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Abstract

In this paper we use the notion of a *housing bubble* as an equilibrium in which some *investors* hold houses only for resale purposes and not for the expectation of a dividend, either in the form of rents or utility. We provide a life-cycle model where households face collateral constraints that tie their credit capacity to the value of their houses and examine the conditions under which housing bubbles can emerge. In such equilibria, the total housing stock is held by owners that extract utility from their homes, landlords that obtain rents, and investors. We show that an economy with tighter collateral constraints is more prone to bubbles which, in turn, tend to have a larger size but are less fragile in face of fund-draining shocks. Our environment also allows for *pure bubbles* on useless assets. We find that multiple equilibria in which the economy moves endogenously from a pure bubble to a housing bubble regime and vice versa are possible. This suggests that high asset price volatility may be a natural consequence of asset shortages (or excess funding) that depress interest rates sufficiently so as to sustain an initial bubble. We also examine some welfare implications of the two types of bubbles and discuss some mechanisms to rule out equilibria with housing bubbles.

Keywords: collateral constraints, buy-to-let investment, housing bubbles, switching bubbles, welfare.

JEL numbers: G21, R21, R31.

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"A tendency to view housing as an investment is a defining characteristic of a housing bubble. Expectations of future appreciation of the home are a motive for buying that deflects consideration from how much one is paying for housing services. That is what a bubble is all about: buying for the future price increases rather than simply for the pleasure of occupying the home. And it is this motive that is thought to lend instability to bubbles, a tendency to crash when the investment motive weakens." *Case and Shiller (2003).*

1 Introduction

Although recently we have witnessed an increasing number of papers about the birth, growth and bust of housing bubble episodes, theoretical underpinnings, useful to understand the macroeconomic consequences of these events, are largely underdeveloped. We offer a theoretical framework to analyze housing bubbles.

Following Case and Shiller, we use the concept of housing bubble as a rational expectations equilibrium in which some investors hold houses only for resale purposes and not for the expectation of a dividend, either in the form of rents or utility from occupancy.[1] There are many channels through which the economy can end up generating bubbles. Our contribution is to expose how the interaction between financial frictions and distortions in the rental market across different households give rise to a set of conditions under which housing bubbles can naturally emerge, grow, and burst.

We develop our arguments in the context of an overlapping generations (OLG) economy where a continuum of households live for three periods. Thus, at each date there coexist three *generations*: (i) the young who receive an exogenous endowment of consumption goods and make their house tenure and financing decisions; (ii) the middle-aged that repay mortgages and accumulate assets aimed at financing consumption later in their life; this group provides the bulk of loanable funds in the economy; and, (iii) the old that simply consume the proceeds of previous savings.

The model incorporates frictions in both the credit and the rental markets. First, we consider minimum collateral requirements that tie the households' credit capacity to the value of their houses. Borrowing limits, when binding, generate a mistmatch between desired and available funds to finance housing purchases. Second, we introduce a distortion in the rental price, which in reality could be related e.g. to informational asymmetries or tax distortions, that produces a bias against renting vis-à-vis home-ownership. Heterogenous borrowing capacity across households gives rise to individual heterogeneity regarding both housing tenure (i.e. home renting versus property) and financial position (constrained or unconstrained). In this context, the rental market allows households with lower credit capacity to consume a larger amount of housing services than if they were to rely only on home-ownership. In particular, we show that within the same young generation there may coexist three groups of households: (i)

[1] A bubble defined in this way can satisfy the conditions usually invoked in the macroeconomic literature on bubbles (see Tirole (1985)). Later in this Introduction we will provide a link of our paper with a recent revival of this literature (see, e.g., Caballero (2006)).

renters, (ii) landlords (i.e. buy-to-let investors), and (iii) homeowners that do not participate in the rental market.

A critical feature of the model is that under some conditions there coexist two alternative regimes without bubbles: (i) a *low valuation* regime, in which landlords do not exhaust their borrowing limits; and (ii) a *high valuation* regime where every member of the young generation is financially constrained. In turn, in a high valuation regime, relative to a low valuation one, the volume of buy-to-let investment, the price-rent ratio and the housing price are higher, and the interest rate is lower. A high valuation regime featuring a low interest with a large volume of lending is possible because low interest rates go hand to hand with a high level of net wealth for the middle-aged generation and, hence, with a higher aggregate supply of funding —i.e. the aggregate savings schedule in a high valuation regime is negatively related to the interest rate. Key to this result is the fact that in a high valuation regime a fall in the interest rate does not lead to more borrowing by the young (constrained) households. Yet, lower interest rates raise net wealth of the middle-aged generation—as the same volume of debt now requires a lower repayment cost. This extra wealth sustains higher levels of current and *future* consumption with the latter raising the funding supply. A critical condition for this mechanism to operate is that borrowing limits are not too loose, for otherwise the economy can not generate sufficient funds to meet a large demand for funding generated by a low interest rate.

The previous funding mechanism that gives rise to multiple bubbleless regimes is at the heart of our analysis of housing bubbles. We show that, in addition to low and high valuation regimes, where middle-aged households only maintain loanable funds in their portfolios, there may exist a *third* type of equilibrium where they also hold assets that do not pay any dividend (i.e. bubbles). To gain further insight into the existence of bubbles, let us imagine that the economy is initially located in a high valuation regime and, suddenly, the interest rate falls. For a given stock of maturing debts, net wealth of the middle-aged and their savings increase. In the bubbleless economy, such extra funding cannot be fruitfully invested, as potential borrowers (the young) can not commit to repay more than they are already borrowing. Investment in houses only for resale purposes by middle-age households which, within the logic of the model act as *speculators* or *bubble-holders*, thus appears as a natural solution for a problem of excess saving (or insufficient asset production).

In our setup, nothing prevents, in addition to housing bubbles, the existence of *pure bubble* equilibria —where intrinsically useless assets have a positive price. We then show that, regardless of the nature of the bubble (pure or housing), its steady state size grows as the severity of the collateral constraint rises and argue that in a pure bubble steady state everybody is better off relative to a housing bubble steady state. The reason for this latter result is intuitive, since in a housing bubble some dwellings are purchased only for a pure investment motive and do not produce housing services to anyone.

As regards the analysis of the dynamic properties of the economy, there are two questions that attract most of our attention. First, under which conditions may the economy jump from a low valuation regime toward an equilibrium with bubbles (and vice versa)? Second, are housing bubbles fragile? With regard to the first question, we show that the once the economy is a low valuation regime, the equilibrium is *unique* and the economy cannot generate a bubble endogenously. Given this sharp result, we study the consequences of *temporary* shocks to the availability of funding and show that for a sufficiently large positive funding shock all possible equilibria must contain a bubble. An environment with tighter collateral constraints is more prone to generate equilibria that contain bubbles, in the sense that relatively smaller shocks may give rise to a bubble regime. In turn, we show that economies with tighter collateral constraints will fuel larger stationary bubbles, that in turn are less fragile to negative fund-draining shocks.

Against the previous result of equilibrium uniqueness following an initial condition within a low valuation regime, we show that our framework allows for *multiple equilibria* in the sense that if the economy is initially in a pure bubble regime, then a purely endogenous jump towards a housing bubble regime is also possible. The same result applies with respect to changes of regime from housing bubbles towards pure bubbles. This suggests that high asset price volatility may be a natural consequence of a problem of asset shortages or excess funding that depresses interest rates sufficiently so as to sustain an initial bubble.

Given that housing bubbles may convey important social costs, we discuss some alternative policies aimed at reducing their size and, eventually, ruling them out. In particular, we argue that removing frictions in the rental market is conducive to smaller bubbles and higher general welfare, as this tends to solve the asset shortage. We then analyze the effects of a Tobin tax on housing speculative investments and show that creating a bias against speculative housing investments may be ineffective to remove bubbles if it is not supplemented with a parallel policy that, as the previous one, attacks the underlying problem of insufficient investment options.

Links to the literature. —The paper is related to several strands of literature. As the model crucially hinges on the existence of borrowing constraints to deliver equilibria with bubbles, our contribution is related to a recent body of the macro-literature that focuses on the role of bubbles as natural market-based solutions to an underlying problem of asset shortages, as extensively discussed in Caballero (2006). In particular, the theory of this paper shares an important feature which recent work by Ventura (2003, 2004) and Caballero, Farhi, and Hammour (2006), where bubbles can emerge even if the initial ("bubbleless") interest rates exceed the growth rate of the economy. The empirical counterpart of this can be summarized as follows: bubbles in our set up only take place if the interest rate falls *below* some threshold and, as the bubble approaches its steady state value the interest rate tends to fall unlike the conclusion in Tirole (1985).

As regards the existence of potential bubble-holders, our model can be re-interpreted along the lines of the segmented-investors framework developed by Ventura (2003).[2] That is, an important ingredient of our theory is that there is a form of segmentation which implies that some households (here, the middle-aged) do not extract utility or rents from their housing stock and, rather, are happy to keep unoccupied houses expecting a capital gain only. In this sense, we do not aim at providing a deep theory of why some households can act as landlords, who obtain rents in addition to eventual capital gains, while others can not. Rather, we assume that the former group exists and study the conditions under which they choose to hold houses for a resale motive only, and then explore a rich set of macroeconomic consequences.

We also extend the recent literature on bubbles in the following dimensions. First, we allow for bubbles on an intrinsically useful asset, an issue which has received scant attention before.

[2] More generally, the notion of segmentation among investors according to the return of their available investment projects is at the heart of the literature initiated by Kiyotaki and Moore (1997).

Jovanovic (2007) has recently provided a parallel working case of a bubble on a genuine useful object (old French wine). His notion of a bubble is close in spirit to ours, as in both cases a fraction of the total stock of the asset is actually used while other fraction is retained for resale purposes only. However, his article abstracts from any form of financial imperfection and, hence, bubbles never solve a problem of insufficient asset supply, an issue which is central to our story.

While Caballero and Krishnamurthy (2006) note that bubbles on useless assets can be welfare reducing in the context of a dynamically inefficient economy in which agents undervalue the aggregate risk of a bubble-crash, we argue that a stationary housing bubble is always a Pareto-dominated equilibrium even if such a bubble does not burst. We also highlight the fact that bubbles (of any kind) tend to amplify welfare differences across households when these face different credit conditions. In short, as bubbles only occur in environments of low interest rates and binding collateral constraints, households with looser collateral constraints tend to win with the surge of a bubble and, partially, at the expense of households with tighter borrowing limits.

Finally, by putting bubbles on useless and useful assets under the same umbrella we are able to extract conclusions regarding, for instance, the relative fragility of housing and pure bubbles and the conditions under which the crash of a bubble leads either to the surge of another bubble or to a bubbleless regime.

Plan of the paper. —In section 2 we present the model and describe how housing, rental, and credit markets are endogenously segmented in the absence of bubbles. In section 3 we analyze the existence and coexistence of the two alternative bubbleless steady states. In section 4 we focus on the conditions under which there exists a steady state with bubbles and analyze how prices and allocations change in the presence of a bubble. In section 5 we discuss some welfare implications of the existence of bubbles. In Section 6 we focus on the dynamic properties of the model and investigate the existence of unique or multiple equilibria. In section 7 we discuss some policy proposals to rule-out housing bubbles. A final section presents the main conclusions of the paper. An appendix contains all the formal proofs.

2 The model

In this section we present an OLG model where a continuum of households of measure one live for three periods. At each date t there coexist three *generations*: (i) *young* who make their house tenure and financing decisions; (ii) *middle-aged* who repay mortgages and build up stocks of financial assets aimed at financing consumption later in their life; this group provides the bulk of loanable savings in the economy; and, (iii) *old* who consume the proceeds of previous savings. After dying, each household is replaced by an identical newly born one, so that total population remains constant.

2.1 Timing and assumptions

In the first period of life, each household receives an exogenous endowment of one unit of consumption goods.

This endowment is net of a lump sum tax levied by the government, T.[3] The household then decides the amount of consumption, housing services, loanable savings (henceforth, deposits) and debt. During this period a household may accumulate assets by purchasing houses (for owner-occupancy or buy-to-let) or deposits. As for the debt decision, we assume that only collateralized debt is available, and we impose the existence of borrowing limits at the time of purchasing a house. Specifically, a minimum downpayment proportional to the value of the housing investment is required. Regarding the minimum downpayment we consider that it is household-specific, i.e. some households can have a higher leverage than others. We think of this assumption as a pragmatic modelling device aimed at capturing individual heterogeneity in the degree of mismatch between *desired* (i.e. in the absence of a downpayment requirement) and *available funds* to finance housing purchases.[4] As for the source of the bias towards ownership, we consider that landlords pay a tax proportional to their rental income.[5]

For simplicity, the two tenure alternatives, renting and owner occupancy, are assumed to provide perfectly substitutable services in an amount equal to the size of the dwelling. Moreover, the stock of houses is thought to be perfectly divisible and each household can simultaneously enjoy housing services from owing a house and renting another.[6]

In the second period, households decide how to allocate their net worth (stock of houses acquired in the previous period plus maturing deposits net of debt repayments) between consumption and deposits. Finally, in the last period they consume all their financial wealth.

Taken together, these assumptions deliver a simple structure that allows us to focus on equilibria that resemble the fact that early in life households borrow to buy houses and, as time goes by, agents who have already acquired houses start to increase their holdings of financial assets. Such a life-cycle portfolio composition pattern has been documented in several recent empirical studies.[7]

It is worth noting that the environment just described embeds some assumptions that are more restrictive than needed to generate the aforementioned life cycle facts. In particular, the model structure implies that the households sell their entire stock of houses in the second period to the newly born generation, and that they do not participate in the rental market at that time either. This assumption allows us to circumvent the problem of having to deal with the segmentation of the housing-renting and credit markets for the generation born in the previous period.[8] Indeed, as we will show later in detail, the critical feature of this structure is that it

[3] The lump sum tax plays a minor role in our story. Later on we introduce government debt and allow for changes on its volume, in which case a sufficiently large T ensures that at some point the government may reduce its demand for funding without hitting the zero-bound constraint on government consumption.

[4] In order to derive such a household-specific mismatch as an endogenous outcome we would have to consider non-homothetic preferences, individual specific dynamic income patterns (as in e.g. Kiyotaki et al. 2007), or heterogenous initial collateralizable endowments (say, bequests in the form of houses, as in Stein 1995). Our assumption allows us to achieve the same outcome in terms of delivering the aforementioned mismatch in a way that preserves analytical tractability. Evidence supporting the effects of individual characteristics, including wealth and income, on the conditions of access to credit is provided by Haurin, Hendershott and Wachter (1996).

[5] See, for instance, Henderson and Ioannides (1983) and Poterba (1984) for some early analyses on frictions in the rental market.

[6] These assumptions, perfect divisibility and non-excludability in the two forms of tenure, greatly simplify the analysis by eliminating non-convexities in the household's choice set. Indeed, the objective of the household's problem is concave and the constraint set convex, and hence the first order conditions are necessary and sufficient.

[7] See e.g., Fernández-Villaverde and Krueger (2007) and Yang (2006).

[8] In the model we solve below, we find equilibria in which the members of the generation born at the current

retains the idea that a household first accumulates wealth in the form of housing stocks and afterwards tends to save in the form of loanable assets.

This OLG structure would also be consistent with an economy in which, even when the households would ideally wish to adjust the size of their housing stock continuously, they do not do so but they rather choose to adjust the size of their financial assets portfolios. Thus, instead of modelling explicitly some frictions in the housing and rental markets that could justify such an asset accumulation pattern (e.g. transaction costs, minimum house-size constraints, non-Walrasian elements, etc.), we employ the following assumptions that keep the model analytically tractable.[9]

2.2 Households and market segmentation

A household born at time t maximizes the following flow of utility:

$$U_t = \log\left(c_t^y\right) + \beta\left[\log\left(c_{t+1}^m\right) + \log\left(h_{t+1} + s_{t+1}\right)\right] + \beta^2 \log\left(c_{t+2}^o\right) \tag{1}$$

subject to the following set of constraints:

$$c_t^y + p_t h_{t+1} + \left[p_t - (1-\tau)\, q_t\right] g_{t+1} + q_t s_{t+1} + a_t^y - d_t \leq 1 \tag{2}$$

$$c_{t+1}^m + a_{t+1}^m \leq p_{t+1}\left(h_{t+1} + g_{t+1}\right) + \left(1 + r_{t+1}\right)\left(a_t^y - d_t\right) \tag{3}$$

$$c_{t+2}^o \leq \left(1 + r_{t+2}\right) a_{t+1}^m \tag{4}$$

$$d_t \leq \left(1 - \theta_z\right) p_t \left(h_{t+1} + g_{t+1}\right) \tag{5}$$

$$c_t^y \geq 0;\ c_{t+1}^m \geq 0;\ c_{t+1}^o \geq 0;\ h_{t+1} \geq 0;\ g_{t+1} \geq 0 \tag{6}$$

$$s_{t+1} \geq 0;\ a_t^y \geq 0;\ a_{t+1}^m \geq 0;\ d_t \geq 0; \tag{7}$$

where $\beta \leq 1$ is the subjective discount factor; c represents consumption of goods and h and s the flow of housing services obtained from owner-occupied and rented houses, respectively. Utility flows are assumed to be identical to the size of the housing stock being occupied. The superscripts y, m, and o, stand for young, middle-aged and old, respectively; a represents deposits, d is the volume of debt, g is the stock of buy-to-let houses, which are rented out within the same period, p is the housing price, q is the rental price (both p and q are given in terms of consumption goods), and r is the interest rate. Rental income is taxed at a rate $\tau > 0$.

According to (5) the borrowing limit is such that the household must satisfy a minimum downpayment requirement, θ_z.[10] The subscript z (henceforth, the "household index") indicates

period are grouped into three distinct segments. Thus, relaxing the assumptions about the endowment, and housing and rental market participation of the generation born in the preceding period would potentially lead to equilibria in which nine segments coexist. While such a sort of analysis is interesting in its own, since it would allow us to add 'property-ladder' considerations into our general equilibrium framework, it goes beyond the scope of this paper.

[9] Indeed, exogenous intermittent individual market participation is not an uncommon assumption in general equilibrium models with collateral constraints (see e.g. Kiyotaki and Moore 2005).

