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Dynamics of the Price Distribution in a General Model of State-Dependent Pricing

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(PRELIMINARY AND INCOMPLETE)

Abstract

This paper analyzes the effects of monetary policy shocks in a DSGE model that allows for a general form of smoothly state-dependent pricing by firms. As in Dotsey, King, and Wolman (1999) and Caballero and Engel (2007), our setup is based on one fundamental property: firms are more likely to adjust their prices when doing so is more valuable. The exogenous timing (Calvo 1983) and fixed menu cost (Golosov and Lucas 2007) models are nested as special cases of our setup.

Our model is calibrated to match the steady-state distribution of price adjustments in microdata; realism calls for firm-specific productivity shocks. Computing a dynamic general equilibrium requires us to calculate how the distribution of prices and productivities evolves over time. We solve the model using the method of Reiter (2006), which is well-suited to this type of problem because it combines a fully nonlinear characterization of the steady state value function and distribution with a linearization of the aggregate dynamics.

In our calibrated model, increased money growth causes a persistent rise in inflation and output. Uncorrelated money growth shocks have only a small effect on output in the menu cost model, so our calibrated model is closer to the Calvo specification in this case. Correlated money shocks, on the other hand, cause a large increase in consumption on impact in all three specifications, though the persistence of consumption is twice as large in the Calvo specification as it is in the other two.

Impulse responses differ depending on the distribution at the time the shock occurs. In particular, a surprise increase in money growth has different effects starting from the steady state distribution than it does if all firms have recently adjusted their prices.

Keywords: Price stickiness, state-dependent pricing, stochastic menu costs, generalized (S,s), heterogeneous agents, distributional dynamics

JEL Codes: E31, E52, D81

1 Introduction[1]

- Sticky prices: crucial for contemporary DSGE modelling, but controversial
- Golosov and Lucas claim Calvo model is misleading for policy analysis, exaggerates effects of money shocks
- Reason Calvo is influential is its tractability (aggregation properties), not its theoretical appeal or realism
- The problem with more realistic state-dependent models is that they imply a heterogeneous agent problem: the state variable is the distribution of prices and productivity
- Methods for calculating distributional dynamics are still new, and Krusell-Smith (1998) leading method is poorly suited for nonlinearities of menu cost model

In this paper we analyze the effects of monetary policy by nesting a model of smoothly state-dependent pricing in a general equilibrium monetary economy. There are two main innovations. First, we base our simulations on a state-dependent pricing mechanism that can be flexibly calibrated to nest many standard pricing models, and we choose parameters to match data from microeconomic studies. As Golosov and Lucas (2007) and others have emphasized, matching these data requires us to allow for firm-specific shocks. But therefore calculating the general equilibrium requires us to solve for the dynamics of the distribution of prices and productivities. Thus, our second innovation is to characterize the distributional dynamics using the computational method of Reiter (2006). This method is well suited for problems in which idiosyncratic shocks are more relevant than aggregate shocks for the individual decision maker, because it is fully nonlinear in idiosyncratic factors even though it imposes linearity in aggregate factors. Moreover, it is easy to implement because each step in the calculation is a familiar numerical procedure. First, it requires calculating the steady state equilibrium, which involves repeating a backwards induction problem on a grid. Second, the aggregate dynamics are solved linearly, which can be done with standard methods (*e.g.* Klein 2000; Sims 2001) in spite of the fact that this involves a very large system of equations representing values and densities at all points on the grid.

1.1 Related literature

The literature on state-dependent pricing has rarely attempted to calculate the full dynamics, with idiosyncratic shocks, in general equilibrium. Some important intuition has been obtained from partial equilibrium studies.

- Partial equilibrium studies, *e.g.* Caballero and Engel (1993, 2007)
- General equilibrium models that eliminate idiosyncratic shocks, *e.g.* Dotsey, King, and Wolman (1999)
- Strong restrictions on shock processes to obtain tractable distributional dynamics, *e.g.* Caplin and Spulber (1987), Gertler and Leahy (2005)

Recently a few studies have tried to go further.

[1] The authors wish to thank Michael Reiter and seminar participants at the Vienna Institute for Advanced Studies for helpful comments. The views expressed in this paper are those of the authors and do not necessarily coincide with those of the Bank of Spain or the Eurosystem.

- Golosov and Lucas (2007) have approximately characterized transition paths and some aspects of the dynamics by means of assumptions that some general equilibrium variables are roughly constant.
- Midrigan (2006) has simulated distributional dynamics using an assumption like that of Krusell and Smith (1998), namely, that the mean price level gives sufficient information to characterize the effect of the price distribution on the value function. However, he focusses mostly on the aggregate steady state, providing only limited information on the dynamics, with no impulse response functions.

Our computational method, in contrast, allows the distribution to affect decisions in a fairly arbitrary way, so we need not make very strong assumptions to calculate equilibrium. The computational method is quick and tractable, allowing us to report general equilibrium transition paths and impulse responses of many variables, including moments of the distribution. Moreover, our model of price adjustment nests most of the common state-dependent pricing frameworks, so we can easily compare alternative models.

2 Partial equilibrium: the firm's problem

Before we describe the full general equilibrium structure of our economy, it is helpful to study the partial equilibrium pricing decision of a monopolistic producer, to see how our model nests a variety of popular pricing frameworks.

Like Golosov and Lucas (2007), we assume price changes are driven primarily by idiosyncratic shocks. Thus, if firms are entirely rational, fully informed, and capable of frictionless adjustment, they will adjust their prices every time a new shock is realized. We instead assume that prices are "sticky", in a well-defined sense: the probability of adjusting is less than one, but is greater when the benefit from adjusting is greater. What we mean by "the benefit from adjusting" becomes clear as soon as we write down the Bellman equations that describe the firm's decision. There is a value associated with optimally choosing a new price today (while bearing in mind that prices will not always be adjusted in the future); likewise there is a value associated with leaving the current price unchanged today (likewise bearing in mind that prices will not always be adjusted in the future). The difference between these two values is the benefit from adjusting (or the loss from failing to adjust). The function $\lambda(L)$ that gives the adjustment probability as a function of the loss L from failing to adjust is taken as a primitive of the model. We will soon show precisely how these assumptions nest, as special cases, several leading models of price stickiness.

There are at least two ways of interpreting this framework. It could be seen as a model of stochastic menu costs, as in Dotsey, King, and Wolman (1999) or Caballero and Engel (1999). If rational, fully-informed firms draw an *iid* adjustment cost x every period, with cumulative distribution function $\lambda(x)$, then they will adjust their behavior whenever the adjustment cost x is less than or equal to the loss L from failing to adjust. Therefore, their probability of adjustment is $\lambda(L)$ when the loss from nonadjustment is L.

But perhaps this is an unnecessarily literal interpretation of the model. Alternatively, following Akerlof and Yellen (1985) for example, "stickiness" can be regarded as a minimal deviation from rational expectations behavior. Under full rationality, full information, and zero adjustment costs, economic agents adjust to a new optimal setting of their control variables in every period; here instead we assume they sometimes fail to make this adjustment. Perhaps failure to adjust occurs because information itself is "sticky" (as in Reis, 2006); or perhaps because managers face information processing constraints (as in Woodford 2008); rather than taking a stand on this, we simply regard our assumption as an axiom that should be imposed on near-rational, near-full-information behavior. Our framework is "close" to full rationality both because we can choose a λ function that is close to one for most L, and more importantly because large mistakes are less likely than trivial ones. In this sense, our framework permits us to smoothly deviate from standard rational decision making, to nest and compare other nearby forms of behavior.

2.1 The monopolistic competitor's decision

Suppose then, following Golosov and Lucas (2007), that each firm i produces output Y under a constant returns technology, with labor N as the only input, and faces idiosyncratic productivity shocks A_{it}:

$$Y_{it} = A_{it} N_{it}$$

We assume firms are monopolistic competitors, facing the demand curve $Y_{it} = \xi_t P_{it}^{-\epsilon}$, where ξ_t represents aggregate demand, and that they must fulfill all demand at the price they set. They hire in competitive labor markets at wage rate W_t, so the period profit function is

$$\Pi_{it} = P_{it} Y_{it} - W_t N_{it} = \left(P_{it} - \frac{W_t}{A_{it}} \right) Y_{it} = \left(P_{it} - \frac{W_t}{A_{it}} \right) \xi_t P_{it}^{-\epsilon}$$

We call the state of the economy Ω_t. We will not yet specify the structure of this state variable, other than to say that it is a Markov process that determines all the aggregate shocks: $\xi_t = \xi(\Omega_t)$, $W_t = W(\Omega_t)$. We assume the productivity shocks A_{it} are given by a constant Markov process, *iid* across firms and unrelated to the aggregate state. Thus A_{it} is correlated with A_{it-1} but is uncorrelated with all other shock processes in the model. The assumption that demand shocks are related to aggregate conditions, while productivity shocks are purely idiosyncratic, is inessential for our methodology; we ignore more general cases only to keep notation simple. The case we focus on is equivalent to that considered in Golosov and Lucas (2007), and is also similar to that in Reis (2006A).

To implement our assumption that adjustment is more likely when it is more valuable, we must define the values of adjustment and nonadjustment. If a firm fails to adjust (so that $P_{it} = P_{it-1}$), then its current profits and its future prospects will both depend on its productivity A_{it} and on its price P_{it}. Therefore these both enter as state variables in the value function of a nonadjusting firm, $V(P_{it}, A_{it}, \Omega_t)$, which also depends on the aggregate state of the economy. When a firm adjusts, we assume it chooses the best price conditional on its current productivity shock and on the aggregate state. Therefore, the value function of an adjusting firm, after netting out any costs that may be required to make the adjustment, is just $V^*(A_{it}, \Omega_t) \equiv \max_P V(P, A_{it}, \Omega_t)$. The value of adjusting to the optimal price, written in the same units as the value function, is then

$$D(P_{it}, A_{it}, \Omega_t) \equiv \max_P V(P, A_{it}, \Omega_t) - V(P_{it}, A_{it}, \Omega_t)$$

Of course, we don't want the real probability of adjustment to differ when values are denominated in euros instead of pesetas. In order to take the function λ that maps the value of adjusting into the probability of adjusting as a primitive of the model, we must be sure to write it in the appropriate units. Under either interpretation of the model, the most natural units are those of labor time. Under the stochastic menu cost interpretation, the labor effort of changing price tags or rewriting the menu is likely to be a large component of the cost. Under the bounded rationality interpretation, even though we don't explicitly model the computation process, we take the probability of adjustment to be related to the labor effort associated with obtaining new information and/or recomputing the optimal price. Therefore, the function λ should depend on the loss from failing to adjust, converted into units of labor time by dividing by the wage rate. That is, the probability of adjustment is $\lambda(L(P_{it}, A_{it}, \Omega_t))$, where $L(P_{it}, A_{it}, \Omega_t) = \frac{D(P_{it}, A_{it}, \Omega_t)}{W(\Omega_t)}$ and λ is a given weakly increasing function which we take as a primitive of the model.

Conditional on adjustment, we have assumed that the firm sets the optimal price, $P^*(A_{it}, \Omega_t) \equiv \arg\max_P V(P, A_{it}, \Omega_t)$. For clarity, we will distinguish between the firm's beginning-of-period price, $\widetilde{P}_{it} = P_{it-1}$, and the price at which it produces and sells at time t, P_{it}, which may or may not differ from $\widetilde{P}_{it}$. The adjustments are determined by the function λ:

$$P_{it} = \begin{cases} P^*(A_{it}, \Omega_t) & \text{with probability } \lambda\left(\frac{D(\widetilde{P}_{it}, A_{it}, \Omega_t)}{W(\Omega_t)}\right) \\ \widetilde{P}_{it} = P_{i,t-1} & \text{with probability } 1 - \lambda\left(\frac{D(\widetilde{P}_{it}, A_{it}, \Omega_t)}{W(\Omega_t)}\right) \end{cases}$$

Function λ must satisfy $\lambda' \geq 0$. In particular, we will consider the class

$$\lambda(L) \equiv \frac{L^\varsigma}{\alpha^\varsigma + L^\varsigma}$$

with α and ς positive. This function equals 0.5 when $L = \alpha$, and is concave for $\varsigma \leq 1$ and S-shaped for $\varsigma > 1$. It has fatter tails than the normal *cdf*, which may help it match the fat tails of the observed adjustment distribution emphasized by Midrigan (2006).

We can now state the Bellman equation that defines the value of producing at any given price. It differs somewhat depending on whether we impose the stochastic menu cost interpretation of our model or the bounded rationality interpretation; we begin with the latter because it is slightly simpler. Given the firm's price P and its productivity shock A, current profits are $\left(P - \frac{W(\Omega)}{A}\right)\xi(\Omega)P^{-\epsilon}$. The firm anticipates adjusting or not adjusting in the next period depending on the benefits of adjusting at that time. Therefore, using primes to denote next period's values, the Bellman equation is:

$$V(P, A, \Omega) = \left(P - \frac{W(\Omega)}{A}\right)\xi(\Omega)P^{-\epsilon} +$$
$$E\left\{ Q(\Omega, \Omega')\left[\left(1 - \lambda\left(\frac{D(P, A', \Omega')}{W(\Omega')}\right)\right)V(P, A', \Omega') + \lambda\left(\frac{D(P, A', \Omega')}{W(\Omega')}\right)\max_{P'} V(P', A', \Omega')\right] \middle| A, \Omega\right\}$$

where $Q(\Omega, \Omega')$ is the firm's discount factor and the expectation refers to the distribution of A' and Ω' conditional on A and Ω. Note that on the left-hand side of this equation, and in the term that represents current profits, P refers to a given firm i's price P_{it} at the time of production. In the expectation on the right, P represents the price $\widetilde{P}_{it+1}$ at the beginning of period $t+1$, which may (probability λ) or may not (probability $1 - \lambda$) be adjusted prior to time $t+1$ production.

