveals that for positive default costs there is less default in the future, whenever the current productivity state happens to be low already today. Indeed, if the current productivity state is low, then default will never be optimal for the low state in the next period for $\lambda = 20\%$. This is optimal because the persistence of the low productivity state is larger than one minus the default costs, so that it it is cheaper to transfer wealth into future low states via precautionary savings.

Overall, figure 1 shows that moderate levels for the default costs shift the optimal policy strongly towards using adjustments in the international wealth position to insure domestic consumption. Moreover, for default costs of $\lambda = 20\%$ (and for higher values) the assumption of full repayment of non-contingent debt provides a reasonably good approximation to the optimal repayment policy: as the graphs in the last row of figure 1 reveal, default is then virtually never optimal and the borrowing limits approach those implied by an economy where default is ruled out by assumption.

The assumption of full repayment, however, provides an exact characterization of optimal repayment policies only for significantly higher default cost levels. This issue is explored further in figure 2, which depicts the marginally binding natural borrowing limits for the two productivity states as a function of the assumed default cost. The response of the borrowing limits to the default costs is characterized by two discontinuities. The first shows up at a default cost level of about $\lambda = 20\%$, when it becomes suboptimal to default in the next period if a low productivity state materializes and the current state is low. As mentioned before, for this cost level self-insurance becomes more attractive than insurance via a default, as the default costs exceed one minus reaching the state against which insurance is desirable. The borrowing limit for the low productivity state then tightens to the value emerging in the economy where repayment is assumed. The second discontinuity arises in the high productivity state at a default cost level of about $\lambda = 0.80\%$. At this point, the probability of reaching the low state also falls below one minus the probability of reaching the low state. Self-insurance is then again more desirable than default in a future low state. For the present setting, the assumption of no default is thus correct in an exact sense whenever default costs exceed a value of $\lambda = 80\%$. More generally, it will be correct if default costs exceed a level of one minus the probability of reaching an undesirable state. Thus in a setting where the probability of reaching an undesirable state is very low, default costs likely need to be close to the prohibitive level of 100% to make repayment optimal in all states.

5 Optimal Default and Economic Disasters

The previous section showed that for default cost levels close to our lower bound estimate it is suboptimal to default on government debt in a setting with business cycle shocks, provided the country is not too close to its borrowing limit. This section quantitatively evaluates to what extent this conclusion continues to be true in a setting with much larger economic shocks. Consideration of larger shocks is motivated by the observation that countries occasionally experience very large negative shocks, as previously argued by Rietz (1988) and Barro (2006), and that such shocks tend to be associated with a government default in the data.[26] To capture the possibility of large shocks, we augment the model by including disaster like shocks to aggregate productivity and then explore the quantitative implications of disaster risk for optimal sovereign default decisions.

5.1 Calibrating Economic Disasters

To capture economic disasters we introduce two disaster sized productivity levels to our aggregate productivity process. We add two disaster states rather

[26] Barro (2006) and Gourio (2010) allow for sovereign default in disaster states, but - due to the different focus of their analysis - consider exogenous default probabilities and default rates.

than a single one to capture the idea that the size of economic disasters is uncertain ex-ante. The inclusion of two disaster states also allows us to calibrate the disaster shocks in a way that they match both the mean and the variance of GDP disaster events. Using a sample of 157 GDP disasters, Barro and Jin (2011) report a mean reduction in GDP of 20.4% and a standard deviation of 12.64%. Assuming that it is equally likely to enter both disaster states from the 'normal' business cycle states, this yields the productivity levels $z^d = 0.9224$ for a medium sized disaster and $z^{dd} = 0.6696$ for a severe disaster. Our vector of possible productivity realizations thus takes the form $Z = \left\{z^h, z^l, z^d, z^{dd}\right\}$, where the parameterization of the business cycle states $\left(z^h, z^l\right)$ is the same as in section 4.2.

The state transition matrix for the shock process is described by the matrix

$$\pi = \begin{pmatrix} 0.7770 & 0.1850 & 0.019 & 0.019 \\ 0.1850 & 0.7770 & 0.019 & 0.019 \\ 0.1429 & 0.1429 & 0.3571 & 0.3571 \\ 0.1429 & 0.1429 & 0.3571 & 0.3571 \end{pmatrix},$$

where the transition probability from the business cycle states into the disaster states is chosen to match the unconditional disaster probability of 3.8% per year, as reported in Barro and Jin (2011). We thereby assume that both disaster states are reached with equal likelihood. The persistence of the disaster states is set to match the average duration of GDP disasters, which equals 3.5 years. For simplicity we assume that conditional on staying in a disaster, the medium and severe disaster state are equally likely to materialize. Finally, the transition probabilities of the business cycle states (13) are rescaled to reflect the transition probability into a disaster state.

Since the presence of disaster risk strongly affects the marginally binding natural borrowing limits (they become much tighter and potentially require even positive net foreign asset positions in all states), we recalibrate the subsistence level for consumption $\bar{c}$. As in section 4 before, we choose $\bar{c}$ such that in an economy where bonds must be repaid always, the economy can sustain a maximum net foreign debt position of 100% of average GDP in the business cycle states (z^h, z^l).[27] Choosing tighter limits does not affect the shape of the optimal default policies but only shifts the policies reported in the next subsection 'further to the right'.

5.2 Optimal Default with Disasters: Quantitative Analysis

Figure 3 reports the optimal default policies for the economy with disaster shocks, assuming default costs of $\lambda = 10\%$. Each panel in the figure corresponds to a different productivity state today and reports the intended amount of default for tomorrow's states z^l, z^d and z^{dd} as a function of the country's net foreign asset position today.[28]

[27] This yields an adjusted value of $\bar{c} = 0.1900$.

[28] Recall that default is never optimal if z^h realizes in the next period.

