

Can Expectations Be Informative about Interest Rate Volatility?

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Abstract

Forecasting volatility of interest rates remains a challenge in finance. An important aspect of any dynamic model of volatility is the fact that volatility is a positive process, not only with respect to the historical measure, but also with respect to the risk neutral measure. As a consequence, risk neutral forecasts of volatility must also always remain positive. One way for this admissibility condition to hold is for volatility to represent an autonomous process under the risk neutral measure. In contrast to the historical time series, the cross section of bond yield provides very precise information about risk neutral forecasts and therefore strong guidance to which combinations of yields give are autonomous. We conclude that in the stochastic volatility setting, the no arbitrage assumption provides strong over-identifying constraints which, when the model is correctly specified, improve inference on the volatility instrument. We also consider alternative models which relax these restrictions.

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Introduction

Forecasting volatility of interest rates remains an important challenge in finance.¹ A rich body of literature has shown that the volatility of the yield curve is, at least partially, related to the shape of the yield curve. For example, volatility of interest rates is usually high when interest rates are high and when the yield curve exhibits higher curvature (see [Cox, Ingersoll, and Ross \(1985\)](#), [Litterman, Scheinkman, and Weiss \(1991\)](#), and [Longstaff and Schwartz \(1992\)](#), among others). This suggests that the shape of the yield curve is a potentially useful instrument for forecasting volatility. In this paper, we assess whether standard no arbitrage affine term structure models can improve the precision of estimation of volatility instruments.

We find that instruments for a positive volatility process are strongly identified by the cross section of yields. The key to our results is that volatility must be a positive process not only under the historical measure but also under the risk neutral measure. Any linear combination of yields which is forecasted to be negative (or has a positive probability of achieving a negative value) under the risk neutral measure *cannot* be an instrument for volatility and maintain positivity. We show that since the risk neutral distribution is estimated much more precisely than the historical distribution, these conditions precisely identify which instruments maintain positive volatility. Moreover, risk neutral forecasts of yields are largely invariant to any volatility considerations (see, e.g. [Campbell \(1986\)](#) or [Joslin \(2010\)](#).) This implies the striking conclusion that at a Gaussian term structure model – which has constant volatility – can reveal which instruments would be admissible for a stochastic volatility model.

To develop some intuition for our results, consider a simple time series model of the yield curve. Consider a standard vector autoregression of the principle components of the yield curve (level, slope and curvature), denoted \mathcal{P} :

$$\Delta\mathcal{P}_t = K_0 + K_1\mathcal{P}_t + \epsilon_{t+1}. \quad (1)$$

In order to capture time-variation in volatility of the level, we can project squared changes in the level of interest rates onto the principal components along the lines of [Campbell \(1987\)](#) and [Harvey \(1989\)](#) through the regression

$$(\Delta\mathcal{P}_t)^2 = \alpha + \beta \cdot \mathcal{P}_t + e_t. \quad (2)$$

This simple instrumentation procedure suggests that the variance of the level factor could be instrumented by $V_t \equiv \alpha + \beta \cdot \mathcal{P}_t$.²

Assuming that volatilities of all the principle components are perfectly correlated, we could then generate a fully dynamic affine model of the term structure by modifying the VAR in (1) so that the conditional covariance of the innovation, ϵ_{t+1} , is affine in $\alpha + \beta \cdot \mathcal{P}_t$. However, this process may not be well-defined since volatility may become negative. Following [Dai and Singleton \(2000\)](#), we will refer to any condition required for a well-defined positive volatility process as an admissibility condition. A sufficient condition for volatility to be a positive process is that the variance defines an autonomous (discrete-time) Cox Ingersoll Ross process.

¹See [Poon and Granger \(2003\)](#) or [Andersen and Benzoni \(2008\)](#) for recent surveys of volatility forecasting.

²In principle, the projection in (2) could result in negative variance. This is an additional constraint.

When there is a single volatility factor and the other risk factors are conditionally Gaussian, this condition is also necessary.³ Among other things, this means that the conditional mean of innovations to V_t , $\beta'(K_0 + K_1\mathcal{P}_t)$, should depend only V_t . This happens only when $\beta'K_1$ is a multiple of β' . That is, β' is a left-eigenvector of K_1 .

The fact that admissibility requires that volatility to be an autonomous process under the historical measure represents a tension between the conditional first and second moments given by (1) and (2). Unless the estimate of β from (2) is a left eigenvector of K_1 from (1), a joint estimation procedure designed to produce an admissible process will necessarily trade off the fit of these two equations. However, as is commonly observed and we verify in our data, estimates of these two equations are imprecise typically with large standard errors.

A starker version of these over-identifying restrictions can be seen by considering the risk neutral measure, \mathbb{Q} . [Joslin, Singleton, and Zhu \(2010\)](#) show that the risk-neutral distribution can be characterized through the equation

$$\Delta\mathcal{P}_t = K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}}\mathcal{P}_t + \epsilon_{t+1}^{\mathbb{Q}}, \quad (3)$$

where $(K_0^{\mathbb{Q}}, K_1^{\mathbb{Q}})$ are appropriately constrained to maintain the internal consistency of no arbitrage. Admissibility requires that volatility must be autonomous under \mathbb{Q} . Thus β' must be a left eigenvector of $K_1^{\mathbb{Q}}$. While this is analogous to the condition for the historical measure, an important difference is the precision of statistical inference regarding K_1 versus $K_1^{\mathbb{Q}}$. Although we observe only one historical time series, each observation of the yield curve effectively represents a term structure of risk neutral expectations of \mathcal{P} . Thus $K_1^{\mathbb{Q}}$ can be estimated very precisely. Given a precise estimate of $K_1^{\mathbb{Q}}$, the left eigenvectors are set and thus so too are the potential volatility instruments.

We show that estimates of $K_1^{\mathbb{Q}}$ are very similar for both the stochastic volatility model and the Gaussian model. Moreover, we also show that these estimates are close to model-free estimates obtained by the regression technique in [Joslin \(2011\)](#). The model-free regression indicate that the $K_1^{\mathbb{Q}}$ is estimated very precisely from our data. As a consequence, very reliable inference is obtained for the admissible volatility instruments. We demonstrate that the admissible volatility instruments implied by the unconstrained estimates of K_1 from (3) are estimated much less precisely.

Our results help clarify the relationship between volatility instruments extracted from the cross-section of bond yields documented by several recent studies. For example, [Collin-Dufresne, Goldstein, and Jones \(2009\)](#) find an extracted volatility factor from the cross-section of yields through a no arbitrage model to be negatively correlated with model-free estimates. [Jacobs and Karoui \(2009\)](#) in contrast generally find volatility extracted from affine models are generally positively related though in some cases they also find a negative correlation. [Almeida, Graveline, and Joslin \(2011\)](#) also find a positive relationship. Our results show that for the $A_1(N)$ class of model, the cross-section of bonds will reveal up to N linear combinations of yield (given by the left eigenvectors of $K_1^{\mathbb{Q}}$) that can serve as instruments for

³More precisely, this result is the for the continuous time $A_1(N)$ class of models in the notation of [Dai and Singleton \(2000\)](#). Analogous conditions maintain in the discrete time stochastic volatility models of [Bansal and Shaliastovich \(2009\)](#) and [Le, Singleton, and Dai \(2010\)](#).

volatility. The no arbitrage structure then essentially implies nothing more for the properties of volatility beyond the assumed one factor structure and the admissibility conditions. To the extent that the instrument obtained from $K_1^{\mathbb{Q}}$ is optimal, the model implied-volatility will match the analogous results obtained from a model without the no arbitrage structure.

Our results also address a number of recent papers that suggest that no arbitrage restrictions within a Gaussian DTSM are completely or nearly irrelevant for the estimation of dynamic term structure models. See, for example, [Duffee \(2011\)](#), [Joslin, Singleton, and Zhu \(2010\)](#), and [Joslin, Le, and Singleton \(2011\)](#). These papers leave open the question of whether the no arbitrage restrictions are useful in DTSMs with stochastic volatility. Our results show that the answer to this question is a resounding yes. The cross-section of yields strongly identifies $K_1^{\mathbb{Q}}$ and therefore strongly identifies the admissible instruments for volatility. To the extent that the admissibility conditions hold in the true data generating process, these over-identifying restrictions will be very useful for inferring the true process of the yield curve. The admissibility conditions though are motivated not by economic consideration so this seems a potentially dubious assumption.