We can simplify substantially by noticing that the value on the right-hand side of the equation is just the value of continuing without adjustment, plus the expected capital gains due to adjustment:

$$V(P, A, \Omega) = \left(P - \frac{W(\Omega)}{A}\right)\xi(\Omega)P^{-\epsilon} +$$
$$+ E\left\{ Q(\Omega, \Omega')\left[V(P, A', \Omega') + \lambda\left(\frac{D(P, A', \Omega')}{W(\Omega')}\right)D(P, A', \Omega')\right] \middle| A, \Omega\right\}$$

or equivalently, our most compact expression:

Bellman equation in partial equilibrium, with aggregate shocks:

$$V(P, A, \Omega) = \left(P - \frac{W(\Omega)}{A}\right)\xi(\Omega)P^{-\epsilon} + E\left\{ Q(\Omega, \Omega')\left[V(P, A', \Omega') + G(P, A', \Omega')\right] \middle| A, \Omega\right\} \tag{1}$$

where $G(P, A', \Omega') \equiv \lambda\left(\frac{D(P, A', \Omega')}{W(\Omega')}\right)D(P, A', \Omega')$ represents the expected gains due to adjustment.

The difficult aspect of this model is seen in the fact that the wage, the aggregate demand factor, the stochastic discount factor, and therefore also the value function all depend on the aggregate state of the economy, Ω. In general equilibrium, there will be many firms i facing different idiosyncratic shocks A_{it} and stuck at different prices P_{it} at any time t. The state of the economy will therefore include the entire distribution of prices and productivities. The reason for the popularity of the Calvo model is that in spite of the fact that firms have many different prices, up to a first-order approximation only the average price matters for the equilibrium of the economy. Unfortunately, this property does not hold in general. In the current context, we need to recognize that all equilibrium quantities are explicitly functions of the distribution of prices and productivity across the economy. We therefore need to calculate equilibrium with an algorithm that keeps track of the distributional dynamics.

We will follow the algorithm of Reiter (2006), which begins by solving the steady state general equilibrium before solving equilibrium with aggregate shocks. So consider an aggregate steady state, so that Ω, W, and ξ are constant. We indicate the steady state value function by dropping Ω as an argument, and the Bellman equation becomes

Bellman equation in partial equilibrium steady state:

$$V(P,A) = \left(P - \frac{W}{A}\right)\xi P^{-\epsilon} + R^{-1}E\left\{V(P,A') + G\left(P,A'\right)|A\right\} \tag{2}$$

where R^{-1} is the steady state of the stochastic discount factor Q, and

$$G(P,A') \equiv \lambda\left(\frac{D(P,A')}{W}\right)D(P,A'), \qquad D(P,A') \equiv \max_{P'} V(P',A') - V(P,A')$$

This steady state Bellman equation is a standard dynamic programming problem, except for the timing of the max operator. A natural solution method is backwards induction on a two-dimensional grid $\Gamma \equiv \Gamma^P \times \Gamma^A$, where Γ^P is a finite grid of possible values of P_i, and Γ^A is a grid of possible values of A_i. However, before we restrict the dynamics of the model to a grid we will describe the general equilibrium and detrend the model with respect to money growth in order to express all quantities in real terms.

2.2 Alternative sticky price frameworks

We will want to compare our simulation results to a number of alternative pricing frameworks. This is straightforward to do, because the Bellman equation above is quite similar to the Bellman equations under several important alternatives.

1. **Calvo pricing**: Suppose prices adjust each period with probability $\bar{\lambda}$, where $\bar{\lambda}$ is an exogenous constant. Then the Bellman equation is the same as (1), except that we must define $\lambda(D/W) \equiv \bar{\lambda}$.

2. **Fixed menu costs**: Suppose the cost of adjusting prices in any given period is κ units of labor (where κ is an exogenous constant called the "menu cost"). Then the Bellman equation is the same as (1), except that the term $G = \lambda(D/W)D$ is replaced by $G = \mathbf{1}\left\{D \geq \kappa W\right\}(D - \kappa W)$, where $\mathbf{1}\left\{D \geq \kappa W\right\}$ is an indicator function taking value one when $D \geq \kappa W$ and zero otherwise.

3. **Stochastic menu costs**: Suppose the cost of adjusting prices in any given period is κ units of labor, where κ is an *i.i.d.* random variable with *c.d.f.* $\lambda(\kappa)$. Then the Bellman equation is the same as (1), except that the term $G = \lambda(D/W)D$ is replaced by $G = \lambda(D/W)[D - WE(\kappa|D > \kappa W)]$.

4. **Information-constrained pricing**: Woodford (2008) proposes a mode in which managers decide on when to review a price based on imprecise awareness of current market conditions. In his model, $G = \lambda(D/W)(D - \kappa W)$, where

$$\lambda(D/W) = \frac{\bar{\lambda}\exp\left(\frac{D/W-\kappa}{\theta}\right)}{(1-\bar{\lambda}) + \bar{\lambda}\exp\left(\frac{D/W-\kappa}{\theta}\right)}$$

and θ is an information cost parameter.

3 General equilibrium

We next embed this partial equilibrium decision framework in an otherwise standard New Keynesian general equilibrium, following the setup of Golosov and Lucas (2007). In addition to the firms, there is a representative household and a monetary authority that chooses the money supply.

3.1 Households

The household's period utility function is

$$u(C_t) - x(N_t) + v(M_t/P_t)$$

discounted by factor β per period. Consumption C_t is a Spence-Dixit-Stiglitz aggregate of differentiated products:

$$C_t = \left[\int_0^1 C_{it}^{\frac{\epsilon-1}{\epsilon}} di\right]^{\frac{\epsilon}{\epsilon-1}} \tag{3}$$

N_t is labor supply, and M_t/P_t is real money balances. The household's period budget constraint is

$$\int_0^1 P_{it}C_{it}di + M_t + R_t^{-1}B_t = W_tN_t + M_{t-1} + T_t + B_{t-1} + \Pi_t$$

where $\int_0^1 P_{it}C_{it}di$ is total nominal spending on the differentiated goods. B_t is nominal bond holdings, with interest rate $R_t - 1$; T_t represents lump sum transfers received from the monetary authority, and Π_t represents dividend payments received from the firms.

Assuming households choose $\{C_{it}, N_t, B_t, M_t\}_{t=0}^{\infty}$ so as to maximize expected discounted utility subject to the budget constraint, we obtain the following necessary conditions. Optimal allocation of consumption across the differentiated goods implies

$$C_{it} = (P_t/P_{it})^{\epsilon}C_t \tag{4}$$

where P_t is the following price index:

$$P_t \equiv \left\{\int_0^1 {P_{it}}^{1-\epsilon}di\right\}^{\frac{1}{1-\epsilon}} \tag{5}$$

which also lets us rewrite nominal spending as $P_tC_t = \int_0^1 P_{it}C_{it}di$.[2] Optimal labor supply and money holdings imply the first-order conditions

$$x'(N_t) = u'(C_t)W_t/P_t \tag{6}$$

$$v'\left(\frac{M_t}{P_t}\right) = u'(C_t)(1 - R_t^{-1}) \tag{7}$$

[2] One of the preceding equations is superfluous: (4) plus (3) implies (5), and likewise (4) plus (5) implies (3).

and the Euler equation is

$$R_t^{-1} = \beta E_t \left(\frac{P_t u'(C_{t+1})}{P_{t+1} u'(C_t)} \right) \tag{8}$$

3.2 Monetary policy

As in Golosov and Lucas (2007), the money supply follows an exogenous, nonstationary stochastic process:

$$M_t \mu_t = M_{t-1} \tag{9}$$

where $\mu_t = \widehat{\mu} \exp(z_t)$, and z_t is an AR(1) process:

$$z_t = \phi_z z_{t-1} + \epsilon_t^z \tag{10}$$

where $0 \leq \phi_z < 1$ and $\epsilon_t^z \sim i.i.d.N(0, \sigma_z^2)$ is a money growth shock. If we set $\widehat{\mu} \equiv \mu \exp\left(-\sigma_z^2/2\right)$, where $\mu \leq 1$, then

$$E_t \left[\frac{M_t}{M_{t+1}} \right] = E_t \mu_t = \mu \exp(\phi_z z_{t-1})$$

so that the money supply trends upward by factor $1/\mu > 1$ each period.

Seignorage revenues are paid to the household as a lump sum transfer, and the government budget is balanced each period. Therefore the government's budget constraint is

$$M_t = M_{t-1} + T_t$$

3.3 Aggregate consistency

Bond market clearing is simply $B_t = 0$. Market clearing for good i implies the following demand and supply relations for firm i:

$$Y_{it} = A_{it} N_{it} = C_{it} = P_t^{\epsilon} C_t P_{it}^{-\epsilon}$$

Since C_t and P_t must both be functions of the aggregate state Ω_t, this has the form we assumed when discussing the firm's problem if we set $\xi_t = P_t^{\epsilon} C_t$. We can therefore also calculate total labor demand:

$$N_t = \int_0^1 \frac{C_{it}}{A_{it}} di = P_t^{\epsilon} C_t \int_0^1 P_{it}^{-\epsilon} A_{it}^{-1} di \tag{11}$$

At this point, we have spelled out all equilibrium conditions: the equilibrium conditions for the household and monetary authority have been stated in this section, and the firms' decision was described in Section 2. Thus we are now ready to identify the aggregate state variable Ω_t. We have only included one aggregate shock in the model, namely, the money supply M_t. There is also a continuum of idiosyncratic shocks, namely the productivity shocks A_{it}, $i \in [0, 1]$. Finally, since firms cannot instantly adjust their prices, they are state variables too. More precisely, the state includes the joint distribution of prices and productivity shocks at the beginning of the period, prior to adjustment.

We will use the notation $\widetilde{P}_{it}$ to refer to firm i's price at the beginning of period t, prior to adjustment; this may of course differ from the price P_{it} at which it produces, because the price may be adjusted before production. Therefore we will distinguish the distribution of production prices and productivity, which we write as $\Phi_t(P_{it}, A_{it})$, from the distribution of beginning-of-period prices and productivity, $\widetilde{\Phi}_t(\widetilde{P}_{it}, A_{it})$. Since beginning-of-period prices and productivities determine all equilibrium decisions at t, we can define the state at time t as $\Omega_t \equiv (M_t, \widetilde{\Phi}_t)$.

3.4 The firm's problem in general equilibrium

The setup of sections 3.1-3.3 holds regardless of how firms set prices. In particular, regardless of the price-setting mechanism, C_t, N_t, P_t, W_t, R_t, C_{it}, P_{it}, and M_t must obey equations (3) - (11). Next, we adapt our pricing setup to this general equilibrium environment. We write the model in the boundedly rational interpretation where the gains from adjustment in state (P, A, Ω) are $G(P, A, \Omega) \equiv \lambda \left(\frac{D(P,A,\Omega)}{W(\Omega)} \right) D(P, A, \Omega)$, but it is straightforward to rewrite it for other types of price stickiness.

We assume that the representative household owns the firms. Therefore the appropriate stochastic discount factor is

$$Q(\Omega_t, \Omega_{t+1}) = \beta \frac{P(\Omega_t) u'(C(\Omega_{t+1}))}{P(\Omega_{t+1}) u'(C(\Omega_t))}$$

In partial equilibrium, the firm's demand function was $\xi_t P_{it}^{-\epsilon}$. Using (4), in general equilibrium we have $\xi_t = C_t P_t^{\epsilon}$. Therefore, for a firm that sets sticky prices, the value of producing with price P_{it} and productivity A_{it} is

Bellman equation in general equilibrium:

$$V(P_{it}, A_{it}, \Omega_t) = \left(P_{it} - \frac{W(\Omega_t)}{A_{it}} \right) P(\Omega_t)^{\epsilon} C(\Omega_t) P_{it}^{-\epsilon} +$$
$$+ \beta E_t \left\{ \frac{P(\Omega_t) u'(C(\Omega_{t+1}))}{P(\Omega_{t+1}) u'(C(\Omega_t))} \left[V(P_{it}, A_{i,t+1}, \Omega_{t+1}) + G(P_{it}, A_{i,t+1}, \Omega_{t+1}) \right] \middle| A_{it}, \Omega_t \right\}$$

where

$$G(P_{it}, A_{i,t+1}, \Omega_{t+1}) \equiv \lambda \left(\frac{D(P_{it}, A_{i,t+1}, \Omega_{t+1})}{W(\Omega_{t+1})} \right) D(P_{it}, A_{i,t+1}, \Omega_{t+1})$$

and

$$D(P_{it}, A_{i,t+1}, \Omega_{t+1}) \equiv \max_{P'} V(P', A_{i,t+1}, \Omega_{t+1}) - V(P_{it}, A_{i,t+1}, \Omega_{t+1})$$

Notice that these equations involve only P, C, W, and P_i. Therefore these equations give us the information needed to determine the idiosyncratic price process. Letting $P^*(A_{i,t+1}, \Omega_{t+1})$ denote the optimal choice in the maximization problem above, the price process is

$$P_{i,t+1} = \begin{cases} P^*(A_{i,t+1}, \Omega_{t+1}) & \text{with probability } \lambda \left(\frac{D(P_{it}, A_{i,t+1}, \Omega_{t+1})}{W(\Omega_{t+1})} \right) \\ P_{it} & \text{with probability } 1 - \lambda \left(\frac{D(P_{it}, A_{i,t+1}, \Omega_{t+1})}{W(\Omega_{t+1})} \right). \end{cases}$$

3.5 Detrending

The value function and all prices have been written so far in nominal terms. It is natural to assume that the model can be rewritten in real terms. Thus suppose we deflate all prices by the nominal money stock, defining $p_t \equiv P_t/M_t$, $p_{it} \equiv P_{it}/M_t$, and $w_t \equiv W_t/M_t$. Given the nominal distribution $\Phi_t(P_i, A_i)$ and the money stock M_t, let us denote by $\Psi_t(p_i, A_i)$ the distribution over real production prices $p_{it} \equiv P_{it}/M_t$. Likewise, let $\widetilde{\Psi}_t(p_i, A_i)$ be the distribution of real beginning-of-period prices $\widetilde{p}_{it} \equiv \widetilde{P}_{it}/M_t$, in analogy to the beginning-of-period distribution of nominal prices $\widetilde{\Phi}_t(\widetilde{P}_i, A_i)$. If it is true that the model can be rewritten in real terms, the distribution $\widetilde{\Psi}_t(\widetilde{p}_i, A_i)$ is a sufficient aggregate state variable to determine real quantities. That is, to describe the real equilibrium, it is not necessary to condition functions on M_t, only on the distribution $\widetilde{\Psi}_t(\widetilde{p}_i, A_i)$.