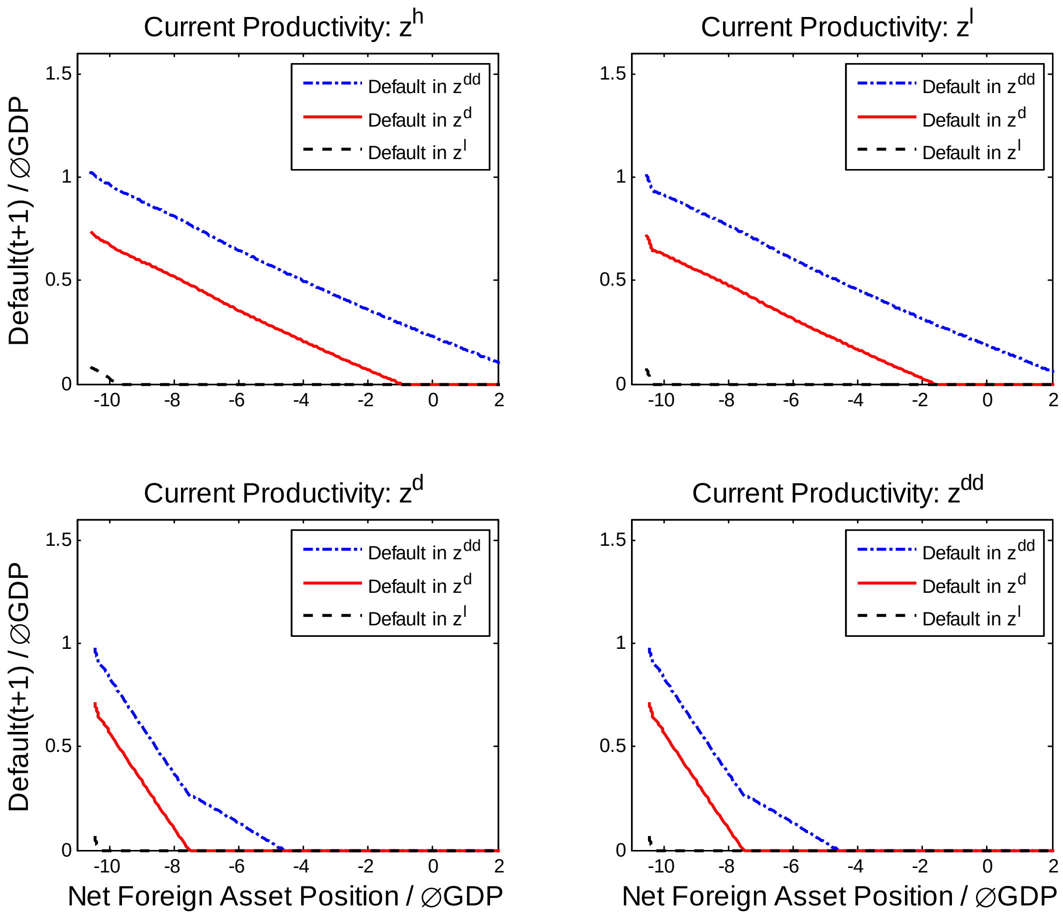

Figure 3: Optimal Default Policies with Disaster States ($\lambda = 0.1$)

Figure 3 reveals that in the presence of large shocks, the introduction of optimal repayment decisions relaxes the marginally binding borrowing limits even more than occurs in a setting with small shocks only. For the considered calibration the borrowing limit drops from -100% of average GDP for the case where repayment must occur always to a level of about -1,000% of average GDP. Figure 3 also shows that it is virtually never optimal to default in the low business cycle state (z^l), unless the net foreign debt position is very close to its maximally sustainable level, similar to section 4 where we considered business cycle shocks only. Furthermore, for a wide range of values for the net foreign asset position, it is optimal to default if the economy makes a transition from a business cycle state to a disaster state, see the top panels in the figure. Default is optimal for a transition to the severe disaster state (z^{dd}), even when the country's net foreign asset position is positive before the disaster. With a positive net foreign asset position, default can occur because the country chooses to issue sufficient amounts of own debt, which are invested in the foreign bond, with the sole purpose to be able to default on it, in case a transition to a severe disaster occurs. The optimal amount of default is thereby increasing as the country's net foreign asset position worsens. Interestingly, once the economy is in a disaster state, a further default in the event that the economy remains in the disaster state is optimal only if the net foreign asset position is very low, see the bottom panels of figure 3. Since the likelihood of staying in a disaster state is quite high, choosing not to repay if the disaster persists would have very high effects on interest rate costs ex-ante. As a result, serial default in case of a persistent disaster is only optimal if the net foreign asset position is sufficiently negative.

The overall shape of the optimal default policies is fairly robust to assuming different values for the default costs λ. Larger costs shift the default policies 'towards the left', i.e., default occurs only for more negative net foreign asset positions. However, higher costs also tighten the maximally sustainable net foreign asset positions, thereby reducing the range of net foreign asset positions over which default occurs. Lower cost have the opposite effect, i.e., they induce a rightward shift and allow to sustain more negative net foreign asset positions.

Figure 4 reports a typical sample path for the net foreign asset position and the amount of default induced by the optimal default policies shown in figure 3. The economy starts at a zero net foreign asset position and each model period corresponds to one year. The figure shows that it is optimal to improve the net foreign asset position when the economy is in the business cycle states, with faster improvements being optimal in the high business cycle state, see for example the dynamics occurring in years 150-180 in figure 4. This induced by precautionary motives and occurs even if the international risk free rate is set 5 basis points below the inverse of the domestic discount factor, so as to insure that the international wealth position remains stationary.

As figure 4 shows, a transition to the severe disaster state usually leads to a default, provided the transition does not occur via the medium disaster

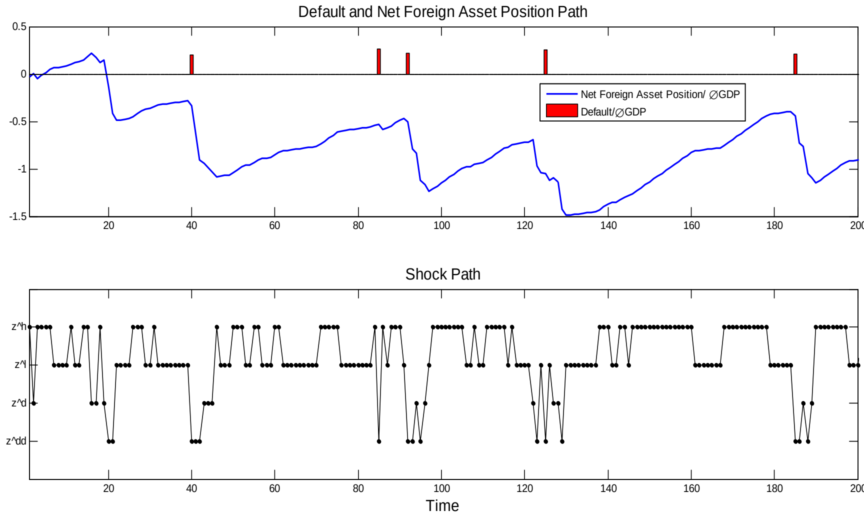

Figure 4: Net Foreign Asset Dynamics under Optimal Default Policy ($\lambda = 0.1$)