The rest of the paper is organized as follows. Good stuff. In Section XXX, we consider alternative models which relax the admissibility constraints by consider non-affine (but nearly affine) alternative models.

To fix notation, suppose that a *MTSM* is to be evaluated using a set of J yields $y_t = (y_t^{m_1}, \dots, y_t^{m_J})'$ with maturities (m_1, \dots, m_J) in periods and with $J \geq \mathcal{N}$, where \mathcal{N} is the number of pricing factors. To be consistent with our empirical work, we fix the period length to be one month. We introduce a fixed, full-rank matrix of portfolio weights $W \in \mathbb{R}^{J \times J}$ and define the “portfolios” of yields $\mathcal{P}_t = Wy_t$ and, for any $j \leq J$, we let \mathcal{P}_t^j and W^j denote the first j portfolios and their associated weights. The modeler’s choice of W will determine which portfolios of yields enter the *MTSM* as risk factors and which additional portfolios are used in estimation. Throughout, we assume a flat prior on the initial observed data.

1 Stochastic Volatility Term Structure Models

This section gives an overview of the stochastic volatility term structure models that we consider. As we elaborate, a key consideration is the admissibility conditions required to maintain a positive volatility process under both the historical and risk neutral measures. The conditions for a positive volatility process can be considered even without a formal term structure model by considering a simple time series model with stochastic volatility. This formulation is also of interest in its own right since it also allows us to assess the role of the no arbitrage assumption in stochastic volatility models. For simplicity, we focus in the main text on the case of a single volatility factor; the general case, as well as modifications for discrete time processes and more technical details, are described in the Appendix.

1.1 Affine models with autonomous volatility

We first review the condition required for a well-defined positive volatility process within a multi-factor setting. Following [Dai and Singleton \(2000\)](#), hereafter DS, we refer to these conditions as admissibility conditions. Recall the N -factor $A_1(N)$ process of DS. This process has an N -dimensional state variable composed of a single volatility factor, V_t , and $(N - 1)$ conditionally Gaussian state variables, X_t . The state variable $Z_t = (V_t, X_t)'$ follows the Itô diffusion

$$d \begin{bmatrix} V_t \\ X_t \end{bmatrix} = \mu_{Z,t} dt + \Sigma_{Z,t} dB_t^{\mathbb{P}}, \quad (4)$$

where

$$\mu_{Z,t} = \begin{bmatrix} K_{0V} \\ K_{0X} \end{bmatrix} + \begin{bmatrix} K_{1V} & K_{1VX} \\ K_{1XV} & K_{1X} \end{bmatrix} \begin{bmatrix} V_t \\ X_t \end{bmatrix}, \quad (5)$$

and

$$\Sigma_{Z,t} \Sigma'_{Z,t} = \Sigma_0 + \Sigma_1 V_t, \quad (6)$$

and $B_t^{\mathbb{P}}$ is a standard N -dimensional Brownian motion under the historical measure, \mathbb{P} . [Duffie, Filipovic, and Schachermayer \(2003\)](#) show that this is the most general affine process on $\mathbb{R}^+ \times \mathbb{R}^{N-1}$.

In order to ensure that the volatility factor, V_t , remains positive, we need that when V_t is zero (a) the expected change of V_t is non-negative and (b) the volatility of V_t becomes zero. Otherwise there is a positive probability that V_t will become negative. Imposing additionally the Feller condition for boundary non-attainment, our admissibility conditions are then

$$\Sigma_{0,11} = 0, \quad (7)$$

$$K_{0V} \geq \frac{1}{2} \Sigma_{1,11} \quad (8)$$

$$K_{1VX} = 0. \quad (9)$$

A consequence of these conditions is that volatility must follow an autonomous process under \mathbb{P} since the conditional mean and variance of V_t depends only on V_t and not on X_t . We now show how to embed the $A_1(N)$ specification into generic term structure models and re-interpret these admissibility constraints in terms of conditions on the volatility instruments.

1.2 Stochastic volatility with generic factor term structure models

Before turning to no arbitrage term structure models with stochastic volatility, we first consider a more general class of term structure models with an underlying $A_1(N)$ factor structure. We can consider the state-space model with the hidden state variable Z_t following the state dynamics given by (4–6) which we augment with the observation equation

$$y_t = A_Z + B_Z Z_t + \epsilon_t, \quad (10)$$

where (A_Z, B_Z) are conformable matrices and ϵ_t are measurement errors which we suppose are independent and normally distributed. We refer to such models as a stochastic volatility

factor vector autoregression and denote the class of models as SV-FVAR₁(N). Notice that the no arbitrage $A_1(N)$ model of DS represents a restricted version of these models since in these models the loadings in (10) are restricted by no arbitrage.

Although maximally flexible, the parameters of this model is not econometrically identified. Following [Joslin, Singleton, and Zhu \(2010\)](#), hereafter JSZ, we can identify the model by observing that $\mathcal{P}_t \equiv Wy_t = (WA_Z) + (WB_Z)Z_t$. After changing to use \mathcal{P}_t as the state variable, the SV-FVAR₁(N) model can be written as an identified state-space model which is parameterized by $\Theta_{SV-FVAR} \equiv (K_0, K_1, \Sigma_0, \Sigma_1, \alpha, \beta, A, B)$. We have the state equation

$$d\mathcal{P}_t = (K_0 + K_1\mathcal{P}_t)dt + \sqrt{\Sigma_0 + \Sigma_1 V_t}dB_t^{\mathbb{P}}, \quad (11)$$

where $V_t = \alpha + \beta \cdot cP_t$, which serves as the volatility instrument.⁴ We have the observation equation

$$y_t = A + B\mathcal{P}_t + \epsilon_t. \quad (12)$$

For identification of the parameters, we also impose that $WA = 0$ and $WB = I_N$, as in JSZ.⁵ We can confirm that the admissibility conditions (7–9) are now

$$\beta' \Sigma_0 \beta = 0, \quad (13)$$

$$\beta' K_0 \geq \frac{1}{2} \beta' \Sigma_1 \beta, \quad (14)$$

$$\beta' K_1 = c\beta', \quad (15)$$

where c is an arbitrary constant. The last admissibility conditions follow by applying Itô's lemma to V_t and observing that its conditional mean ($\beta' K_0 + \beta' K_1 \mathcal{P}_t$) must be affine in V_t or, equivalently, affine in $\beta' \mathcal{P}_t$. The condition in (15) can be restated as the condition that β' is a left eigenvector of K_1 , which is required so that V_t is an autonomous process under \mathbb{P} .

1.3 Stochastic volatility with no arbitrage term structure models

Following DS, we can embed the state dynamics of (4–6) into an affine no arbitrage term structure model by defining the short rate to be affine in Z_t :

$$r_t = \rho_0 + \rho_1 \cdot Z_t.$$

We can then price zero coupon bond of all maturities by specifying the risk neutral dynamics (or equivalently, the market prices of risk). We consider the general market prices of risk of [Cheridito, Filipovic, and Kimmel \(2007\)](#) so that Z_t follows an admissible affine process under \mathbb{Q} :

$$dZ_t = (K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}} Z_t)dt + \sqrt{\Sigma_0 + \Sigma_1 Z_t}dB_t^{\mathbb{Q}}. \quad (16)$$

⁴Here $\alpha = -e_1'(WB_Z)^{-1}(WA_Z)$ and $\beta = e_1'(WB_Z)^{-1}$. This relies on our assumption that a volatility instrument fully spans volatility: our logic can be extended to consider an unspanned component of volatility where our result now apply to the spanned component.

⁵This is slightly overidentifying. See JSZ.