[10] Thus, we are assuming that a home buyer must put up a fixed fraction of the value of the house as a downpayment. A similar constraint can be found e.g. in Ortalo-Magné and Rady (2006). In an environment in which default on mortgaged debt is possible, Stein (1995) shows that a credit limit of this form arises endogenously

that the minimum downpayment is specific to each household. In particular, $z \in [\underline{z}, \overline{z}]$ is distributed according to a time-invariant continuos and differentiable function $F(z)$ with support on $[\underline{z}, \overline{z}]$. The following assumption specifies how θ_z varies across households.

Assumption 1 (Heterogenous financial mismatch). $\theta_z = \theta - z$, and $0 < \theta - z < 1$ $\forall z \in [\underline{z}, \overline{z}]$.

Thus, the individual minimum downpayment, θ_z, is a combination of a common term θ, aimed at capturing the economy-wide conditions on collateral requirements and an individual-specific term, z.[11]

We next introduce the different groups of households that will coexist in an interior equilibrium (defined later), as classified according to their tenure choice, and then characterize in detail how the housing, rental, and credit markets are accordingly segmented.

Definition 1 *(i) Renter, if $s_{t+1} > 0$ and $g_{t+1} = 0$, (ii) Constrained Buyer, if $g_{t+1} = s_{t+1} = 0$, (iii) Landlord, if $s_{t+1} = 0$ and $g_{t+1} > 0$.*

The following proposition characterizes the solution of the problem faced by the different types of households.

Proposition 1 *(i) All landlords are simultaneously either constrained or unconstrained; (ii) $g_{t+1} \cdot s_{t+1} = 0$; (iii) $s_{t+1} = 0$, if the borrowing constraint is not binding; (iv) $h_{t+1} > 0$; (v) There exist a non-empty set of constrained households with $z \in [z_1, z_2]$ such that $g_{t+1} = s_{t+1} = 0$, where $\theta(z_1) = \frac{2+\beta}{\eta_t}$, and $\theta(z_2) = (1-\tau)\frac{2+\beta}{\eta_t}$, where $\eta_t \equiv \frac{p_t}{q_t}$.*
Proof: *See Appendix.*

We present here the outline and intuition behind the results. To understand the first result (i), notice that the gross return for a landlord born at time t (constrained or not) in terms of time $t+1$ units of consumption is equal to $\frac{p_{t+1}}{p_t-(1-\tau)q_t}$, which includes capital gains plus rents. If the landlord is unconstrained, then such return must coincide with the cost of external funds, i.e.

$$\frac{p_{t+1}}{p_t - (1-\tau)\, q_t} = 1 + r_{t+1} \tag{8}$$

Now, imagine that there is a constrained landlord. Then his portfolio's gross return must be larger than the interest rate, i.e.

$$\frac{p_{t+1}}{p_t - (1-\tau)\, q_t} > 1 + r_{t+1} \tag{9}$$

Thus, a simple arbitrage argument rules out the coexistence of these two landlords. The second result (ii) states that no household would optimally participate on both sides of the tax-distorted rental market. From (iii) it follows that only households exhausting their borrowing capability

as an optimal choice for lenders. An alternative modelling choice, which we do not explore here, is to assume that a borrower can get external funding up to a fixed fraction of the discounted resale value of the collateralized item, in which case the borrowing constraint would read as $d_t \leq (1-\theta_z)\, p_{t+1}\,(h_{t+1} + g_{t+1}) \,/\, (1 + r_{t+1})$. Interestingly, as it will become clear later, if there is a housing bubble both specifications have similar implications.

[11] The assumption that the combination of these two terms takes a linear form is made for simplicity.

will demand rental units. That is, if the household is not constrained it will never get services in the rental market, since these services could be self-produced by exploiting the borrowing capacity and, hence avoiding the tax. In this sense, the existence of a borrowing limit is a necessary condition for a rental market to be operative. Result (iv) is a direct consequence of previous result (iii), and it states that a household that optimally decides to demand a positive amount of rental services must simultaneously be an owner. Finally, from (v) it follows that, as long as $\tau > 0$, there is a fraction of households that do not participate in the rental market, and we refer to them as *constrained buyers* in definition 1. These households exhaust their borrowing limits to finance the purchase of their homes. This is so because, on the one hand, they face a borrowing limit low enough so as to prefer not to demand housing services in the rental market; but on the other hand, such a limit is sufficiently tight so as to preclude them from investing in buy-to-let dwellings. Finally, from the expressions for $\theta(z_1)$ and $\theta(z_2)$, it follows that the measure of households (i.e. the extensive margin) that demand (supply) rental services increases (decreases) with the price-rent ratio, which we denote by η_t.

Interior equilibrium. Before characterizing the equilibrium of this economy, we briefly refer to the supply of houses and the actions of the government. First, we assume a fixed stock of houses, denoted by H. Second, the government consumes all the proceeds obtained from lump sum taxes and the tax on rental income. For the moment, we do not consider government debt, so that the government flow of funds constraint at time t is

$$C_t^g = T + \tau q_t G_{t+1} \tag{10}$$

where C_t^g is government consumption and G_{t+1} is the aggregate amount of rental services produced by landlords. In the remaining we look at interior equilibria where the rental market is operative.

Definition 2 *A perfect foresight competitive interior equilibrium is a set of allocations,* $\{c_t^y, c_t^m, c_t^o, a_t^y, a_{t+1}^m, d_t, h_{t+1}, s_{t+1}, g_{t+1}, C_t^g\}$*, for all household types* $z \in [\underline{z}, \overline{z}]$*, prices, rents and interest rates sequences* $\{p_t, q_t, r_t\}$ *, and a taxes* T *and* τ*, such that: (i) households maximize utility (1) subject to constraints (2)-(6) given the sequences* $\{p_t, q_t, r_t\}$ *and the government policy; (ii) the government satisfies its flow of funds constraint (10); and (iii) markets (goods, houses, rental, and credit) clear.*

3 Multiple *bubbleless* steady states

In this section we analyze the properties of the model focusing on steady state equilibria without bubbles. The latter are analyzed extensively in the forthcoming sections. Given that all landlords are simultaneously either constrained or unconstrained (proposition 1.i), we introduce the following definition of alternative "bubbleless" steady states.

Definition 3 *A Low Valuation Steady State -LVSS- (High Valuation Steady State -HVSS-) is a set of time invariant allocations* $\{c^y, c^m, c^o, a^y, a^m, d, h, s, g, C_t^g\}$*, prices and rents* $\{p, q\}$ *, a positive interest rate* r*, and taxes* $\{T, \tau\}$ *satisfying the requirements listed in definition 2 and such that all landlords are unconstrained (constrained).*

In principle, absent the possibility of storage in this economy, there is no reason to rule out steady states with *negative* interest rates. However, as shown later, the possibility of a housing bubble removes any steady state with $r < 0$. We postpone this discussion to section 4 and focus now on stationary equilibria with $r > 0$. Also, we proceed by assuming that the tax distortion is arbitrarily small, i.e. $\tau \to 0^+$. The limiting case of a "frictionless" rental market greatly simplifies the analysis that follows as it removes distributional issues. Specifically, as $\tau \to 0^+$, the measure of constrained buyers becomes negligible and this allows us to focus on an economy in which there only exist landlords and renters in the young generation.[12] Nevertheless, sections 5 and 7 contain discussions on several aspects of the model with a non-negligible distortion in the rental market.

In the remaining of this section, we first characterize the steady state equilibrium conditions in the rental, housing, and credit markets and then analyze the conditions of existence and coexistence of the two types of steady states outlined in definition 3.

3.1 Housing, rental and credit markets

The housing market clearing condition can be written in compact form as

$$pH = H^T(\eta) \tag{11}$$

where total expenditure on housing services (measured in consumption goods) is denoted by $H^T(\eta)$ in order to emphasize that it can be expressed as a function whose only argument is the price-rent ratio, η. The following lemma will help later on to rank housing prices across the alternative steady states.

Lemma 1 *Total expenditure on housing services is an increasing function of* η*,*

$$\frac{\partial H^T}{\partial \eta} = \gamma\beta \int_{\underline{z}}^{z_1} dF(z) + \gamma\beta(1-\tau)^{-1}\int_{z_2}^{\overline{z}} dF(z) \overset{\tau \to 0^+}{=} \gamma\beta > 0, \tag{12}$$

where $\gamma \equiv (1+\beta)^{-2}$.
Proof: *See Appendix.*

The effects of changes in η on total expenditure in housing services only hinge on the expenditure switch along the intensive margin of renters and landlords, which, at the aggregate level, correspond to the two terms in the right hand side of (12), respectively.[13] For renters, the positive effect of η on housing expenditure is the result of two opposite effects. On the one hand, an increase in η generates a reduction in the resources devoted to purchase houses, but

[12] Notice that if $\tau \to 0^+$, then $z_1 \to z_2^-$ (see proposition 1.v)..

[13] The merit of assuming that the objective function is continuous (i.e. any individual can be an owner and a renter simultaneously) is that, by the theorem of the maximum, the maximizing choices are also continuous in z, as the set of of constraints is also continuous in the choice variables. In practical terms this implies that the variation along the extensive margins will cancel out. This result will apply in all subsequent calculations, unless otherwise noted. We also notice that, even if the tax τ is not negligible, changes in η are irrelevant for the expenditure of constrained buyers since they do not participate in the rental market.

on the other, the increase in η also raises the amount of renting services. The later positive dominates the former since substituting renting for constrained ownership allows a household to enjoy a larger total amount of housing services. For a landlord, an increase in η makes the buy-to-let investment relatively less profitable, thus reducing the opportunity cost of occupying a larger dwelling. This, in turn, pushes up his own level of expenditure on houses services.

We now turn to the analysis of the rental market. In a LVSS the supply of houses for renting is governed by the steady state counterpart of (8),

$$\eta = \frac{1+r}{r}, \tag{13}$$

which is written under the assumption that $\tau \to 0^+$. As this condition holds, every landlord is willing to elastically supply houses for rent up to the limit dictated by his borrowing constraint. But, from proposition 1(v) and lemma 1 we know that for a sufficiently high η the demand for renting will be relatively high while the supply will be low. Taking this argument to the limit, for a sufficiently high value of η the pool of landlords can only meet that demand for renting by hitting their borrowing capacity. Such a limiting case corresponds to what we have defined as a HVSS, in which every landlord is constrained so that the steady state version of (9),

$$\eta < \frac{1+r}{r}, \tag{14}$$

holds and, as a result, the supply of houses for renting is not longer fully elastic. The following lemma formalizes this discussion regarding the price-rent ratio and extends it to the interest rate and the housing price. (Hereafter, we use the superscripts LV and HV with reference to the corresponding steady state, e.g. η^{LV} denotes the price-rent ratio in a LVSS).

Lemma 2 *If both steady states, LVSS and HVSS, exist then* $\eta^{LV} < \eta^{HV}$, $r^{HV} < r^{LV}$ *and* $p^{HV} > p^{LV}$.
Proof: *See Appendix.*

First, the fact that $r^{HV} < r^{LV}$ reflects that landlords will only hit their borrowing limit if the interest rate is sufficiently low since their (unconstrained) demand for funds is a decreasing function of r. Second, as a HVSS goes hand to hand with a higher price-rent ratio and the latter is positively related to total expenditure on housing services (lemma 1), we learn that $p^{HV} > p^{LV}$.

Remark 1. We notice that for some particular parameter configurations it is possible to find a stationary equilibrium in which $\eta = \eta^{HV}$, with each landlord's borrowing constraint (5) holding as an equality, while its associated multiplier being equal to zero, which implies that the non-arbitrage condition (13) holds. For the ease of the exposition, we label such limiting case as a HVSS.

We now turn our attention to the characterization of the credit market across the two alternative steady states. As with the aggregate expenditure on housing services, imposing equilibrium in the rental market we can write the aggregate demand for funds $D(\eta)$, as a function of the price-rent ratio only. As stated in the following lemma, $D(\eta)$ also displays similar comparative statics to those of $H^T(\eta)$.

Lemma 3 *The aggregate demand for funding is an increasing function of η. Thus, in a LVSS (relative to a HVSS) the aggregate volume of lending is lower.*
Proof: *See Appendix.*

Much as in the case of the demand for housing services, the positive effects of η on the demand for debt hinge on the movements along the intensive margin of renters and landlords. In fact, the intuition behind the positive sign above is reminiscent from the one arising in the discussion of the housing market before. That is, an increase in η raises the overall demand for housing services, thus increasing the demand for funding. Then, given that $\eta^{HV} > \eta^{LV}$, it trivially follows that the volume of lending in a HVSS is larger than in a LVSS. Furthermore, the fact that η and r are negatively related to each other around a LVSS (see (13)) implies that $\frac{\partial D^{LV}}{\partial r} < 0$. However, the lack of feedback from r to η around a HVSS, which reflects the fact that the interest rate is already sufficiently low so that all members of the young generation are at their borrowing limit, implies that $\frac{\partial D^{HV}}{\partial r} = 0$. This latter feature of the aggregate demand for funds turns out to be critical for the possibility of multiple steady states. We now investigate the mechanism that allows the economy to generate a sufficient supply of funding, A, so as to sustain a high volume of lending and low interest rates in a HVSS.

Lemma 4 *The aggregate supply of funds function has a negative slope in a HVSS and a positive slope in a LVSS i.e.* $\frac{\partial A^{HV}}{\partial r} < 0$ *and* $\frac{\partial A^{LV}}{\partial r} > 0$, *respectively.*
Proof: *See Appendix.*

This lemma contains a result that lies at the core of the high valuation funding mechanism of this paper, namely the reversal of the slope of the supply of funds in a HVSS.[14] To better understand this mechanism, it is helpful to differentiate between, first, the effects of changes in r on the resources required to repay a given volume of outstanding mortgages and, hence, on each middle-aged household's net wealth and, second, the influence of r on the aggregate demand for funds by the young. As described before, this latter effect is not present around a HVSS since the young find optimal to exhaust their borrowing capacity, thus making the aggregate demand for debt independent of the interest rate. Hence, a reduction in the interest rate raises every middle-aged household's net wealth, which induces a larger accumulation of deposits.

In contrast, around a LVSS changes in the interest rate matter for some middle-aged households' net wealth through the two effects just mentioned. In particular, the total amount of debt held by renters and landlords decreases with the interest rate. Moreover, the amount of

[14] Herein we are implicitly associating the supply of funds of the middle-aged generation with the aggregate supply. In so doing, we note that young households who are renters and constrained are net (constrained) borrowers so they do not save in the form of deposits. But since in a LVSS the landlords' supply of houses for renting is completely elastic and the interest rate paid on deposits and debt is the same, we can not compute their deposits and debt, a^y and d, separately, but only the aggregate net position for the whole group of landlords. For the ease of the exposition, we follow the convention of treating the (negative of) aggregate net financial position of landlords as debt. Thus, the aggregate supply of loanable funds corresponds just with the savings of the middle-aged generation.

resources devoted to repay outstanding mortgages by these two groups one period later is a decreasing function of the interest rate. This result follows from the substitution effects previously described: a reduction in the interest rate induces, given (13), an increase in the relative price η, which translates into a higher demand for debt due to a larger equilibrium volume of buy-to-let investment and a higher demand for housing services by landlords. As the interest rate falls (around a LVSS) the resulting increase in the demand for housing services by landlords and renters and the corresponding rise in their (joint) demand for funds implies that total mortgage payments increase, thus reducing net wealth in their second period of life. As a result, their supply of loanable funds falls too.

In sum, the preceding analysis provides an answer to the following question: how can an economy have a low interest rate in a high housing price equilibrium when it requires more funding to be sustained? In our model, this answer crucially depends upon the presence of collateral constraints which may drastically change the pattern of the aggregate supply of funds.

Remark 2. The heterogeneity in the returns of alternative investment choices is a crucial condition for the existence of a HVSS. In this steady state, there coexist landlords holding portfolios that earn a return greater than the interest rate together with other households (middle-aged) whose savings only yield the interest rate. Less crucial, however, is the particular modelling strategy followed to segment households between those who can act as landlords and those how cannot. For example, we could equally allow for a positive fraction of households within each cohort to act as landlords over their entire life. Then, as such fraction is not too high, a HVSS will still exist.[15]

3.2 Existence and co-existence

We next investigate the conditions for the existence of both types of bubbleless steady state.

Proposition 2 *(i) There exists a unique LVSS if* $\theta < \overline{\theta}$*, and (ii) there exits a unique HVSS if* $\underline{\theta} \leq \theta < \overline{\theta}$*.*
Proof: *See Appendix.*

Thus, both steady states will coexist whenever θ takes intermediate values. We next outline the main intuition behind this result. Let us first focus on the LVSS. Uniqueness here is warranted since the aggregate excess demand for funds is a monotonically decreasing function of the interest rate. This implies that if a LVSS exists then equilibrium r is unique and so are the rest of prices and quantities. But existence of a LVSS will require a not too high value of θ, for otherwise the economy cannot channel savings to borrowers at an interest rate sufficiently high to preclude landlords from exhausting their credit capacity.

We now turn to the analysis of the HVSS. Uniqueness here also follows from the monotonicity of the aggregate excess demand for credit, which now is increasing in the interest rate. Further, this function is negatively related to θ, so that around a HVSS we obtain that $\frac{dr}{d\theta} > 0$. Hence, there exists a lower bound $\underline{\theta}$, such that for $\theta < \underline{\theta}$ the excess demand of credit is strictly positive even as r approaches zero. In other words, too low values of θ imply that the economy can

[15] A similar exogenous segmentation can be found in Caballero et al. (2006, sec. III.B).

not generate enough funding to sustain an equilibrium in which every household exhausts his borrowing capacity. Likewise, if $\theta > \overline{\theta}$, then there is no η such that both rental and credit markets clear, and $\eta < \frac{1+r}{r}$ so that landlords optimally choose the highest amount of credit.

[FIGURE 1]

The four panels of Figure 1 correspond to the alternative credit market scenarios implicit in the preceding discussion. The top-left panel depicts a situation in which the borrowing limit is too low to sustain a HVSS (i.e. $\theta < \underline{\theta}$). Thus, as r approaches zero there persists a positive excess demand for credit. The top-right panel is consistent with a value for the downpayment lying inside the coexistence range, i.e. $\underline{\theta} < \theta < \overline{\theta}$, where the demand and supply of funds cross twice, and henceforth the two steady states exist. The bottom-left panel displays the corner case corresponding to $\theta = \overline{\theta}$, for which there exists a unique possible steady state equilibrium. Finally, the bottom-right panel displays a case with $\theta > \overline{\theta}$, and hence the excess demand for credit is always strictly negative regardless of the value of the interest rate and, as a result, for sufficiently high θ there is not any interior steady state equilibrium.