state first (as is the case for year 20). Also, following a disaster, the net foreign asset position deteriorates, whenever the disaster persists for more than one period (see for example year 40), otherwise the net foreign asset position is largely unaffected or improves even slightly (see year 85). This shows that for the considered level of default cost, default provides only partial insurance against disaster risk. As a result, the net foreign asset dynamics are typically characterized by rapid deteriorations during persistent disaster periods and gradual improvements during normal times, with the latter accelerating during high business cycles states.

6 Welfare Analysis

This section determines the welfare effects of letting the government choose whether or not to repay its debt compared to a situation where repayment is simply forced upon the government (or assumed) in each state.

We base our welfare comparison on the model with disaster states from section 5 and consider a broad range of default costs. We evaluate the utility consequences in terms of welfare equivalent consumption changes over the first 500 years, comparing our optimal default model to a situation in which repayment of bonds is assumed to occur in all contingencies. Specifically, letting c_t^1 denote the optimal state contingent consumption path in the no-default economy and c_t^2 the corresponding consumption path with (costly) optimal default, we report for each level of default costs the welfare equivalent

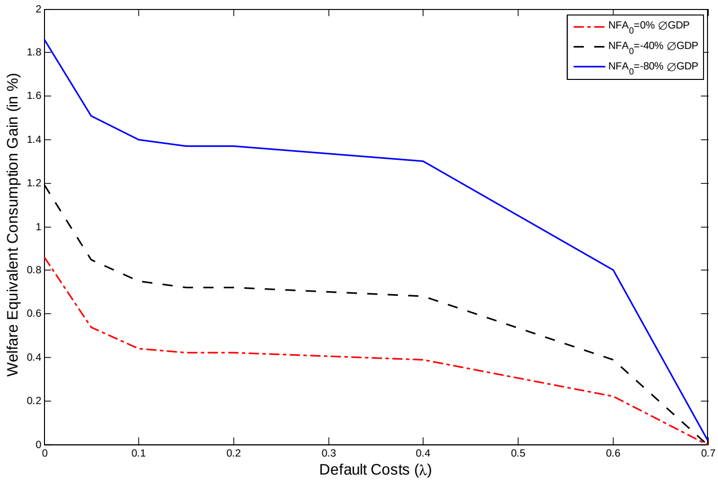

Figure 5: Welfare Gains from Using the Default Option

consumption change ω solving

$$E_0\left[\sum_{t=0}^{500}\beta^t\frac{((c_t^1(1+\omega)-\bar{c}))^{1-\gamma}}{1-\gamma}\right]=E_0\left[\sum_{t=0}^{500}\beta^t\frac{(c_t^2-\bar{c})^{1-\gamma}}{1-\gamma}\right]$$

where the expectations are evaluated by averaging over 10000 sample paths. To highlight the effects of the country's initial international wealth position for the welfare results, we consider three scenarios with different values for the net foreign asset to average GDP ratio: 0%, -40% and -80%. Figure 5 reports the consumption equivalent welfare gains implied by optimal default position for these different starting values of the debt ratio as a function of the default costs, which are shown on the x-axis of the figure. Depending on the initial wealth position, the welfare gains amount to 0.5-1.5% of consumption each period for a broad range of default costs, with the gains increasing further at low default cost levels. The welfare gains are surprisingly robust over a wide range of default costs, instead are more sensitive to the initial net foreign asset position. For default costs $\lambda \geq 0.5$ the welfare gains from default decrease steeply and for $\lambda \geq 0.7$ there exist no welfare gains anymore. For such high levels of the default cost it becomes suboptimal to insure against a future disaster state when the economy is already in a disaster, independently of the country's net foreign asset position. This is shown in the lower panel of figure 6 which reports the optimal default policies when $\lambda = 0.7$. For $\lambda = 0.7$, the government receives only 0.3 units of consumption for each unit of default. Since the likelihood of a specific disaster state (either z^d or z^{dd}) to re-occur is 0.3571, the cost of using the default option for any of these states is $0.3571/(1 + r) > 0.3$. Therefore, use of the default option is dominated by use of precautionary savings for transferring resources into a future

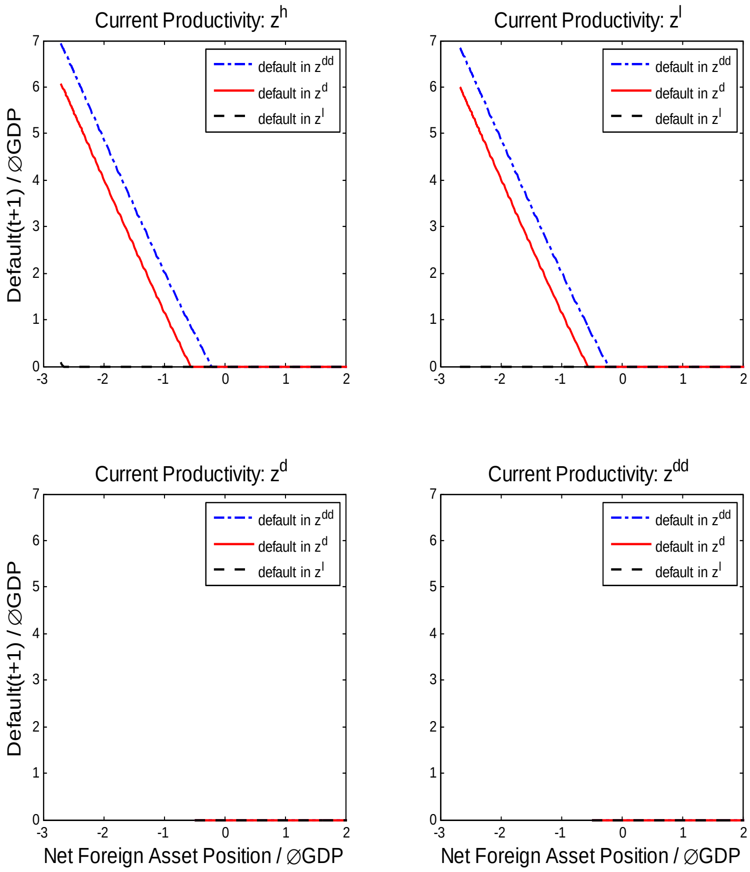

Figure 6: Optimal Default Policy with Disaster States and High Default Cost ($\lambda = 0.7$)