In order to define an admissible process under \mathbb{Q} , the analogous conditions to (8–9). A key implication of these restrictions is that the volatility factor must be an autonomous factor under \mathbb{Q} . Before exploring these implications, we first re-parameterize the no arbitrage model in a similar spirit to JSZ in order to have observable factors.

A result of JSZ characterizes Gaussian no arbitrage models in terms of parameters governing the dynamics of \mathcal{P}_t (which would be $(K_0, K_1, \Sigma_0, \Sigma_1, \alpha, \beta)$ in the present SV-FVAR₁(N) case) but to replace the general observation equation loadings (A, B) with parameters which characterize no arbitrage loadings in the observation equation (12). They show that in the Gaussian case, the no arbitrage loadings form an $(N + 1)$ -dimensional subspace of all potential loadings which can be parameterized by N eigenvalues and an intercept term. In the stochastic volatility under present consideration, their method does not directly apply since due to stochastic convexity effects, the loadings will depend on the volatility parameter, Σ_1 .

We now show how we can apply a similar parameterization in the spirit of JSZ with stochastic volatility. For an alternative characterization, see the Appendix. As noted, a difficulty in directly applying their method is the stochastic convexity effect. To this end, define $y_t^{c,m}$ to be the m -year average risk-neutral expected short rate:

$$y_t^{c,m} \equiv \frac{1}{m} E_t^{\mathbb{Q}} \left[\int_t^{t+m} r_\tau d\tau \right].$$

Similarly, let y_t^c denote the vector $(y_t^{c,m_1}, \dots, y_t^{c,m_J})$ and let us also denote $W y_t^c$ by \mathcal{P}_t^c . To the extent that convexity effects are small, this will approximate the usual zero coupon yield. For longer maturities, convexity effects will be larger but may show less variation due to mean reversion of volatility.

Convexity-adjusted yields allow use to impliment the JSZ formulation with minor modifications. Specifically, we can now parameterize the $A_1(N)$ no arbitrage term structure model in terms of the parameters $(\lambda^{\mathbb{Q}}, i_{\lambda^{\mathbb{Q}}, CIR}, r_\infty^{\mathbb{Q}}, \alpha_c, s_V, \Sigma_0^c, \Sigma_1^c, K_0^{\mathbb{P}}, K_1^{\mathbb{P}})$. Here $\lambda^{\mathbb{Q}}$ gives the eigenvalues of $K_1^{\mathbb{Q}}$. $i_{\lambda^{\mathbb{Q}}, CIR}$ selects which of the N eigenvalues under \mathbb{Q} corresponds to the CIR factor. $r_\infty^{\mathbb{Q}}$ gives the long run mean of the short rate. α_c gives the intercept in the volatility instrument in terms of the convexity adjusted yields. $s_V \in \{-1, +1\}$ gives the sign of the volatility instrument, as we elaborate below. (Σ_0^c, Σ_1^c) gives the affine loadings for the conditional variance of \mathcal{P}_t^c in terms of the volatility instrument. The general construction follows similar lines to JSZ and we leave complete specification and formulas to recover the exact parameters are summarized in the Appendix. Additionally, we provide the corresponding admissibility conditions.

We highlight some important components of the construction. First, similar to the development in JSZ, we will generically have

$$d\mathcal{P}_t^c = (K_0^{\mathbb{Q}c} + K_1^{\mathbb{Q}c} \mathcal{P}_t^c) dt + \sqrt{\Sigma_0^c + \Sigma_1^c (\alpha_c + s_V \beta_c \cdot \mathcal{P}_t^c)} dB_t^{\mathbb{Q}}. \quad (17)$$

As in JSZ, $K_1^{\mathbb{Q}c}$ depends only on $\lambda^{\mathbb{Q}}$.

A number of results will hold for the volatility structure. First, since s_V can either be +1 or -1, the sign of the correlation of variance with any variable can theoretically be either

positive or negative. Second, β_c will also depend only on $\lambda^{\mathbb{Q}}$ and $i_{\lambda^{\mathbb{Q}}, CIR}$. While this fact may be surprising at first, the reason is quite intuitive. Admissibility requires the volatility instrument to be an autonomous process so that its mean does not depend on other factors. This occurs only when $\beta'_c K_1^{\mathbb{Q}c}$ is a multiple of β'_c as in (15). This requires that β'_c is a left eigenvector of $K_1^{\mathbb{Q}c}$. Since fixing $\lambda^{\mathbb{Q}}$ fixes $K_1^{\mathbb{Q}c}$, this also fixes the set of left eigenvectors of $K_1^{\mathbb{Q}c}$. Thus the only remaining choice is which of the N left eigenvectors of $K_1^{\mathbb{Q}c}$ is to be the volatility instrument.

We see a very stark contrast in the flexibility of potential volatility instrument in the SV-FVAR₁(N) model and the no arbitrage $A_1(N)$ model. In the SV-FVAR₁(N), the volatility instrument has arbitrary loadings on the yield factors, \mathcal{P}_t . In contrast, in the no arbitrage model, the volatility loadings (in terms of the convexity adjusted yield factors, \mathcal{P}_t^c), have loadings that are determined almost entirely by the cross-sectional properties of the yield curve. We now turn to explore the implications of this observation.

2 Positive volatility under the risk neutral measure

In this section, we provide intuitive examples why the no-arbitrage restrictions can strongly pin down the volatility factor, without reference to the time series properties of the data.

We have seen in Section 1.2 that in order to have a well-defined autonomous volatility process we must have the condition (15). This condition can be restated as that β' is a left eigenvector of K_1 . This means that $\beta \cdot \mathcal{P}_t$ is a sufficient statistic to forecast future volatility. This provides some guidance on potential volatility instruments. For example, although level is known to be related to volatility, it is also well-known (for example Campbell and Shiller (1987)), that the slope of the yield curve predicts future changes in the level of interest rates. Up to the associated uncertainty of such statistical evidence, this suggests that the slope of the yield curve predicts the level and thus also that the level of interest rates is not an autonomous process. We now show that the risk neutral measure provides strong guidance for which combinations of yields form autonomous processes, as required for a valid volatility instrument.

For illustrative purposes, we consider the case of three factors $N = 3$, but the same intuition applies to any N .

2.1 Example 1

Let's define the convexity-adjusted forward rate by:

$$f_t(n) = E_t^{\mathbb{Q}}[r_{t+n}]. \quad (18)$$

In the spirit of Collin-Dufresne, Goldstein, and Jones (2008) we can write the following risk-neutral dynamics:

$$E_t^{\mathbb{Q}} \begin{pmatrix} V_{t+1} \\ r_{t+1} \\ f_{t+1}(1) \end{pmatrix} = \text{constant} + \begin{pmatrix} a_1 & 0 & 0 \\ 0 & 0 & 1 \\ a_2 & a_3 & a_4 \end{pmatrix} \begin{pmatrix} V_t \\ r_t \\ f_t(1) \end{pmatrix}. \quad (19)$$

The first row is due to the autonomous nature of V_t . The second row is the definition of the forward rate in (18). The last row is obtained from the fact that in a three factor affine model, $(V_t, r_t, f_t(1))$ are informationally equivalent to the three underlying states at time t . From the last row and by applying the law of iterated expectation to (18), we have:

$$a_2 V_t = \text{constant} + f_t(2) - a_3 r_t - a_4 f_t(1). \quad (20)$$

Using (20) to substitute V_t out from (19), we have:

$$E_t^{\mathbb{Q}} \begin{pmatrix} r_{t+1} \\ f_{t+1}(1) \\ f_{t+1}(2) \end{pmatrix} = \text{constant} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \begin{pmatrix} r_t \\ f_t(1) \\ f_t(2) \end{pmatrix}. \quad (21)$$

where $\alpha_1 = -a_1 a_3$, $\alpha_2 = a_3 - a_1 a_4$, and $\alpha_3 = a_4 + a_1$. It follows from the last row of (21) that:

$$f_t(3) = \text{constant} + \alpha_1 r_t + \alpha_2 f_t(1) + \alpha_3 f_t(2). \quad (22)$$

Equation (22) reveals that if the short rate as well as the forward rates can be empirically observed, the loadings α can in principle be pinned down simply by regressing $f_t(3)$ on r_t , $f_t(1)$, and $f_t(2)$. Based on the mappings from (a_1, a_3, a_4) to α , it follows that the regression implied by (22) will also identify all the a coefficients, except for a_2 . In the context of equation (20), it means that the volatility factor is already determined up to a translation and scaling effect.