Lemma 5 *Within the range of coexistence the distance between both equilibria, as measured by the differences in the price-rent ratio, the housing price, and the interest rate, is decreasing in the size of the downpayment, θ.*

This is an intuitive result that emphasizes the fact that, inside the range of coexistence, the size of the θ-multiplier on p, η and r is higher in the HVSS than in the LVSS. In the HVSS changes in θ will affect *every* young household, while in the LVSS there is always a fraction of households that are not financially constrained (landlords) so their optimal behavior is unaffected by marginal changes in the downpayment. In other words, in the HVSS the extra capacity of the whole economy to "produce" collateral following a fall in θ is fully exploited, with a greater subsequent impact on equilibrium prices, while only a fraction of it (the one corresponding to renters) is actually used in the LVSS.

4 Steady states with bubbles

In this section we show that under some conditions there exists a steady state with bubbles. We stick at the notion of a bubble as the price of an asset that exceeds the present discounted value of the dividends that the investor extracts from the asset. In short, we will show that in addition to the steady states in which the middle-aged households only maintain deposits in their portfolios, as in the LVSS and HVSS analyzed earlier, there is a steady state in which these households also hold assets that do not pay any dividend.

Later on we will be in a position to prove the claim that only the middle-aged hold bubbles. For the moment, let us introduce the necessary conditions for the existence of a steady state with bubbles (see e.g. Tirole 1985). First, as we assume free-disposal of the asset, $Q_t^b \geq 0$, where Q_t^b is the asset unit price in terms of goods. Second, arbitrage between the bubble and deposits implies that $\frac{Q_{t+1}^b}{Q_t^b} = 1 + r_{t+1}$, which simply states that the price of the bubble must grow at a rate equal to the interest rate. Third, since the growth rate of the economy is zero (by assumption), a bubble in a steady state must have a constant size too, i.e. $Q^b_{t+1} = Q^b_t = Q^b$. Hence, combining this latter condition with the previous non-arbitrage condition we learn that a steady state with a positive bubble exists only if $r = 0$. Notice the previous condition follows from the assumption that the growth rate of the economy is precisely equal to zero. None of the forthcoming arguments hinge on such particular normalization. Finally, the economy must generate sufficient funding to sustain a stationary bubble. This implies that at $r = 0$ there must exist an excess of saving (or asset shortage), defined as the difference between the total supply of funds, A, and the aggregate demand for funding, D.[16] The bubble, whose value (or size) is denoted by B, then fills such a gap. Thus, at $r = 0$ we must have,

$$A - D = B > 0 \tag{15}$$

Before entering into more formal arguments, it is helpful to go back to Figure 1. The top-right panel in this figure, which contains the unique scenario under which a HVSS exists, shows clearly that, given the uniformly negative (zero) slope of the aggregate supply (demand) of funds function for interest rates within the interval $[0, r^*)$, the set of necessary conditions for the existence of a HVSS and a steady state with a bubble are the same. This allows us to state an important result, namely, that a HVSS exists if and only there exists a steady state with a bubble.

To gain further insight into the existence of bubbles in this economy, it is instructive to consider the effects of a fall in the interest rate from r^{HV} towards zero. As r falls, the net wealth of the middle-aged, who where constrained in the previous period (this follows from $r < r^{HV}$), increases, which, in turn, raises their savings. In the bubbleless economy of the previous section, such extra funding cannot be fruitfully invested, as potential borrowers (the young) cannot commit to repay more than they are already borrowing, i.e. D^{HV}. Investment in a bubble by middle-age households which, within the logic of the model act as *speculators* or *bubble-holders*, thus appears as a natural solution for a problem of insufficient asset production (Caballero 2006). Indeed, as illustrated in the bottom-right panel of Figure 1, as collateral constraints are sufficiently loose, in the sense that $\theta < \underline{\theta}$, the maximum amount of debt that borrowers can sell is high enough so that a bubble cannot emerge. A result that naturally follows is that the size of the bubble is positively linked to the value of θ. In other words, as the asset shortage problem is more severe, the amount of directed funds to the bubble investment increases.

An interesting result is that a bubble, that entails the investment on an asset with a positive price and zero dividends, *must* coexist with a situation in which the inequality $p < \frac{1+r}{r} q$ holds, i.e. a price of a house that falls strictly *below* the discounted value of dividends, in form of rents, that a landlord extracts from that house. Recall that the previous inequality is an essential feature of a HVSS, that obviously survives in a steady state with a bubble insofar low interest rates lead landlords to stay at their borrowing limit. Does this mean that landlords hold negative bubbles? No, that inequality simply reflects the fact that the landlords are collateral constrained, that is, the return of the buy-to-let activity is higher than

[16] Given that in the presence of bubbles the savings of the middle-aged have more than one possible destination (deposits and the bubble) we now interpret A in a wider sense, and refer to it generally as the *supply of funds*.

the interest rate. Indeed, the fact that collateral constraints are binding for landlords precludes the possibility that the housing price rises enough so as to match the present discounted value of rents. In an environment without collateral constraints landlords would borrow as much as necessary to bring the price of a dwelling sufficiently high so that the return of the buy-to-let investment would equal the interest rate. In such equilibrium, the unlimited capacity of landlords to issue debt would rule out the asset shortage problem and a bubble would not arise.

We are now in a position to prove our claim that only the middle-aged will have the incentive to hold the bubble. In a bubble equilibrium, the interest rate is so low that the young generation will have the incentive to hit their borrowing constraint. But then the young will never hold a bubble since their intertemporal marginal rate of substitution will be higher than the interest rate, which is the return of a bubble.

The following proposition summarizes the above results.

Proposition 3 *There exists a steady state with a bubble if and only if there exits a HVSS. The size of the bubble depends positively on θ.*
Proof: *See Appendix.*

We next consider three alternative types of bubbles.

Pure bubbles. Imagine that there is a fixed stock of m *intrinsically useless* objects. We henceforth define a pure bubble as an equilibrium in which such objects have a positive price, Q_t^b. Let B_t^{PB} denote the time t total value of the bubble, i.e. $B_t^{PB} = mQ_t^b$. The counterpart of the general non-arbitrage condition listed above is now $\frac{Q_{t+1}^b}{Q_t^b} = 1 + r_{t+1}$. Thus, according to (15), a steady state with a pure bubble (*PBSS*) with size B^{PB} is possible if $A - D \geq B^{PB} > 0$ at $r = 0$. (We henceforth use the PB and HB superscripts to denote prices and quantities in a PBSS and in a HBSS, respectively.)

Housing bubbles. From the perspective of a middle-aged, the purchase of a house is not different from the investment in a pure bubble as it only constitutes a vehicle of savings that does not deliver him any dividend (either in the form of rents or utility from occupancy). This is the notion of a housing bubble maintained in this paper. The existence of an equilibrium in which middle-aged buy houses just for their resale value then requires that $\frac{p_{t+1}}{p_t} = 1 + r_{t+1}$. Let B_t^{HB} denote the time t value of the investment in houses by the middle-aged. Thus, a steady state with a housing bubble (*HBSS*) requires that $A - D \geq B^{HB} > 0$ when $r = 0$.

Notice that, as informally claimed before (see definition 3), there cannot exist a HVSS with $r < 0$. If this were the case, a middle-aged household could just save using only houses which in such steady would yield a net return of zero. As such profitable opportunity persists at the aggregate, no middle-aged would be willing to supply loanable funds and, hence, a HVSS cannot be sustained. Importantly, one cannot provide a symmetric argument regarding pure bubbles as in that case nothing ensures that such a bubble will emerge to rule out $r < 0$.

Mixed bubbles. Given that pure and housing bubbles yield the same return, a speculator must be indifferent between them. Thus, although the total size of the bubble in a steady state, B, which is the sum of B^{PB} and B^{HB}, is unique, its composition is indeterminate. However, for the sake of the exposition we henceforth focus on the two extreme cases in which $B = B^{PB}$

and $B = B^{HB}$, and refer to them as *pure* and *housing bubbles*, respectively.

We next discuss how the different nature of the bubble, pure versus housing, affects prices and allocations while in the next section we focus on the effects of bubbles on welfare.

We first concentrate on the consequences of a bubble for the equilibrium of the rental market. In particular, the fact that bubble goes hand to hand with $r = 0$, does not bear any consequence on either the aggregate supply or the aggregate demand of rental services, since every young in either side of that market is also borrowing constrained (recall that neither the middle-aged nor the old participate in that market). In short, the choices by the young generation in the presence of a bubble are the same as in a HVSS. This, in turn, implies that the equilibrium price-rent ratio is identical in a HVSS and in a steady-state with a bubble, regardless of its nature (pure or housing).

If the bubble appears in the housing market, then the equilibrium in this market is different relative to a HVSS. House prices will be higher in the HBSS as a result of the excess of saving, $A - D = B^{HB}$, that will be devoted to invest in houses for resale purposes only. To see this, notice that the counterpart of the housing market clearing condition (11) then becomes:

$$pH = H^T(\eta) + B^{HB} \tag{16}$$

Then, combining the previous result that in any steady state with bubbles $\eta = \eta^{HV}$ with the market clearing condition (11) evaluated at a HVSS, we can rewrite (16) as

$$p^{HB} = p^{HV} + \frac{B^{HB}}{H} \tag{17}$$

where p^{HB} is the housing price in a housing bubble. From (17), it follows that $p^{HB} > p^{HV}$, which combined with $\eta^{HB} = \eta^{HV}$ implies that $q^{HB} > q^{HV}$. Intuitively, as regards the equilibrium determination of prices and rents, a housing bubble is tantamount to a reduction of the available stock of houses for occupancy, with the natural effect of rising prices and rents in the same proportion. In a PBSS, as B^{HB} is zero, it is straightforward to notice that the corresponding housing price and rents coincide with the ones in a HVSS. The following proposition summarizes the previous findings.

Proposition 4 *If there exists a steady state with a bubble, then the composition of the bubble (pure vs housing) is indeterminate. Prices and rents are higher in a HBSS than in a HVSS, while the price-rent ratio remains as in the HVSS, i.e.* $\eta^{HB} = \eta^{HV}$.
Proof: *See Appendix.*

5 Welfare

In this section we first analyze the implications for welfare of a housing bubble versus a pure bubble. Specifically, we evaluate the lifetime utility function (1) across different steady states. Then, we provide a welfare comparison of a LVSS with a steady state with a pure bubble. Our analysis focuses on unconditional welfare comparison across steady states thus leaving aside transitional dynamics issues. For the most part of the section we maintain the frictionless rental market assumption ($\tau \to 0^+$). This assumption is specially convenient here because it will allow us to treat the two forms of producing housing services, ownership-occupancy and renting, as virtually perfect substitutes, as the user cost for an unconstrained owner and for a renter become identical. We later on discuss some consequences from relaxing that assumption.

The following proposition formally states that a housing bubble is always strictly dominated, in terms of welfare, by a pure bubble.

Proposition 5 *For any* $\theta \in (\underline{\theta}, \overline{\theta})$, *(i)* $U_z^{PB} > U_z^{HB}$, *and (ii)*$\frac{d\left(U_z^{PB} - U_z^{HB}\right)}{d\theta} > 0$, $\forall z \in [\underline{z}, \overline{z}]$.
Proof: *See Appendix.*

The first result reflects the fact that in a HBSS some dwellings are purchased only for a pure investment motive and, as such, do not produce housing services. The second result links the welfare differences across the two types of bubbles to the size of the credit constraint, θ. A reduction in θ alleviates the problem of scarcity of destinations for available funds. That is, as θ falls, the middle-aged increase their lending to the young thus reducing the potential size of the housing bubble and its adverse effects on welfare. At this point it is important to notice the fact that a change in θ has no a direct effect on the demand for houses by speculators, since they are not constrained (i.e. they earn a return equal to the interest rate).

We next compare welfare in a LVSS vis-à-vis a PBSS. In these two steady states the (total) housing stock is employed in producing housing services. Since landlords are unconstrained in a LVSS and in an equilibrium with $\tau \to 0^+$ make housing services available to renters at the same user cost paid by an unconstrained household (i.e. the rental price q), the renters demand functions for goods and total housing services ($h + s$) collapse into those of unconstrained households. In short, this implies that differences in θ_z *per se* do not convey differences in individual utility.

Yet, as emphasized in the following proposition, a frictionless rental market does not imply that relative differences in the households' life cycle consumption path disappear when landlords are constrained, which is necessarily the case in any steady state with a bubble. The reason for this asymmetric result can be understood in terms of the different returns of *internal* and *external* funding. In a LVSS, unconstrained landlords face the same return for internal and external funds, namely, the interest rate. However, when collateral constraints bind, there is a positive premium for internal funds which implies that a higher credit capacity goes hand to hand with higher net wealth after debt repayment and, hence, higher utility. Formally, using the steady state versions of the flow of funds constraint of a middle-aged (3) and the borrowing constraint (5) holding as an equality, we can write

$$c^m + a^m = p\,(h+g)\,[1-(1+r)\,(1-\theta_z)]\,, \tag{18}$$

where the right hand side of this expression represents net wealth. Identical housing user cost for renters and landlords implies that $h+s$ takes the same value for every household, with $s = 0$ for landlords. The value of total house purchases, $h + g$ (with $g = 0$ for renters), is inversely related to θ_z, as dictated by the borrowing constraint. Putting things together, (18) makes clear that net wealth relative to initial endowment depends negatively on θ_z.

Proposition 6 *Assume that there exists a household* $\widetilde{z} \in (\underline{z}, \overline{z})$ *for whom* $U_{\widetilde{z}}^{LV} = U_{\widetilde{z}}^{PB}$. *Then, the downpayment of that household is given by*

$$\theta_{\widetilde{z}} = \frac{\left(1 + r^{LV}\right)^{\frac{1+2\beta}{1+\beta}}}{\left(1 + r^{LV}\right)^{\frac{1+2\beta}{1+\beta}} - 1} \frac{1}{\eta^{HV}} \tag{19}$$

Then, for any household with $\theta_z < (>)\, \theta_{\widetilde{z}}$ *we find that* $U_z^{LV} < (>)\, U_z^{PB}$.
Proof: *See Appendix.*

The proposition states that households with looser credit constraints will be better off in a steady state with a pure bubble relative to a LVSS steady state and vice versa. Thus, in an environment with heterogeneous credit capacity, bubbles tend to amplify welfare differences across household that are mitigated in a LVSS, where higher interest rates and slacking borrowing constraints for landlords result in allocations that resemble the ones that would obtain in an economy without collateral constraints and hence, with no welfare dispersion induced by differences in θ_z.[17]

Frictions in the rental market. How robust are the previous findings when we relax the assumption that $\tau \to 0^+$? Relative to the previous discussion, a first consequence from considering a distorted rental market (i.e. a non negligible τ) is that even in a LVSS heterogeneity in access to credit will generate *per se* consumption and welfare heterogeneity. Specifically, households with lower θ_z will tend *ceteris paribus* to enjoy higher utility than households with tighter credit limits that, in equilibrium, obtain a larger share of their housing services from the distorted rental market. Thus, allowing for non negligible rental market frictions does not alter the central message of the previous proposition 6. In a bubble equilibrium, a low θ_z still results in higher utility via higher net wealth at the beginning of the second period of life, as in the frictionless case.

Furthermore, distortions in the rental market tend to accentuate individual differences between a LVSS and a steady state with a pure bubble. This follows directly from the fact that the price-rent ratio in a bubble is higher than in a LVSS, which according to proposition 1 implies a larger measure of renters and a higher share of their total housing services obtained from renting. Thus, in addition to the relative loss of net wealth, households with high θ_z are forced to rely more heavily on the rental market if there is a bubble, with the subsequent utility loss that imposes a large τ.

The last claim above contains a subtle implication for aggregate welfare. If the rental market is distorted, even a pure bubble may entail significant welfare costs, in the sense that low interest rates and high price-rent ratios tend to increase the proportion of the total flow of housing services that are produced with the less efficient "technology" (i.e. through the rental market).

[17] The assumption in proposition 6 that there exists a pivotal household $\widetilde{z} \in (\underline{z}, \overline{z})$ or, equivalently, that $\theta_{\widetilde{z}}$ obtained from (19) lies within the interval $(\theta_{\overline{z}},\ \theta_{\underline{z}})$, is tantamount to imposing a minimum degree of dispersion in the distribution function $F(z)$.

6 Dynamics

In this section we first analyze the dynamic properties of the model around the different steady states and then study the conditions under which the economy can switch across steady states. To keep the analysis simple and tractable we do not invoke additional persistence mechanisms (say, adjustment costs), so that in all cases we can reduce the equilibrium dynamics to a single non-linear difference equation. Overall, here we use the model developed so far to shed light on two important issues. First, we study the conditions under which the economy jumps from a bubbleless regime toward an equilibrium with bubbles (and vice versa). Second, we perform an analysis of bubbles fragility.

6.1 Dynamics around steady states

In what follows we characterize the properties of the system governing the equilibrium dynamics around the four steady states considered so far (i.e. LVSS, HVSS, PBSS and HBSS). We maintain the assumption of frictionless rental market throughout. The most technical details of this subsection are contained in the appendix.

Dynamics around a LVSS. The credit market clearing condition around a LVSS can be written as follows:

$$\gamma\beta\left[\eta_t - (2+\beta)\right] = \gamma\beta^2\left(1+r_t\right), \tag{20}$$

where the left-hand-side of this equation corresponds to the net demand for funding for the young which, as discussed in section 3, can be expressed as an increasing function of η (see lemma 3). The right-hand-side corresponds to the supply of funds by the middle aged which under log utility is a fraction $\frac{\beta}{1+\beta}$ of their net wealth. The latter depends positively on the return earned by the time $t-1$ portfolios of the young generation, i.e. r_t given that landlords of that generation are unconstrained. The housing market clearing condition is simply

$$p_t H = \gamma\beta\eta_t \tag{21}$$

Equations (20) and (21) together with the no-arbitrage condition (8) can be combined to obtain the following non-linear first-order difference equation describing the evolution of the house prices, p_t, along a *low valuation regime* as follows:

$$p_t = (2+\beta)\,\frac{p_{t-1} - \frac{\gamma\beta}{H}}{p_{t-1} - (1+\beta)\,\frac{\gamma\beta}{H}} \tag{22}$$

with $\frac{dp_t}{dp_{t-1}} < 0$ and $\frac{d^2p_t}{dp_{t-1}^2} > 0$. In the appendix we show that $\lim\limits_{p_{t-1}\to p^{LV}}\left|\frac{dp_t}{dp_{t-1}}\right| < 1$, so the system exhibits oscillatory convergence toward the steady state.