The "real" value function v should likewise be the nominal value function, divided by the current money stock, and should be written as a function of real prices. That is,

$$V(P_{it}, A_{it}, \Omega_t) = M_t v\left(\frac{P_{it}}{M_t}, A_{it}, \widetilde{\Psi}_t\right) = M_t v\left(p_{it}, A_{it}, \widetilde{\Psi}_t\right)$$

Deflating in this way, the system can be rewritten as follows (see the appendix for details).
Detrended Bellman equation, general equilibrium:

$$v(p_{it}, A_{it}, \widetilde{\Psi}_t) = \left(p_{it} - \frac{w(\widetilde{\Psi}_t)}{A_{it}}\right)\left(\frac{p_{it}}{p(\widetilde{\Psi}_t)}\right)^{-\epsilon} C(\widetilde{\Psi}_t) +$$

$$+\beta E_t\left\{ \tfrac{p(\widetilde{\Psi}_t)u'(C(\widetilde{\Psi}_{t+1}))}{p(\widetilde{\Psi}_{t+1})u'(C(\widetilde{\Psi}_t))} \left[v\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) + g\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)\right] \Big| A_{it}, \widetilde{\Psi}_t\right\}$$

where

$$g\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) \equiv \lambda\left(\tfrac{d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)}{w(\widetilde{\Psi}_{t+1})}\right) d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)$$

$$d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) \equiv \max_{p'} v(p', A_{t+1}(i), \widetilde{\Psi}_{t+1}) - v\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)$$

Let $p^*(A_{i,t+1}, \widetilde{\Psi}_{t+1})$ denote the optimal choice in the maximization problem above. Taking into account the fact that the firm starts period $t+1$ with the eroded price $\widetilde{p}_{i,t+1} \equiv \mu_{t+1}p_{it}$, the price process is

$$p_{i,t+1} = \begin{cases} p(A_{i,t+1}, \widetilde{\Psi}_{t+1}) & \text{with probability } \lambda\left(\tfrac{d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)}{w(\widetilde{\Psi}_{t+1})}\right) \\ \mu_{t+1}p_{it} & \text{with probability } 1-\lambda\left(\tfrac{d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)}{w(\widetilde{\Psi}_{t+1})}\right). \end{cases}$$

In other words, when the firm's nominal price is not adjusted at time $t+1$, its real price is deflated by factor μ_{t+1}.

4 Computing general equilibrium: steady state

4.1 Discrete numerical model

In order to actually solve the model numerically, we approximate the economy by assuming that individual states, in real terms, always lie in a finite grid. For a discrete model of this type, the firm's decision can be calculated by backwards induction. This is an entirely standard solution method, but we will now spell it out in detail, both to see how it fits into our general equilibrium and in order to clarify our calculations later when we solve the more difficult case of aggregate shocks.

Thus, suppose we approximate our model on a two-dimensional grid $\Gamma \equiv \Gamma^p \times \Gamma^A$, where $\Gamma^p \equiv \{p^1, p^2, ...p^{\#^p}\}$ is a logarithmically-spaced grid of length $\#^p$ of possible values of p_i, and $\Gamma^A \equiv \{a^1, a^2, ...a^{\#^A}\}$ is a logarithmically-spaced grid of length $\#^A$ of possible values of A_i. Also, define $\Delta^p \equiv \log(p^{j+1}/p^j)$ as the logarithmic step size in grid Γ^p. From here on, we will frequently use superscripts to identify notation related to various grids. In this approximation, we can think of the distributions Ψ and $\widetilde{\Psi}$ as matrices of size $\#^p \times \#^A$ in which the (j,k) element represents the fraction of the population of firms in state (p^j, a^k).

Likewise, we can now think of the value function as a $\#^p \times \#^A$ matrix $\mathbf{V}$ of values $v^{jk} \equiv v(p^j, a^k)$ associated with the prices and productivities $\left(p^j, a^k\right) \in \Gamma$. We can then construct splines to evaluate the value function at points $p \notin \Gamma^p$ off the price grid, when necessary. In particular, we define the policy function

$$p^*(A) \equiv \arg\max_p v(p, A) \tag{12}$$

without requiring that it be chosen from the grid Γ^p, because (as we will see below) our solution method requires policies to vary continuously with their arguments. It is useful to collect the policies at the productivity grid points $a^k \in \Gamma^A$ as a row vector $\mathbf{p}^* \equiv \left\{p^{*1}...p^{*\#^A}\right\} \equiv \left\{p^*(a^1)...p^*(a^{\#^A})\right\}$. Next, we define the $\#^p \times \#^A$ matrix of adjustment values $\mathbf{D}$ in which the (j,k) element is

$$d^{jk} \equiv \max_p v(p, a^k) - v^{jk} \tag{13}$$

that is, the value of adjustment when the idiosyncratic state is $\left(p^j, a^k\right) \in \Gamma$. (Here again, the chosen p is *not* constrained to lie in Γ^p.) Finally, we can define expected gains by a $\#^p \times \#^A$ matrix $\mathbf{G}$ in which the (j,k) element is

$$g^{jk} \equiv \lambda\left(d^{jk}/w\right) d^{jk} \tag{14}$$

We can now write down the discrete Bellman equation and the discrete distributional dynamics in a precise way. The dynamics involve three main steps. First, consider a firm with beginning-of-t price $\widetilde{p}_{it} = p^j \in \Gamma^p$ and $A_{it} = a^k \in \Gamma^A$. This firm's production price will be $p_{it} = p^{*k}$ with probability $\lambda(d^{jk}/w)$, or will remain unchanged ($p_{it} = \widetilde{p}_{it} = p^j$) with probability $1 - \lambda(d^{jk}/w)$. If adjustment occurs, we maintain our grid-based approximation by rounding p^{*k} up or down stochastically to the nearest grid points. Therefore, we calculate the production distribution Ψ_t from the beginning-of-t distribution $\widetilde{\Psi}_t$ as follows:

$$prob(p_{it} = p^l | \widetilde{p}_{it} = p^j, A_{it} = a^k) = \begin{cases} 1 - \lambda(d^{jk}/w) & \text{if } p^j = p^l \\ \lambda(d^{jk}/w)\left(\frac{p^{*k} - p^{l-1}}{p^l - p^{l-1}}\right) & \text{if } p^l = \min\{p \in \Gamma^p : p \geq p^{*k}\} \\ \lambda(d^{jk}/w)\left(\frac{p^{l+1} - p^{*k}}{p^{l+1} - p^l}\right) & \text{if } p^l = \max\{p \in \Gamma^p : p < p^{*k}\} \\ 0 & \text{otherwise} \end{cases}$$

In these formulas, we assume Γ^p is chosen wide enough so that $p^1 < p^{*k} < p^{\#^p}$ for all k.

Equivalently, we can summarize this set of equations in matrix notation. Let $\mathbf{E}$ be a $\#^p \times \#^p$ matrix of ones. Let $\lambda(\mathbf{D}/w)$ be a $\#^p \times \#^A$ matrix with element (j,k) equal to $\lambda(d^{jk}/w)$. Also, given $a^k \in \Gamma^A$, define $l(k)$ so that $p^{l(k)} = \min\{p \in \Gamma^p : p \geq p^{*k}\}$. Then let $\mathbf{P}$ be the $\#^p \times \#^A$ matrix taking the value $\left(\frac{p^{*k} - p^{l(k)-1}}{p^{l(k)} - p^{l(k)-1}}\right)$ in row $l(k)$, column k; and taking value $\left(\frac{p^{l(k)} - p^{*k}}{p^{l(k)} - p^{l(k)-1}}\right)$ in row $l(k) - 1$, column k; and taking value zero elsewhere. Then the relation between distributions Ψ_t and $\widetilde{\Psi}_t$ is:

$$\Psi_t = (\mathbf{E} - \lambda(\mathbf{D}/w)). * \widetilde{\Psi}_t + \mathbf{P}. * (\mathbf{E} * (\lambda(\mathbf{D}/w). * \widetilde{\Psi}_t)) \tag{15}$$

where (as in MATLAB) the operator $.*$ represents element-by-element multiplication, and $*$ represents ordinary matrix multiplication.

The second step in the distributional dynamics is to adjust real prices to take into account steady state money growth. Suppose for simplicity that deflating by the money growth rate causes real prices to decrease by an integer number of "steps" in the price grid Γ^p; in other words, suppose $\#^\mu \equiv \log \mu / \Delta^p$ is a nonnegative integer. Then, rounding up to p^1 when prices fall off bottom of the grid, $p_{it} = p^l$ implies $\widetilde{p}_{i,t+1} = \max\{p^1, p^{l-\#^\mu}\}$.[3] We can think of this as defining a matrix $\mathbf{R}$ of size $\#^p \times \#^p$ which equals one in row $m(l) = \max\{1, l - \#^\mu\}$ of column l, and is zero elsewhere.[4] This allows us to interpret the row m, column l element of $\mathbf{R}$ as

$$R^{ml} = prob(\widetilde{p}_{i,t+1} = p^m | p_{it} = p^l)$$

In other words, $\mathbf{R}$ is a Markov matrix that governs the transition from real prices in period t at the time of production, to real prices at the beginning of $t+1$. Note that since the number of points in the policy grid is large, $\mathbf{R}$ is likely to be enormous, but the fact that it is also extremely sparse will make our calculations feasible.

The third and final step in the distributional dynamics is to take into account the Markov matrix $\mathbf{S}$ that governs the idiosyncratic productivity shocks A_i. The row m, column k element of $\mathbf{S}$ is

$$S^{mk} = prob(A_{i,t+1} = a^m | A_{it} = a^k)$$

Combining the second and third steps, we can calculate the beginning-of-period distribution $\widetilde{\Psi}_{t+1}$ at $t+1$ as a function of the time t distribution of production prices Ψ_t:

$$\widetilde{\Psi}_{t+1} = \mathbf{R} * \Psi_t * \mathbf{S}' \tag{16}$$

The simplicity of this equation comes partly from the fact that the exogenous shocks to $A_{i,t+1}$ are independent of the inflation adjustment that links $\widetilde{p}_{i,t+1}$ with p_{it}. Also, exogenous shocks are represented from left to right in the matrix Ψ_t, so that their transitions can be treated by right multiplication, while policies are represented vertically, so that transitions related to policies can be treated by left multiplication.

The same transition matrices show up when we write the Bellman equation in matrix form. Let $\mathbf{U}$ be the $\#^p \times \#^A$ matrix of current payoffs, with element $u^{jk} = \left(p^j - \frac{w}{a^k}\right) C \left(\frac{p_j}{p}\right)^{-\epsilon}$ for $\left(p^j, a^k\right) \in \Gamma$. Then the Bellman equation is

Steady state general equilibrium Bellman equation, matrix version:

$$\mathbf{V} = \mathbf{U} + \beta \mathbf{R}' * (\mathbf{V} + \mathbf{G}) * \mathbf{S} \tag{17}$$

Since the Bellman equation iterates backwards in time, it involves probability transitions represented by $\mathbf{R}'$ and $\mathbf{S}$, whereas the distributional dynamics iterate forward in time and therefore contain $\mathbf{R}$ and $\mathbf{S}'$.

The functional equations (15), (16), and (17) can equivalently be seen as a system of $4\#^p\#^a + \#^a$ scalar equations. These equations involve the unknown matrices $\mathbf{V}$, $\mathbf{D}$, Ψ, and $\widetilde{\Psi}$; the vector $\mathbf{p}^*$; and the scalars w, p, and C: a total of $4\#^p\#^a + \#^a + 3$ unknown scalars. The remaining $4 + \#^p$

[3] In other words, we assume that any nominal price that would have a real value less than p^1 after inflation is automatically adjusted upwards so that its real value is p^1. This assumption is made for numerical simulation reasons only, and has a negligible impact on the equilibrium as long as we choose a sufficiently wide grid Γ^p. If we were to compute examples with trend deflation, we would need to make an analogous adjustment to prevent real prices from surpassing the maximum grid point $p^{\#^p}$.

[4] It is straightforward to generalize the definition of $\mathbf{R}$ to the case where $\#_\mu$ is not an integer; see section 5.

equations needed to close the general equilibrium are

$$u'(C) = \frac{px'(N)}{w} \tag{18}$$

$$1 - \frac{v'(1/p)}{u'(C)} = \beta\mu \tag{19}$$

$$N = \sum_{j=1}^{\#^p}\sum_{k=1}^{\#^a} \Psi^{jk} C^j / A^k \tag{20}$$

$$C^j = (p^j/p)^{-\epsilon} C \tag{21}$$

$$p = \left[\sum_{j=1}^{\#^p}\sum_{k=1}^{\#^a} \Psi^{jk}\left(p^j\right)^{1-\epsilon}\right]^{\frac{1}{1-\epsilon}} \tag{22}$$

In these equations, $j \in \{1, 2, ...\#^p\}$ indicates possible prices, and $k \in \{1, 2, ...\#^a\}$ indicates possible productivities. Note that since the last equations are related to goods demand and supply, the relevant distribution is Ψ, which is associated with the time of production. We have dropped the equation that defines the consumption aggregator C since it is implied by the others. Overall, then, we have $4\#^p\#^a + \#^p + \#^a + 4$ scalar equations to determine an equivalent number of unknowns: v^{jk}, d^{jk}, Ψ^{jk}, $\widetilde{\Psi}^{jk}$, p^{*k}, C^j, C, N, w, and p.

4.2 Results: steady state

This steady state model can be calibrated by comparing its predictions to cross-sectional data on price changes, like those reported in Klenow and Kryvstov (2005), Midrigan (2006), and Nakamura and Steinsson (2007). In a companion paper, Costain and Nakov (2008), we report detailed results from a variety of specifications, and compare the model's behavior under low and high steady-state inflation rates. Here we simply briefly discuss our preferred estimate from that paper. We will simulate our model at monthly frequency, for consistency with the results reported in the empirical literature. Also, since these papers all attempt to remove price changes attributable to temporary "sales", our simulation results should be interpreted as a model of "regular" price changes unrelated to sales.