Repeated iterations of the above steps allow us to write any forward rate $f_t(n)$ as a linear function of $(r_t, f_t(1), f_t(2))$. Suppose that we use J forwards in $(f_t(1), \dots, f_t(J))$ in estimation, then:

$$\begin{pmatrix} r_t \\ f_t(1) \\ f_t(2) \\ f_t(3) \\ f_t(4) \\ \vdots \\ f_t(J) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ & g_4(\alpha) & \\ & \dots & \\ & g_J(\alpha) & \end{pmatrix} \begin{pmatrix} r_t \\ f_t(1) \\ f_t(2) \end{pmatrix}$$

where (g_4, \dots, g_J) represent the cross-sectional restrictions of no-arbitrage. So we can think of no-arbitrage as having two facets. First, it imposes a cross-section to time series link through the fact that fixing α constrains what the volatility factor must look like, through a_3 and a_4 . Second, it induces cross-sectional restrictions on the loadings (g_4, \dots, g_J) , just as is seen with pure gaussian term structure models.

2.2 Example 2

We can also use the approach of Joslin (2011) to consider arbitrary linear combinations of yields as factors. The insight again is that risk-neutral expectations are, up to convexity,

observed as forward rates. The m -term forward rate the begins in one period, $f_t^{1,m}$ is given by

$$f_t^{1,m} = \frac{1}{m}((m+1)y_t^{m+1} - r_t). \quad (23)$$

Thus we can use (23) to approximate $E_t^{\mathbb{Q}}[y_{t+1}^m]$ whereby we simply ignore any convexity term. Notice that since our primary interest is not in the level of expected-risk neutral changes but in their variation (i.e. $K_1^{\mathbb{Q}}$), it is only stochastic convexity effects that will violate this approximation. Thus to the extent that changes in convexity effects are small this approximation will be valid for inference on $K_1^{\mathbb{Q}}$.

From this method, we extract observations on $E_t^{\mathbb{Q}}[y_{t+1}^m]$ from forward rates which we can then convert into estimates of $E_t^{\mathbb{Q}}[\mathcal{P}_{t+1}]$ using the weighting matrix W . We denote this approximation of $E_t^{\mathbb{Q}}[\mathcal{P}_{t+1}]$ by \mathcal{P}_t^f . We can consider the regression of the form

$$\mathcal{P}_t^f = \text{constant} + K_1^{\mathbb{Q}}\mathcal{P}_t + u_t. \quad (24)$$

Given the validity of our convexity approach and repeated cross-sections of yields, we should be able to obtain estimates of $K_{1\mathcal{P}}^{\mathbb{Q}}$ even without estimating a term structure model.

2.3 Example 3

Consider the $J \times 1$ vector of yields y_t used in estimation:

$$y_t = A + BX_t, \quad (25)$$

where X_t denotes the underlying states. To have affine bond pricing, the essential requirement is that the risk-neutral dynamics of X_t be affine. Focusing on the conditional mean, and applying standard rotations to X_t , we can always write:⁶

$$E_t^{\mathbb{Q}}[X_{t+1}] = \text{diag}(\lambda^{\mathbb{Q}})X_t. \quad (26)$$

In the spirit of [Joslin, Singleton, and Zhu \(2010\)](#), we can rotate X_t to any N yields portfolios: $\mathcal{P}_t = Wy_t$. Rewriting (25) and (26), replacing X_t with \mathcal{P}_t , we have:

$$y_t = \text{constant} + B(WB)^{-1}\mathcal{P}_t, \quad (27)$$

$$E_t^{\mathbb{Q}}[\mathcal{P}_{t+1}] = \text{constant} + \underbrace{(WB)\text{diag}(\lambda^{\mathbb{Q}})(WB)^{-1}}_{K_{1\mathcal{P}}^{\mathbb{Q}}}\mathcal{P}_t. \quad (28)$$

Assuming the effect of convexity on the loadings B is minimal (which we confirm subsequently in our empirical exercises), to the first-order approximation, the loadings B are only dependent on $\lambda^{\mathbb{Q}}$.⁷ Consequently, equation (27) implies that we could estimate $\lambda^{\mathbb{Q}}$ by regressing y_t on \mathcal{P}_t ,

⁶Suppose that we start from $E_t^{\mathbb{Q}}[X_{t+1}] = K_0 + K_1X_t$. The appropriate rotation is $\tilde{X}_t = C + DX_t$ where D is the inversion of the eigenvector matrix of K_1 such that DK_1D^{-1} is diagonal. And $C = (DK_1D^{-1} - I)^{-1}DK_0$. Here for ease of exposition, we assume that the eigenvalues of K_1 are distinct and real-valued.

⁷Specifically, for the loading on the i^{th} factor for yield y_t^n with n periods to maturity is approximately $\frac{1 + \lambda^{\mathbb{Q}}(i) + \dots + \lambda^{\mathbb{Q}}(i)^{n-1}}{n\Delta t}$.

forcing the loadings to be of the form $B(WB)^{-1}$, which is only dependent on $\lambda^{\mathbb{Q}}$. Once $\lambda^{\mathbb{Q}}$ is identified, the risk-neutral feedback matrix $K_{1\mathcal{P}}$ in (28) is in turn pinned down. Notably, this must be true of all canonical affine term structure models regardless of whether a volatility factor is included or not.

Focusing on term structure models with one stochastic volatility factor V_t , affine bond pricing implies that we can write:

$$V_t = \text{constant} + b'\mathcal{P}_t \quad (29)$$

for some b . It follows that:

$$E_t^{\mathbb{Q}}[V_{t+1}] = \text{constant} + b'K_{1\mathcal{P}}^{\mathbb{Q}}\mathcal{P}_t. \quad (30)$$

Since V_{t+1} must be autonomous, its conditional mean can only depend on its lagged value. Consequently, we must have:

$$b'K_{1\mathcal{P}}^{\mathbb{Q}} = eb' \quad (31)$$

for some scalar e . In other words, b' must be a left eigenvector of $K_{1\mathcal{P}}^{\mathbb{Q}}$.

3 Empirical Results

In this section, we empirically evaluate the implications of our theory, using the US Treasury data spanning the period from March 1984 until December 2006. The sample is chosen such that our analysis is not materially affected by either the Fed experiment regime or the current financial crisis. We use the monthly unsmoothed Fama Bliss zero yields and the following seven maturities in our estimation: 3-month, 6-month, one-out to five-year. We first provide evidence that for our sample period, conditional volatilities of yields can be captured to a large extent by linear combinations of yields. Next, we examine the extent to which the requirement of autonomous volatility can impact the identification of volatility under both the physical and risk-neutral measures.