Dynamics around a HVSS. Around the HVSS the price-rent ratio, η^{HV}, is time-invariant, reflecting the fact that the members of the young generation, who are the participants in the rental market, are constrained so that eventual movements in r_t will bear no effects on the demand and supply of rental services. From the dynamic counterpart of expression (11), it follows that p_t can be expressed as a function of η_t only, which implies that p_t is also constant. Finally, the credit market clearing condition can be solved for a unique and constant equilibrium interest rate. Thus, we understand the HVSS as a limiting single point that separates the regime of LVSS and the regime of bubbles.

Dynamics around a steady state with a housing bubble. In the appendix we show that the time t size of a housing bubble, B_t^{HB}, can be expressed as a function of p_t, r_t, η_t, p_{t-1}, η_{t-1}. Notice that according to proposition 4, the price-rent ratio in an equilibrium with bubbles is η^{HV} and, hence, time-invariant. Furthermore, r_t can be expressed as a function p_t and p_{t-1}, according to the no-arbitrage condition $\frac{p_t}{p_{t-1}} = 1 + r_t$. Thus, we can write B_t^{HB} as a function of p_t and p_{t-1} only. Plugging B_t^{HB} into the housing market clearing condition (16) we obtain a non-linear first-order difference equation for p_t, which can be expressed in compact form as

$$p_t = \frac{K_0 p_{t-1}}{p_{t-1} - K_1} \tag{23}$$

In the appendix we show that K_0 and K_1 are strictly positive and time-invariant, with $K_0 > K_1$. As in the low valuation regime, we find that $\frac{dp_t}{dp_{t-1}} < 0$ and $\frac{d^2 p_t}{dp_{t-1}^2} > 0$ and $\lim_{p_{t-1} \to p^{LV}} \left| \frac{dp_t}{dp_{t-1}} \right| < 1$, so housing prices in a *housing bubble regime* converge oscillatory to the steady state too. Then, using the time t counterpart of the housing market clearing condition (16), we can write the size of the housing bubble as

$$B_t^{HB} = \frac{K_0 p_{t-1}}{p_{t-1} - K_1} H - p^{HV} H \tag{24}$$

which has the same dynamic features of (23).

Dynamics around a steady state with a pure bubble. As in the housing bubble, the price-rent ratio in a pure bubble is η^{HV}. In contrast to the case of a housing bubble, now housing prices are constant and equal to p^{HV}, as can be seen directly from (17) after imposing $B^{HB} = 0$. Hence, the key equation that captures the dynamic properties of this regime relates changes in the size of the pure bubble, B_t^{PB}, with changes in the interest rate, r_t. Then, exploiting the no-arbitrage condition $\frac{B_t^{PB}}{B_{t-1}^{PB}} = 1 + r_t$, we can express B_t^{PB} as a non-linear function of B_{t-1}^{PB},

$$B_t^{PB} = \frac{K_3 B_{t-1}^{PB}}{B_{t-1}^{PB} + K_4} \tag{25}$$

where K_3 and K_4 are strictly positive and time-invariant, with $K_3 > K_4$. This last inequality, which must hold for a steady state with a pure bubble to exist, also implies that the system exhibits local monotonic convergence. That is, the bubble grows uniformly as it approaches the steady state with the interest rate converging to zero from above. Formally, $\frac{dB_t^{PB}}{dB_{t-1}^{PB}} > 0$ and $\frac{d^2 B_t^{PB}}{d(B_{t-1}^{PB})^2} < 0$ and $\lim_{B_{t-1}^{PB} \to B} \left| \frac{dB_t^{PB}}{dB_{t-1}^{PB}} \right| < 1$.

The difference in the dynamic paths of pure and housing bubbles is intuitive. To see why the system converges non-monotonically in the latter (as in a low valuation regime), let consider a relative high p_{t-1}. From the borrowing constraint (5) we see that a high p_{t-1} goes hand to hand with a high ratio $\frac{d_t}{(1-\theta_z)h_{t-1}}$, which tends to reduce net wealth of the middle aged at time t. This, in turn, implies less funding available to the middle-aged to purchase houses, which generates a relatively lower p_t and, by the same argument, a higher p_{t+1} and so on. In contrast to this, given that middle aged do not hold pure bubbles at the beginning of the period, Q_t^b does not bear any wealth effect on them, thus removing the source of oscillations.

[FIGURE 2]

The top panel of Figure 2 contains the laws of motion of the housing price corresponding to a low valuation regime (equation (22) which applies for $p_t < p^{HV}$) and to a housing bubble (equation (23) for $p_t > p^{HV}$). The bottom panel of figure 2 plots the laws of motion of the value of housing the housing bubble (equation (24)) and the pure bubble (equation (25)).

6.2 An exogenous regime switching mechanism

Suppose that up to $t = 0$, inclusive, the economy was in a low valuation regime, thus with p_0 obeying the dynamic equation (22), $\eta_0 < \eta^{HV}$ and $D_0 = A_0 < D^{HV}$. Can such an economy *endogenously* jump at $t = 1$ toward the region of a bubble? Also, is the reverse jump possible? Can the economy jump from a housing bubble regime to a pure bubble regime and vice versa? In the remaining of this section we deal with these questions.

The following proposition states the impossibility of an endogenous jump from a low valuation regime to any bubble regime and vice versa.

Proposition 7 *Assume that $\underline{\theta} < \theta < \overline{\theta}$ holds so that multiple steady states (some with bubbles) coexist. Then,*

(i) if the economy is at $t = 0$ in a low valuation regime, then the equilibrium is unique.

(ii) if the economy is at $t = 0$ in a bubble regime (pure or housing), then the economy cannot endogenously switch toward a low valuation regime.
Proof: *See Appendix.*

The following case illustrates the content of this proposition. Imagine that the economy is at $t = 0$ in the vicinity of a LVSS. Then, in the absence of any exogenous disturbance, a bubble will never emerge. Moving from a LVSS regime toward, say, an equilibrium with a pure bubble would require increasing funding by the middle aged, so that the young may obtain more credit and eventually hit their borrowing limits. From the flow of funds constraint of the middle-aged (3), we see that any increase in housing price, Δp_t, due to higher demand for housing services by the young raises net wealth, thus increasing total available funding. However, this positive effect on funding is not sufficient to sustain the price increase, Δp_t. The reason is that the rise in prices rises the total demand for funding by the young in an amount (in terms of goods) $\Delta p_t H$, which coincides with the increase in the net wealth of the middle-aged. But the middle-aged generation only increase their savings by $\frac{\beta}{1+\beta}\Delta p_t H < \Delta p_t H$, and consume the remaining fraction of their wealth increase, $\frac{1}{1+\beta}\Delta p_t H$. A similar argument applies for the case of a housing bubble. Here, the increase in the net demand for funding by the young can be written as $\Delta p_t H - p_t H_t^B$, where H_t^B is the stock of houses purchased by the middle-aged for a speculative motive. But now the increase in the supply of funding by the middle-aged only amounts to $\frac{\beta}{1+\beta}\Delta p_t H - p_t H_t^B$.

A symmetric argument applies to rule out the possibility that the economy jumps from a bubble to the LVSS regime. To see this, notice that the switch from a bubble to a LVSS regime implies that p falls from a level equal (pure bubble) or above (housing bubble) p^{HV} towards a level strictly below p^{HV}. Yet, the negative effect on impact on the middle-aged supply of funding is of a smaller magnitude than the fall in the demand by the young.

Funding shocks. Given the previous findings, we next allow for the possibility that an unexpected temporary shock hits the economy, fueling or draining funds available to the households. A straightforward example of such shock in the context of a closed economy could come from a shift in the demand for funding by the government. To clarify the argument, we next introduce government debt so that the government flow of funds constraint (26) now becomes:

$$D_t^g = C_t^g - T - \tau q_t G_t + (1 + r_t)\, D_{t-1}^g \tag{26}$$

where D_t^g is the time t demand for funds by the government. For simplicity, we assume that government and private debt pay the same interest rate. We must also modify the credit market clearing condition accordingly, so that in its most general form it can be written as follows:[18]

$$A_t = D_t + D_t^g + B_t$$

We consider a simple fiscal rule designed to maintain a constant level of government debt at (almost) all times, thus adjusting G_t as dictated by the government budget constraint (26). We then introduce a time t unexpected temporary shock to the funding available to households as a one-period deviation of the government funding demand from the long-run level D^g, and denote such deviation by $F_t \equiv D_t^g - D^g$, with $F_s = 0$ for every $s \neq t$. The following proposition contains the necessary and sufficient conditions under which the economy switches from a low valuation to a regime with bubbles in face of a given shock F_t.

Proposition 8 *Assume that at $t = 0$ the economy is in a LVSS. At $t = 1$ an unexpected one-period shock $F > 0$ takes place. It follows that*

$$\text{if } F \begin{cases} < \widetilde{F} & \text{then the economy remains in a low valuation regime} \\ = \widetilde{F} & \text{then the economy switches to a HVSS} \\ > \widetilde{F} & \text{then the economy switches to a bubble regime} \end{cases}$$

where $\widetilde{F} = \frac{\gamma\beta}{1+\beta}\left(\eta^{HV} - \eta^{LV}\right)$.
Proof: *See Appendix.*

Thus, in principle, only if there is a sufficiently large funding shock (i.e. $F > \widetilde{F}$) will the economy embark into a bubble path. Otherwise, a smaller shock will make the economy return

[18] It is clear that now the exact values of $\underline{\theta}$ and $\overline{\theta}$ are also functions of the steady state level of government debt, B^G.

to the LVSS and, only if the size of the shock is $\widetilde{F}$ the economy will be trapped into a HVSS. The exact size of $\widetilde{F}$ depends positively on the difference between the minimum volume of funding required to sustain an equilibrium with bubbles and the one consistent with a LVSS, which is expressed in the proposition in terms of the gap between η^{HV} and η^{LV}.[19] Linking the minimum size of the shock to the gap between η^{HV} and η^{LV} is useful since, by lemma 5, such a difference is a decreasing function of θ. This last finding has an important practical implication, namely that tighter borrowing constraints make the economy more prone to the surge of a bubble (i.e. lower $\widetilde{F}$), in which case, according to proposition 3, the bubble becomes also bigger as time passes.

The previous proposition remains silent with respect the type of bubble that emerges if $F > \widetilde{F}$. That is, conditional on a shock sufficiently large so as to generate funds beyond the level consistent with a HVSS, all possible equilibria must contain a bubble, but either a housing or a pure bubble are equally possible. The following proposition, however, emphasizes that housing and pure bubbles must be treated differently when analyzing the magnitude of the shock that can take the economy out of a bubble regime.

Proposition 9 *Assume that at $t = 0$ the economy is in a HBSS and at $t = 1$ an unexpected one-period negative (fund-draining) shock with absolute value F takes place. Then, if $F > \widetilde{F}^{HB}$, the economy moves toward the low valuation regime. Now, suppose that at $t = 0$ the economy is in a PBSS. Then, following a negative shock with absolute value F_1, the economy jumps into the low valuation regime if $F > \widetilde{F}^{PB}$. Where $\widetilde{F}^{HB} = \left(1 - \frac{\beta}{1+\beta}\frac{p^{HV}}{p^{HV} + \frac{B^{HB}}{H}}\right) B^{HB} < B^{HB}$ and $\widetilde{F}^{PB} = B^{PB}$. Since $B^{HB} = B^{PB} = B$, it follows that $\widetilde{F}^{PB} > \widetilde{F}^{HB}$.*
Proof: *See Appendix.*

This proposition implies that the size of the smallest negative (in absolute value) fund-draining shock required to move the economy toward a LVSS regime is positively related to the (steady state) size of the bubble. Thus, small bubbles tend to be more *fragile* to fund-draining shocks than larger ones, in the sense that the latter will survive some shocks that will necessarily rule out the former.

The fact that $\widetilde{F}^{PB} > \widetilde{F}^{HB}$ follows directly from proposition 3, i.e. the total size of a bubble in a steady state does not depend on the nature of the bubble, so that $F^{PB} = B^{HB} = B^{PB} > F^{HB}$. This is an important result, as it emphasizes that a housing bubble is more fragile than a pure one. The intuition behind this is as follows. As the housing demand for speculative reasons falls, following the burst of a housing bubble, so does the housing price. This, in turn, reduces the net wealth of the middle-aged generation and, hence, their supply of funds. This negative effect on the availability of funds adds to the (exogenous) negative fund-draining shock. That is, the shock generates negative spillovers on the net worth of those households that had not purchased houses for speculative reasons,[20] that work as an amplifying mechanism of the initial shock. These spillovers are not present in the case of a pure bubble, since the young never buy it so that the fall in the price of that asset does not convey any loss of wealth for the current

[19] See the appendix for details.

[20] Recall that the middle-aged generation at the time of the shock purchased houses in the previous period not for speculative reasons but to occupy or rent them out.

middle-aged generation. As a result, the minimum negative funding shock needed to crash a housing bubble is smaller.

Finally, combining the result in proposition 9 with the one in proposition 8 it follows that an economy with tighter collateral constraints, on the one hand, is more prone to bubbles which, in turn, tend to have a larger size but; and on the other, such bubbles tend to be less fragile to a given negative fund-draining shock.

[FIGURE 3]

Figure 3 illustrates the previous discussion. The top panel represents the demand and supply for funds schedules and shows how the size a stationary bubble relates to the degree of collateral requirements. The bottom panel aims at depicting the idea that a relatively small positive funding shock may be sufficient to move the economy from a low valuation regime towards a regime containing housing bubble whenever θ is large. In such a case, the distance between p^{LV} and p^{HV}, which according to the housing market clearing condition common to both steady states, LVSS and HVSS, $p_t H = \gamma\beta\eta_t$, is proportional to the difference between η^{LV} and η^{HV} and, hence, to $\widetilde{F}$, is small. On the contrary, the size of the bubble is then large and so it is the smallest fund draining shock $\widetilde{F}^{HB}$. The bottom right panel shows that the opposite is true when collateral constraints are looser.

6.3 Switching bubbles

Can the burst of a pure bubble give rise to the surge of a housing bubble (and vice versa)? In the previous section we have shown that an endogenous jump from a low valuation regime to a bubble one (and the other way around) is not possible. Contrary to this, the following proposition shows that a purely endogenous change of regime across the two types of bubbles is possible.

Proposition 10 *Assume that at $t = 0$ the economy is in the steady with a housing (pure) bubble. Then, an endogenous switch towards a pure (housing) bubble regime is possible.*
Proof: *See Appendix.*

This proposition opens the possibility for multiple equilibria in the sense that, given an initial bubble regime, the economy may jump into the other bubble regime, even in the absence of an exogenous funding shock. From this perspective high asset price volatility may appears as a natural result of the economy being located into the region in which bubbles exist. The argument works as follows. Suppose that at $t = 0$ the economy is in a steady state with a housing bubble and at the beginning of $t = 1$, the middle-aged coordinate on their portfolio decisions and, rather than invest in houses, all of them purchase the useless asset (with the expectation that the future generations will also do so).[21] The impact effect of such portfolio switch is a fall in housing prices which tends to depress both the funding supply by the current middle-aged and the demand for funds by the young. But the former falls less than the latter,

[21] We assume that the pure bubble is initially sold by the old generation at time 1.

according to the argument in proposition 7(ii). Hence the burst of a housing bubble does not remove the excess of funding (with respect to the volume of debt) typical of a bubble regime. Rather, such an excess of funding flows to the useless asset, giving rise to a pure bubble. Yet the fact that the middle-aged suffer a wealth loss on impact implies that the emerging bubble has an initial size smaller than in the steady state, toward which it converges monotonically from below. The top panel of figure 4 illustrates this.

[FIGURE 4]

Now consider the reverse situation. Assume that at $t = 0$ the economy is in a steady state with a pure bubble and at $t = 1$ the middle-aged generation suddenly stop buying the useless asset and begin to invest in houses. This portfolio reshuffling gives rise to an unexpected rise in housing prices which raises the middle-aged supply of funding (via wealth effect) by more than the demand. Now, contrary to the previous case, the fact that the middle-aged enjoy a once-for-all wealth gain implies that the emerging housing bubble has an initial size larger than in the steady state. Since B_t^{HB} tracks the dynamic pattern of p_t, following the initial peak, the bubble approaches its steady state oscillating around it. The bottom panel of figure 4 depicts the transition from a PBSS towards a HBSS.

7 Ruling-out housing bubbles

In this section we present alternative government policies to rule out housing bubbles (or reduce their size). We first discuss the consequences of a policy aimed at lessening the home-ownership versus renting bias and then analyze the effects of a Tobin tax on housing speculative investments.[22] The first policy pertains to a general class of policies aimed at removing the shortage of assets by improving the menu of non-speculative investments and provides an answer to the following question: what can the government do to reduce the likelihood of *any* bubble? In contrast, the second policy does not solve the underlying problem of assets scarcity. Given that a funding shock inevitably moves the economy towards a bubble regime, our second policy offers a resolution to the following key question: what can the government do to avoid the socially detrimental use of houses as a pure store of value? We note that this last query is of special interest given the previous results that stationary housing bubbles are strictly dominated in terms of welfare by pure bubbles (proposition 5).

7.1 Reducing frictions in the rental market

We now focus on a reform that seeks to reduce the gap between the user cost for a tenant and the net rental price obtained by a landlord (i.e. the owner). In the context of our model, a simple way of reducing the previous distortion is through a fall in τ, but more generally this might be interpreted as any improvement in the protection of property rights or alleviation of moral hazard problems that might be at the heart of the aforementioned gap.

[22] Of course, there may be other policies that are equally helpful to rule out housing bubbles, though we focus on these two as they involve direct action on policy tools genuinely akin to the housing market.

The key formal result is contained in the following lemma.

Lemma 6 *Under fairly general conditions,* $\frac{dB}{d\tau} > 0$.
Proof: *See Appendix.*

The previous lemma states that the size of the bubble increases with the magnitude of the distortion τ. To see this, suppose that the economy is initially located in a steady state with a bubble and at some point τ falls. The reduction in τ will directly increase the aggregate demand for rental services which, in turn, will lead to a larger buy-to-let investment, and therefore to a higher housing price. The associated increase in the value of the stock of houses leads to a proportional increase in the volume of credit demanded by the young generation and a corresponding fall in the volume of funds invested in the bubble.[23] In other words, a reduction in τ tends to raise the profitability of the buy-to-let investment vis-à-vis the return of purely speculative investments which unchains a portfolio re-composition by the middle-age generation.[24] The argument applies equally to both housing and pure bubbles, since in either case the size and the return of the bubble do not hinge on its nature.