We take the steady state growth rate of money, 0.64% per quarter, and our utility parameterization, from Golosov and Lucas (2007). Therefore we set the discount factor to $\beta = 1.04^{-1}$ per year. Consumption utility is CRRA, $u(C) = \frac{1}{1-\gamma}C^{1-\gamma}$, with $\gamma = 2$. Labor disutility is linear, $x(N) = \xi N$, with $\xi = 6$. The utility of real money holdings is logarithmic, $v(m) = \nu \log(m)$, with $\nu = 0.0323$ (this value is chosen so that the wage is one in steady state). The elasticity of substitution in the consumption aggregator is $\epsilon = 7$.

Given these utility parameters, we can next calibrate the idiosyncratic productivity shock process and the adjustment process to match data on the distribution of regular price changes. We assume productivity is AR(1) in logs:[5]

$$\log A_{it} = \rho \log A_{it-1} + \varepsilon^a_{it}$$

[5] Our numerical method requires us to treat A as a discrete variable, so we use Tauchen's method (Mertens, 2006) to approximate this AR(1) process on the discrete grid Γ^A. We use a grid of 101 points representing five standard deviations of the process A. Our price grid of 501 logarithmically-spaced points extends 10% past the prices that would be chosen at the highest and lowest values of A if prices were fully flexible. This results in price steps of 0.28% in our baseline estimate.

where ε_{it}^{a} is a mean-zero, normal, *iid* shock. The only free parameters in the productivity process are ρ and σ_{ε}^{2}, the variance of ε_{it}^{a}. The adjustment process has two free parameters, α and ζ, in the function class we have imposed, $\lambda(d) = d^{\zeta}/(\alpha^{\zeta} + d^{\zeta})$.

Our utility specification makes the steady state calculation especially simple, because for given parameters it can be reduced to a fixed point problem in the real price level p. Guessing p permits us to invert the Euler equation (19) to calculate $C = \left(\frac{1-\beta\mu}{p}\right)^{1/\gamma}$, and we can then calculate the wage from (18) as $w = \xi p C^{\gamma}$. This gives us enough information to construct the matrix $\mathbf{U}$, so we can solve the Bellman equation (17) and then find the steady state price distribution Ψ from (15) and (16). Knowing the price distribution Ψ, we can calculate the price level p from (22). Finding a fixed point in p thus allows us to construct the steady state equilibrium.

Table 1 reports our preferred estimate from Costain and Nakov (2008), (labelled SDSP, for 'state-dependent sticky prices'), together with data from several empirical papers.[6] It also reports a Calvo (1983) version of the model and a fixed menu cost version (as in Golosov and Lucas 2007). These alternative versions use the same productivity process, while fixing the rate of price adjustment at its average in the data (the Calvo version) or calibrating the fixed menu cost to best fit the data. The SDSP calibration is chosen to match three moments: the median frequency of price adjustment, the median absolute price adjustment, and the fraction of price adjustments less than 5% in absolute value. As the table shows, the fixed menu cost version of the model fits the data poorly because it generates no small price adjustments. The Calvo model, instead, generates far too many small price adjustments. As our previous paper shows, this failure of the Calvo model can be corrected by choosing a more variable productivity process, but if so then the Calvo model leads to much larger welfare losses (the average welfare loss rises to over 3% of the median value of the firm.) By contrast, the average welfare loss in the SDSP calibration is only one half of one percent of the median value of the firm.

Figure 1 graphs a variety of objects that characterize the equilibrium. In the first plot we see the value function, as a function of prices and marginal cost (one over productivity); the lowest value occurs when the highest marginal cost is paired with the lowest price. The fourth and sixth plots show the distributions $\widetilde{\Psi}$ at the beginning of the period and Ψ at the time of production. The production distribution Ψ looks rather like a sail-backed dinosaur: the "sail" represents the mass of firms that have adjusted to the optimal price conditional on current productivity. At the beginning of the next period, this mass gets spread out by the productivity shock process, resulting in the smooth distribution $\widetilde{\Psi}$ seen in the fourth graph. Graphing the policy function in the eighth plot shows that the firm sets prices closer to the mean than would be the case under flexible prices, in anticipation of mean reversion of the technology process. The last graph shows the distribution of nominal price adjustments, which is mildly bimodal around zero, and resembles quite closely the distributions of supermarket price changes shown in Midrigan (2006), Figure 1.

Finally, it is helpful to consider the computational implications of the relatively large but infrequent price adjustments seen in the data. With a median absolute price change of 8%, a typical price movement by firms in our baseline SDSP simulation is a jump of 30 steps in the price grid Γ^{p}. Clearly then, at any point in time most firms lie many steps away from their optimal prices; the table shows that the typical deviation from the optimal price ranges from 3.3% (in terms of the median) to 5.4% (on average), depending on the model.[7] This suggests that constraining price adjustment to a finite grid is relatively unimportant both for price dynamics and for welfare

[6] The column marked "target" is a simple average of the numbers from the three empirical papers. The simulations reported here are the same as those shown in Table 2 of Costain and Nakov (2008).

[7] The reader might expect the median (or average) price adjustment to coincide with the median (or average) distance from the optimal price in the case of the Calvo model. The only reason they are not equal in the table is that the distribution of price adjustments is determined by the beginning-of-period distribution $\widetilde{\Psi}$, whereas we report the distance from the optimal price at the time of production. That is, we report the distance from the optimal price with respect to the distribution Ψ that pertains *after* adjustments have taken place.

analysis. We confirm this fact in Table 1 by recomputing the model (under the SDSP calibration) on a much coarser grid, with only 25 possible productivities (spanning ± 2.5 standard deviations instead of ± 5 standard deviations) and only 25 possible prices. Thus, in the coarser grid, each price step represents a 3.2% price change, instead of 0.32% in the previous calculation.

This dramatic coarsening of the grid has only minor consequences for the performance of the model. The statistic that changes most is the standard deviation of price adjustments, which rises from 10.4% to 11.2%. The other statistics are barely altered, including the welfare losses caused by price stickiness. Thus, computing the dynamics on a finite grid seems unimportant for the results, even when the grid is quite coarse. This is very helpful for our purposes, because it suggests that the more numerically challenging problem of characterizing the distributional dynamics can also be studied on a coarse grid.

5 Computing general equilibrium: dynamics

To characterize our model's distributional dynamics in general equilibrium under aggregate shocks, we implement the algorithm of Reiter (2006). Reiter's method recognizes that the large system of nonlinear equations we solved to calculate the general equilibrium steady state can also be interpreted as a system of nonlinear first-order autonomous difference equations that describe the dynamics of a grid-based approximation of our general equilibrium with aggregate shocks. In the absence of strong strategic complementarities or an inappropriate Taylor rule that might give rise to indeterminacy, such an equation system can be solved by perfectly standard linear simulation techniques. We will solve for the saddle-path stable solution of our linearized model using the QZ decomposition, following Klein (2000).

The crucial thing to notice about Reiter's method is that it combines linearity and nonlinearity in a way appropriate for the model at hand. In our model, idiosyncratic shocks are likely to be larger and more economically important for individual firms' decisions than aggregate shocks. This is true in many macroeconomic contexts (e.g. precautionary saving) and in particular Klenow and Kryvstov (2005), Golosov and Lucas (2007), and Midrigan (2006) argue that firms' pricing decisions appear to be driven primarily by idiosyncratic shocks. Therefore, to deal with large idiosyncratic shocks, we treat functions of idiosyncratic states in a fully nonlinear way, by calculating them on a grid. As we emphasized above, this grid-based solution can be regarded as a large system of nonlinear equations: separate equations for all grid points. By linearizing each of these equations with respect to the aggregate dynamics, we recognize that aggregate changes are unlikely to affect individual value functions in a strongly nonlinear way. That is, we are implicitly assuming that both money supply shocks μ and changes in the distributions Ψ and $\widetilde{\Psi}$ have sufficiently smooth effects on individual values that a linear treatment of these effects is sufficient.

Thus, we will write the general equilibrium dynamics as a system of difference equations. For parsimonious notation in this context, we indicate dependence on the aggregate state by time subscripts, instead of by writing endogenous variables as functions of Ω_t. We will see that the difference equation system is a straightforward generalization of the steady state equations from the previous section. First, the time t money growth process is $\mu_t = \widehat{\mu}\exp(z_t)$, where

$$z_t = \phi_z z_{t-1} + \epsilon_t^z \tag{23}$$

where ϵ_t^z is an *iid* normal shock with mean zero and standard deviation σ_z.

Second, the firms' Bellman equation can be written as a $\#^p \times \#^A$ matrix system of equations for each $(p^j, a^k) \in \Gamma$. Let $\mathbf{U}_t$ be the matrix of current profits, so that the (j,k) element of $\mathbf{U}_t$ is

$$u_t^{jk} \equiv \left(p^j - \frac{w_t}{a^k}\right) C_t \left(\frac{p^j}{p_t}\right)^{-\epsilon} \equiv \left(p^j - \frac{w(\widetilde{\Psi}_t)}{a^k}\right) C(\widetilde{\Psi}_t) \left(\frac{p^j}{p(\widetilde{\Psi}_t)}\right)^{-\epsilon} \tag{24}$$

Write the value function as a matrix $\mathbf{V}_t$, with (j,k) element equal to $v_t^{jk} \equiv v_t(p^j, a^k) \equiv v(p^j, a^k, \widetilde{\Psi}_t)$ for $(p^j, a^k) \in \Gamma$. We can write the Bellman equation as

Dynamic general equilibrium Bellman equation, matrix version:

$$\mathbf{V}_t = \mathbf{U}_t + \beta E_t \left\{ \frac{p_t u'(C_{t+1})}{p_{t+1} u'(C_t)} \mathbf{R}'_{t+1} * (\mathbf{V}_{t+1} + \mathbf{G}_{t+1}) * \mathbf{S} \right\} \tag{25}$$

All quantities in the Bellman equation are analogous to corresponding quantities in the steady state equilibrium. The matrix $\mathbf{G}_{t+1}$ is defined by

$$\mathbf{G}_{t+1} \equiv \lambda(\mathbf{D}_{t+1}/w_{t+1}). * \mathbf{D}_{t+1} \tag{26}$$

where the (l,m) element of $\mathbf{D}_{t+1}$ is

$$d_{t+1}^{lm} \equiv d_{t+1}(p^l, a^m) \equiv \max_{p'} v_{t+1}(p', a^m) - v_{t+1}(p^l, a^m) \tag{27}$$

The expectation E_t in the Bellman equation refers only to the effects of the time $t+1$ money shock μ_{t+1}, because the shocks and dynamics of the idiosyncratic state $(p^j, a^k) \in \Gamma$ are completely described by the matrices $\mathbf{R}'_{t+1}$ and $\mathbf{S}$. Note that $\mathbf{S}$ has no time subscript, and is exactly the same matrix described in the previous section. The Markov matrix $\mathbf{R}_{t+1}$ differs from the steady state matrix $\mathbf{R}$ only because in the fully dynamic equilibrium we must detrend by the realized money shock μ_{t+1} instead of trend money growth μ. The row n, column l element of $\mathbf{R}_{t+1}$, which we will call R_{t+1}^{nl}, is

$$R_{t+1}^{nl} = prob(\widetilde{p}_{i,t+1} = p^n | p_{it} = p^l, \mu_{t+1}) = \begin{cases} 1 & \text{if } \mu_{t+1} p^l \leq p^1 = p^n \\ \frac{\mu_{t+1} p^l - p^{n-1}}{p^n - p^{n-1}} & \text{if } p^1 < p^n = \min\{p \in \Gamma^p : p \geq \mu_{t+1} p^l\} \\ \frac{p^{n+1} - \mu_{t+1} p^l}{p^{n+1} - p^n} & \text{if } p^1 \leq p^n = \max\{p \in \Gamma^p : p < \mu_{t+1} p^l\} \\ 0 & \text{otherwise} \end{cases}$$

As for the dynamics of the distribution, the two steps are analogous to the steady state case:

$$\Psi_t = (\mathbf{E} - \lambda(\mathbf{D}_t/w_t)). * \widetilde{\Psi}_t + \mathbf{P}_t. * (\mathbf{E} * (\lambda(\mathbf{D}_t/w_t). * \widetilde{\Psi}_t)) \tag{28}$$

$$\widetilde{\Psi}_{t+1} = \mathbf{R}_{t+1} * \Psi_t * \mathbf{S}' \tag{29}$$

Matrix $\mathbf{P}_t$ is constructed from the policy function

$$p_t^{*k} \equiv p_t^*(a^k) \equiv p^*(a^k, \Omega_t) \tag{30}$$

in the same way as in the steady state. If $p^{l(k)}$ is the first price grid point greater than or equal to p_t^{*k}, then $\mathbf{P}_t$ takes value $\left(\frac{p_t^{*k} - p^{l(k)-1}}{p^{l(k)} - p^{l(k)-1}}\right)$ in row $l(k)$, column k; and value $\left(\frac{p^{l(k)} - p_t^{*k}}{p^{l(k)} - p^{l(k)-1}}\right)$ in row $l(k)-1$, column k; and is zero elsewhere.