disaster state. Full repayment, therefore, is optimal in all future states, once the economy has hit a disaster state. As a result, the borrowing constraints tighten significantly in the disaster states[29] and the required amount of insurance in the business cycle state (z^l, z^h) increases strongly as the net foreign asset position deteriorates, see figure 6. While some insurance via default is still optimal for such high cost levels, it does not lead to sizable welfare gains because high default costs imply that is it also very costly to using the insurance mechanism.

[29] They reach the levels applying in the economy with non-defaultable bonds.

7 Long Maturities and Optimal Bond Repurchase Programs

We now discuss the effects of introducing domestic bonds with longer maturities.[30] Issuing long bonds can offer an advantage over issuing just short bonds, as has been assumed thus far, whenever the market value of long bonds reacts to domestic conditions in a way that allows the government to insure against domestic shocks in a way that is less costly than a default on short bonds. It would be desirable, for example, that the market value of outstanding long bonds decreases following a disaster shock, so that the government realizes a capital gain that lowers the overall debt burden. Capital gains, however, do not materialize if repayment occurs in all future states, unlike in the setting studied by Angeletos (2002). The depreciation of the debt's market value, thus, can only be induced via the anticipation of a default event in the future.

Issuing long bonds will offer an advantage against the outright default on maturing short bonds, whenever the (transaction) costs associated with repurchasing outstanding long bonds at the devaluated market price are lower than the costs induced by a default event on maturing short bonds. If both costs are identical, i.e., if the repurchase in a situation where default is anticipated in the future is associated with the same costs as a default on maturing bonds, then there will be of no additional value associated with issuing long bonds. The same holds true, if the costs of a repurchase are higher than those of an outright default. When these costs are lower, however, then there exist advantages from issuing long bonds. Consider, for example, the situation where a repurchase of long bonds does not give rise to any costs. The government will then find it optimal to fully insure domestic consumption, i.e., achieve the first best allocation, independently of the costs associated with an outright default on maturing bonds. The optimal bond issuance strategy will then have the feature that the government issues each period long bonds that (partially) default at maturity, depending on the productivity realization in some earlier period. The default at maturity needs to be calibrated such that the interim capital gains that realize fully insure domestic consumption against domestic productivity shocks, i.e., satisfies the proportionality restriction (10). The government can then repurchase the existing stock of long bonds, fully avoid default costs, and issue new long bonds with a new implicit contingent repayment profile so as to insure against future shocks. In this way outright default on maturing bonds never occurs.

8 Conclusions

In a setting with incomplete government bond markets, debt default is part of the optimal government policy under commitment. The choice whether or not to repay maturing debt allows for increased international risk sharing and significantly relaxes the net foreign debt positions that a country can sustain. It also considerably increases welfare, even when default costs are sizable.

[30] Introducing longer maturities of risk-free foreign debt has no consequences for the optimal outcomes.

Besides providing a normative benchmark, the present analysis with a committed sovereign may also offer a more plausible positive description of actual government default policies than is generally recognized in the economics literature. First, although there now exists a voluminous literature on potential mechanisms supporting sovereign debt in the absence of commitment, these mechanisms have received somewhat limited empirical support and have difficulties in quantitatively accounting for the large volume of outstanding government debt.[31] From the viewpoint of the present setting with commitment, outstanding debt levels do not appear implausibly high. Second, unlike in a setting without commitment, where default incentives are strongest following a positive innovation to domestic income (the threat of a market exclusion is then least deterring) and where consequently default tends to occur following positive income shocks [32], default incentives in the present setting are strongest following a negative shock to domestic output. As a result default occurs following a negative shocks, which is more in line with the empirical evidence. Finally, pointing towards the fact that a country has sufficient resources to repay but nevertheless chooses to default, is fully consistent with the optimal default policy of a fully committed government.

In the light of these facts, it appears natural to deduce that governments can issue debt simply because they can in fact credibly commit to repay debt in some future states of the world, although they might optimally *choose* not to repay in some states in which repayment turns out to be excessively costly.

A Appendix

A.1 Default Costs Born by Lender

This appendix shows that if an allocation is feasible in a setting in which the settlement costs associated with a default are born by the borrower, then it is also feasible in a setting in which some or all of these costs are born by the lender. For simplicity, we only consider the extreme alternative where all costs are born by the lender. Intermediate cases can be covered at the costs of some more cumbersome notation.

Consider a feasible choice $\{G_t^L \geq 0,\, G_t^S \geq 0,\, \Delta_t \in [0,1]^N,\, k_t \geq 0,\, c_t \geq \bar{c}\}_{t=0}^{\infty}$, i.e., a choice that satisfies the constraints of the government's problem (4), which assumes $\lambda^l = 0$ and $\lambda^b = \lambda$. Let variables with a tilde denote the corresponding choices in a setting in which the lender bears the settlement costs, i.e., where $\lambda^l = \lambda$ and $\lambda^b = 0$. We show below that it is then feasible to choose the same real allocation, i.e., to choose $\widetilde{k}_t = k_t$ and $\widetilde{c}_t = c_t$, provided one selects appropriate values for $\widetilde{G}_t^L$, $\widetilde{G}_t^S$ and $\widetilde{\Delta}_t$.

[31] In the words of Panizza et al. (2009): *'Almost three decades after Eaton and Gersovitz' pathbreaking contribution there still exists no fully satisfactory answer to how sovereign debt can exist in the first place. None of the default punishments that the classic theory of sovereign debt has focused on appears to enjoy much empirical backing' (p. 692).*

[32] In some papers assuming limited commitment, e.g., Arellano (2008), this feature is masked by the presence of state-congtingent default costs, which tilt the default profile towards low output states.