More generally, we can gain further insights on the previous mechanism by thinking of the rental market as an institution that allows transferring resources from "house-savers" towards "house-borrowers", much as the credit market does. Indeed, in the context of our model, the existence of borrowing limits make the credit market operative so that for some young renting a house is a market mechanism used to relax the effects of credit constraints without sacrifying much housing consumption. In this regard, a fall in τ brings about similar qualitative aggregate effects as those arising from a reduction of the size of the collateral constraints. That is, the size of the bubble will tend to be smaller when the borrowing limits are looser or even vanishing if collateral constraints become weak enough (see proposition 3).

7.2 A Tobin tax

Conditional on the economy embarking on a bubble path, one possible way to eliminate housing bubbles is for the government to create a bias against speculative investment in houses and in favor of speculation in a socially unproductive asset. Such a bias could arise from the combination of the following government actions. First, imagine that the government can commit to levy a proportional tax, τ^h, on speculative home purchases. Then, the net return of that investment is $(1 - \tau^h)p_{t+1}/p_t$. This implies that a steady state with a housing bubble will only be possible if the interest rate is negative, i.e. $r = -\tau^h$. However, absent any other form of bubbleless investment, an equilibrium with a housing bubble and negative interest rate is possible even though the economy could equally converge to a steady state with a pure bubble and $r = 0$. Intuitively, the "fiat" nature of a pure bubble equilibrium implies that nothing

[23] The general conditions we refer to are mild requirements which ensure that the direct positive effect on the demand for funding dominates an eventual positive effect on the funding supply by the middle-aged. We have shown, in the context and proof of proposition 8, that this is necessarily the case when $\tau \to 0^+$. When τ is not negligible the exact shape of the distribution function $F(z)$ matters and hence we have to invoke some implicit "general conditions" on $F(z)$ and/or the size of τ.

[24] It is worth noting that for some configuration of parameters of the model the possibility of a bubble vanishes since the condition $A > D$ at $r = 0$ may not longer hold.

ensures that it will arise. If the middle-aged at time t expect that the middle-aged at time $t+1$ will not purchase the useless asset, then the former will prefer to invest in houses even in the presence of a tax. Hence, this tax scheme, taken in isolation, will not suffice to rule out a housing bubble. Instead, it can be argued that a Tobin tax tends to make the original assets shortage more acute.

Thus, in addition to generate a bias against the housing speculative investment the government must provide an alternative vehicle for savings whose effectiveness does not hinge only on coordination of private expectations. In principle, the intrinsically useless asset could serve this purpose if the government commits to implement a backing scheme as follows. Suppose that the government stands ready to redeem each unit of the useless asset in exchange for consumption goods at a positive price $\underline{Q}^b < Q^b$, i.e. the deal offered by the government is dominated by the steady state market price of the asset. Then, even an infinitesimal $\underline{Q}^b$ will rule out any steady state with a housing bubble. To see this imagine that, on the contrary, there is a housing bubble, so that the return of the middle-aged portfolios is $-\tau^h$. Then, any member of such generation at $T-1$ could buy a unit of the intrinsically useless asset at a price $\underline{Q}^b$ (i.e. the same paid by the government) and sell it to the government in the next period obtaining a return of zero, that is, strictly above the yield of the speculative housing investment.[25] As a result, the only possible steady state equilibrium under this government strategy resembles the pure bubble one. That is, the interest rate is zero and the excess supply of funding, $A-D$, is devoted to purchase the intrinsically useless asset.[26]

The previous mechanism, however, may be difficult to enforce if, for instance, the government cannot verify that a given house is being held for speculative reasons. In such a case, we conjecture that levying a tax on the *entire* volume of housing transactions, irrespectively of their nature, speculative or not, may perform similarly in terms of welfare to the Tobin tax scheme presented above. To gain intuition on this, let us consider an arbitrarily small τ^h that is paid by *each* homeowner. As regards the housing demand by the young for non-speculative motives a small tax will convey a *continuos* small welfare loss. As regards home purchases for speculative reasons, the tax will remove them, thus giving rise to *discrete* increase in the degree of utilization of the housing stock and a fall in the housing price. The overall welfare gain derived from the latter must overwhelm the deadweight utility loss from the tax.

8 Conclusions

In this paper we use the notion of a *housing bubble* as an equilibrium in which some *investors* hold houses only for resale purposes and not for the expectation of a dividend, either in the form of rents or utility. We provide a life-cycle model where households face collateral constraints that tie their credit capacity to the value of their houses and examine the conditions under

[25] This result, and the underlying government backing scheme, is reminiscent of some mechanisms studied to rule out speculative hyperinflations in monetary economies (see, .e.g. Wallace (1981) and Obstfeld and Rogoff (1983)). Caballero and Krishnamurthy (2006) study a similar device based on the supply of government collateralized bonds.

[26] We note that such an equilibrium is not a pure bubble, since the intrinsically useless asset is now backed by the government, even though in equilibrium the households never turn the asset into goods given by the government.

which housing bubbles can emerge. In such equilibria, the total housing stock is held by owners that extract utility from their homes, landlords that obtain rents, and investors. We show that an economy with tighter collateral constraints is more prone to bubbles which, in turn, tend to have a larger size but are less fragile in face of fund-draining shocks. Our environment also allows for *pure bubbles* on useless assets. We find that multiple equilibria in which the economy moves endogenously from a pure bubble to a housing bubble regime and vice versa are possible. This suggests that high asset price volatility may be a natural consequence of asset shortages (or excess funding) that depress interest rates sufficiently so as to sustain an initial bubble. We also examine some welfare implications of the two types of bubbles and discuss some mechanisms to rule out (welfare-reducing) housing bubbles.

Appendix

Proof of claims in section 2

Solution of the households optimization problem. The first order conditions of the problem solved by a household born at time t are:

$$\begin{aligned}
(c_t^y) &: \quad (c_t^y)^{-1} = \lambda_t && (27)\\
(s_{t+1}) &: \quad -\lambda_t q_t + \beta\,(h_{t+1}+s_{t+1})^{-1} + \varphi_{t+1}^s = 0 && (28)\\
(a_t^y) &: \quad -\lambda_t + \beta\lambda_{t+1}\,(1+r_{t+1}) + \varphi_t^a = 0 && (29)\\
(d_t) &: \quad \lambda_t - \beta\lambda_{t+1}\,(1+r_{t+1}) + \varphi_t^d - \mu_t = 0 && (30)\\
(h_{t+1}) &: \quad -\lambda_t p_t + \beta\,(h_{t+1}+s_{t+1})^{-1} + \beta\lambda_{t+1}p_{t+1} + \varphi_{t+1}^h + (1-\theta)\,p_t\mu_t = 0 && (31)\\
(g_{t+1}) &: \quad -\lambda_t\,[p_t - (1-\tau)\,q_t] + \beta\lambda_{t+1}p_{t+1} + \varphi_{t+1}^g + (1-\theta)\,p_t\mu_t = 0 && (32)\\
\left(a_{t+1}^m\right) &: \quad -\lambda_{t+1} + \beta\lambda_{t+2}\,(1+r_{t+2}) + \varphi_{t+1}^a = 0 &&\\
\left(c_{t+1}^m\right) &: \quad \left(c_{t+1}^m\right)^{-1} = \lambda_{t+1} &&\\
\left(c_{t+2}^o\right) &: \quad \left(c_{t+2}^o\right)^{-1} = \lambda_{t+2} &&
\end{aligned}$$

where λ's are the Lagrange multipliers associated with restrictions (2)-(4), μ is the Lagrange multiplier of (5), and the φ´s are the Lagrange multipliers of the non-negativity constraints in (6) and (7).

Below we provide the **individual demand functions** for the types of households in definition 2:

i. Renters

$$\begin{array}{lll}
c_t^y = \gamma; & c_{t+1}^m = \gamma\beta\frac{\delta_{t+1}}{p_t\left(\theta_z - \frac{1}{\eta_t}\right)}; & c_{t+2}^o = \gamma\beta^2\frac{\delta_{t+1}(1+r_{t+2})}{p_t\left(\theta_z - \frac{1}{\eta_t}\right)}\\
h_{t+1} = \gamma\beta\,(1+\beta)\,\frac{1}{p_t\left(\theta_z - \frac{1}{\eta_t}\right)}; & s_{t+1} = \gamma\beta\frac{\theta_z\eta_t - (2+\beta)}{p_t\left(\theta_z - \frac{1}{\eta_t}\right)}; & g_{t+1} = 0\\
d_t = \gamma\beta\,(1+\beta)\,\frac{1-\theta_z}{\theta_z - \frac{1}{\eta_t}}; & a_t^y = 0; & a_{t+1}^m = \gamma\beta^2\frac{\delta_{t+1}}{p_t\left(\theta_z - \frac{1}{\eta_t}\right)}
\end{array}$$

where $\delta_{t+1} \equiv p_{t+1} - (1+r_{t+1})\,(1-\theta_z)\,p_t$.

ii. Constrained buyers

$$\begin{array}{lll}
c_t^y = \gamma; & c_{t+1}^m = \gamma\beta\frac{2+\beta}{1+\beta}\frac{\delta_{t+1}}{\theta_z p_t}; & c_{t+2}^o = \gamma\beta^2\frac{2+\beta}{1+\beta}\frac{\delta_{t+1}(1+r_{t+2})}{\theta_z p_t}\\
h_{t+1} = \gamma\beta\,(2+\beta)\,\frac{1}{\theta_z p_t}; & s_{t+1} = 0; & g_{t+1} = 0\\
d_t = \gamma\beta\,(2+\beta)\,\frac{1-\theta_z}{\theta_z}; & a_t^y = 0; & a_{t+1}^m = \gamma\beta^2\frac{2+\beta}{1+\beta}\frac{\delta_{t+1}}{\theta_z p_t}
\end{array}$$

iii-a. Unconstrained *landlords*

$$\begin{array}{lll}
c_t^y = \gamma; & c_{t+1}^m = \gamma\beta\,(1+r_{t+1})\,; & c_{t+2}^o = \gamma\beta^2\,(1+r_{t+1})\,(1+r_{t+2})\\
h_{t+1} = \gamma\beta\frac{1}{(1-\tau)q_t}; & s_{t+1} = 0; & g_{t+1} \text{ fully elastic if } \frac{p_{t+1}}{1+r_{t+1}} = p_t - (1-\tau)\,q_t\\
(a-d)_t^y = \gamma\beta\left(2+\beta\text{-}\frac{\eta_t}{1-\tau}\right)\text{-}\frac{p_{t+1}}{1+r_{t+1}}g_{t+1}; & & a_{t+2}^o = \gamma\beta^2\,(1+r_{t+1})
\end{array}$$

iii-b. Constrained landlords

$$c_t^y = \gamma; \qquad c_{t+1}^m = \gamma\beta \frac{\delta_{t+1}}{p_t\left(\theta_z - \frac{1-\tau}{\eta_t}\right)}; \qquad c_{t+2}^o = \gamma\beta^2 \frac{\delta_{t+1}(1+r_{t+2})}{p_t\left(\theta_z - \frac{1-\tau}{\eta_t}\right)}$$

$$h_{t+1} = \gamma\beta \frac{1}{(1-\tau)q_t}; \qquad s_{t+1} = 0; \qquad g_{t+1} = \gamma\beta \frac{2+\beta-\frac{\eta_t\theta_z}{1-\tau}}{p_t\left(\theta_z - \frac{1-\tau}{\eta_t}\right)}$$

$$d_t = \gamma\beta(1+\beta)\frac{1-\theta_z}{\theta_z - \frac{1-\tau}{\eta_t}}; \qquad a_t^y = 0; \qquad a_{t+1}^m = \gamma\beta^2 \frac{\delta_{t+1}}{p_t\left(\theta_z - \frac{1-\tau}{\eta_t}\right)}$$

Proof of Proposition 1. (i) We prove it by contradiction. Assume that there is an unconstrained landlord for whom $\mu_t = \varphi_{t+1}^g = 0$ must hold. Then from (29) and (30) it follows that $\varphi_t^a + \varphi_t^d = 0$, and $\varphi_t^a = \varphi_t^d = 0$. Then, using (29) and (32) it follows that $\frac{p_{t+1}}{p_t-(1-\tau)q_t} = 1+r_{t+1}$. Now assume that there is a constrained landlord for whom $\varphi_{t+1}^g = 0$ and $\mu_t > 0$ must hold. Then from (5) we learn that $\varphi_t^d = 0$. Combining (29) and (30), it follows that $\varphi_t^a > 0$; i.e. if the borrowing constraint is binding then $a_t^y = 0$ and $\frac{\lambda_t}{\beta\lambda_{t+1}} > 1 + r_{t+1}$. Next, by combining (30) and (32) it follows that $\frac{p_{t+1}}{p_t-(1-\tau)q_t} > 1 + r_{t+1}$, which contradicts that $\frac{p_{t+1}}{p_t-(1-\tau)q_t} = 1 + r_{t+1}$.

(ii) This follows directly from $\tau > 0$.

(iii) According to (8) the cost of renting for a tenant is $q_t = \frac{1}{1-\tau}\left(p_t - \frac{p_{t+1}}{1+r_{t+1}}\right)$, which for $\tau > 0$ is always greater than the cost of buying the same unit of housing services, i.e. $p_t - \frac{p_{t+1}}{1+r_{t+1}}$. This proves the claim.

(iv) We prove it by contradiction. Assume that $h_{t+1} = 0$. Hence $s_{t+1} > 0$ and, according to previous point (ii) of this proposition, $g_{t+1} = 0$. Given that collateral is zero, then $d_t = 0$ and $a_t^y > 0$. Now, consider the two cases in point (i): (a) All landlords are unconstrained. Thus, from (8) we can write

$$p_{t+1} - (1 + r_{t+1})(p_t - q_t) = (1 + r_{t+1})\tau q_t > 0 \tag{33}$$

Now combining (28), (29), and (31) it follows that

$$p_{t+1} - (1 + r_{t+1})(p_t - q_t) = -\varphi_{t+1}^h > 0 \tag{34}$$

and the last inequality, which follows from (33), is inconsistent with $\varphi_{t+1}^h \geq 0$. (b) All landlords are constrained. Thus, from (9) we can write

$$p_{t+1} - (1 + r_{t+1})(p_t - q_t) > (1 + r_{t+1})\tau q_t > 0$$

which again leads to contradiction.

(v) We first identify the household with the lowest index z_1, such that at time t it optimally chooses not to rent, i.e. $s_{t+1}(z_1) = \varphi_{t+1}^s(z_1) = 0$. Then, we identify the household with the index z_2, such that it optimally chooses a zero supply of renting services, i.e. $g_{t+1}(z_2) = \varphi_{t+1}^g(z_2) = 0$. We finally show that $z_2 > z_1$.
In order to pin down z_1, we impose $s_{t+1}(z_1) = \gamma\beta\left(\frac{1}{q_t} - \frac{1+\beta}{\theta_{z_1}p_t - q_t}\right) = 0$, where $\gamma \equiv (1+\beta)^{-2}$. Using this equality, we can write

$$\theta_{z_1} = (2+\beta)\frac{q_t}{p_t} = \frac{(2+\beta)}{\eta_t} \tag{35}$$

where $\eta_t \equiv \frac{p_t}{q_t}$ is the price-rent ratio. To compute z_2, we proceed as before distinguishing two cases: every landlord is (a) unconstrained, or (b) constrained:

(a) We take an unconstrained landlord and obtain the maximum amount of buy-to-let investment such that the borrowing limit is reached. To pin down the household with z_2, we impose that $g(z_2)$ is equal to zero. Formally, we notice that for this household the following hold: $a_t^y(z_2) = 0$, and $d_t(z_2) = (1 - \theta_{z_2}) p_t h_{t+1}(z_2)$, which can be used to find that:

$$\theta_{z_2} = \frac{(2+\beta)(1-\tau)}{\eta_t}$$

(b) When every landlord is constrained, we use the following condition:

$$g_{t+1}(z_2) = \gamma\beta \left[\frac{1+\beta}{\theta_{z_2} p_t - (1-\tau) q_t} - \frac{1}{(1-\tau) q_t} \right] = 0$$

which allows us to obtain

$$\theta_{z_2} = \frac{(2+\beta)(1-\tau)}{\eta_t}. \tag{36}$$

Finally, from assumption 1 and (35) and (36) it follows that: $\theta_{z_1} < \theta_{z_2} \Rightarrow z_2 > z_1 \; \forall t$.

Proof of claims in section 3

Proof of lemma 1. Imposing equilibrium in the rental market, the aggregate expenditure functions for the three groups of young households, which are obtained from the individual demand functions for housing services, are

$$\begin{aligned} H^R &= \gamma\beta(1+\beta) \int_{\underline{z}}^{z_1} \left(\theta_z - \frac{1}{\eta} \right)^{-1} dF(z) \\ H^{CB} &= \gamma\beta(2+\beta) \int_{z_1}^{z_2} \theta_z^{-1} dF(z) \\ H^L &= \gamma\beta(1-\tau)^{-1} \int_{z_2}^{\overline{z}} \eta dF(z) + \gamma\beta \int_{\underline{z}}^{z_1} \left[\theta_z \eta - (2+\beta) \right] \left(\theta_z - \frac{1}{\eta} \right)^{-1} dF(z), \end{aligned} \tag{37}$$

where H^R, H^{CB}, and H^L stand for the level of expenditure (measured in consumption goods) devoted to housing purchases by the group of renters, constrained buyers, and landlords (owner-occupancy plus buy-to-let), respectively, and $H^T(\eta) = H^R + H^{CB} + H^L$. Now, exploiting the differentiability of the functions in expression (37), it follows that

$$\frac{\partial H^T(\eta)}{\partial \eta} = \gamma\beta \int_{\underline{z}}^{z_1} dF(z) + \gamma\beta(1-\tau)^{-1} \int_{z_2}^{\overline{z}} dF(z) > 0$$

which is expression (12) in the main text.