Finally, the remaining equations that must be satisfied by the dynamic general equilibrium are

$$x'(N_t) = \frac{w_t}{p_t} u'(C_t) \tag{31}$$

$$1 - \frac{v'(1/p_t)}{u'(C_t)} = \beta E_t \left(\mu_{t+1} \frac{p_t u'(C_{t+1})}{p_{t+1} u'(C_t)} \right) \tag{32}$$

$$N_t = \sum_{j=1}^{\#^p} \sum_{k=1}^{\#^a} \Psi_t^{jk} C_t^j / A^k \tag{33}$$

$$C_t^j = (p^j/p_t)^{-\epsilon} C_t \tag{34}$$

$$p_t = \left[\sum_{j=1}^{\#^p} \sum_{k=1}^{\#^a} \Psi_t^{jk} \left(p^j\right)^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}} \tag{35}$$

5.1 Linearization

In Appendix B we further simplify this equation system for the case of linear labor disutility, $x(N) = \xi N$.[8] We show that in this case the system can be expressed in terms of the 'jump' variables $\mathbf{V}_t$, C_t, and p_t; the endogenous, predetermined state variable $\widehat{\Psi}_t \equiv \Psi_{t-1}$ (the lagged distribution of idiosyncratic states); and the exogenous state μ_t. All other variables can easily be eliminated from the system, leaving us with $\#^{TOT} \equiv 2\#^p\#^a+3$ nonlinear, first-order, autonomous difference equations governing the same number of processes. If we then collapse all the endogenous variables into a vector of the form

$$\overrightarrow{X}_t \equiv \left(vec\left(\mathbf{V}_t\right)', \quad C_t, \quad p_t, \quad vec\left(\widehat{\Psi}_t\right)' \right)'$$

then the entire system of expectational difference equations governing the equilibrium has the following form:

$$E_t \mathcal{F}\left(\overrightarrow{X}_{t+1}, \overrightarrow{X}_t, z_{t+1}, z_t\right) = 0 \tag{36}$$

where E_t is an expectation conditional on μ_t and all previous shocks. If we linearize system $\mathcal{F}$ with respect to all arguments by constructing the Jacobian matrices $\mathcal{A} \equiv D_{\overrightarrow{X}_{t+1}} \mathcal{F}$, $\mathcal{B} \equiv D_{\overrightarrow{X}_t} \mathcal{F}$, $\mathcal{C} \equiv D_{\mu_{t+1}} \mathcal{F}$, and $\mathcal{D} \equiv D_{\mu_t} \mathcal{F}$, then we obtain the system

$$E_t \mathcal{A} \overrightarrow{X}_{t+1} + \mathcal{B} \overrightarrow{X}_t + E_t \mathcal{C} \mu_{t+1} + \mathcal{D} \mu_t = 0 \tag{37}$$

This equation system has the form considered by Klein (2000), so we solve our model using his QZ decomposition method.[9]

6 Results: dynamics

6.1 Effects of money growth shocks

We now study the impulse response functions implied by money supply shocks in several versions of our model, as illustrated in Figures 3 and 4. The figures show the response to a 1% increase

[8] The assumption $x(N) = \xi N$ is not essential; the more general case with nonlinear labor disutility simply requires us to simulate a larger equation system that includes N_t.

[9] Alternatively, the equation system can be rewritten in the form of Sims (2001). We chose to implement the Klein method because it is especially simple and transparent to program.

in money supply growth in the SDSP calibration of our model, and compares it with the response in the Calvo and fixed menu cost specifications. Each specification is calculated starting from the ergodic distribution of prices and productivities associated with that specification (trend inflation is assumed to be 0.64% per quarter). Figure 3 shows impulse responses under the assumption that money supply growth is *iid*, while Figure 4 assumes money growth has monthly autocorrelation of 0.8 (0.51 at quarterly frequency). For numerical tractability we compute equilibrium on the coarse grid of 25 productivities and 25 price levels analyzed in Table 1 and Figure 2, which yields a distribution of price changes similar to that on a much finer grid.

The impulse responses show how an increase in money growth causes both inflation and consumption to rise. The response of our calibrated model (lines with disks) mostly lies between the responses seen in the Calvo (lines with squares) and fixed menu cost (lines with crosses) specifications. All three versions are simulated under the same parameters, and the same aggregate and idiosyncratic shock processes, changing only the specification of the adjustment probability function λ. As Golosov and Lucas (2007) have emphasized, prices adjust more quickly in the fixed menu cost specification, so there is a larger spike in the inflation rate and smaller, less persistent changes in real variables in that specification than in the others. This difference is especially pronounced under *iid* money growth, where the SDSP calibration resembles the Calvo specification more closely than the fixed menu cost specification. However, under the more realistic assumption of autocorrelated money growth, the three models differ much less. While the total response of inflation over time is necessarily larger with autocorrelated money, there are also much larger real effects in this case. Consumption and labor jump strongly on impact in all three specifications, though they are substantially more persistent in the Calvo specification than in the other two.

To better understand these differences in adjustment, we also show the impulse responses of some statistics related to the distributional dynamics. In Fig. 3, we see that following an uncorrelated 1% rise in the money supply, the fraction of firms adjusting rises by 2.5 percentage points on impact in the menu cost specification (from 10% to 12.5% monthly), compared with a rise of just 0.3% in the SDSP specification. With autocorrelated money growth (Fig. 4), the fraction of adjusters reacts substantially in the SDSP specification too (rising persistently by one percentage point), making the inflation adjustment more similar to that under fixed menu costs. In the Calvo case, of course, the fraction adjusting is unchanged, so all the change in inflation goes through an increase in the average price change. We also plot the response of price dispersion (the standard deviation of prices), which moves in opposite directions in the Calvo and menu cost frameworks. Price dispersion here is partly caused by the presence of idiosyncratic productivity shocks. But additional price dispersion, above that derived from the shocks themselves, causes inefficient variation in demand across goods, thus decreasing the consumption aggregate (3). In the Calvo model, price dispersion rises after a money growth shock, since those firms that are able to adjust their prices now choose a greater increase. In the menu cost model, instead, price dispersion falls, because adjustment speeds up substantially.[10] In the SDSP model, these two effects largely offset each other, so that price dispersion varies less; the overall effect is a 1% rise in price dispersion in the case of autocorrelated shocks.

We have also repeated these impulse response calculations starting from a 10% annual inflation rate, as in the late 1970s in the US. There is little change in the results (so they are not shown). A 1% money supply shock has a slightly stronger effect on inflation, and thus a slightly weaker effect on consumption, when the baseline inflation rate is higher.Also, higher inflation makes the responses of the three models slightly more similar.

In their paper, Golosov and Lucas (2007) show that money supply shocks calibrated to be

[10] This is similar to the effect of steady state inflation in the menu cost specification in Costain and Nakov (2008). In that paper, the standard deviation of price changes falls with inflation because it causes the probability of adjustment to increase. In the limit, as the probability of adjustment approaches one, differences in price adjustments reflect differences in idiosyncratic productivity only, with no remaining effect from price stickiness.

sufficiently large to match the standard deviation of inflation would explain a very small part of output fluctuations. But our impulse response calculations cast doubt on this claim, because the response of output to money is much smaller in the fixed menu cost model than it is in our preferred SDSP specification if shocks are uncorrelated, and moreover all the output responses are much larger when money shocks are correlated. Therefore, like Golosov and Lucas, we compare the output variability implied by all three versions of the model in Table 2. We set $\phi_z = 0.5$, and choose the standard deviation of the money shock for each version of the model so that each version matches 100% of observed US inflation volatility. We then calculate the implied variability of output. Remarkably, under the SDSP specification, these money shocks would explain essentially all US output fluctuations, so if anything, the effect of money supply shocks is too strong in this model. Under the Calvo specification, the calibrated money shocks would cause 181% of the output fluctuations observed in the US. With menu costs, the figure is much lower, slightly under 50%, but even this is much higher than Golosov and Lucas found, since they focused on the case of uncorrelated shocks. We also calculate a "Phillips curve" by regressing output on money growth; we obtain a coefficient of approximately one, much larger than that reported by Golosov and Lucas.

6.2 The role of the distribution

Figures 5 and 6 illustrate the distributional dynamics of the model, and some of the implied nonlinearities. Figure 5 shows transitional dynamics of the SDSP calibration of the model, starting from a variety of different initial conditions; Fig. 6 shows how the impulse response to a money supply shock changes when starting from different initial conditions.

Figure 5.1 shows the impact of a large monetary shock, but the calculation is carried out in a different way from the calculations above. Instead of starting at the steady state distribution and feeding in a money shock, we simply shift the distribution of real prices two grid points to the left. That is, we decrease real prices by 6.4%, which is equivalent to an uncorrelated increase in money supply growth by 6.4%. By calculating the effects of the shock in this way, we take nonlinearities into account, since our computational method allows full nonlinearity between one grid point and the next. Some of the impulse responses are proportional to those shown before; for example, the response of inflation in Fig. 5.1 is roughly six times larger than that shown for the SDSP calibration in the second panel of Fig. 2. But the fraction of firms adjusting increases in a more-than-linear way, rising more than tenfold, which makes sense since the value of adjustment should increase nonlinearly in the distance from the optimal price. Price dispersion also increases more than linearly, and labor, surprisingly, falls strongly on impact. Fig. 5.2 performs a similar exercise, shifting the distribution of productivities by two grid points, which is equivalent to an uncorrelated 6.4% increase in productivity. Inflation falls, consumption rises, and labor falls, as expected in a sticky-price model.

Figures 5.3 and 5.4 instead show the transitional dynamics starting from degenerate distributions. In Fig. 5.3, we graph the dynamics under the assumption that, for some reason, all prices are initially set to the mean price. Initially, therefore, the fraction of firms adjusting is 2.5 percentage points above its steady state, and price dispersion rapidly increases from far below its steady state. The degenerate initial price distribution is inefficient, so consumption also starts 1% below its steady state. Fig. 5.4 shows a related exercise: all firms start from the mean productivity and the mean price. Therefore this experiment can be seen as the effect of unexpectedly "turning on" the idiosyncratic shock process. In this case most firms initially feel no need to adjust their prices, until their productivity begins to drift away from its mean value. Therefore, the fraction of firms adjusting is initially below its steady state value, and price dispersion converges more slowly to steady state than it did in Fig. 5.3.

Figures 5.5 and 5.6 temporarily suppress the effects of frictions, in two ways. Fig. 5.5 studies the transition path starting from a distribution in which all firms have set their optimal flexible prices. So this experiment can be seen as the effect of unexpectedly "turning on" price stickiness. Unsurprisingly, as the second-to-last panel of Fig. 1 shows, firms choose more price dispersion under flexible prices; the pricing policy function is flatter under sticky prices. Therefore, when price stickiness is turned on, price dispersion gradually falls by 22%. The fraction of firms adjusting is initially below trend, since firms are nonetheless quite close to the prices they would prefer to choose under price stickiness. Also, the flexible prices are more efficient than the steady state distribution of sticky prices, so initially in this experiment consumption is above trend while labor is below. Fig. 5.6 is similar, but instead of starting all firms from their preferred flexible prices, we start them from their preferred sticky prices. This might be seen as the effect of "introducing the euro": it is as if all firms are forced to change prices at one point in time, taking into account the fact that their prices will be sticky in the future. Conditional on this one-time change, we see that the fraction of firms adjusting is thereafter below steady state, since they start out at their preferred sticky price. The one-time adjustment also allows them to "catch up" with past inflation, so inflation is thereafter below steady state. Price dispersion is also temporarily above trend. That is, because of price stickiness, prices in the steady state distribution vary less strongly with productivity than the sticky-price policy function itself does.

Since the value of adjustment varies greatly with initial conditions, the effects of a monetary policy shock will also vary. Fig. 6 shows an example. For the SDSP calibration, it shows the effect of an uncorrelated 1% increase in money supply, either starting from the steady state distribution (as seen previously in Fig. 3), or with all firms starting from the optimal sticky price. In both cases, we aim to show only the effect of the money supply shock itself, so to graph the blue circled curve in Fig. 6 we first compute a path starting from optimal sticky prices plus a money shock; then we compute a path starting from optimal sticky prices without a money shock (as seen in Fig. 5.6), and then we take the difference between the two. Several aspects of the impulse response are unchanged; in particular, the response of real consumption to the money shock is not altered by the starting distribution. A few more firms adjust and the average price change is slightly larger when starting from optimal sticky prices. But some impulse responses differ more: a money shock increases price dispersion twice as much when starting from the optimal sticky price than it does starting from the steady state distribution, and labor must also increases substantially more in this case.

7 Conclusions

We have computed the impact of money supply shocks in a model of state-dependent pricing, characterizing the dynamics of the distribution of prices while allowing this distribution to have general equilibrium effects on firms' and consumers' decisions. State-dependent pricing somewhat weakens the real effects of monetary shocks, compared with the Calvo model. Nonetheless, these real effects remain important under the calibration that best fits microeconomic pricing data, especially when money supply shocks are autocorrelated (0.8 monthly), as they are in reality. With this autocorrelation, the initial effect of money supply on output in our state-dependent model is approximately the same as in the Calvo model, with half the persistence.

8 Appendix A: Detrending

Suppose the model can be rewritten in real terms by deflating all prices by the nominal money stock, defining $p_t \equiv P_t/M_t$, $p_{it} \equiv P_{it}/M_t$, and $w_t \equiv W_t/M_t$. Given the nominal distribution $\Phi_t(P_i, A_i)$ and the money stock M_t, we denote by $\Psi_t(p_i, A_i)$ the distribution over real production prices $p_{it} \equiv P_{it}/M_t$. Likewise, let $\widetilde{\Psi}_t(p_i, A_i)$ be the distribution of real beginning-of-period prices

$\widetilde{p}_{it} \equiv \widetilde{P}_{it}/M_t$, in analogy to the beginning-of-period distribution of nominal prices $\widetilde{\Phi}_t(\widetilde{P}_i, A_i)$. If it is true that the model can be rewritten in real terms, the distribution $\widetilde{\Psi}_t(\widetilde{p}_i, A_i)$ is a sufficient aggregate state variable to determine real quantities. That is, to describe the real equilibrium, it is not necessary to condition functions on M_t, only on the distribution $\widetilde{\Psi}_t(\widetilde{p}_i, A_i)$.