3.1 Conditional Volatility from Linear Projections

In this section, we investigate whether interest rate volatility can be approximately spanned by yields. In doing this, we focus on the conditional volatility of the level factor – the most important determinant of yields for all maturities. We first construct the first three principal factors, \mathcal{P}_t , for our data and regress the level factor one period ahead, $\mathcal{P}_{1,t+1}$, on the lagged \mathcal{P}_t :

$$\mathcal{P}_{1,t+1} = K_0 + K_1\mathcal{P}_t + \epsilon_{t+1}. \quad (32)$$

To estimate the conditional volatility of $\mathcal{P}_{1,t+1}$, we project the fitted residuals, $\hat{\epsilon}_{t+1}$ from (32) onto \mathcal{P}_t :

$$\hat{\epsilon}_{t+1}^2 = \alpha + \beta\mathcal{P}_t + e_{t+1}. \quad (33)$$

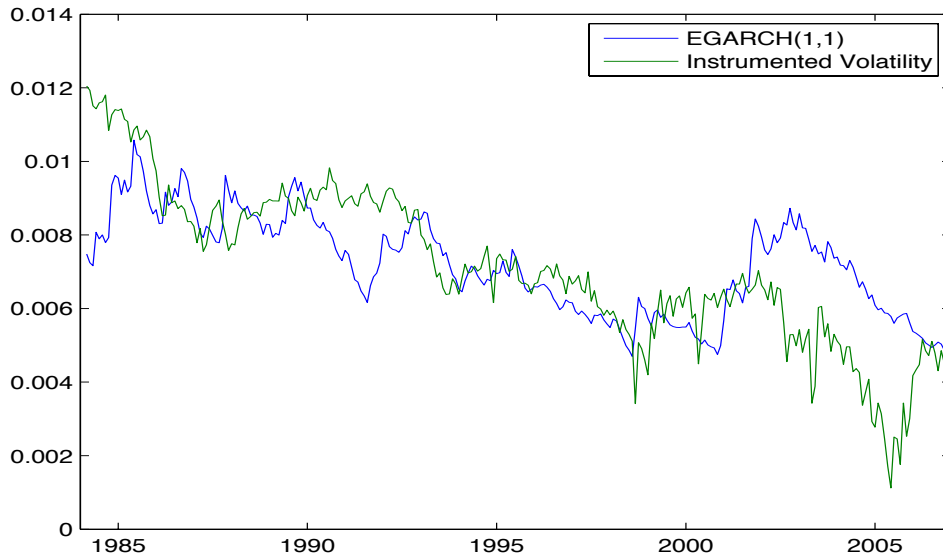


Figure 1: Volatility of the level factor estimated from the EGARCH(1,1) model and a two-stage regression.

The square root of the predicted component of (33) is a measure of the conditional volatility of the level factor implied by the affine structure of standard term structure models. In running (32) and (33), we do not require that the volatility factor be autonomous. In doing this, we can see whether this spanned measure of volatility, free of admissibility constraints, can reasonably capture interest rate volatility. While the true volatility process is not known, we proxy for volatility by using estimates from an EGARCH(1,1) model as a benchmark. Specifically, using the fitted residuals $\hat{\epsilon}_{t+1}^2$ as input, we obtain and plot the EGARCH(1,1) estimates in Figure 1 along with the predicted component of (33), labeled “Instrumented Volatility.” It is visually clear that the two series track each other quite well with a sample correlation of 66% – a similar correlation is reported by Jacobs and Karoui (2009) for their longer sample period.

Additionally, using a multi-period version of (33),⁸ Le and Singleton (2011) show that spanned volatility is considerably rich of information relevant for predicting future excess returns. Together, these suggest that volatilities instrumented by yields can potentially reflect realistic fluctuations in yields as well as meaningful information about priced risks. As we overlay the estimation of (33) with the requirement that volatility be autonomous, there is a possibility that the estimates of the loading β can be altered in such a way that the information content of the resulting volatility estimates can be quite different. We take on these issues at greater depth in the subsequent sections. Here with the aim of providing some initial descriptive evidence of the impact of the autonomous requirement, we perform the

⁸In particular, they stack $\hat{\epsilon}_{t+h}^2$ over multiple horizons and use the technique of rank reduced regression to estimate α and β .

following simple test. Using the estimated loading $\hat{\beta}$ from (33), we regress $\hat{\beta}\mathcal{P}_{t+1}$ on \mathcal{P}_t :

$$\hat{\beta}\mathcal{P}_{t+1} = \text{constant} + \gamma\mathcal{P}_t + \text{noises}. \quad (34)$$

Under the null hypothesis that the volatility factor $\alpha + \beta\mathcal{P}_t$ is autonomous, it must be true that γ is a scaled version of β . This amounts to a χ^2 test with two degrees of freedom. Using twelve lags to construct the robust asymptotic covariance matrix of the estimates, accounting for estimation errors with sequential inferences, we obtain a $\chi^2(2)$ statistics of 11.2, statistically significant at the 1% level. This evidence shows that while there exists a linear combination of yields that can give a reasonable account of the risks in yields, such a combination does not give rise to an autonomous volatility factor. This suggests the potential tension in fit when autonomy is imposed in estimation.

3.2 Autonomy requirement under the historical measure

We first consider the requirement that volatility forms an autonomous process in the SV-FVAR₁(N) specification. This requires (15) which states that the volatility instrument, β' be a left eigenvector of $K_1^{\mathbb{P}}$. To the extent that the conditional mean is strongly identified by the time-series, this condition will pin down the admissible volatility instruments up to a sign choice and the choice of which of the N left eigenvectors instruments volatility. However, in general even with a moderately long time series, such as our 22 year sample, inference on the conditional mean is not very precise. At the same time however, this condition does provide some guidance as we just seen that there is statistical evidence that the linear projection of the squared residuals does not seem to follow an autonomous process.

We quantify this uncertainty as follows. First we estimate an unconstrained VAR on the first three principal factors, \mathcal{P}_t . Ignoring the intercepts, the estimates for our sample period are:

$$\mathcal{P}_{t+1} = \underbrace{\begin{pmatrix} 0.9805 & -0.0190 & -0.5530 \\ -0.0027 & 0.9581 & 0.3928 \\ 0.0014 & -0.0005 & 0.8433 \end{pmatrix}}_{K_1^{\mathbb{P}}} \mathcal{P}_t + \text{noises} \quad (35)$$

Then, for a given volatility instrument β_0 , we re-estimate the VAR under the constraint that β_0 is a left eigenvector of $K_1^{\mathbb{P}}$. The VAR is easily estimated under this constraint after a change of variables so the the eigenvector constraint becomes a zero constraint (compare the constraints in (9) and (15)). We then conduct a likelihood ratio test of the unconstrained versus the constrained alternative and compute the associated p-value. A p-value close to 1 indicates that the evidence is consistent with such an instrument being an autonomous process while a p-value close to 0 indicates evidence against the instrument being an autonomous process.⁹

The results are plotted in Figure 2. We see that there are three peaks which correspond to the three left eigenvectors of the maximum likelihood estimate of $K_1^{\mathbb{P}}$. As our intuition

⁹We view this test as an approximation since it assumes volatility is constant.

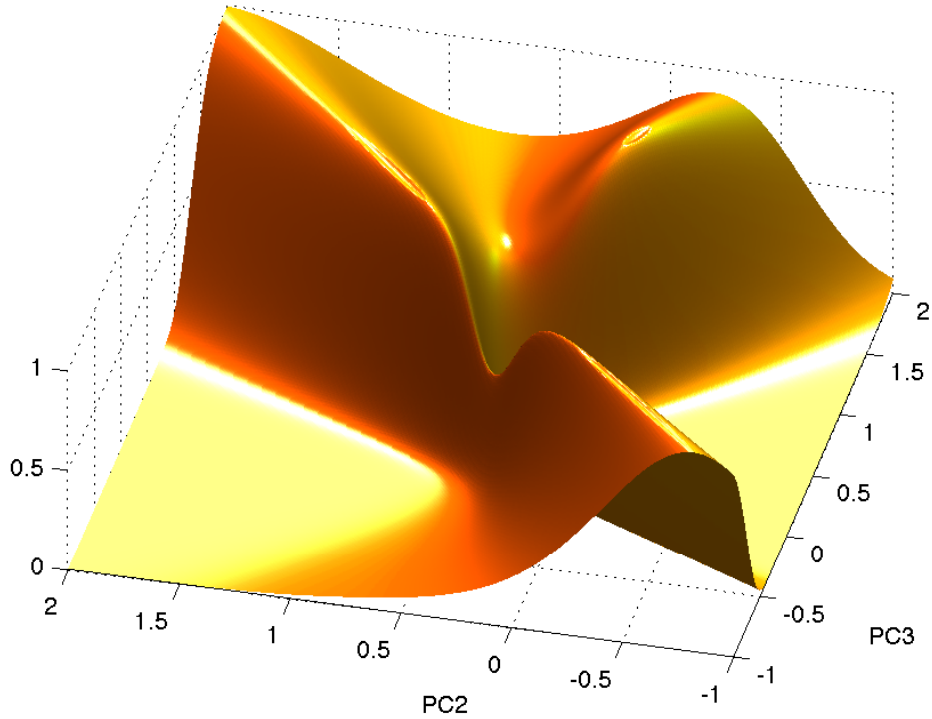


Figure 2: Likelihood Ratio Tests of \mathbb{P} -Autonomous Restrictions. This figure reports the p -values of the likelihood ratio test of whether a particular linear combination of yields, $\beta\mathcal{P}_t$, is autonomous under \mathbb{P} , plotted against the loadings of PC2 and PC3. The loading of PC1 is one minus the loadings on PC2 and PC3 ($\beta(1) = 1 - \beta(2) - \beta(3)$). PC1, PC2, and PC3 are scaled to have in-sample variances of one.

suggests, many, though not all, instruments appear to potentially satisfy the requirement of (15) according to the metric that we are considering. Thus we conclude that there is a great deal of flexibility in general the SV-FVAR₁(N) model in forming the volatility instrument.