Proof of lemma 2. (i) Let $g^{\max}$ be the stationary supply of houses for rent by a landlord such that (5) holds as an equality, i.e. $g^{\max}$ is implicitly defined by $d = (1-\theta) p (h + g^{\max})$ and $a^y = 0$. Using the individual demand functions we obtain the aggregate counterpart of supply of renting, denoted by $G^{\max}$:

$$G^{\max} = \frac{\gamma\beta}{p}\int_{z_2}^{\overline{z}} \frac{2+\beta-\frac{\eta}{1-\tau}\theta_z}{\theta_z-\frac{1-\tau}{\eta}} dF(z) \tag{38}$$

Now we can write the aggregate demand for renting services in a steady state, denoted by S, as

$$S = \frac{\gamma\beta}{p}\int_{\underline{z}}^{z_1} \frac{\eta\theta_z-(2+\beta)}{\theta_z-\frac{1}{\eta}} dF(z) \tag{39}$$

Then, exploiting (11), (12), (38), and (39), we learn that $\frac{\partial[G^{\max}-S]}{\partial\eta} < 0$ (specifically, $\frac{\partial G^{\max}}{\partial\eta} < 0$ and $\frac{\partial S}{\partial\eta} > 0$). From definition 3, it follows that two necessary conditions for the existence of a LVSS are (13) and $G^{\max} - S > 0$. Given that the excess capacity, $G^{\max} - S$, is a decreasing function of η, then there exists a threshold value η^*, such that $\eta^{LV} < \eta^*$, where η^* is implicitly defined as the price-rent ratio for which $G^{\max} = S$. Also, a necessary condition for the existence of a HVSS is that $G^{\max} = G^{HV} = S^{HV}$. Thus, $\eta^{HV} = \eta^*$. Thus, $G^{HV} = G^{\max} > G^{LV}$.

Then, let us define r^* as the interest rate that satisfies (13) for $\eta = \eta^*$, i.e. $r^* = (1-\tau)\left[\eta-(1-\tau)\right]^{-1}$. Then, given the monotonic negative relationship between r and η implied by (13), we learn that if a LVSS exists (recall that $\eta^{LV} < \eta^*$, as just shown), then $r^{LV} > r^*$. As for the HVSS, we combine the previous result, $\eta^{HV} = \eta^*$, with the inequality in (14) to conclude that $r^{HV} < r^*$. Finally, $p^{HV} > p^{LV}$ follows directly by combining (11) and (12) with lemma 2.

Proof of lemma 3. From the individual demand functions given above and imposing equilibrium in the rental market, it follows that the aggregate demand for funds in a LVSS can be written as

$$D^{LV} = \gamma\beta\left\{\begin{array}{c} \underbrace{\int_{\underline{z}}^{z_1}(1+\beta)(1-\theta_z)\left(\theta_z-\frac{1}{\eta}\right)^{-1} dF(z)}_{D^{R,LV}} + \underbrace{\int_{z_1}^{z_2}(2+\beta)\frac{1-\theta_z}{\theta_z} dF(z)}_{D^{CB,LV}} + \\ \underbrace{\int_{\underline{z}}^{z_1}\left(1-\frac{1-\tau}{\eta}\right)(\eta\theta_z-(2+\beta))\left(\theta_z-\frac{1}{\eta}\right)^{-1} dF(z) + \int_{z_2}^{\overline{z}}\left(\frac{\eta}{1-\tau}-(2+\beta)\right) dF(z)}_{D^{L,LV}} \end{array}\right\} \tag{40}$$

Notice that the last component, $D^{L,LV}$, contains two separate terms: debt financing (i) buy-to-let investment (left), and (ii) owner-occupancy (right). Similarly, we can write the aggregate demand for funds in the HVSS as

$$D^{HV} = \gamma\beta\left\{\begin{array}{c} \underbrace{\int_{\underline{z}}^{z_1}(1+\beta)(1-\theta_z)\left(\theta_z-\frac{1}{\eta}\right)^{-1} dF(z)}_{D^{R}} + \underbrace{\int_{z_1}^{z_2}(2+\beta)\frac{1-\theta_z}{\theta_z} dF(z)}_{D^{CB}} + \\ \underbrace{\int_{\underline{z}}^{z_1}(\eta\theta_z-(2+\beta))\left(\theta_z-\frac{1}{\eta}\right)^{-1} dF(z) + \int_{z_2}^{\overline{z}}\left(\frac{\eta}{1-\tau}-(1+\beta)\theta_z\left(\theta_z-\frac{1-\tau}{\eta}\right)^{-1}\right) dF(z)}_{D^{L}} \end{array}\right\} \tag{41}$$

Some algebra reveals that $\frac{\partial D^{LV}}{\partial\eta} > 0$ and $\frac{\partial D^{HV}}{\partial\eta} > 0$. Exploiting a continuity argument it follows that $\lim_{\eta\to\eta^{*-}} D^{LV}{=}D^{HV}$, i.e. $D^{LV} < D^{HV} \Leftrightarrow \eta^{LV} < \eta^*$.

Proof of lemma 4. We reproduce below the individual saving function for each type of constrained middle-aged households in a HVSS:

$$a^{m,k} = \begin{cases} \gamma\beta^2 \frac{1-(1+r)(1-\theta_z)}{\theta_z - \frac{1}{\eta}} & if \quad k = R \\ \gamma\beta^2 \frac{2+\beta}{1+\beta}\frac{1-(1+r)(1-\theta_z)}{\theta_z} & if \quad k = CB \\ \gamma\beta^2 \frac{1-(1+r)(1-\theta_z)}{\theta_z - \frac{(1-\tau)}{\eta}} & if \quad k = L \end{cases}. \tag{42}$$

As the value of η that clears the rental market does not vary with the interest rate, from (42) we learn that

$$\left.\frac{\partial a^{m,k}}{\partial r}\right|_{r<r^*} < 0, \quad \text{for all } k = R, CB, L.$$

Thus, at the aggregate we get $\left.\frac{\partial A}{\partial r}\right|_{r<r^*} < 0$.

Now consider a LVSS. For a renter, after some simple algebra it can be shown that $\left.\frac{\partial a^{m,R}}{\partial r}\right|_{r>r^*} > 0$ iff

$$\frac{1-(1+r)(1-\theta_z)}{(\theta_z - \frac{1}{\eta})(1+r)} > (1-\theta_z)(1-\tau)(1+r). \tag{43}$$

We next show that this inequality necessarily holds. To see this, notice that since a renter must be constrained, then the following inequality holds: $\frac{1-(1+r)(1-\theta_z)}{(\theta_z - \frac{1}{\eta})} > 1+r$, where the term in the left hand side is the intertemporal marginal rate of substitution between period 1 and 2 which is greater than $1+r$. Hence, a sufficient condition for (43) is $\theta_z > 1 - \frac{1}{(1-\tau)(1+r)}$. But, according to (13) and proposition 1(v), $\theta_z > (2+\beta)\frac{r}{(1-\tau)(1+r)}$. Then, it trivially follows that $(2+\beta)\frac{r}{(1-\tau)(1+r)} > 1 - \frac{1}{(1-\tau)(1+r)}$ holds. Thus, $\left.\frac{\partial a^R}{\partial r}\right|_{r>r^*} > 0$. For a constrained buyer $\frac{\partial a^{CB}}{\partial r} < 0$ still holds for $r > r^*$. Finally, for an unconstrained landlord, whose intertemporal marginal rate of substitution is simply $1+r$, it follows that $\left.\frac{\partial a^L}{\partial r}\right|_{r>r^*} > 0$.

Proof of Proposition 2. i) We rewrite below the aggregate supply of funds, denoted A^{LV}, after imposing (13),

$$A^{LV} = \gamma\beta^2 \left\{ \int_{\underline{z}}^{z_1} \frac{1\text{-}\frac{\eta}{\eta-(1\text{-}\tau)}(1\text{-}\theta_z)}{\theta_z\text{-}\frac{1}{\eta}} dF(z) + \int_{z_1}^{z_2} \frac{2+\beta}{1+\beta} \frac{1\text{-}\frac{\eta}{\eta\text{-}(1\text{-}\tau)}(1\text{-}\theta_z)}{\theta_z} dF(z) + \int_{z_2}^{\overline{z}} \frac{\eta}{\eta\text{-}(1\text{-}\tau)} dF(z) \right\}$$

A bit of algebra shows that $\lim_{\tau\to 0^+} \frac{\partial D^{LV}}{\partial\theta} = \lim_{\tau\to 0^+} \frac{\partial A^{LV}}{\partial\theta} = 0$. Then, exploiting lemma 4 and the implicit function theorem we learn that $\lim_{\tau\to 0^+} \frac{dr}{d\theta} = 0$. Thus, the interest rate that clears the credit market is independent of the value taken by θ as τ becomes arbitrarily small. Then, according to lemma 2, for a LVSS to exist, the credit market clearing rate, which we denote by r_0, must satisfy $r_0 > r^*$, or, equivalently, $(1-\tau)\frac{1+r_0}{r_0} = \eta_0 < \eta^* = (1-\tau)\frac{1+r^*}{r^*}$. As shown in the same lemma, $\frac{\partial[G^{\max}-S]}{\partial\eta} < 0$, while some extra simple algebra also shows that $\frac{\partial[G^{\max}-S]}{\partial\theta} < 0$, which together imply that $\frac{dr^*}{d\theta} > 0$. Thus, the necessary condition $r_0 > r^*$ will hold only if $\theta < \overline{\theta}$, where $\overline{\theta}$ is implicitly defined in the equality $r^* = r\left(\overline{\theta}\right)$. Provided θ falls below that upper bound, monotonicity of the excess demand for credit function (see lemma 2) ensures equilibrium uniqueness, since unique r implies unique equilibrium η, p and q (by (13) and (12)).

ii) The excess demand function in the credit market now becomes:

$$D^{HV} - A^{HV} = \gamma\beta \left\{ \underbrace{\int_{\underline{z}}^{z_1} \frac{\kappa_z}{\theta_z - \frac{1}{\eta}} dF(z)}_{x_1} + \underbrace{\int_{z_1}^{z_2} \frac{2+\beta}{1+\beta} \frac{\kappa_z}{\theta_z} dF(z)}_{x_2} + \underbrace{\int_{z_2}^{\overline{z}} \frac{\kappa_z}{\theta_z - \frac{1-\tau}{\eta}} dF(z)}_{x_3} \right\} \tag{44}$$

where $\kappa_z \equiv (1+2\beta+\beta r)(1-\theta_z) - \beta$. Below, we show that $\frac{dr}{d\theta} > 0$, so that for a sufficiently low θ there is no positive interest rate that clears the credit market. The complete proof involves several steps:

Step 1. Evaluate the sign of $\frac{d\eta}{d\theta}$. To do this, we totally differentiate the rental market clearing condition,

$$\int_{\underline{z}}^{z_1} \frac{\eta\theta_z - (2+\beta)}{\theta_z - \frac{1}{\eta}} dF(z) = \int_{z_2}^{\overline{z}} \frac{2+\beta - \frac{\eta}{1-\tau}\theta_z}{\theta_z - \frac{1-\tau}{\eta}} dF(z) \tag{45}$$

to find that

$$\eta_\theta \equiv \frac{d\eta}{d\theta} = -\frac{\int_{\underline{z}}^{z_1} \frac{1+\beta}{\left(\theta_z - \frac{1}{\eta}\right)^2} dF(z) + \int_{z_2}^{\overline{z}} \frac{1+\beta}{\left(\theta_z - \frac{1-\tau}{\eta}\right)^2} dF(z)}{\int_{\underline{z}}^{z_1} \left(1 + \frac{1+\beta}{\left(\theta_z - \frac{1}{\eta}\right)^2} \frac{1}{\eta^2}\right) dF(z) + \int_{z_2}^{\overline{z}} \left(\frac{1}{1-\tau} + \frac{1+\beta}{\left(\theta_z - \frac{1-\tau}{\eta}\right)^2} \frac{1-\tau}{\eta^2}\right) dF(z)} < 0 \tag{46}$$

Step 2. We next evaluate the sign of partial derivative $\frac{\partial(D-A)}{\partial\theta}$, treating η as a (negatively related) function of θ, as juts shown in the previous step. We do this separately for the three terms in the right side of (44).

First, by differentiating x_1 with respect to θ, we learn that

$$\frac{\partial x_1}{\partial \theta} < 0 \Leftrightarrow -\left[(1+2\beta+\beta r)\left(1 - \frac{1}{\eta}\right) - \beta\right] - \frac{\eta_\theta}{\eta^2}\kappa_z < 0 \tag{47}$$

We next check that the latter sign condition necessarily holds in equilibrium. Since over this range $\theta_z > \frac{1}{\eta}$, a sufficient condition for that sign condition to be met is that $-\frac{\eta_\theta}{\eta^2} < 1$, or, exploiting (46) above,

$$\int_{\underline{z}}^{z_1} dF(z) + \int_{z_2}^{\overline{z}} \left(\frac{1}{1-\tau} - \frac{1+\beta}{\left(\theta_z - \frac{1-\tau}{\eta}\right)^2} \frac{\tau}{\eta^2}\right) dF(z) > 0$$

which clearly holds as $\tau \to 0^+$.

Second, for any constrained buyer, we obtain that $\frac{\partial x_2}{\partial \theta} < 0$.

Third, following an argument similar to the one employed for renters, it can be shown that

$$\frac{\partial x_3}{\partial \theta} < 0 \Leftrightarrow -\left[(1+2\beta+\beta r)\left(1 - \frac{1-\tau}{\eta}\right) - \beta\right] - (1-\tau)\frac{\eta_\theta}{\eta^2}\kappa_z < 0$$

As in any interior equilibrium $\theta_z > \frac{1-\tau}{\eta}$, a sufficient condition for the inequality in the right side above to hold is that $-(1-\tau)\frac{\eta_\theta}{\eta^2} < 1$. A simple manipulation of (46) shows that this last inequality necessarily holds. Thus, $\lim_{\tau\to 0^+}\frac{\partial\left(D^{HV}-A^{HV}\right)}{\partial\theta} < 0$.

Step 3. Given that around a HVSS $\frac{\partial(D-A)}{\partial r} > 0$, it directly follows that $\lim_{\tau\to 0^+}\frac{dr}{d\theta} > 0$. Let us then define $\underline{\theta}$, as the unique solution for θ in the credit market clearing condition evaluated at $r = 0$. As we look at steady states with non-negative interest rates, no HVSS exists if $\theta < \underline{\theta}$.

We now turn to the analysis of the upper bound for θ. Let us consider a value for θ, call it θ^+, such that $\theta^+ > \overline{\theta}$, and denote by η^+ the price-rent ratio that would clear the rental market, i.e. the solution to (45) when $\theta = \theta^+$. Our claim here is that for such a pair $\left(\theta^+,\eta^+\right)$ there is a strictly negative excess demand in the credit market when the necessary condition for the existence of a HVSS holds (i.e. $\eta^+ < (1-\tau)\frac{1+r}{r}$). To prove it, we first notice that, as just shown, for $\theta^+ > \overline{\theta}$ there is no LVSS, the reason being that for an interest rate, r, satisfying $\eta^+ = (1-\tau)\frac{1+r}{r}$ there would be a positive excess demand in the rental market and, symmetrically, a negative excess demand in the credit market, as the interest rate that satisfies this last equality would lie strictly above the one that would clear the credit market. Thus, the following inequality holds when $\eta^+ = (1-\tau)\frac{1+r}{r}$,.

$$\begin{aligned} y_1 &\equiv \gamma\beta\left\{\int_{\underline{z}}^{z_1}(1+\beta)\frac{1\text{-}\theta_z^+}{\theta_z^+\text{-}\frac{1}{\eta^+}}dF(z)+\int_{z_1}^{z_2}(2+\beta)\frac{1\text{-}\theta_z^+}{\theta_z^+}dF(z)+\int_{z_2}^{\overline{z}}(1+\beta)\frac{1\text{-}\theta_z^+}{\theta_z^+\text{-}\frac{1-\tau}{\eta^+}}dF(z)\right\} \\ &< \gamma\beta\left\{\int_{\underline{z}}^{z_1}\beta\frac{1\text{-}\frac{\eta^+\left(1-\theta_z^+\right)}{\eta^+-(1-\tau)}}{\theta_z^+\text{-}\frac{1}{\eta^+}}dF(z)+\int_{z_1}^{z_2}\beta\frac{2+\beta}{1+\beta}\frac{1\text{-}\frac{\eta^+\left(1-\theta_z^+\right)}{\eta^+-(1-\tau)}}{\theta_z^+}dF(z)+\int_{z_2}^{\overline{z}}\beta\frac{\eta^+}{\eta^+\text{-}(1\text{-}\tau)}dF(z)\right\}\equiv y_2 \end{aligned}$$

where the first line is the highest demand for funds given a pair $\left(\theta^+,\eta^+\right)$ and the second line is the total supply of funds when $\eta^+ = (1-\tau)\frac{1+r}{r}$. We next ask whether there exists a r^+ such that $\eta^+ < (1-\tau)\frac{1+r^+}{r^+}$ that clears the credit market. The aggregate supply of funds for a vector $\left(\theta^+,\eta^+,r^+\right)$ is then given by the first line of the expression below

$$\begin{aligned} y_3 &\equiv \gamma\beta\left\{\int_{\underline{z}}^{z_1}\beta\frac{1\text{-}\left(1+r^+\right)\left(1\text{-}\theta_z^+\right)}{\theta_z^+\text{-}\frac{1}{\eta^+}}dF(z)+\int_{z_1}^{z_2}\beta\frac{2+\beta}{1+\beta}\frac{1\text{-}\left(1+r^+\right)\left(1\text{-}\theta_z^+\right)}{\theta_z^+}dF(z)+\int_{z_2}^{\overline{z}}\beta\frac{1\text{-}\left(1+r^+\right)\left(1\text{-}\theta_z^+\right)}{\theta_z^+\text{-}\frac{1-\tau}{\eta^+}}dF(z)\right\} \\ &> \gamma\beta\left\{\int_{\underline{z}}^{z_1}\beta\frac{1\text{-}\frac{\eta^+\left(1-\theta_z^+\right)}{\eta^+-(1-\tau)}}{\theta_z^+\text{-}\frac{1}{\eta^+}}dF(z)+\int_{z_1}^{z_2}\beta\frac{2+\beta}{1+\beta}\frac{1\text{-}\frac{\eta^+\left(1-\theta_z^+\right)}{\eta^+-(1-\tau)}}{\theta_z^+}dF(z)+\int_{z_2}^{\overline{z}}\beta\frac{\eta^+}{\eta^+-(1-\tau)}dF(z)\right\}\equiv y_2 \end{aligned}$$

where the inequality follows directly from $1+r^+ < \frac{\eta^+}{\eta^+-(1-\tau)}$. Combining the last two expressions above, we learn that $y_3 > y_1$, and, hence, for $\theta^+ > \overline{\theta}$ there is no r^+ such that the credit market clears and (14) holds. Finally, equilibrium uniqueness follows from the strict monotonicity of the excess demand function in the rental, credit and housing markets, with respect to η, r and p, respectively.