Therefore, if there exists a real equilibrium, the real aggregate functions can be written in terms of $\widetilde{\Psi}_t$ only, and must satisfy $C_t = C(\widetilde{\Psi}_t) = C(\Omega_t)$, $N_t = N(\widetilde{\Psi}_t) = N(\Omega_t)$, $p_t = p(\widetilde{\Psi}_t) = P(\Omega_t)/M_t$ and $w_t = w(\widetilde{\Psi}_t) = W(\Omega_t)/M_t$. Deflating from one period to the next will depend on the growth rate of money supply from one period to the next (but not on the level of the money supply). Thus the stochastic discount factor will be

$$q(\mu_{t+1}, \widetilde{\Psi}_t, \widetilde{\Psi}_{t+1}) = \beta \frac{M_t p(\widetilde{\Psi}_t) u'(C(\widetilde{\Psi}_{t+1}))}{M_{t+1} p(\widetilde{\Psi}_{t+1}) u'(C(\widetilde{\Psi}_t))} = \beta \mu_{t+1} \frac{p(\widetilde{\Psi}_t) u'(C(\widetilde{\Psi}_{t+1}))}{p(\widetilde{\Psi}_{t+1}) u'(C(\widetilde{\Psi}_t))}$$

The "real" value function v should likewise be the nominal value function, divided by the current money stock, and should be written as a function of real prices. Therefore we have

$$V(P_{it}, A_{it}, \Omega_t) = M_t v\left(\frac{P_{it}}{M_t}, A_{it}, \widetilde{\Psi}_t\right) = M_t v\left(p_{it}, A_{it}, \widetilde{\Psi}_t\right)$$

If a firm's nominal price at time t is P_{it}, then the value of maintaining this price fixed at time $t+1$ can be written in nominal or real terms as

$$\begin{aligned} V(P_{it}, A_{i,t+1}, \Omega_{t+1}) &= M_{t+1} v\left(\frac{P_{it}}{M_{t+1}}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) \\ &= M_{t+1} v\left(\frac{M_t p_{it}}{M_{t+1}}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) = M_{t+1} v\left(\mu_{t+1} p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) \end{aligned}$$

Likewise, if for any time t nominal price P_{it} we define

$$D(P_{it}, A_{i,t+1}, \Omega_{t+1}) \equiv \max_{P'} V(P', A_{i,t+1}, \Omega_{t+1}) - V(P_{it}, A_{i,t+1}, \Omega_{t+1})$$

$$G\left(P_{it}, A_{i,t+1}, \Omega_{t+1}\right) \equiv \lambda\left(\frac{D\left(P_{it}, A_{i,t+1}, \Omega_{t+1}\right)}{W(\Omega_{t+1})}\right) D\left(P_{it}, A_{i,t+1}, \Omega_{t+1}\right)$$

then we can define

$$D(P, A_{i,t+1}, \Omega_{t+1}) \equiv M_{t+1} d\left(\frac{P}{M_{t+1}}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) = M_{t+1} d\left(\mu_{t+1} p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)$$

$$G(P, A_{i,t+1}, \Omega_{t+1}) \equiv M_{t+1} g\left(\frac{P}{M_{t+1}}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) = M_{t+1} g\left(\mu_{t+1} p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)$$

Using this deflated notation, we can rewrite the Bellman equation as

$$M_t v(p_{it}, A_{it}, \widetilde{\Psi}_t) = M_t \left(p_{it} - \frac{w(\widetilde{\Psi}_t)}{A_{it}}\right)\left(\frac{p_{it}}{p(\widetilde{\Psi}_t)}\right)^{-\epsilon} C(\widetilde{\Psi}_t) +$$

$$\beta E_t \left\{ \tfrac{M_t p(\widetilde{\Psi}_t) u'(C(\widetilde{\Psi}_{t+1}))}{M_{t+1} p(\widetilde{\Psi}_{t+1}) u'(C(\widetilde{\Psi}_t))} \; M_{t+1} \left[v\left(\mu_{t+1} p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) + g\left(\mu_{t+1} p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)\right] \Big| \, A_{it}, \widetilde{\Psi}_t \right\}$$

Note that M_t cancels from both sides of the equation, and M_{t+1} cancels inside the expectation. Therefore we obtain

$$v(p_{it}, A_{it}, \widetilde{\Psi}_t) = \left(p_{it} - \frac{w(\widetilde{\Psi}_t)}{A_{it}}\right)\left(\frac{p_{it}}{p(\widetilde{\Psi}_t)}\right)^{-\epsilon} C(\widetilde{\Psi}_t) +$$

$$\beta E_t \left\{ \frac{p(\widetilde{\Psi}_t)u'(C(\widetilde{\Psi}_{t+1}))}{p(\widetilde{\Psi}_{t+1})u'(C(\widetilde{\Psi}_t))} \left[v\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) + g\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)\right] \middle| A_{it}, \widetilde{\Psi}_t \right\}$$

where

$$g\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) \equiv \lambda\left(\frac{d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)}{w(\widetilde{\Psi}_{t+1})}\right) d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)$$

$$d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right) \equiv \max_{p'} v(p', A_{t+1}(i), \widetilde{\Psi}_{t+1}) - v\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)$$

Let $p^*(A_{i,t+1}, \widetilde{\Psi}_{t+1})$ denote the optimal choice in the maximization problem above. Taking into account the fact that the firm starts period $t+1$ with the eroded price $\widetilde{p}_{i,t+1} \equiv \mu_{t+1}p_{it}$, the price process is

$$p_{i,t+1} = \begin{cases} p(A_{i,t+1}, \widetilde{\Psi}_{t+1}) & \text{with } prob = \lambda\left(\frac{d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)}{w(\widetilde{\Psi}_{t+1})}\right) \\ \mu_{t+1}p_{it} & \text{with } prob = 1 - \lambda\left(\frac{d\left(\mu_{t+1}p_{it}, A_{i,t+1}, \widetilde{\Psi}_{t+1}\right)}{w(\widetilde{\Psi}_{t+1})}\right). \end{cases}$$

In other words, when the firm's nominal price is not adjusted at time $t+1$, its real price is deflated by factor μ_{t+1}.

9 Appendix B. Implementing the Klein (2000) method

Under linear labor disutility, $x(N) = \xi N$, general equilibrium dynamics are summarized by the following equations:

$$\mathbf{V}_t = \mathbf{U}_t + \beta E_t \left\{ \frac{p_t u'(C_{t+1})}{p_{t+1}u'(C_t)} \mathbf{R}'_{t+1} * (\mathbf{V}_{t+1} + \mathbf{G}_{t+1}) * \mathbf{S} \right\}$$

$$1 - \frac{v'(1/p_t)}{u'(C_t)} = \beta E_t \left(\mu_{t+1} \frac{p_t u'(C_{t+1})}{p_{t+1}u'(C_t)} \right)$$

$$p_t^{1-\epsilon} = \sum_{j=1}^{\#^p} \sum_{k=1}^{\#^a} E_t\left(\widehat{\Psi}_{t+1}^{jk}\right)\left(p^j\right)^{1-\epsilon}$$

$$E_t\left(\widehat{\Psi}_{t+1}\right) = (\mathbf{E} - \lambda(\mathbf{D}_t/w_t)). * \widetilde{\Psi}_t + \mathbf{P}_t. * (\mathbf{E} * (\lambda(\mathbf{D}_t/w_t). * \widetilde{\Psi}_t))$$

$$z_{t+1} = \phi_z z_t + \epsilon^z_{t+1} \tag{38}$$

where $\widehat{\Psi}_t \equiv \Psi_{t-1}$ is the lagged distribution of idiosyncratic states from the time of production.

These equations form a system of $\#^{TOT} \equiv 2\#^p\#^a + 3$ first-order expectational difference equations in the jump variables $\mathbf{V}_t$, C_t, and p_t, the endogenous state variables $\widehat{\Psi}_t$, which are predetermined in the sense that $\widehat{\Psi}_{t+1} = E_t\widehat{\Psi}_{t+1}$, and the exogenous state process z_t. To see this, note that we can construct μ_t, and μ_{t+1}, and thus $\mathbf{R}_t$ and $\mathbf{R}_{t+1}$, from z_t and z_{t+1}. Given

$\mathbf{R}_t$, we can construct $\widetilde{\Psi}_t = \mathbf{R}_t * \widehat{\Psi}_t * \mathbf{S}'$ from $\widehat{\Psi}_t$. Under linear labor disutility, we can construct $w_t = \xi p_t / u'(C_t)$, which gives us all the information needed to construct $\mathbf{U}_t$. Finally, given $\mathbf{V}_t$ and $\mathbf{V}_{t+1}$ we can construct $\mathbf{D}_t$ and $\mathbf{D}_{t+1}$ and thus $\lambda(\mathbf{D}_t/w_t)$ and $\mathbf{G}_{t+1}$. Thus we have all the quantities that appear in the equation system, which can be summarized as a nonlinear system of the form

$$E_t \mathcal{F}\left(\overrightarrow{X}_{t+1}, \overrightarrow{X}_t, z_{t+1}, z_t\right) = 0 \tag{39}$$

using the notation

$$\overrightarrow{X}_t \equiv \left(vec\,(\mathbf{V}_t)', \quad C_t, \quad p_t, \quad vec\left(\widehat{\Psi}_t\right)'\right)'$$

By linearization we then obtain a system of the form (36), which Klein's (2000) method is designed to solve.

10 Appendix C. Reduction of dimension

The linearization of (36) can be performed numerically by considering small deviations in all elements of $\overrightarrow{X}_t$, $\overrightarrow{X}_{t+1}$, z_t and z_{t+1},and the software of Klein (2000) or Sims (2001) can then be applied directly to solve the linearized system. The simulations reported in the paper apply this method to the model approximated on a coarse grid with 25 possible shocks A and 25 possible prices p. As we argued in the text, using a coarse grid is unlikely to be economically important in the context of our model.

In future calculations we intend to check this assertion by combining a fine representation of the steady state with a coarse representation of the dynamics. There are two reasons why the dynamic calculation suffers from a severe curse of dimensionality. First, the Jacobians $\mathcal{A}$ and $\mathcal{B}$ have size $\#^{TOT} \times \#^{TOT}$ (so that if there are hundreds of gridpoints, the Jacobians will contain millions of elements) and are not sparse. The other major problem is that the computation time for QZ decomposition is cubic in the number of equations $\#^{TOT}$. Both of these issues make the system too large for ordinary desktop computers when a fine grid is used in the second step of Reiter's algorithm.

However, the fact that firms' decisions are likely to depend less strongly on aggregate variations than on idiosyncratic shocks again helps us out, because it means that the grid used to characterize the steady state response to idiosyncratic shocks need not be the same as the grid used to characterize the deviation of the value function from the steady state value function. Thus, we will calculate the steady state value function on the fine policy grid $\Gamma^p \times \Gamma^A$, but perturb it on a coarser grid $\gamma^p \times \gamma^A$, where $\gamma^p \subset \Gamma^p$ and $\gamma^A \subset \Gamma^A$. We can then treat the value function as the sum of the steady state value plus a deviation which is piecewise linear over the coarser grid $\gamma^p \times \gamma^A$. So for example, if γ^p and γ^A consist of each tenth grid point of Γ^p and Γ^A, then the number of equations and variables related to the value function in the linearized system (37) is reduced by a factor of one hundred.[11] Similar tricks can be used to reduce the dimensionality of the linearized distributional dynamics.

[11] An alternative way of understanding this procedure is that it is a collocation method that imposes piecewise linear form on the deviations of the value function from the steady state value function, and treats each tenth point in Γ^p as a collocation point.

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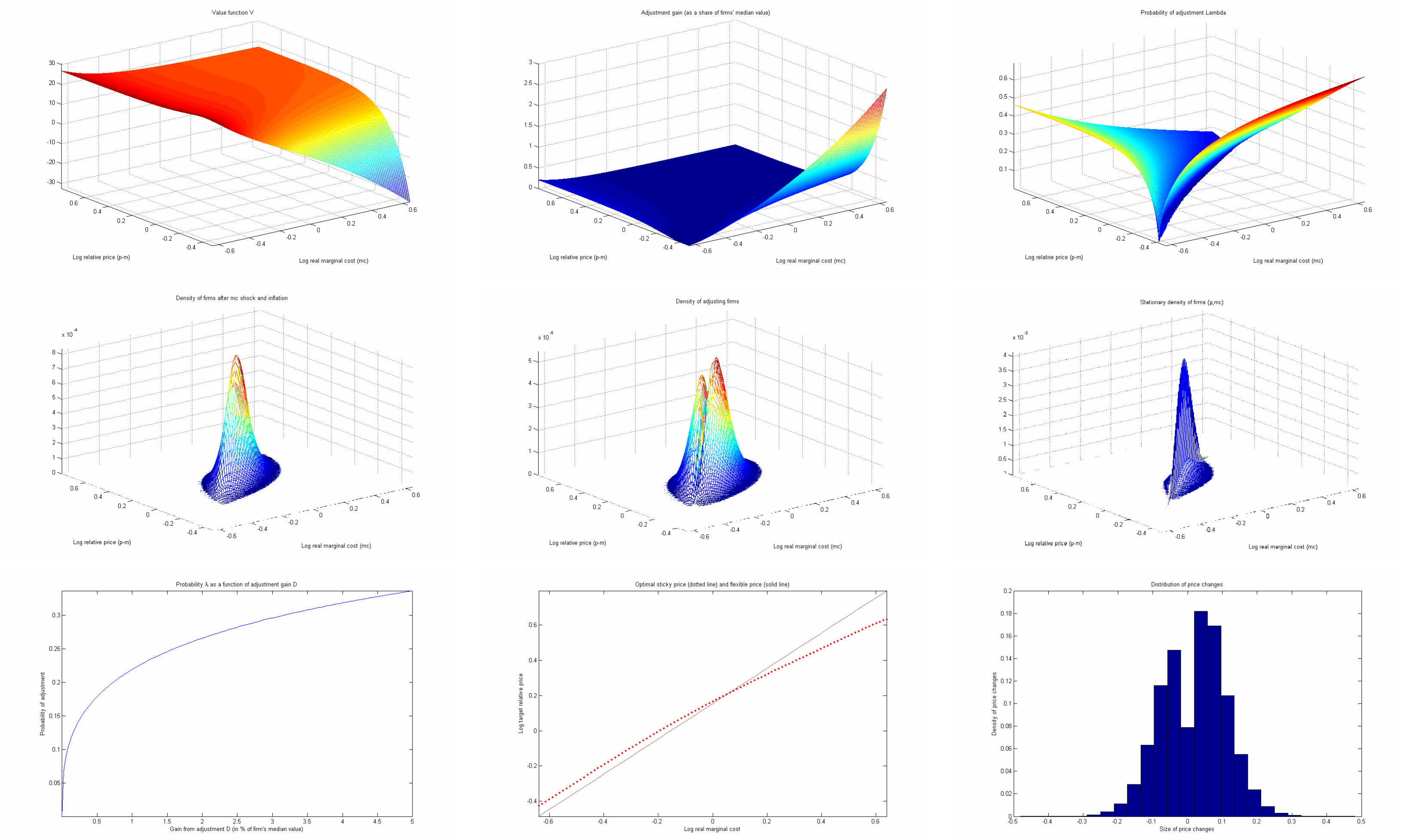
Figure 1. Steady-state value function, distributions, and related objects (SDSP)



Simulations from SDSP model, calculated on fine grid. First line: value V, loss D, and adjustment probability λ, as functions of real price p and cost shock A^{-1}. Second line: beginning of period distribution, adjustment distribution, and distribution at time of production. Third line: adjustment probability λ as a function of D, policy function, and distribution of monthly non-zero price changes.