3.3 Autonomy requirement under the risk-neutral measure

Having seen the precision with which the historical measure guides the set of admissible volatility instruments, we now turn to the information gained from the risk-neutral measure. We follow the model-free approach of Joslin (2011) as in Section 2.2 to study this restriction in a similar manner to our observation on the historical measure. Specifically, we consider that, ignoring constants, the regression

$$E_t^{\mathbb{Q}}[\mathcal{P}_{t+1}] = K_{1P}^{\mathbb{Q}}\mathcal{P}_t + u_t,$$

where, as before, we proxy for $E_t^{\mathbb{Q}}[\mathcal{P}_{t+1}]$ using the linear combination of forwards \mathcal{P}_t^f . As in our analysis of the autonomy constraints under the historical measure, we consider both the unconstrained version of this regression and the same regression with the constraint that a particular β_0 is a left eigenvector of $K_{1\mathcal{P}}^{\mathbb{Q}}$. Intuitively, although we observe only a single time series under the historical measure with which to draw inferences, we observe repeated term structures of risk-neutral expectations every month and this allows us to draw much more precise inferences.

Figure 3 plots the p-values for this test of the restrictions of various instruments to be autonomous under \mathbb{Q} . In contrast to **Figure 2** and in accordance with our intuition, we see that the risk-neutral measure provides very strong evidence for which instruments are able to be valid volatility instruments. Bond prices essentially pin down $K_{1\mathcal{P}}^{\mathbb{Q}}$ and the only degrees of freedom are the sign choice and choosing which of the three left eigenvectors is the volatility instrument. We confirm also later in **Section 4** that $K_{1\mathcal{P}}^{\mathbb{Q}}$ estimated from the $A_1(N)$ model is essentially identical to the estimate from the Gaussian $A_0(N)$ model (which has constant volatility). Thus we see the surprising conclusion that the $A_0(N)$ model allows us to essentially identify (up to choice of which eigenvector) the volatility instrument in the $A_1(N)$ model.

4 Factor model estimates of volatility

We now turn to see the estimation results for volatility in both the general SV-FVAR₁(N) factor model and the $A_1(N)$ model. The time series of model-implied volatilities are plotted in **Figure 4**. We see that in general both the factor model (which imposes only admissibility without restrictions on the cross-sectional loadings) and the no arbitrage model produce a volatility time series for the level of interest rates which is generally consistent with the EGARCH(1,1) estimates. The correlations between the SV-FVAR₁(N) and $A_1(N)$ volatility estimates and the EGARCH(1,1) estimates are XXX and XXX, respectively. We note again that we should anticipate a positive correlation since both specifications are free to change the sign of the volatility instrument so to the extent that the maximum likelihood criterion function is maximized with a positive sign, this positive correlation should generally occur.

5 Comparison of Gaussian and Stochastic Volatility Models

We have argued that variation in the risk-neutral expectations, as determined by $K_1^{\mathbb{Q}}$, is well approximated by ignoring convexity effects. This suggests that $K_1^{\mathbb{Q}}$ (and thus also the eigenvalues of $K_1^{\mathbb{Q}}$) are approximately equal in the $A_0(N)$ and $A_1(N)$ no arbitrage term structure models. ?? provides the estimates for our data and verifies that this is indeed the case. Also tabulated are the results of the model-free regression for $K_1^{\mathbb{Q}}$ which again is similar.

This observation has a number of implications. First, as stated previously, this allows us to pin down the potential volatility instruments using the cross-section of yields due to the

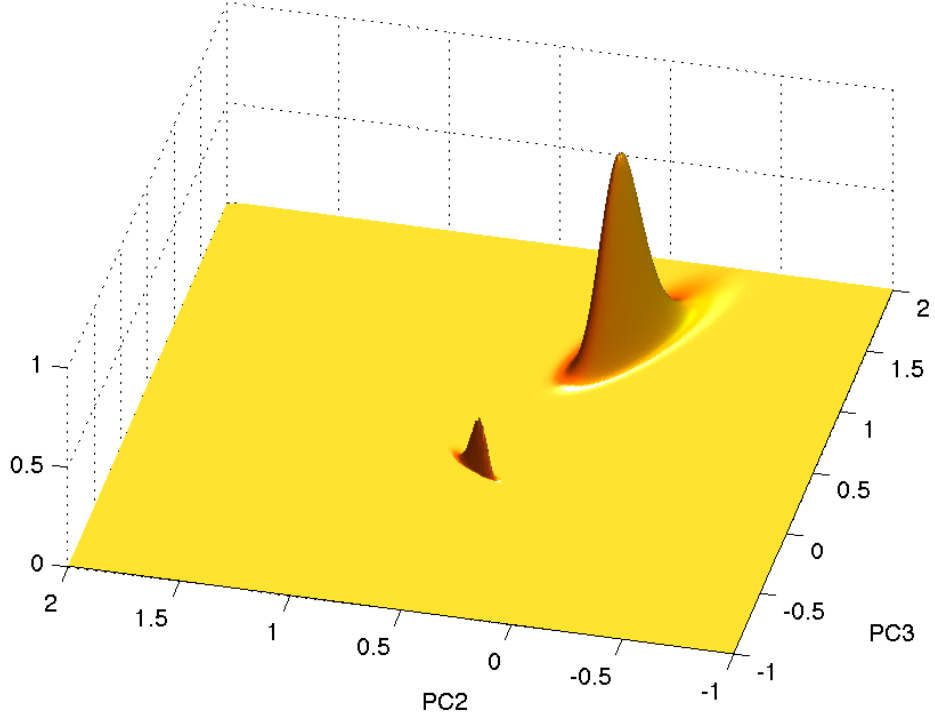


Figure 3: Likelihood Ratio Tests of \mathbb{Q} -Autonomous Restrictions. This figure reports the p -values of the likelihood ratio test of whether a particular linear combination of yields, $\beta\mathcal{P}_t$, is autonomous under \mathbb{Q} , plotted against the loadings of PC2 and PC3. The loading of PC1 is one minus the loadings on PC2 and PC3 ($\beta(1) = 1 - \beta(2) - \beta(3)$). PC1, PC2, and PC3 are scaled to have in-sample variances of one.

Figure 4: This figure plots the model estimates of the volatility from the estimated $A_1(N)$ and SV-FVAR₁(N) with the EGARCH(1,1) estimates.

Table 1: This table contains the estimates of $K_{1\mathcal{P}}^{\mathbb{Q}}$ computed from the $A_0(N)$ and $A_1(N)$ models, along with the estimates from the model-free regression of (24).

admissibility constraint. Essentially the volatility instrument is free in terms of the sign but must be one of the left eigenvectors of $K_1^{\mathbb{Q}}$ which can be computed accurately from either the cross-sectional regression or from estimation of the $A_0(N)$ model which has constant volatility and can be estimated quite quickly as shown in JSZ.

This observation also shows that in some regards, the estimation of the no arbitrage $A_1(N)$ model is more tractable than estimate of the SV-FVAR $_1(N)$ model. In the case of the Gaussian models the opposite holds: the factor model is trivial to estimate as it amounts to a set of ordinary least squares regressions while the no arbitrage model is slightly more difficult to estimate due to the non-linear constraints in the factor loadings. In the stochastic volatility models, the admissibility conditions require a number of non-linear constraints in order to ensure that volatility remains positive. The no arbitrage model essentially determines the volatility instrument up to sign and choice of eigenvector. This actually simplifies the computational burden in estimation since it reduces the set of non-linear constraints that need to be imposed.