Proof of lemma 5. We first focus on the LVSS. From the proof proposition 2.i) above we know that $\lim_{\tau\to 0^+}\frac{dr}{d\theta} = \lim_{\tau\to 0^+}\frac{d\eta}{d\theta} = 0$. From (12), we learn that $\lim_{\tau\to 0^+}\frac{dp}{d\theta} = 0$. Hence, $\lim_{\tau\to 0^+}\frac{dq}{d\theta} = 0$. As regards the HVSS, from the proof of proposition three we know that $\lim_{\tau\to 0^+}\frac{dr}{d\theta} > 0$ and $\lim_{\tau\to 0^+}\frac{d\eta}{d\theta} < 0$. Using the latter in (12), we find that $\lim_{\tau\to 0^+}\frac{dp}{d\theta} < 0$ and $\lim_{\tau\to 0^+}\frac{dq}{d\theta} = 0$. This proves the lemma.

Proof of claims in section 4

Proof of Proposition 3. Given that at $r = 0$ the young are constrained, we know that the size of the bubble in a steady state, B, is simply the difference $A^{HV} - D^{HV}$ evaluated at $r = 0$. If there exists a HVSS then $A^{HV} - D^{HV}$ evaluated at $r = 0$ must be positive, by definition 3. We have shown in proposition 2 that $\lim_{\tau \to 0^+} \frac{d\left(D^{HV} - A^{HV}\right)}{d\theta} < 0$, hence $\lim_{\tau \to 0^+} \frac{dB}{d\theta} > 0$.

Proof of Proposition 4. The claim that $\eta^{HB} = \eta^{HV}$ follows from the fact that the existence of a bubble does not affect the rental market clearing condition (45). Finally, combining the two last results, $p^{HB} > p^{HV}$ and $\eta^{HB} = \eta^{HV}$, it directly follows that $q^{HB} > q^{HV}$.

Proof of claims in section 5

Proof of Proposition 5. As $\tau \to 0^+$, we learn from the steady state counterpart of the individual demand functions above that for a household with index z, $c^y = \gamma$, $c^m = \gamma\beta \frac{\theta_z}{\theta_z - \frac{1}{\eta^{HV}}}$; and $c^o = \gamma\beta^2 \frac{\theta_z}{\theta_z - \frac{1}{\eta^{HV}}}$, when $r = 0$, which is a necessary condition for both a HBSS and a PBSS. Thus, as regards consumption, there is no difference between a HBSS and a PBSS. As regards the remaining argument in the utility function (1), $h + s$, we know from the individual demand functions that $h + s = \gamma\beta\frac{1}{q}$, with $s = 0$ if such household is a landlord. Then, given that $q^{PB} = q^{HV} < q^{HB}$, we learn that $U_z^{PB} > U_z^{HB}$.

Proof of Proposition 6. The derivation of $\theta_{\tilde{z}}$ is done directly by plugging the individual demand function into the equality $U_{\tilde{z}}^{PB} = U_{\tilde{z}}^{LV}$.

Proof of claims in section 6

Dynamics around a LVSS. Combining (20), (21) and (8) we arrive at the following non-linear first difference equation for η,

$$\eta_t = (2 + \beta) \frac{\eta_{t-1} - 1}{\eta_{t-1} - (1 + \beta)} \tag{48}$$

Then, using the housing market clearing condition (21), it is straightforward to rewrite the above equation in terms of p as (22) in the main text. We then use (48) to solve for η^{LV}:

$$\eta^{LV} = \frac{1}{2}\left[3 + 2\beta + \sqrt{(1 + 2\beta)^2 + 4\beta}\right]$$

We rule out the negative-root solution on the basis that it is only consistent with $\eta^{LVE} < 1$, which cannot be part of an equilibrium. As regards the properties of $\frac{d\eta_t}{d\eta_{t-1}}$, we find that $\frac{d\eta_t}{d\eta_{t-1}} = -\beta (2 + \beta) \frac{1}{\left[\eta_{t-1} - (1+\beta)\right]^2} < 0$, and $\lim_{\eta_{t-1} \to \eta^{LV}} \left|\frac{d\eta_t}{d\eta_{t-1}}\right| < 1 \iff \beta (2 + \beta) < \left[\eta_{t-1} - (1 + \beta)\right]^2$. We next check that the last inequality holds by substituting η_{t-1} by its steady state value above, η^{LVE}, i.e.

$$\beta (2 + \beta) < \left[\frac{1}{2} + \frac{1}{2}\sqrt{(1 + 2\beta)^2 + 4\beta}\right]^2$$

which can be rewritten as

$$\sqrt{8\beta + 4\beta^2} < 1 + \sqrt{8\beta + 4\beta^2 + 1}$$

which clearly holds.

Dynamics around a HBSS. We start by combining the expression of the size of the housing bubble at time t, $B_t^{HB} = A_t^{HB} - B_t^{HB}$, with the no-arbitrage condition $1+r_t = p_t/p_{t-1}$, to find that

$$B_t^{HB} = \gamma\beta^2 \frac{p_t}{p_{t-1}} \int_{\underline{z}}^{\overline{z}} \frac{\theta_z}{\theta_z - \frac{1}{\eta^{HV}}} dF(z) - \gamma\beta(1+\beta) \int_{\underline{z}}^{\overline{z}} \frac{1-\theta_z}{\theta_z - \frac{1}{\eta^{HV}}} dF(z)$$

We next plug the above expression for B_t^{HB} into the housing market clearing condition (16) and rearrange terms to obtain

$$p_t = \underbrace{\frac{\gamma\beta(1+\beta)}{H_s} \int_{\underline{z}}^{\overline{z}} \frac{\theta_z}{\theta_z - \frac{1}{\eta^{HV}}} dF(z)}_{K_0} + \underbrace{\frac{\gamma\beta^2}{H_s} \int_{\underline{z}}^{\overline{z}} \frac{\theta_z}{\theta_z - \frac{1}{\eta^{HV}}} dF(z)}_{K_1} \frac{p_t}{p_{t-1}}$$

Thus, $p_t = K_0 + K_1 \frac{p_t}{p_{t-1}}$. Rearranging terms gives expression (23) in the main text. It trivially follows that with $\frac{dp_t}{dp_{t-1}} < 0$. The necessary and sufficient condition for $\lim_{p_{t-1} \to p^{HB}} \left| \frac{dp_t}{dp_{t-1}} \right| < 1$ is that $K_1 < K_0$, which holds given that $\beta \leq 1$.

Dynamics around a PBSS. We combine the expression of the size of the housing bubble at time t, $B_t^{PB} = A_t^{PB} - B_t^{PB}$, with the no-arbitrage condition $1 + r_t = B_t^{PB}/B_{t-1}^{PB}$, to write

$$B_t^{PB} = \underbrace{\gamma\beta \int_{\underline{z}}^{\overline{z}} \frac{(1+\beta)\theta_z - 1}{\theta_z - \frac{1}{\eta^{HV}}} dF(z)}_{K_3} - \underbrace{\gamma\beta^2 \int_{\underline{z}}^{\overline{z}} \frac{1-\theta_z}{\theta_z - \frac{1}{\eta^{HV}}} dF(z)}_{K_4} \frac{B_t^{PB}}{B_{t-1}^{PB}}$$

Hence, $B_t^{PB} = K_3 - K_4 \frac{B_t^{PB}}{B_{t-1}^{PB}}$. Rearranging terms gives the expression (25) in the main text. It follows that with $\frac{dB_t^{PB}}{dB_{t-1}^{PB}} > 0$. The necessary and sufficient condition for $\lim_{B_{t-1}^{PB} \to B} \left| \frac{dB_t^{PB}}{dB_{t-1}^{PB}} \right| < 1$ is that $K_3 > K_4$, which must hold if there exists a steady state with a pure bubble.

Proof of Proposition 7. (i) Given the local stability property analyzed above, we rule out the possibility of a jump from a given point within the low valuation regime toward a new point within that regime that violates the dynamic equation (22). We next check that a jump towards a housing bubble regime is not possible either. Without loss of generality we assume that the initial point is the LVSS. We prove it by contradiction. Suppose that at $t = 1$ the economy jumps to the housing bubble regime. Then, the equilibrium condition $A_1 = D_1 + B_1^{HB}$ is

$$A^{LV} + \frac{\beta}{1+\beta} \left(p_1 - p^{LV}\right) H = D^{HV} + B_1^{HB} \tag{49}$$

where the LHS contains A_1, which equals A^{LV} plus the extra funding following the unexpected wealth gain from higher p. In the RHS we have used the fact that the volume of funding in a bubble regime is D^{HVE}. Combining (49) with the housing market clearing condition $H = \frac{\gamma\beta}{p_1}\eta^{HV} + B_1^{HB}$ we find that

$$B_1^{HB} = (1+\beta) \left\{ A^{LV} - D^{HV} + \frac{\gamma\beta^2}{1+\beta} \left(\eta^{HV} - \eta^{LV}\right) \right\}$$

We next plug $A^{LV} = D^{LV} = \gamma\beta\left[\eta^{LV} - (2+\beta)\right]$ and $D^{HV} = \gamma\beta\left[\eta^{HV} - (2+\beta)\right]$ into the expression above and rearrange terms to find that

$$B_1^{HB} = -\gamma\beta\left(\eta^{HV} - \eta^{LV}\right) < 0 \tag{50}$$

Which proves that it is not possible to jump from a LVSS to a housing bubble regime. A similar argument shows that it is not possible to jump toward a HVSS or a pure bubble regime either. Thus, the equilibrium is unique.

(ii) Imagine that the economy is initially in the regime of a housing bubble. Suppose that at $t = 1$ the economy jumps to the low valuation regime. Then, the equilibrium condition $A_1 = D_1$ is

$$A^{HB} + \frac{p_1 - p^{HB}}{p^{HB}}\gamma\beta^2 \int_{\underline{z}}^{\overline{z}} \frac{1}{\theta_z - \frac{1}{\eta^{HV}}} = D_1$$

using $A^{HB} = D^{HV} + B^{HB}$, $D_1 = \gamma\beta\left[\eta_1 - (2+\beta)\right]$, and $D^{HV} = \gamma\beta\left[\eta^{HV} - (2+\beta)\right]$, and rearranging terms we rewrite the above expression as

$$B + \frac{p_1 - p^{HB}}{p^{HB}}\gamma\beta^2 \int_{\underline{z}}^{\overline{z}} \frac{1}{\theta_z - \frac{1}{\eta^{HV}}} dF(z) = \gamma\beta\left(\eta_1 - \eta^{HV}\right) \tag{51}$$

At this point, it is helpful to rewrite the rental market clearing condition in a HVSS (45), given $\tau \to 0^*$, as

$$(1+\beta)\int_{\underline{z}}^{\overline{z}} \frac{1}{\theta_z - \frac{1}{\eta^{HV}}} dF(z) = \eta^{HV} \tag{52}$$

Plugging (52) into (51), and rearranging we get,

$$\left(1 - \frac{\beta}{1+\beta}\frac{p_1}{p^{HB}}\right) B^{HB} = -\frac{\gamma\beta}{1+\beta}\left(\eta^{HV} - \eta_1\right) \tag{53}$$

Given that in a low valuation regime $p_1 < p^{HB}$ and $\eta_1 < \eta^{HV}$, we find that the RHS of the above expression is negative, which again leads to contradiction.

Now suppose that the economy is initially in the regime of a pure bubble. Suppose that at $t = 1$ the economy jumps to the low valuation regime. Then, the equilibrium condition $A_1 = D_1$ is

$$A^{PB} + \frac{p_1 - p^{HV}}{p^{HV}}\gamma\beta^2 \int_{\underline{z}}^{\overline{z}} \frac{1}{\theta_z - \frac{1}{\eta^{HV}}} = D_1$$

using $A^{PB} = D^{HV} + B$, $D_1 = \gamma\beta\left[\eta_1 - (2+\beta)\right]$, and $D^{HV} = \gamma\beta\left[\eta^{HV} - (2+\beta)\right]$, and rearranging terms we rewrite it as

$$B + \frac{p_1 - p^{HV}}{p^{HV}}\gamma\beta^2 \int_{\underline{z}}^{\overline{z}} \frac{1}{\theta_z - \frac{1}{\eta^{HV}}} dF(z) = \gamma\beta\left(\eta_1 - \eta^{HV}\right) \tag{54}$$

Plugging (52) into (51), and rearranging we get,

$$B = -\frac{\gamma\beta}{1+\beta}\left(\eta^{HV} - \eta_1\right) < 0 \tag{55}$$

which contradicts the initial assumption that $B > 0$.

Proof of Proposition 8. In order to find out the value of the positive funding shock that moves the economy from a LVSS to a HVSS, $\widetilde{F}$, we write the following equilibrium condition at $t = 1$,

$$\widetilde{F} + A^{LV} + \frac{\beta}{1+\beta}\left(p^{HV} - p^{LV}\right) H = D^{HV} \tag{56}$$

After some substitutions, we rewrite the above expression as

$$\widetilde{F} + D^{LV} + \frac{\beta}{1+\beta}\left(D^{HV} - D^{LV}\right) = D^{HV}$$

Thus,

$$\widetilde{F} = \frac{1}{1+\beta}\left(D^{HV} - D^{LV}\right) = \frac{\gamma\beta}{1+\beta}\left(\eta^{HV} - \eta^{LV}\right)$$

It is clear that for larger shocks, the excess of funding $F - \widetilde{F} > 0$ gives rise to a bubble. Consider, for instance, the case in which a housing bubble emerges following a funding shock $F > \widetilde{F}$. In that case, (56) becomes

$$F + A^{LV} + \frac{\beta}{1+\beta}\left(p_1^{HB} - p^{LV}\right) H = D^{HV} + B_1^{HB}$$

Then, we can solve this expression for the size of of the housing bubble in the period of the shock, B_1^{HB}, to find that

$$B_1^{HB} = (1+\beta)\left(F - \widetilde{F}\right) > 0$$

A similar argument can be used for the case of a pure bubble.

Finally, if $F < \widetilde{F}$, the economy cannot even jump to the HVSS, so it remains in the low valuation regime.

Proof of Proposition 9. (In proving this proposition we rely on several results posed in the proof of proposition 7). Starting from a HBSS, we solve the following equilibrium condition for the negative funding shock, F_1, that moves the economy to the HVSS,

$$F_1 + A^{HB} + \frac{p^{HV} - p^{HB}}{p^{HB}}\gamma\beta^2 \int_{\underline{z}}^{\overline{z}} \frac{1}{\theta_z - \frac{1}{\eta^{HV}}} = D^{HV}$$

To find, after some substitutions, that $-F_1 = \left(1 - \frac{\beta}{1+\beta}\frac{p^{HV}}{p^{HV} + \frac{B^{HB}}{H}}\right) B = \widetilde{F}^{HB}$. Thus, given previous results, if the negative shock if larger in absolute value than $\widetilde{F}^{HB}$, it will move the economy into the low valuation regime.

Now, taking the PBSS as the initial condition, we solve the following equilibrium condition for the negative funding shock, F_1, that moves the economy to the HVSS,

$$F_1 + A^{PB} = D^{HV}$$

Using $A^{PB} = D^{HV} + B$, we learn that

$$-F_1 = B = \widetilde{F}^{PB}$$

with similar consequences as in the case of a housing bubble. Also, notice that $1 > \frac{\beta}{1+\beta}\frac{p^{HV}}{p^{HV}+\frac{B^{HB}}{H}}$, which implies that $\widetilde{F}^{PB} > \widetilde{F}^{HB}$.

Proof of Proposition 10. We first show that the economy can jump endogenously from a HBSS to a pure bubble regime. This is tantamount to imposing the following equilibrium condition at the time of the endogenous jump, $t = 1$,

$$A^{HB} + \frac{p_1 - p^{HB}}{p^{HB}}\gamma\beta^2\int_{\underline{z}}^{\overline{z}} \frac{1}{\theta_z - \frac{1}{\eta^{HV}}} = D^{HV} + B_1^{PB}$$

with $B_1^{PB} > 0$. We next use $A^{HB} = D^{HV} + B$ and (52) to rewrite the above expression, after some manipulation, as

$$\left(1 - \frac{\beta}{1+\beta}\frac{p^{HV}}{p^{HB}}\right) B = B_1^{PB}$$

Thus, $B > B_1^{PB} > 0$.

Second, we prove that the economy can jump endogenously from a PBSS to a housing bubble regime. Specifically, we show that the following equilibrium condition holds for some $B_1^{HB} > 0$,

$$A^{PB} + \frac{p_1^{HB} - p^{HV}}{p^{HV}}\gamma\beta^2\int_{\underline{z}}^{\overline{z}} \frac{1}{\theta_z - \frac{1}{\eta^{HV}}} = D^{HV} + B_1^{HB}$$

Using $A^{PB} = D^{HV} + B$ and (52), we find that $B_1^{HB} = (1+\beta) B > B$. Thus, the housing bubble initially overshoots above its steady state value, B.