First, note that in a setting where foreign investors bear the settlement costs, the discount rate $\widetilde{R}(z_t, \widetilde{\Delta}_t)$ on domestic bonds satisfies

$$1 + r = \frac{1 - (1+\lambda)\sum_{n=1}^{n} \widetilde{\delta}_t^n \Pi(z^n|z_t)}{\frac{1}{1+\widetilde{R}(z_t,\widetilde{\Delta}_t)}} \tag{14}$$

where the denominator on the right-hand side denotes the issuance price of the bond and the numerator the expected repayment net of the lender's settlement cost. The previous equation thus equates the expected returns of the domestic bonds with the expected return on the foreign bond.

Next, consider the following financial policies:[33]

$$\widetilde{\Delta}_t = (1-\lambda)\Delta_t \frac{G_t^S}{\widetilde{G}_t^S} \tag{15a}$$

$$\widetilde{G}_t^S = \frac{1+\widetilde{R}(z_t,\widetilde{\Delta}_t)}{\widetilde{R}(z_t,\widetilde{\Delta}_t)} \frac{R(z_t,\Delta_t)}{1+R(z_t,\Delta_t)} G_t^S \tag{15b}$$

$$\widetilde{G}_t^L = G_t^L + \widetilde{G}_t^S - G_t^S \tag{15c}$$

As we show below, in a setting in which settlement costs are born by the lender, the financial policies $\{\widetilde{G}_t^L, \widetilde{G}_t^S, \widetilde{\Delta}_t\}_{t=0}^{\infty}$ give rise to the same state-contingent financial payoffs as generated by the policies $\{G_t^L, G_t^S, \Delta_t\}_{t=0}^{\infty}$ in a setting in which settlement cost are born by the borrower. Therefore, as claimed, the former policies allow to implement the same real allocations.

Consider the financial flows generated by the policy component $(\widetilde{G}_t^L, \widetilde{G}_t^S, \widetilde{\Delta}_t)$. In period t, the financial inflows are given by

$$\frac{\widetilde{G}_t^S}{1+\widetilde{R}(z_t,\widetilde{\Delta}_t)} - \widetilde{G}_t^L$$

Using the definitions (15), it is straightforward to show that these inflows are equal to

$$\frac{1}{1+R(z_t,\Delta_t)} G_t^S - G_t^L$$

which are the inflows under the policy (G_t^L, G_t^S, Δ_t) in a setting where settlement costs are born by the lender.

We show next, that the financial flows in $t+1$ are also identical under the two policies. The financial inflows generated by the policy choices $(\widetilde{G}_t^L, \widetilde{G}_t^S, \widetilde{\Delta}_t)$ in some future contingency $n \in \{1, ...N\}$ in period $t+1$ are given by

$$-\widetilde{G}_t^S(1-\widetilde{\delta}_t^n) + \widetilde{G}_t^L$$

[33] Lengthy but straightforward calcucations, which are available upon request, show that these policies satisfy $\widetilde{\Delta}_t \in [0,1]^N$, although they may imply $\widetilde{G}_t^L < 0$, which requires the government to issue also safe bonds, i.e., bonds that promise full repayment in the explicit and implicit component of their contract.

From the first and last equation in (15), we obtain that these flows are equal to

$$-(1-(1-\lambda)\delta_t^n)G_t^S + G_t^L$$

which are the inflows generated by the policy (G_t^L, G_t^S, Δ_t) in a setting where settlement costs are born by the lender.

Finally, since the policy $\{G_t^L, G_t^S, \Delta_t\}_{t=0}^{\infty}$ satisfies the marginally binding natural borrowing limits in the government's problem (4), it will generate bounded financial flows, so that the same applies for the policy $\{\widetilde{G}_t^L, \widetilde{G}_t^S, \widetilde{\Delta}_t\}_{t=0}^{\infty}$. The latter will thus also satisfy the marginally binding natural borrowing limits of the problem in which settlement cost are born by the lender, which completes the proof.

A.2 Proof of Proposition 1

Consider some state contingent beginning-of-period wealth profile w_t arising from some combination of bond holdings, default decisions and capital investment $(G_{t-1}^L,\ G_{t-1}^S,\ \Delta_{t-1},\ k_{t-1})$ in problem (4). We now show that one can generate the same state contingent beginning-of-period wealth profile $\widetilde{w}_t = w_t$ in problem (7) by choosing $\widetilde{k}_{t-1} = k_{t-1}$ and by choosing an appropriate investment profile (a_{t-1}, b_{t-1}). Moreover, the funds required to purchase (a_{t-1}, b_{t-1}) are the same as those required to purchase $\left(G_{t-1}^L, G_{t-1}^S\right)$ when the default profile is Δ_{t-1}. With the costs of financial investments being the same in both problems, identical physical investments, and identical beginning of period wealth profiles, it then follows from constraints (4b) and (7b) that the implied consumption paths are also the same in both problems, establishing the equivalence between the two problems.

To keep notation as simple as possible we establish the previous claim for the case with 2 productivity states only. The extension to N states is relatively straightforward. Consider the following state contingent initial wealth profile

$$\begin{pmatrix} w_t(z^1) \\ w_t(z^2) \end{pmatrix} = \begin{pmatrix} z^1 k_{t-1}^{\alpha} + G_{t-1}^L - G_{t-1}^S(1-(1-\lambda)\delta_{t-1}^1) \\ z^2 k_{t-1}^{\alpha} + G_{t-1}^L - G_{t-1}^S(1-(1-\lambda)\delta_{t-1}^2) \end{pmatrix}.$$

One can replicate this beginning-of-period wealth profile in problem (7) by choosing $\widetilde{k}_{t-1} = k_{t-1}$ and by choosing the portfolio

$$b_{t-1} = G_{t-1}^L - G_{t-1}^S, \tag{16}$$

$$a_{t-1} = \begin{pmatrix} G_{t-1}^S \delta_{t-1}^1 \\ G_{t-1}^S \delta_{t-1}^2 \end{pmatrix} \tag{17}$$