6 Alternative models

Our primary results are driven by the fact that an affine drift requires a number of constraints in order to assure that volatility stays positive. A number of alternative models could be considered. First, one could consider a model with unspanned or nearly unspanned volatility. This, however, can only partially counteract our results in the sense that the projection of volatility onto yields must still mathematically be a positive process. So several of our insights maintain. Another possible model to consider is a model with non-linear drift. That is, we can suppose that there is a latent state variable Z_t with the drift of Z_t linear in Z_t without any constraints provided that volatility (or its instrument) is far from the boundary. Near the zero boundary, the drift of the volatility may be non-linear in such a way as to maintain positivity. Provided that the probability of entering this non-linear region is small (under \mathbb{Q}), similar pricing equation will be obtained as in the standard affine setting.

7 Conclusion

In this paper, we show that there is a strong link in a no arbitrage affine term structure model between the cross-section of bond yields and the set of positive processes under the risk neutral measure. Since volatility must define a positive process under any measure, this provides a tight link between the cross section of bond yields and the set of valid volatility instruments. To the extent that the no arbitrage model is correctly specified, no arbitrage therefore provides evidence for identifying volatility instruments. Moreover, the information about volatility instruments in the cross section of bond yield can also be approximately obtained either through model-free regressions or by analyzing Gaussian models with constant volatilities. Empirically, we show that broadly the no arbitrage models appear to match volatility with this condition in our sample. Of separate interest, we also provide a general

canonical form for affine models with stochastic volatility (the $A_1(N)$ class) where nearly all of the parameters directly relate to observable variables.

A Canonical Forms

In this section, we lay out canonical forms for discrete-time affine term structure models with one stochastic volatility factor. Our construction draws on the discrete volatility dynamics of [Le, Singleton, and Dai \(2010\)](#) and the canonical setup of [Joslin, Singleton, and Zhu \(2010\)](#) (JSZ) for gaussian term structure models. The continuous time limit of our construction nets the $A_1(N)$ family of models considered in [Dai and Singleton \(2000\)](#).

A.1 Canonical Form for Risk-Neutral Dynamics of Latent Factors

Let's denote the N latent risk factors by $(V_t, G_t)'$ where V_t governs the conditional volatility of the states and G_t represent the remaining $N - 1$ conditionally gaussian risk factors. We first write down the conditional risk-neutral (\mathbb{Q}) dynamics of states, in their most general form, that guarantee affine bond pricing. Next, we choose a set of normalizations to reduce these dynamics to a canonical setup.

Under the measure \mathbb{Q} , we assume that V_t follows a compound autoregressive gamma (CAR) process, characterized by three strictly positive parameters $(\nu^{\mathbb{Q}}, \rho^{\mathbb{Q}}, c)$ such that:¹⁰

$$E_t^{\mathbb{Q}}[V_{t+1}] = \nu^{\mathbb{Q}}c + \rho^{\mathbb{Q}}V_t, \quad \text{and} \quad \text{Var}_t^{\mathbb{Q}}[V_{t+1}] = \nu^{\mathbb{Q}}c^2 + 2\rho^{\mathbb{Q}}cV_t. \quad (36)$$

Here, $\rho^{\mathbb{Q}}$, $\nu^{\mathbb{Q}}$, and c modulate the (risk-neutral) persistence, mean and scale of V_t , respectively. To avoid V_t being absorbed at the zero boundary, we impose the discrete-time counter-part of the Feller condition and require $\nu^{\mathbb{Q}}$ be greater than one. One attractive property of the CAR process is that V_t is always strictly positive, yet its conditional Laplace transform is exponentially affine in V_t ,¹¹ which is essential to obtain affine bond pricing. It is important to note that V_t must be autonomous – that is, the gaussian states G_t must not directly affect the conditional dynamics of V_t . The presence of G_t on the right hand side of either equation in (36) would give a strictly positive probability that either the conditional mean or the conditional variance of V_{t+1} be negative.

The \mathbb{Q} -dynamics of G_t is characterized by:

$$G_{t+1} - \phi V_{t+1} \sim N(L_0^{\mathbb{Q}} + L_1^{\mathbb{Q}}G_t + L_v^{\mathbb{Q}}V_t, \Omega_1 V_t + \Omega_0). \quad (37)$$

Here the full state vector $(V_t, G_t)'$ affects the conditional mean, but only V_t governs the conditional variance of G_{t+1} , both in an affine fashion. To define a valid variance matrix, we require that Ω_1 and Ω_0 be semi-positive definite. Parameter ϕ , if nonzero, induces conditional correlation between G_{t+1} and V_{t+1} .

Coupled with the standard assumption that the short rate r_t is affine in $(V_t, G_t)'$:

$$r_t = r_{\infty} + \varrho_v V_t + \varrho_g' G_t, \quad (38)$$

¹⁰For further details of the CAR process, such as its associated conditional density and Laplace transform, see [Le, Singleton, and Dai \(2010\)](#).

¹¹That is, $E_t^{\mathbb{Q}}[e^{uV_{t+1}}] = e^{a(u)+b(u)V_t}$.

it follows that bond yields y_t^n are affine in the states for all maturities n .

Due to the unobservable nature of the states, many of the parameters defining (36), (37), and (38) are not identified. For example, V_t can be scaled up and down together with the scaling parameter c (and with appropriate adjustments of relevant parameters in (37) and (38)) without affecting the bond prices implied by the model. To avoid this indeterminacy, we fix $c = \frac{1}{2}\Delta t$ where Δt denotes the discrete time interval.¹² Furthermore, using rotations of states similar to those in Joslin (2006), we could set ϕ , $L_0^{\mathbb{Q}}$, and the off-diagonal elements of $L_1^{\mathbb{Q}}$ to zeros.¹³ Adopting the notation of JSZ, we denote the diagonal elements of $L_1^{\mathbb{Q}}$ by a column vector $\lambda^{\mathbb{Q}}$. Likewise, assuming that the gaussian states are non-degenerate, we could normalize the loadings ϱ_g in (38) to ones. Appendix *****XXX***** describes the specific rotations leading to these normalizations. The risk-neutral dynamics of $(V_t, G_t)'$ is fully characterized by $\Theta^{\mathbb{Q}} = (\nu^{\mathbb{Q}}, \rho^{\mathbb{Q}}, \lambda^{\mathbb{Q}}, L_v^{\mathbb{Q}}, \Omega_1, \Omega_0, r_{\infty}, \varrho_v)$.

A.2 Canonical Form with Yields Portfolios

Let y_t denote the $J \times 1$ vector of yields used in estimation. From the canonical setup in the preceding section, we have:

$$y_t = A + BV_t + CG_t \quad (39)$$

where A, B, C are determined through standard recursions given in Appendix *****XXX*****. Notably, A and B depend on $\Theta^{\mathbb{Q}}$, but C only depends on $\lambda^{\mathbb{Q}}$.

Fixing a $N - 1 \times J$ matrix W_g (with full row rank) and the associated yields portfolios:

$$\mathcal{P}_{g,t} = W_g y_t = \underbrace{W_g A}_{\phi_0} + \underbrace{W_g B}_{\phi_v} V_t + \underbrace{W_g C}_{\phi_g} G_t, \quad (40)$$

it follows that one can invert (40) and write G_t as a linear combination of the observable portfolios $\mathcal{P}_{g,t}$ and the volatility factor V_t . Furthermore, using this linear combination in place of G_t in the normalized versions of (37) and (38), we directly obtain the risk-neutral dynamics of $\mathcal{P}_{g,t}$.¹⁴

$$\mathcal{P}_{g,t+1} - \phi_v V_{t+1} \sim N(K_0^{\mathbb{Q}} + K_1^{\mathbb{Q}} \mathcal{P}_{g,t} + K_v^{\mathbb{Q}} v_t, \Sigma_1 V_t + \Sigma_0), \quad (41)$$

as well as how the short rate depends on V_t and $\mathcal{P}_{g,t}$.¹⁵

$$r_t = \rho_0 + \rho_v V_t + \rho'_g \mathcal{P}_{g,t}. \quad (42)$$

¹²This implies that in the continuous time limit, the conditional variance of V_{t+1} will approach $V_t dt$.