Table 1. Steady-state results

Menu cost: $\left(\sigma_\varepsilon^2, \rho\right)$ same as SDSP; $\kappa = 0.03$
Calvo: $\left(\sigma_\varepsilon^2, \rho\right)$ same as SDSP; $\lambda = 0.10$
SDSP: $\left(\sigma_\varepsilon^2, \rho, \varsigma, \alpha\right) = (0.0021, 0.9351, 0.3675, 5.7347)$, computed with 501 or 25 grid points for p

Model:	MC	Calvo	SDSP 501 pt	SDSP 25 pt	Target	Dataset NS	Dataset KK	Dataset VM
Frequency of price changes	10.3	10	10.1	10.2	10.0	10.0		
Mean absolute price change	12.3	5.64	8.9	9.1	8.8		8.5	9.1
Median absolute price change	11.9	4.25	7.9	8.1	8.5	8.5		
Mean price increase	11.9	6.47	9.3	9.6	9.5		9.5	
Median price increase	11.7	4.89	8.3	8.8	7.3	7.3		
Std of price changes	12.4	7.28	10.4	11.2	11.8			11.8
Percent of price increases	60	60	59	58	62	67	61	58
Percent of price changes <5% in abs value	0.03	56.7	29	28	36		40	32
Flow of menu cost as % of firms' revenues	0.72				0.5		0.5	
Median distance from optimal price	3.29	3.65	3.90	3.86				
Average distance from optimal price	3.80	5.09	5.29	5.42				
Median loss as % of median value	0.021	0.083	0.043	0.043				
Average loss as % of median value	0.040	0.45	0.16	0.17				
Std of loss as % of median value	0.045	1.25	0.34	0.34				

Note: All statistics refer to regular consumer price changes excluding sales, and are stated in percent. "Target" is a simple average of the corresponding statistics reported by Nakamura and Steinsson (NS), Klenow and Kryvtsov (KK) and Virgiliu Midrigan's (VM) averages for two supermarket chains.

Table 2. Variance decomposition and Phillips curves of alternative models
Note: $\phi_z = 0.5$; $\sigma_z^2 = 1.98e - 5$

	Data	SDSP II	Calvo	Menu cost
Std of money shock (x100)		0.45	0.80	0.22
Std of quarterly inflation (x100)	0.246	0.246	0.246	0.246
Share explained by μ shock alone		100%	100%	100%
Std of quarterly output growth (x100)	0.51	0.52	0.92	0.25
Share explained by μ shock alone		102%	181%	49%
Slope coefficient in PC regression		1.026	1.16	0.6987
(standard error)		(0.015)	(0.034)	(0.012)
R2		0.74	0.44	0.9742

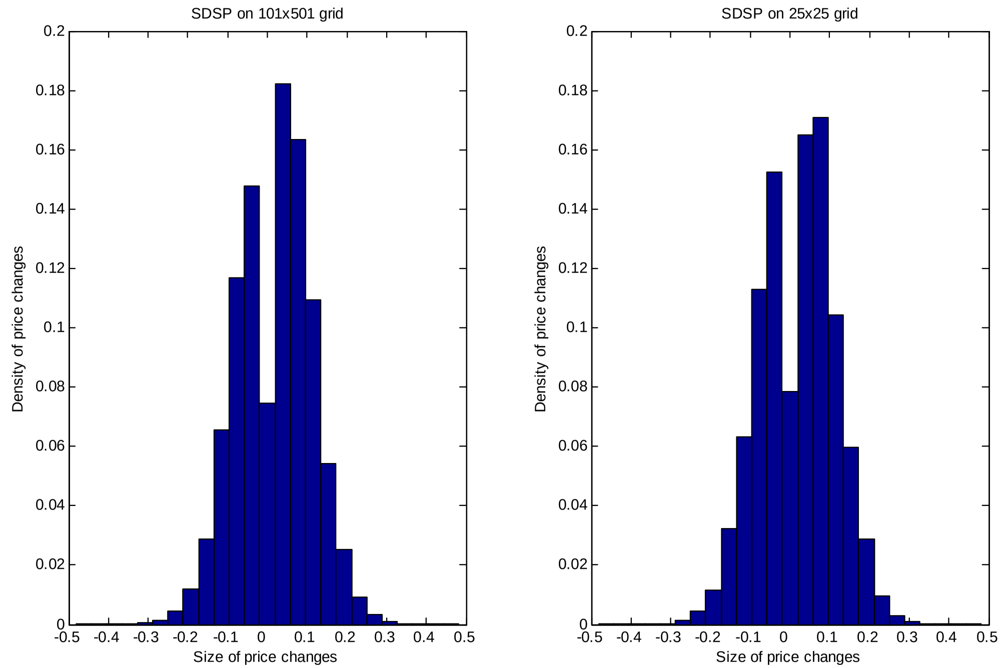


Figure 2. Steady-state distribution of nonzero price adjustments: fine and coarse grids

Horizontal axis: log price change. Vertical axis: frequency

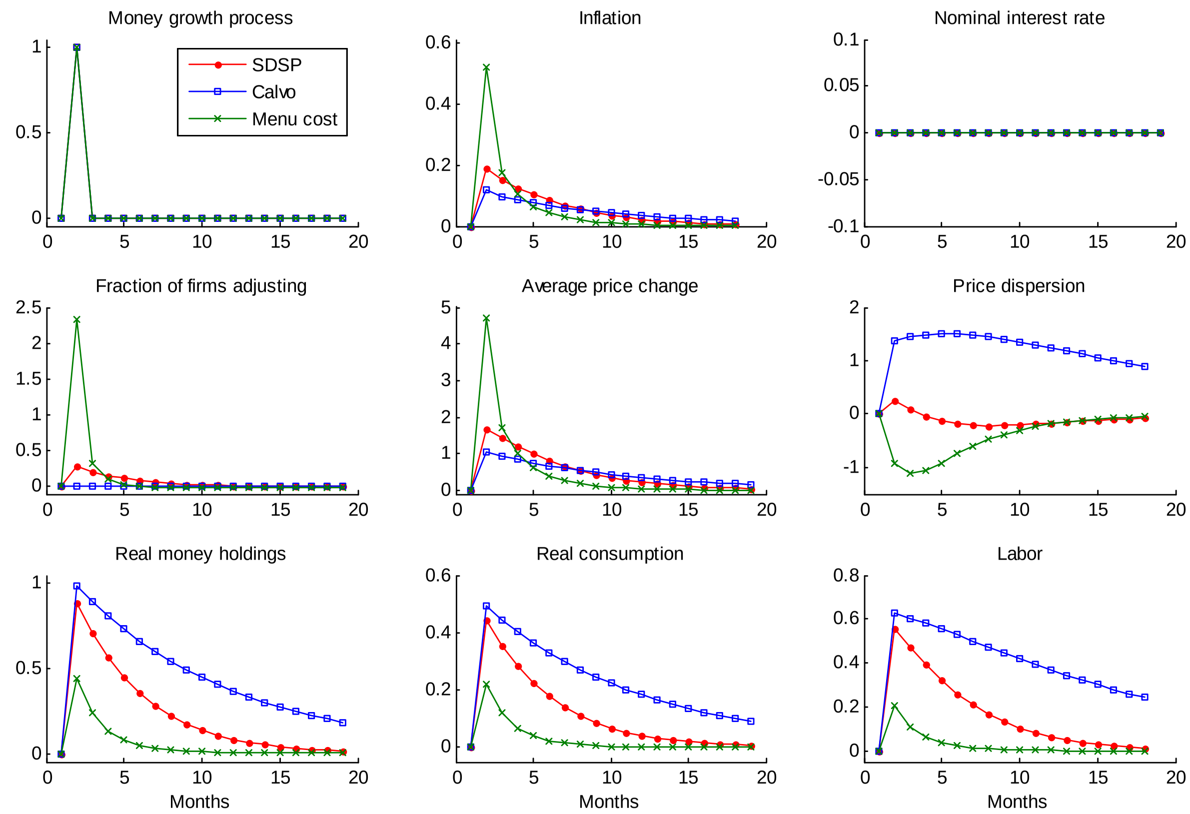


Figure 3. Impulse-response functions.

Panels 1-5: difference from steady state in percentage points. Panels 6-9: percent deviation from steady state.

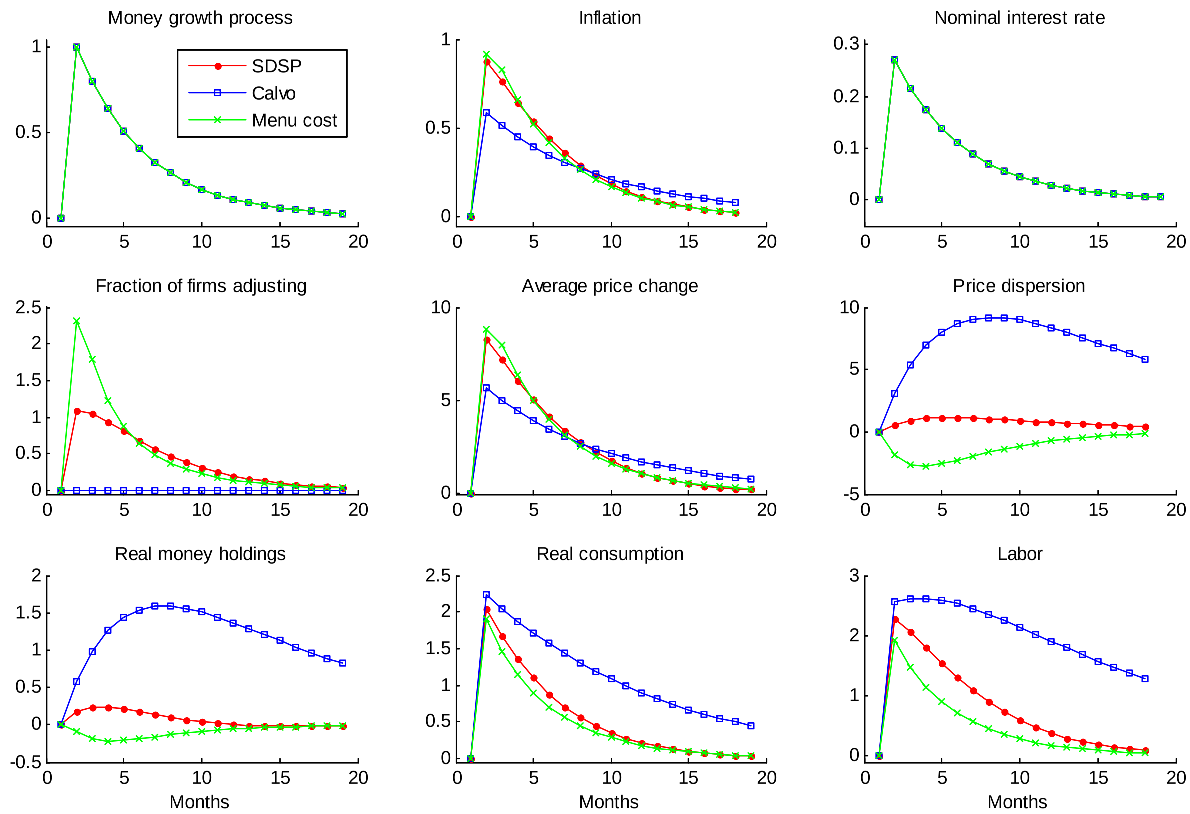


Figure 4. Impulse-response functions.

Panels 1-5: difference from steady state in percentage points. Panels 6-9: percent deviation from steady state.

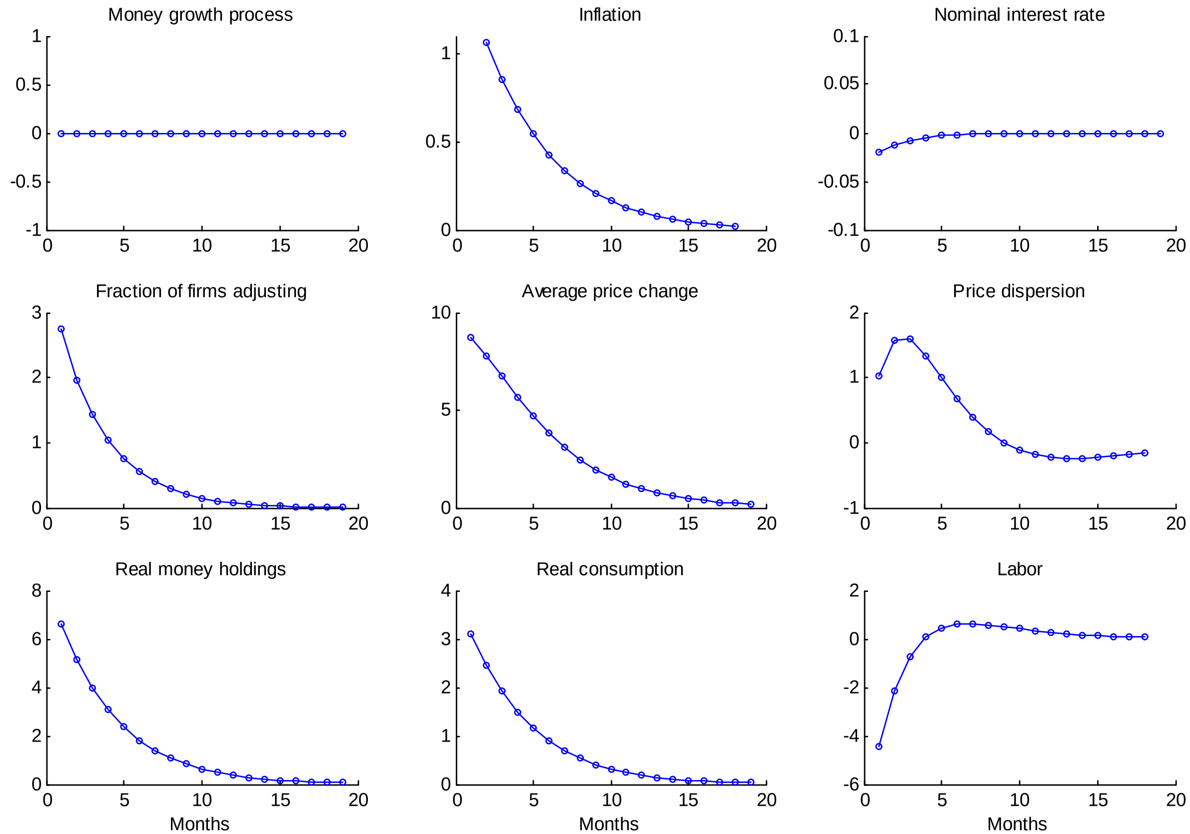


Figure 5.1 Transitional dynamics from shifted distribution: 6.4% increase in money.