The previous equations show that b in problem (7) has an interpretation as the net foreign asset position in problem (4) and that a in problem (7) can be interpreted as the state contingent default on outstanding own bonds. We will make use of this interpretation in the latter part of the paper. The funds f_{t-1} required for (G_{t-1}^L, G_{t-1}^S) under the default profile $(\delta_{t-1}^1, \delta_{t-1}^2)$ are given by

$$f_{t-1} = \frac{1}{1+r} G_{t-1}^L - \frac{1}{1+R(z_{t-1}, (\delta_{t-1}^1, \delta_{t-1}^2))} G_{t-1}^S$$

where the interest rate satisfies

$$\frac{1}{1+R(z_{t-1},(\delta^1_{t-1},\delta^2_{t-1}))} = \frac{1}{1+r}\left((1-\delta^1_{t-1})\pi(z^1|z_{t-1})+(1-\delta^2_{t-1})\pi(z^2|z_{t-1})\right).$$

The funds $\widetilde{f}_{t-1}$ required to purchase (b_{t-1},a_{t-1}) are

$$\widetilde{f}_{t-1} = \frac{1}{1+r}(G^L_{t-1}-G^S_{t-1}) + \frac{1}{1+r}\left(\delta^1_{t-1}\pi(z^1\,|z_{t-1})+\delta^2_{t-1}\pi(z^2\,|z_{t-1})\right)G^S_{t-1},$$

where we used the price of the Arrow security in (5). As is easy to see $\widetilde{f}_{t-1} = f_t$, as claimed.

Finally, note that we need to impose the restriction $a \geq 0$ on problem (7), as otherwise it would follow from equation (17) that one could implement a consumption path in problem (7) that cannot be implemented in problem (4) with values of δ^i satisfying $\delta^i \in [0,1]$ for all i. This completes the equivalence proof.

A.3 First Order Equilibrium Conditions

This appendix derives the first order conditions for problem (7). We first rewrite the problem replacing beginning-of-period wealth by components (see definition (6)):

$$\begin{aligned}
\max_{\left\{b_t,a_t\geq 0,\widetilde{k}_t\geq 0,\widetilde{c}_t\geq\bar{c}\right\}} \quad & E_0\sum_{t=0}^{\infty}\beta^t u(\widetilde{c}_t) \\
s.t.\quad \forall t: \widetilde{c}_t = {} & z_t\widetilde{k}^{\alpha}_{t-1}+b_{t-1}+(1-\lambda)a_{t-1}(z_t) \\
& -\widetilde{k}_t - \frac{1}{1+r}b_t - p_t\cdot a_t \\
z_{t+1}\widetilde{k}^{\alpha}_t + b_t + (1-\lambda)a_t(z_{t+1}) \geq {} & NBL(z_{t+1}) \quad \forall z_{t+1}\in Z \\
\widetilde{w}_0 = w_0, z_0 \quad & given,
\end{aligned}$$

Next, we formulate the Lagrangian and let η_t denote the multiplier on the budget constraint in period t, $\nu^{z^n}_t$ the multiplier for the short-selling constraint on the Arrow security that pays off in state z^n in $t+1$, and ω_{t+1} the multiplier associated with the natural borrowing limits. We drop the inequality constraints for $\widetilde{k}_t$ and $\widetilde{c}_t$, as the Inada conditions guarantee an interior solution for these variables. Differentiating the Lagrangian with respect to the choice variables one obtains

$$\begin{aligned}
\widetilde{c}_t: &\quad u'(\widetilde{c}_t) - \eta_t = 0 \\
b_t: &\quad -\eta_t\frac{1}{1+r} + \beta E_t\eta_{t+1} + \beta E_t\omega_{t+1} = 0 \\
a_t(z^n): &\quad -\eta_t p_t(z^n) + \beta\pi(z^n|z^t)\eta_{t+1}(z^n)(1-\lambda) \\
&\quad +\nu^{z^n}_t + \beta\pi(z^n|z^t)\omega_{t+1}(z^n)(1-\lambda) = 0 \quad \forall n\in N \\
\widetilde{k}_t: &\quad -\eta_t + \alpha\widetilde{k}^{\alpha-1}_t\beta E_t\eta_{t+1}z_{t+1} + \alpha\widetilde{k}^{\alpha-1}_t\beta E_t\omega_{t+1}z_{t+1} = 0
\end{aligned}$$

Using the FOC for consumption to replace η_t in the last three FOCs, one obtains Euler equations for the bond holdings, the Arrow securities and capital investment:

$$
\begin{aligned}
Bond: & -u'(\widetilde{c}_t)\frac{1}{1+r} + \beta E_t u'(\widetilde{c}_{t+1}) + \beta E_t \omega_{t+1} = 0 && \text{(18a)}\\
Arrow: & -u'(\widetilde{c}_t)p_t(z^n) + \beta\pi(z^n|z_t)u'(\widetilde{c}_{t+1}(z^n))(1-\lambda) \\
& +\nu_t^{z^n} + \beta\pi(z^n|z^t)\omega_{t+1}(z^n)(1-\lambda) = 0 \quad \forall n \in N && \text{(18b)}\\
Capital: & -u'(\widetilde{c}_t) + \alpha\widetilde{k}_t^{\alpha-1}\beta E_t u'(\widetilde{c}_{t+1})z_{t+1} + \alpha\widetilde{k}_t^{\alpha-1}\beta E_t \omega_{t+1} z_{t+1} = 0 && \text{(18c)}
\end{aligned}
$$

In addition, the Kuhn-Tucker FOCs include the following complementarity conditions:

$$
\begin{aligned}
& 0 \leq a_t(z^n) \quad \perp \quad \nu_t^{z^n} \geq 0 \ \ \forall n \in N && \text{(18d)}\\
& 0 \leq z^n \widetilde{k}_t^{\alpha} + b_t + (1-\lambda)a_t^{z^n} - NBL(z^n) \quad \perp \quad \omega_{t+1}(z^n)) \geq 0 \ \forall n \in N. && \text{(18e)}
\end{aligned}
$$

Combined with the budget constraint, the Euler equations and the complementarity conditions constitute the optimality conditions for problem (7).

A.4 Proof of Proposition 2

We first show that the proposed consumption solution (8) satisfies the budget constraint, that the inequality constraints $a \geq 0$ are not binding, and that the NBLs are also not binding. Thereafter, we show that the remaining first order conditions of problem (7), as derived in appendix A.3, also hold.