¹³For ease of exposition, we assume that the eigenvalues of $L_1^{\mathbb{Q}}$ are distinct and real-valued. See JSZ for detailed treatments in the case of complex or repeated eigenvalues.

¹⁴It can be shown that: $K_1^{\mathbb{Q}} = \phi_g \text{diag}(\lambda^{\mathbb{Q}}) \phi_g^{-1}$, $K_v^{\mathbb{Q}} = \phi_g L_v^{\mathbb{Q}} - K_v^{\mathbb{Q}} \phi_v$, $K_0^{\mathbb{Q}} = (I_{N-1} - K_1^{\mathbb{Q}}) \phi_0$, $\Sigma_1 = \phi_g \Omega_1 \phi'_g$, and $\Sigma_0 = \phi_g \Omega_0 \phi'_g$.

¹⁵It can be shown that: $\rho'_g = l' \phi_g^{-1}$, $\rho_v = \varrho_v - \rho'_g \phi_v$, $\rho_0 = r_{\infty} - \rho'_g \phi_0$.

Together, (36), (41) and (42) fully characterize the risk-neutral dynamics of a term structure model, with risk factors V_t and $\mathcal{P}_{g,t}$, that is observationally equivalent to one defined by (36), (37) and (38), with risk factors V_t and G_t . No new parameters have been introduced – since there is a unique mapping from $\Theta^{\mathbb{Q}}$ to every parameter that governs (41) and (42). It is straightforward to write down the new term structure in terms of V_t and $\mathcal{P}_{g,t}$:

$$y_t = A_{\mathcal{P}} + B_{\mathcal{P}}V_t + C_{\mathcal{P}}\mathcal{P}_{g,t}, \quad (43)$$

where $C_{\mathcal{P}} = C\phi_g^{-1}$, $B_{\mathcal{P}} = B - C_{\mathcal{P}}\phi_v$, and $A_{\mathcal{P}} = A - C_{\mathcal{P}}\phi_0$.

To complete the model, we specify the physical (\mathbb{P}) dynamics of V_t and $\mathcal{P}_{g,t}$ that are analogous to (36) and (41). Specifically, V_t follows a CAR process and $\mathcal{P}_{g,t}$ is conditionally gaussian such that:

$$E_t^{\mathbb{P}}[V_{t+1}] = \nu^{\mathbb{P}}c + \rho^{\mathbb{P}}V_t, \quad \text{and} \quad \text{Var}_t^{\mathbb{P}}[V_{t+1}] = \nu^{\mathbb{P}}c^2 + 2\rho^{\mathbb{P}}cV_t, \quad (44)$$

$$\mathcal{P}_{g,t+1} \sim^{\mathbb{P}} N(K_0^{\mathbb{P}} + K_1^{\mathbb{P}}\mathcal{P}_{g,t} + K_v^{\mathbb{P}}V_t, \Sigma_1V_t + \Sigma_0), \quad (45)$$

where apart from the Feller condition ($\nu^{\mathbb{P}} > 1$) and conditions that guarantee stationarity ($0 < \rho^{\mathbb{P}} < 1$ and the eigenvalues of $K_1^{\mathbb{P}}$ be within the unit circle), all other parameters are completely unconstrained. Note that the \mathbb{P} and \mathbb{Q} measures share the scaling parameter c and the variance parameters Σ_1 and Σ_0 , hence diffusion invariance is preserved in the continuous time limit.

Our construction so far is the analogue of the JSZ canonical form for the gaussian components of a term structure model with stochastic volatility. In essence, the $N - 1$ gaussian latent factors can be replaced by *any* $N - 1$ (theoretical) yields portfolios.¹⁶ The autonomous nature of V_t however rules out the possibility that *any ex ante* yield portfolio can represent V_t . Nevertheless, from any yield portfolio independent of $\mathcal{P}_{g,t}$, the yield portfolio that represents V_t can be identified.

To see this, consider a yield portfolio $\mathcal{P}_{v,t}$ characterized by loadings W_v ($1 \times J$), we have:

$$\mathcal{P}_{v,t} = W_v A_{\mathcal{P}} + W_v B_{\mathcal{P}}V_t + W_v C_{\mathcal{P}}\mathcal{P}_{g,t} \quad (46)$$

which leads to

$$V_t = (W_v B_{\mathcal{P}})^{-1}(\mathcal{P}_{v,t} - W_v A_{\mathcal{P}} - W_v C_{\mathcal{P}}\mathcal{P}_{g,t}). \quad (47)$$

Clearly, V_t is a linear combination of $\mathcal{P}_{v,t}$ and $\mathcal{P}_{g,t}$. In particular, because $C_{\mathcal{P}}$ only depends on $\lambda^{\mathbb{Q}}$, the relative loadings of the N portfolios in (47) are completely determined by $\lambda^{\mathbb{Q}}$.

¹⁶It is important to see the distinction between theoretical (model implied) yields portfolios and observed yields portfolios. Our canonical form applies to the former since the latter can be different due to measurement errors or other microstructure noises. [Joslin, Le, and Singleton \(2011\)](#) discuss the implications of using different yields portfolios in estimation.

A.3 Unconstrained Counterparts

Fixing the yields portfolios (W_g, W_v) used in estimation, our canonical form in the preceding section clearly reveals that the no-arbitrage restrictions are two-fold. First, as is seen with pure gaussian term structure models, the no-arbitrage restrictions put constraints on the risk loadings $A_{\mathcal{P}}, B_{\mathcal{P}}$, and $C_{\mathcal{P}}$ in (43), linking theoretical yields to the underlying risk factors. These loadings are linked in a non-linear fashion to a lower-dimensional parameter set $\Theta^{\mathbb{Q}}$. Second, the volatility factor V_t is tightly linked to the yields portfolios through (47). Up to a scaling and a translation effect, the risk-neutral persistence $\lambda^{\mathbb{Q}}$ of the gaussian factors completely determine the relative dependence of V_t on the yields portfolios.

These two restrictions aside, the risk-neutral dynamics have no direct impact on the physical dynamics of the volatility factor. The parameters defining the physical dynamics of V_t in (44) are completely distinct from $\Theta^{\mathbb{Q}}$. As for the gaussian factors, the parameters that govern the conditional mean in (45) are also distinct from $\Theta^{\mathbb{Q}}$. However, due to diffusion invariance, the conditional variances under \mathbb{P} and \mathbb{Q} are modulated by the same parameters (Σ_1 and Σ_0).

These observations naturally lead to the following construction of the unconstrained counterparts to our no-arbitrage model. First, we specify the linear dependence of V_t on the yields portfolios:

$$V_t = \gamma_0 + \gamma_v \mathcal{P}_{v,t} + \gamma_g \mathcal{P}_{g,t} \quad (48)$$

where γ_0, γ_v , and γ_g are completely unconstrained.¹⁷ Equation (48) is the unconstrained counterpart to (47). Second, once the volatility factor is determined through (48), we specify a linear dependence of y_t on V_t and $\mathcal{P}_{g,t}$:

$$y_t = A_{\mathcal{U}} + B_{\mathcal{U}} V_t + C_{\mathcal{U}} \mathcal{P}_{g,t}, \quad (49)$$

where $A_{\mathcal{U}}, B_{\mathcal{U}}$, and $C_{\mathcal{U}}$ are again completely unconstrained. Equation (49) is the unconstrained analogue of (43). Finally, we adopt exactly the same physical dynamics described in (44) and (45).

B Concluding Remarks

¹⁷Of course, γ_0, γ_v , and γ_g must be such that V_t implied by (48) be strictly positive in sample. The same requirement applies to $\Theta^{\mathbb{Q}}$ and V_t implied by (47).

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