Proof of claims in section 7

Proof of Lemma 6. First step. By totally differentiating the rental market clearing condition (45), we find that

$$\frac{d\eta}{d\tau} = -\frac{\int_{z_2}^{\overline{z}}\left(\frac{1+\beta}{\left(\theta_z - \frac{1-\tau}{\eta}\right)^2}\frac{1}{\eta} + \frac{\eta}{(1-\tau)^2}\right) dF(z)}{\int_{\underline{z}}^{z_1}\left(1 + \frac{1+\beta}{\left(\theta_z - \frac{1}{\eta}\right)^2}\frac{1}{\eta^2}\right) dF(z) + \int_{z_2}^{\overline{z}}\left(\frac{1+\beta}{\left(\theta_z - \frac{1-\tau}{\eta}\right)^2}\frac{1-\tau}{\eta^2} + \frac{1}{1-\tau}\right) dF(z)} < 0$$

Second step. From (44), it is clear that the effect of a change in τ may have a positive or a negative effect on the individual net demand for funds over the life-cycle, i.e. $d - a^m$, depending, among other things, on the initial sign of $d - a^m$.[27] To deal with this issue, we exploit the following simple result: as the credit market clears, the following sign conditions must hold: $d(\underline{z}) - a^m(\underline{z}) < 0$, and $d(\overline{z}) - a^m(\overline{z}) > 0$. To see this notice that the numerator of the three terms in the right side of (44), i.e. κ_z, are continuos and monotonically decreasing in θ_z. Thus, there exists a household (and only one) with $\widetilde{z} \in (\underline{z}, \overline{z})$, such that $d(\widetilde{z}) = a^m(\widetilde{z})$, whose individual borrowing limit in a steady state with a bubble satisfies: $\widetilde{\theta} = \frac{1+\beta}{1+2\beta}$. In the paragraphs bellow we refer to this household as the *pivotal household in the credit market* and, depending on the identity of this household, we distinguish three possible cases:

[27] For example, this becomes clear by inspection of the first term in the right side of (44). That is, whether a rise in τ, which, as just shown, lowers η, shifts $b - a^m$ up or downwards obviously depends on the sign of $b - a^m$.

Case 1. The pivotal household is a landlord (i.e. $\widetilde{z} > z_2$). This implies that every renter is a net saver over his life-cycle, that is, $d - a^m < 0$. Hence, as $\frac{d\eta}{d\tau} < 0$, we learn that $\frac{dx_1}{d\tau} < 0$. Regarding the group of constrained-buyers, it trivially follows that $\frac{dx_2}{d\tau} = 0$. Finally, in computing the sign of $\frac{dx_3}{d\tau}$, we split the total group of landlords into two sets depending on the sign of $d - a^m$. As $\frac{d\left(\frac{1-\tau}{\eta}\right)}{d\tau} < 0$, the following results are obtained: $\frac{d(d-a^m)}{d\tau} > (<)\, 0$ if $d - a^m < (>)\, 0$. In order to find the sign of the variation of the net demand for credit for the whole group of landlords we first notice that when the credit market clears the following inequalities necessarily hold,

$$\int_{\underline{z}}^{z_1} \frac{\kappa_z}{\theta_z - \frac{1}{\eta}} dF(z) < 0, \text{ and } \int_{z_2}^{\overline{z}} \frac{\kappa_z}{\theta_z - \frac{1-\tau}{\eta}} dF(z) > 0 \tag{57}$$

The second inequality above implies that $\int_{z_2}^{\overline{z}} \frac{d\left[\frac{\kappa_z}{\theta_z - \frac{1-\tau}{\eta}} dF(z)\right]}{d\tau} < 0$. Thus, if $\widetilde{z} > z_2$, then $\lim_{r \to 0} \frac{d(D-A)}{d\tau} < 0$.

Case 2. The pivotal household is a constrained buyer (i.e. $z_1 \leq \widetilde{z} \leq z_2$). This implies that $d - a^m < (>)\, 0$ for every renter (landlord). Thus, from the the results obtained in the previous case, we learn that $\lim_{r \to 0} \frac{d(D-A)}{d\tau} < 0$, as well.

Case 3. The pivotal household is a renter (i.e. $\widetilde{z} \leq z_1$). In this case, $d - a^m > 0$ for every landlord, hence, $\frac{d(d-a^m)}{d\tau} < 0$. For the group of renters for whom $d - a^m < 0$, we still obtain $\frac{d(d-a^m)}{d\tau} < 0$. Now, however, for the remaining renters, i.e. those with $d - a^m > 0$, the effect is the opposite, i.e.$\frac{d(b-a^m)}{d\tau} > 0$. Thus, the only scenario in which a rise in τ does increase the excess demand for credit requires the joint fulfillment of the following two conditions: $\widetilde{z} \leq z_1$ and

$$\int_{\widetilde{z}}^{z_1} \frac{\kappa_z}{\left(\theta_z \text{-} \frac{1}{\eta}\right)^2} dF(z) > \text{-} \int_{\underline{z}}^{\widetilde{z}} \frac{\kappa_z}{\left(\theta_z \text{-} \frac{1}{\eta}\right)^2} dF(z) \text{-} \int_{z_2}^{\overline{z}} \left[\frac{\eta}{\eta_\tau} + (1\text{-}\tau)\right] \frac{\kappa_z}{\left(\theta_z \text{-} \frac{1-\tau}{\eta}\right)^2} dF(z) \tag{58}$$

We think of this inequality as a very unlikely outcome since market clearing as discussed above requires that the first inequality in (57) holds, i.e.,

$$\int_{\widetilde{z}}^{z_1} \frac{\kappa_z}{\theta(z) \text{-} \frac{1}{\eta}} dF(z) < \text{-} \int_{\underline{z}}^{\widetilde{z}} \frac{\kappa_z}{\theta(z) \text{-} \frac{1}{\eta}} dF(z) \tag{59}$$

That is, while (59) requires a large pool of renters with z below $\widetilde{z}$, (58) is basically imposing the opposite, which leads us to believe that the set of parameters and distribution function specifications that satisfy both inequalities simultaneously must be a very narrow one. In this precise sense, we argue in the main text that the conditions under which the result of the proposition hold are general ones.

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Figure 1
Credit market – Steady state

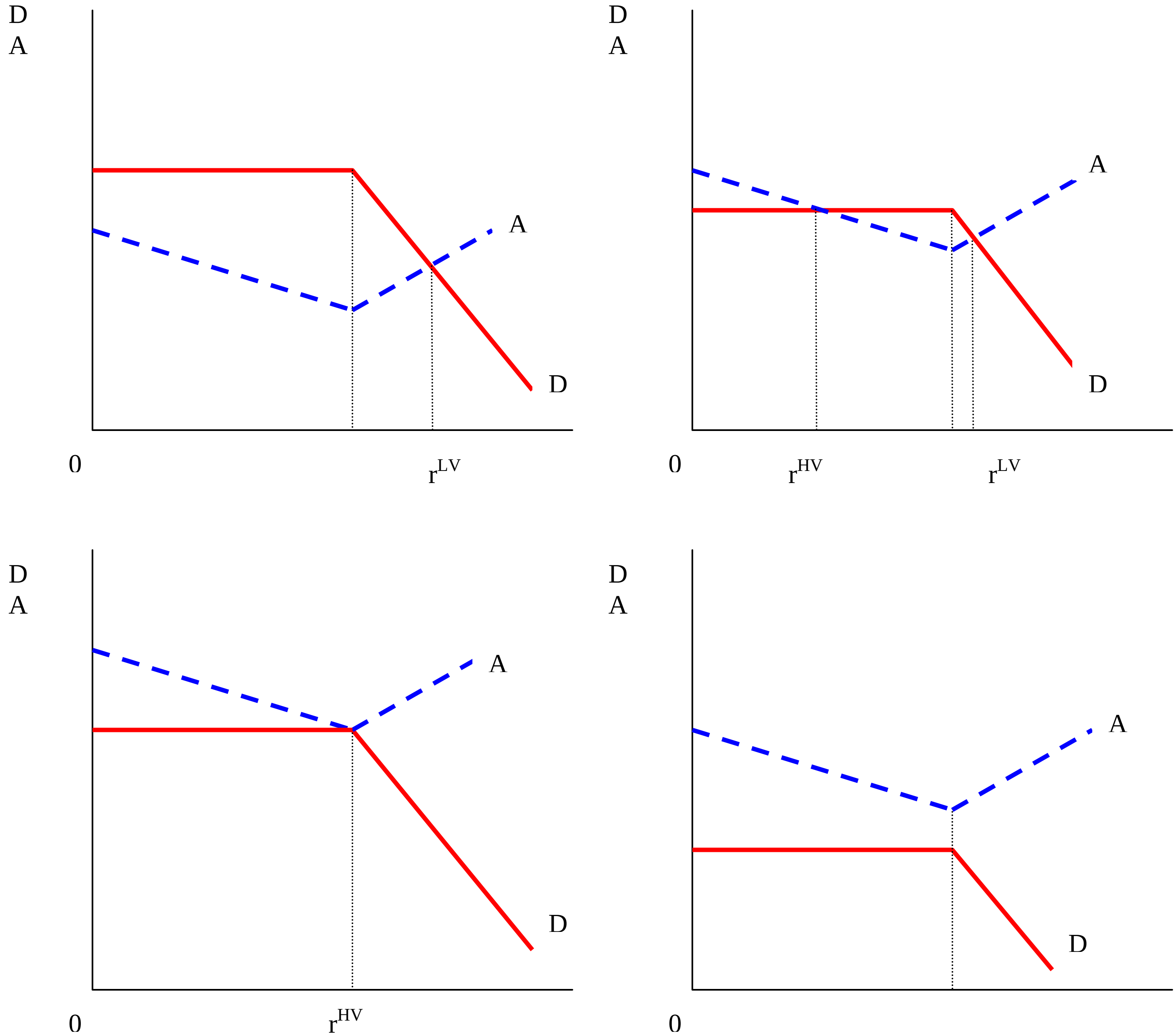


Figure 2
Dynamics: Housing price (top) and bubbles (bottom)

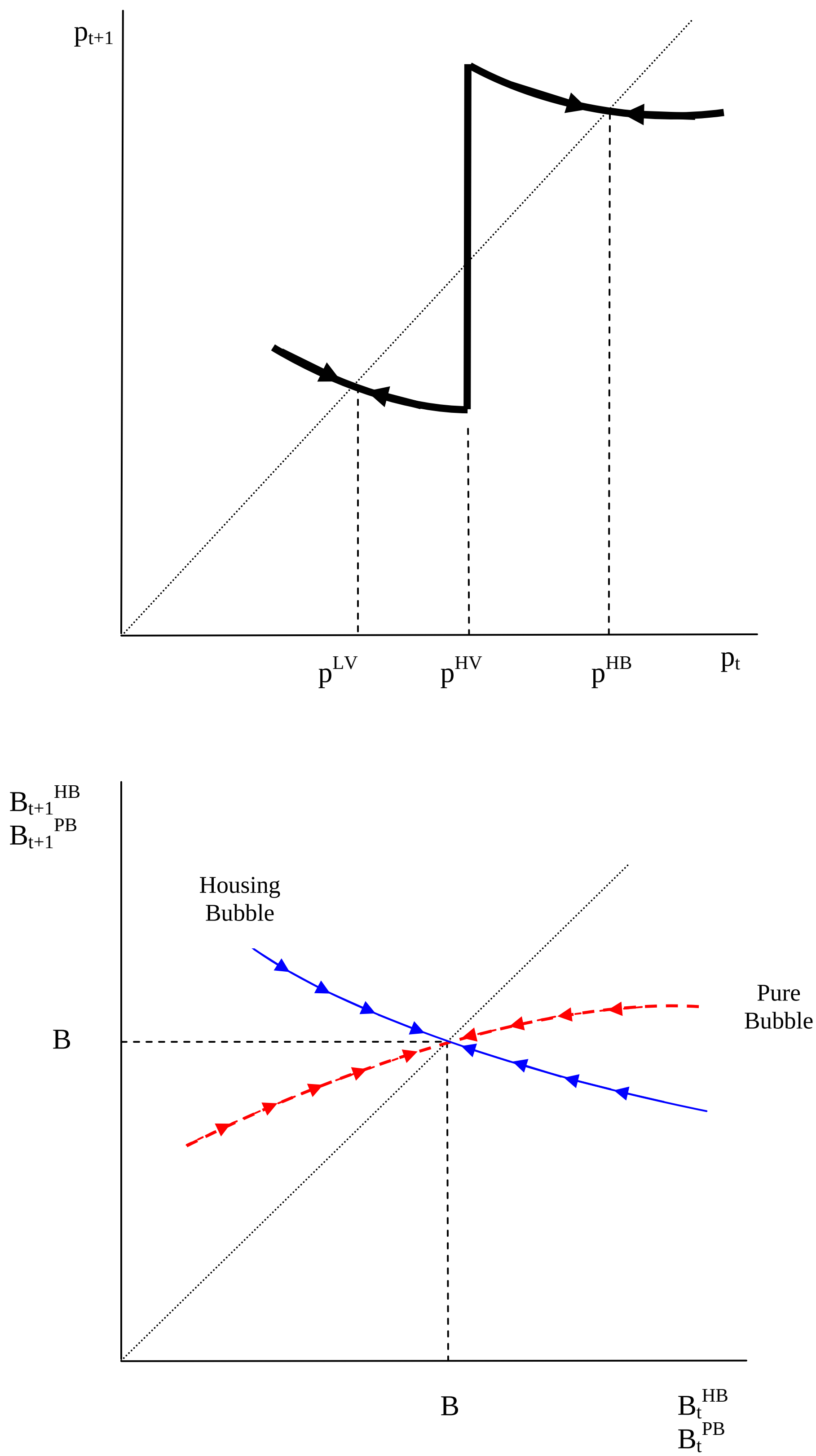


Figure 3
Collateral Constraints and the Size of the Bubbles

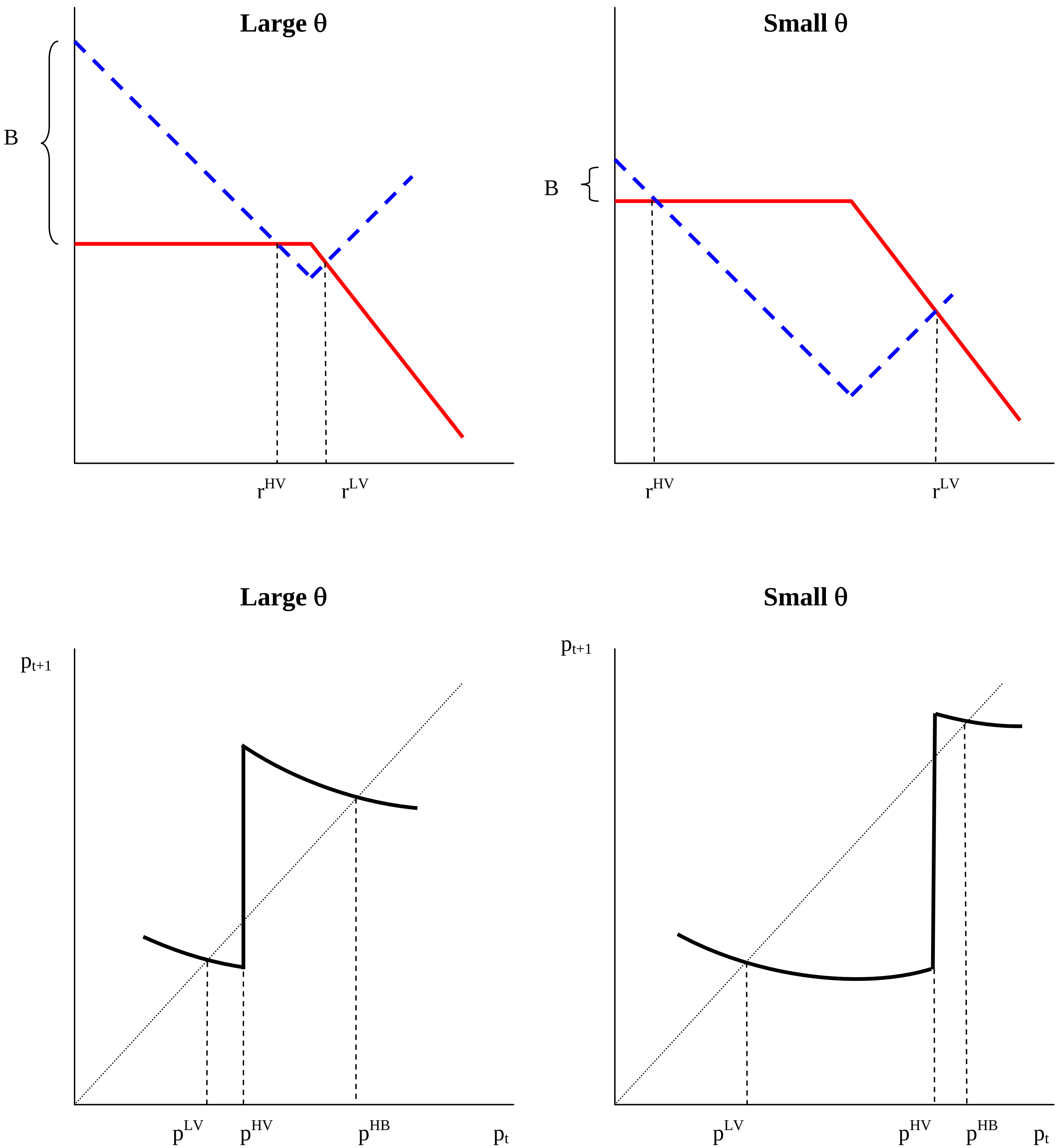
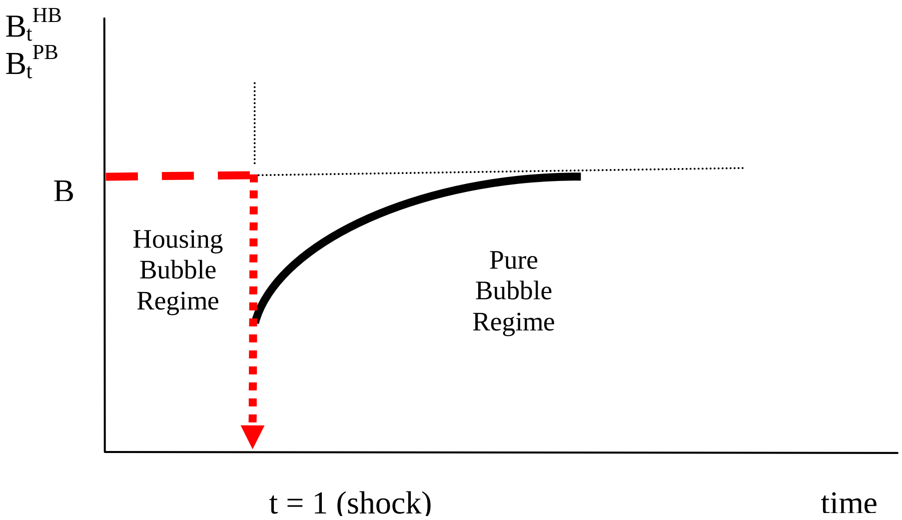


Figure 4
Switching Bubbles

A. From a Housing Bubble to a Pure Bubble



B. From a Pure Bubble to a Housing Bubble