We start by showing that the portfolio implementing (8) in period $t = 1$ is consistent with the flow budget constraint and $a \geq 0$. The result for subsequent periods follows by induction. In period $t = 1$ with productivity state z^n, beginning-of-period wealth under the optimal capital investment strategy (9) is

$$\widetilde{w}_1^n \equiv z^n \left(k^*(z_0)\right)^{\alpha} + b_0 + a_0(z^n) \tag{19}$$

To insure that consumption can stay constant from $t = 1$ onwards we need again

$$c = (1-\beta)(\Pi(z^n) + \widetilde{w}_1^n) \tag{20}$$

for all possible productivity realizations $n = 1, ..N$. This provides N conditions that can be used to determine the $N+1$ variables b_0 and $a_0(z^n)$ for $n = 1, ..., N$. We also have the condition $a_0(z^n) \geq 0$ for all n and by choosing $\min_n a_0(z^n) = 0$, we get one more condition that allows to pin down a unique portfolio (b_0, a_0). Note that the inequality constraints on a do not bind for the portfolio choice, as we have one degree of freedom, implying that the multipliers $v_1^{z^n}$ in appendix A.3 are all zero. It remains to show that the portfolio achieving (20) is feasible given the initial wealth $\widetilde{w}_0$. Using (19) to substitute $\widetilde{w}_1^n$ in equation (20) we get

$$c = (1-\beta)(\Pi(z^n) + z^n \left(k^*(z_0)\right)^{\alpha} + b_0 + a_0(z^n)) \forall n = 1, ...N.$$

Combining with (8) we get

$$\Pi(z^n) + z^n \left(k^*(z_0)\right)^\alpha + b_0 + a_0(z^n) = \Pi(z_0) + \widetilde{w}_0$$

Multiplying the previous equation with $\pi(z^n|z_0)$ and summing over all n one obtains

$$E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + b_0 + \sum_{n=1}^{N} \pi(z^n|z_0) a_0(z^n) = \Pi(z_0) + \widetilde{w}_0.$$

Using $\Pi(z_0) = -k^*(z_0) + \beta E_0 \left[z_1 \left(k^*(z_{t+j})\right)^\alpha\right] + \beta E_0 \left[\Pi(z_1)\right]$ and (5) the previous equation delivers

$$(1-\beta) E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + b_0 + (1+r) p_0 a_0 = -k^*(z_0) + \widetilde{w}_0$$

Using $\beta = 1/(1+r)$ this can be written as

$$\begin{aligned} &(1-\beta) E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] \\ &+ \frac{1-\beta}{\beta} p_0 a_0 + (1-\beta) b_0 + \frac{1}{1+r} b_0 + p_0 a_0 = -k^*(z_0) + \widetilde{w}_0 \end{aligned} \tag{21}$$

From (20) follows that the first terms in the previous equation are equal to

$$(1-\beta) E_0 \left[\Pi(z_1) + z_1 \left(k^*(z_0)\right)^\alpha\right] + \frac{1-\beta}{\beta} p_0 a_0 + (1-\beta) b_0 = c$$

so that (21) is just the budget equation for period zero. The portfolio giving rise to (20) in $t = 1$ thus satisfies the budget constraint of period zero. The results for $t \geq 1$ follow by induction.

It then follows from equation (20) that $\widetilde{w}_t$ is bounded, as $\Pi(z_t)$ is bounded, so that the process for beginning-of-period wealth does not involve explosive debt. The NBLs are thus not binding so that the multipliers $\omega_{t+1} = 0$ for all t and all contingencies.

Using $v_t^{z_n} \equiv 0$, $\omega_{t+1} \equiv 0$, the fact that capital investment is given by (9) and that the Arrow security price is (5), the Euler conditions (18a) - (18c) then all hold when consumption is given by (8). This completes the proof.

A.5 Numerical Solution Approach

To compute recursive equilibria for Problem 7 we apply a global solution method as to account for the non-linear default policies in our model. As endogenous state variable we use beginning-of-period wealth, defined as above. Combined with exogenous productivity shocks we define our state space S to be

$$S = \left\{z^1 \times \left[NBL(z^1), w_{max}\right], ..., z^N \times \left[NBL(z^N), w_{max}\right]\right\}$$

where we set w_{max} such that in equilibrium optimal policies never imply wealth values above this threshold. The NBLs are set such they are marginally binding. How these values are derived is shown in Appendix A.6.

We want to describe equilibrium in terms of time-invariant policy functions that map the current state into current policies. Hence, we want to compute policies

$$\widetilde{f} : (z_t, w_t) \rightarrow (\{c_t, k_t, b_t, a_t\}) \,,$$

where their values (approximately) satisfy the equilibrium conditions derived above. We use a time iteration algorithm where equilibrium policy functions are approximated iteratively. In a time iteration procedure, one takes tomorrow's policy (denoted by f^{next}) as given and solves for the optimal policy today (denoted by f) which in turn is used to update the guess for tomorrow's policy. Convergence is achieved once $||f - f^{next}|| < \epsilon$ and we set $\widetilde{f} = f$. In each time iteration step we solve for optimal policies on a sufficient number of grid points distributed over the continuous part of the state space. Between grid points we use linear splines to interpolate tomorrow's consumption policy. Following Garcia and Zangwill (1981), we can transform the complementarity conditions of our first order equilibrium conditions into equations. To solve for a root of the resulting non-linear equation system at a particular grid point we use Ziena's Knitro, an optimization software that can be called from MatLab. For more details on the time iteration procedure and how one transforms complementarity conditions into equations, see for example, Brumm and Grill (2010). To come up with a starting guess for the consumption policy we use the fact that at the NBLs optimal consumption equals the subsistence level. We therefore guess a convex, monotonically increasing function g which satisfies $g(z^i, NBL(z^i)) = \bar{c} \; \forall i$ and use a reasonable value for $g(z^i, w_{max})$.

A.6 The Marginally Binding Natural Borrowing Limits (NBLs)

This appendix explains how we compute the state-contingent marginally binding NBLs that we use as lower bounds for the state space in our numerical solution approach. We also prove that the marginally binding NBLs are unique and that if beginning-of-period wealth ever falls short of them in some contingency, there is a positive probability that debt dynamics will violate *any* finite debt limit, independently of how lax it is chosen.

To simplify the exposition, we consider a setting with just two productivity levels ($N = 2$) and order these such that $z^1 > z^2$. We also suppress time subscripts for the moment. The extension to more productivity states is straightforward, as we explain below. Let $NBL(z^i)$ denote the marginally binding NBL in productivity state $i = 1, 2$. The marginally binding NLBs solve the following problem

$$\begin{aligned} NBL(z^i) = \arg\max w(z^i) \;\; s.t. & \\ w'(z^j) \geq NBL(z^j) \text{ for } j = 1, 2 & \end{aligned} \tag{22a}$$

for $i = 1, 2$, where $w(z^i)$ denotes beginning-of-period wealth in state z^i and $w'(z^j)$ the beginning-of-period wealth in the next period if the productivity state is z^j. Marginally binding NBLs can thus be interpreted as a set of state-contingent *minimum* beginning-of-period wealth levels, such that beginning-of-period wealth in all future states remains above the limits defined by the set of state-contingent marginally binding NBLs. As we show below, the fixed point problem defined by the system of optimization problems (22a) has a unique solution.

Using the budget constraint from the equivalent formulation of the decision problem (7), beginning of period wealth in state z^i has the following uses

$$w(z^i) = c(z^i) + k(z^i) + \frac{1}{1+r} \cdot b(z^i) + p^2(z^i) \cdot a(z^i)$$

where $p^2(z^i)$ denotes the price of the Arrow security that pays one unit of the consumption good in state z^2.[34] Given the choices $\big(c(z^i), k(z^i), b(z^i), a(z^i)\big)$, future beginning of period wealth levels are

$$\begin{aligned} w'(z^1) &= z^1 k(z^i)^\alpha + b(z^i) \\ w'(z^2) &= z^2 k(z^i)^\alpha + b(z^i) + (1-\lambda) a(z^i) \end{aligned}$$

Since $c(z^i)$ does not affect future beginning-of-period wealth levels, it is optimal to choose the lowest possible level of consumption in (22a), i.e., $c(z^i) = \bar{c}$. The Lagrangian of the maximization problem (22a) can thus be written as

$$\begin{aligned} L = {} & \bar{c} + k(z^i) + \frac{1}{1+r} \cdot b(z^i) + p^2(z^i) \cdot a(z^i) \\ & + \lambda^1(z^i) \left(z^1 k(z^i)^\alpha + b(z^i) - NBL(z^1)\right) \\ & + \lambda^2(z^i) \left(z^2 k(z^i)^\alpha + b(z^i) + (1-\lambda) a(z^i) - NBL(z^2)\right) \end{aligned}$$

The first order conditions of this problem are[35]

$$\begin{aligned} 1 + \lambda^1(z^i) \alpha z^1 k(z^i)^{\alpha-1} + \lambda^2(z^i) \alpha z^2 k(z^i)^{\alpha-1} &= 0 \\ \frac{1}{1+r} + \lambda^1(z^i) + \lambda^2(z^i) &= 0 \\ p^2(z^i) + \lambda^2(z^i)(1-\lambda) &= 0 \end{aligned}$$

and imply after eliminating Lagrange multipliers in the first equation:

$$k^*(z^i) = \left(\frac{1}{\alpha \left(\frac{1}{1+r} z^1 + \frac{p^2(z^i)}{1-\lambda} (z^2 - z^1) \right)} \right)^{\frac{1}{\alpha-1}}$$

[34] Since $z^1 > z^2$, we can ignore the Arrow security for state z^1 as use of this security is either inefficient (if default costs are positive) or the asset is redundant (in the absence of default costs).

[35] Since the objective is linear and the constraint set convext, first order conditions are necessary and sufficient.

where a starred variable denotes the value of the optimal solution. The constraints then imply

$$\begin{aligned} b^*(z^i) &= NBL(z^1) - z^1 k^*(z^i)^\alpha \\ a^*(z^i) &= \frac{NBL(z^2) - z^2 k^*(z^i)^\alpha - b^*(z^i)}{(1-\lambda)} \end{aligned}$$

Since (k^*, b^*, a^*) are either independent of the marginally binding NBLs or linear functions thereof, the system of equations

$$NBL(z^i) = \bar{c} + k^*(z^i) + \frac{1}{1+r} \cdot b^*(z^i) + p^2(z^i) \cdot a^*(z^i)$$

for $i = 1, 2$ has a unique solutions for the marginally binding $NBL(z^i)$. An extension of this approach to more productivity states is straightforward (though notationally more tedious) as the structure of the Lagrangian remains unchanged when adding more productivity states.

We now prove that violating the marginally binding NBLs computed above in some contingency necessarily leads to unboundedly negative beginning-of-period wealth dynamics and violates any finite NBL, no matter how loosely it is set, with positive probability. Suppose that in some productivity state z^i and some period t, beginning-of-period wealth falls short of the limits implied by the marginally binding NBL computed above, i.e.,

$$w_t(z^i) = NBL(z^i) - \varepsilon \tag{23}$$

for some $\varepsilon > 0$. We then prove below that for at least one contingency z^j in $t + 1$ it must hold that

$$w_{t+1}(z^j) \leq NBL(z^j) - \varepsilon(1 + r) \tag{24}$$

so that for this contingency the distance to the marginally binding NBL is increasing. Since the same reasoning applies also for future periods, and since the marginally binding NBLs assume finite values, this implies the existence of a contingency along which future wealth far in the future becomes unboundedly negative, implying that any finite natural borrowing limit will be violated with positive probability.

It remains to prove that if (23) holds in period t and contingency z^i, this implies that (24) holds for some contingency z^j in $t + 1$. Suppose for contradiction that

$$w_{t+1}(z^h) > NBL(z^h) - \varepsilon(1 + r) \tag{25}$$

for all h can be achieved. The cost minimizing way to satisfy the constraints (25) for all h is to choose $a^*(z^i), k^*(z^i)$ and $\widetilde{b} = b^*(z^i) - \gamma$ for some $\gamma < \varepsilon$.[36] This, however, is not a feasible choice because it would require beginning-of-period t wealth strictly larger than (23), which concludes the proof.

[36] To verify this just solve the minimization problem (22a) where the constraints on future wealth are replaced by (25).

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