Panels 1-5: difference from steady state in percentage points. Panels 6-9: percent deviation from steady state.

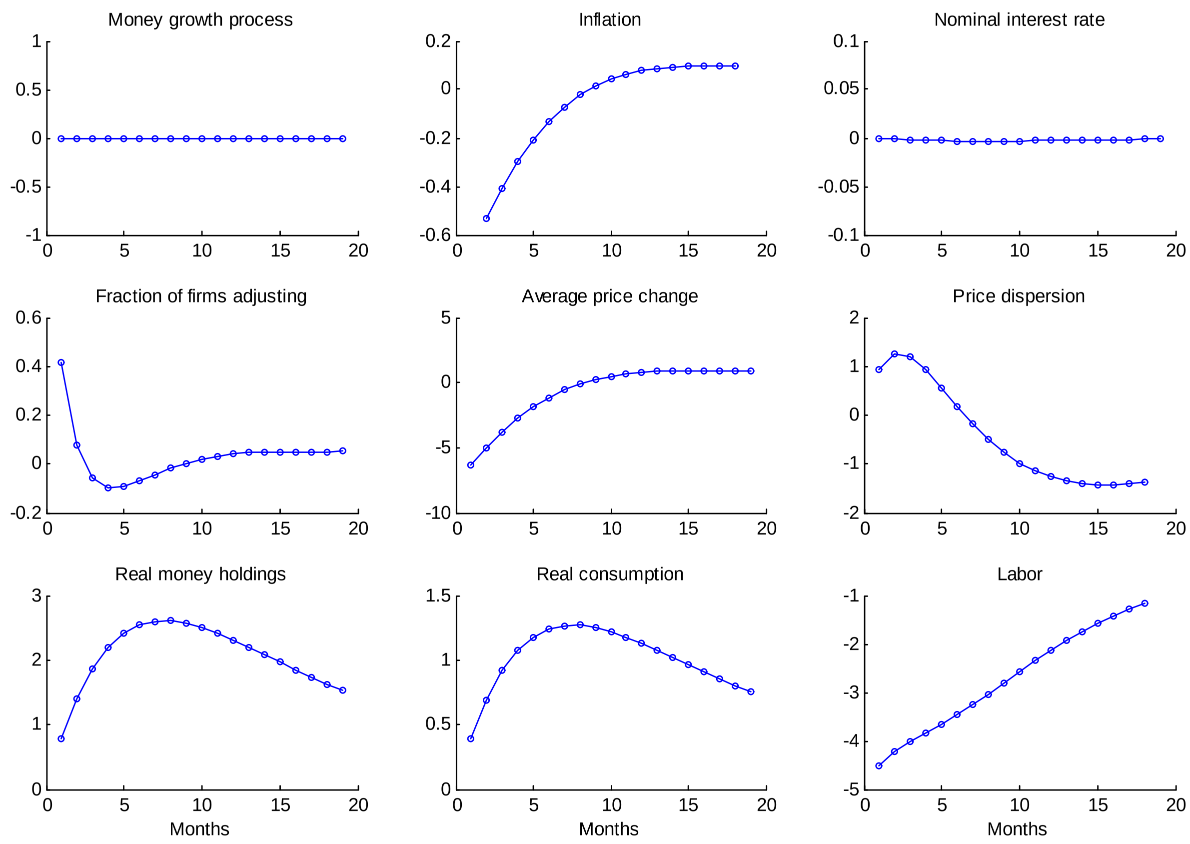


Figure 5.2 Transitional dynamics from shifted distribution: 6.4% increase in aggregate productivity.
Panels 1-5: difference from steady state in percentage points. Panels 6-9: percent deviation from steady state.

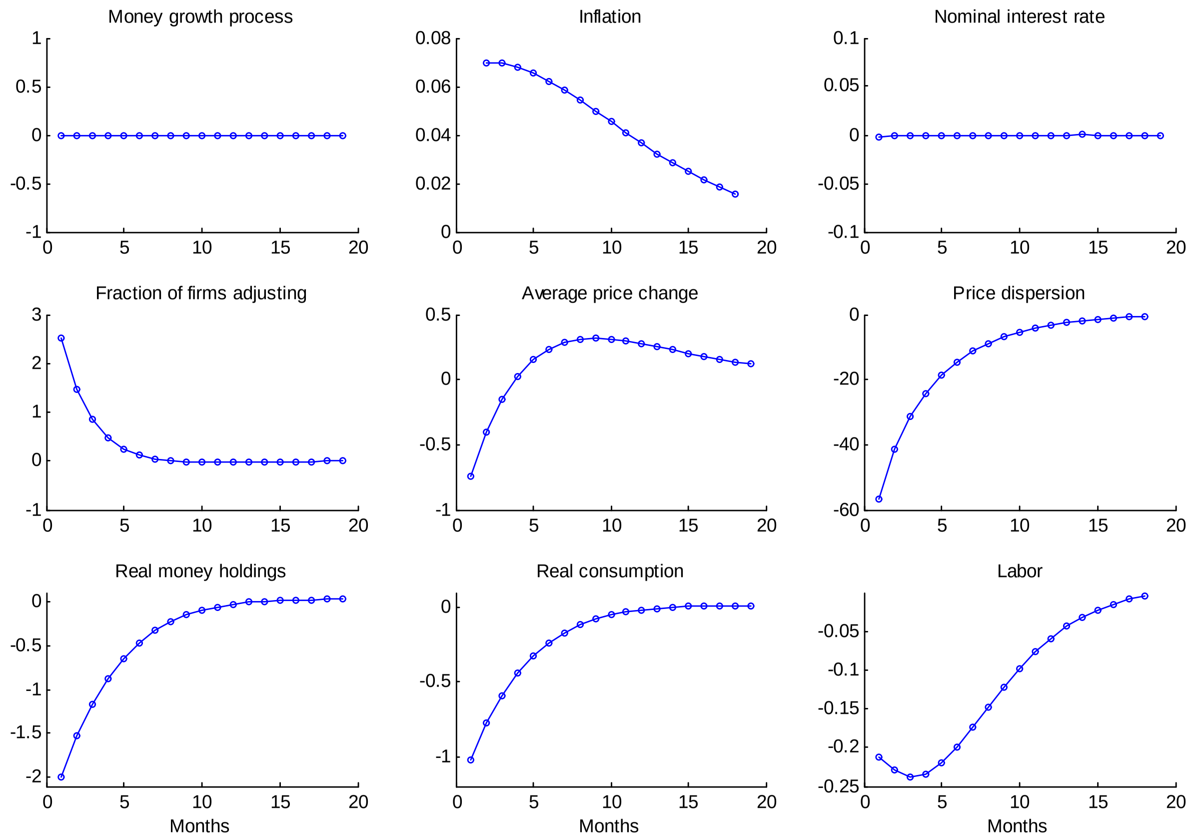


Figure 5.3 Transitional dynamics from shifted distribution: all firms from same price.

Panels 1-5: difference from steady state in percentage points. Panels 6-9: percent deviation from steady state.

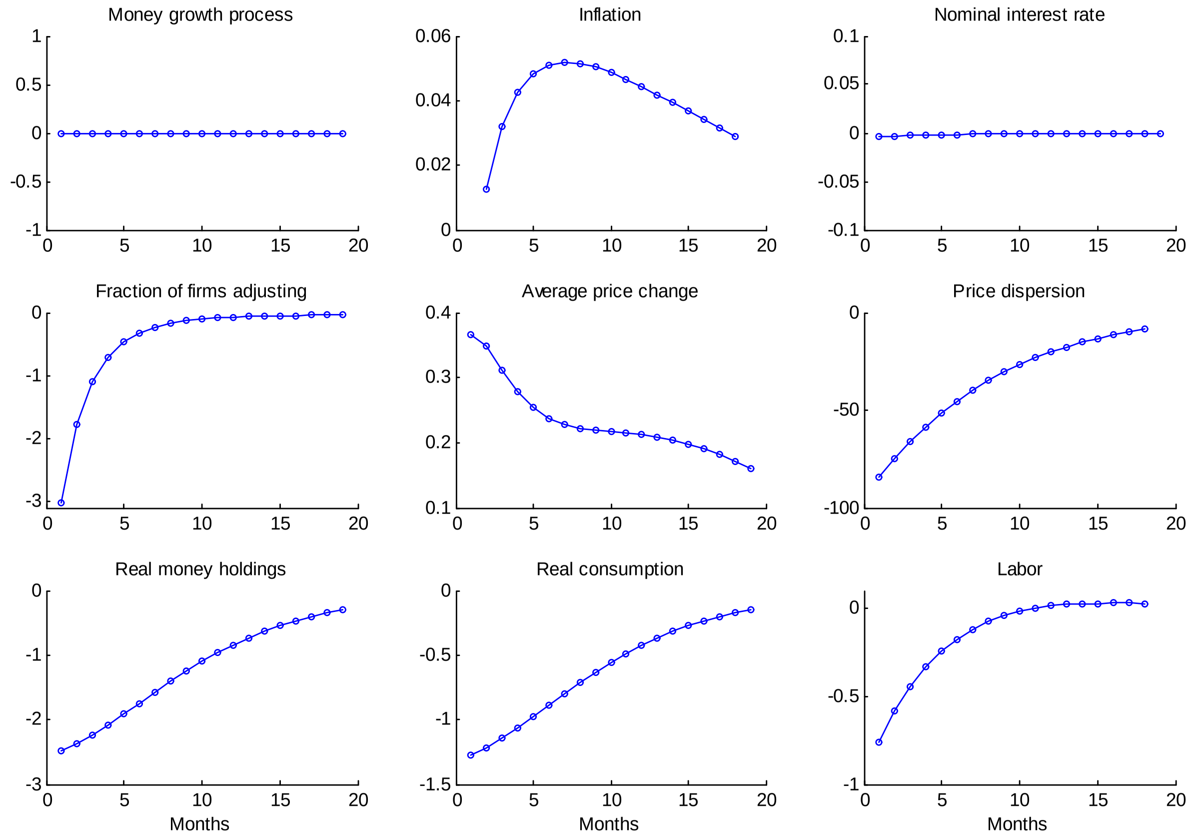


Figure 5.4 Transitional dynamics from shifted distribution: all firms from same price and same cost.
Panels 1-5: difference from steady state in percentage points. Panels 6-9: percent deviation from steady state.

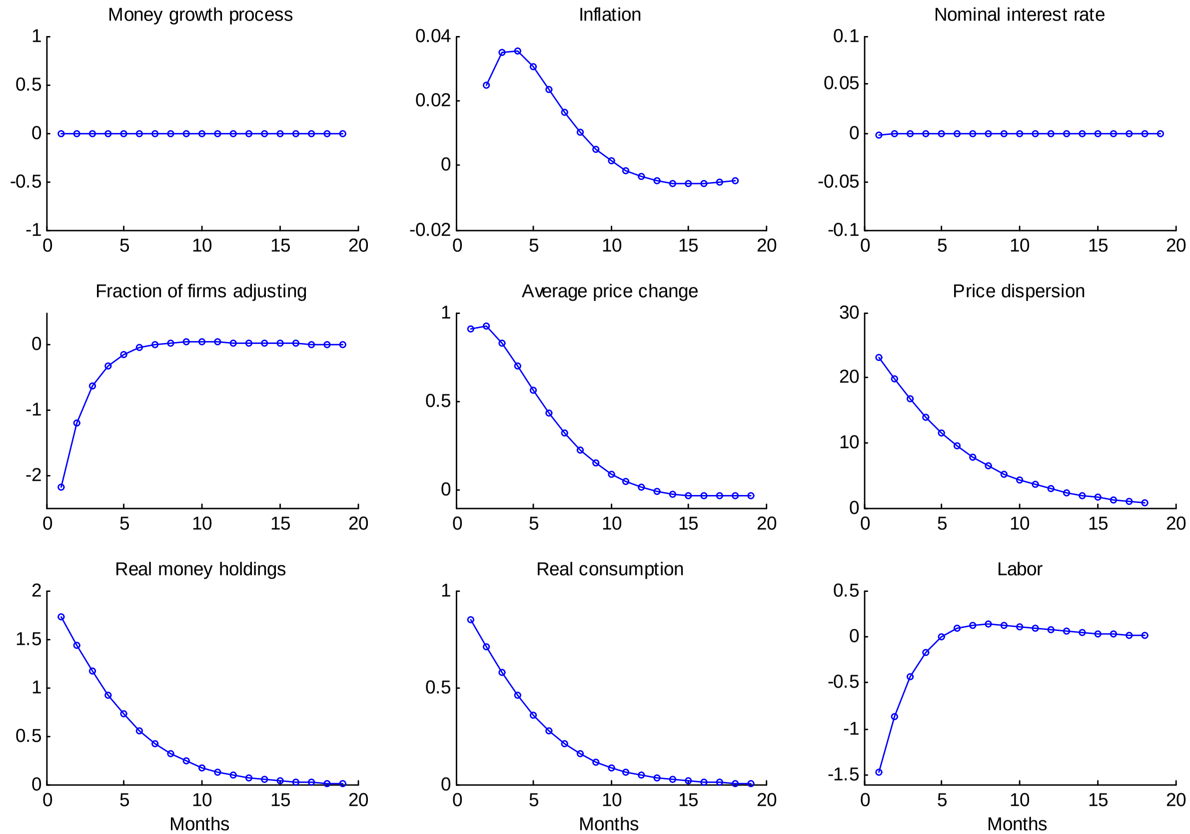


Figure 5.5 Transitional dynamics from shifted distribution: all firms from optimal flexible price.
Panels 1-5: difference from steady state in percentage points. Panels 6-9: percent deviation from steady state.

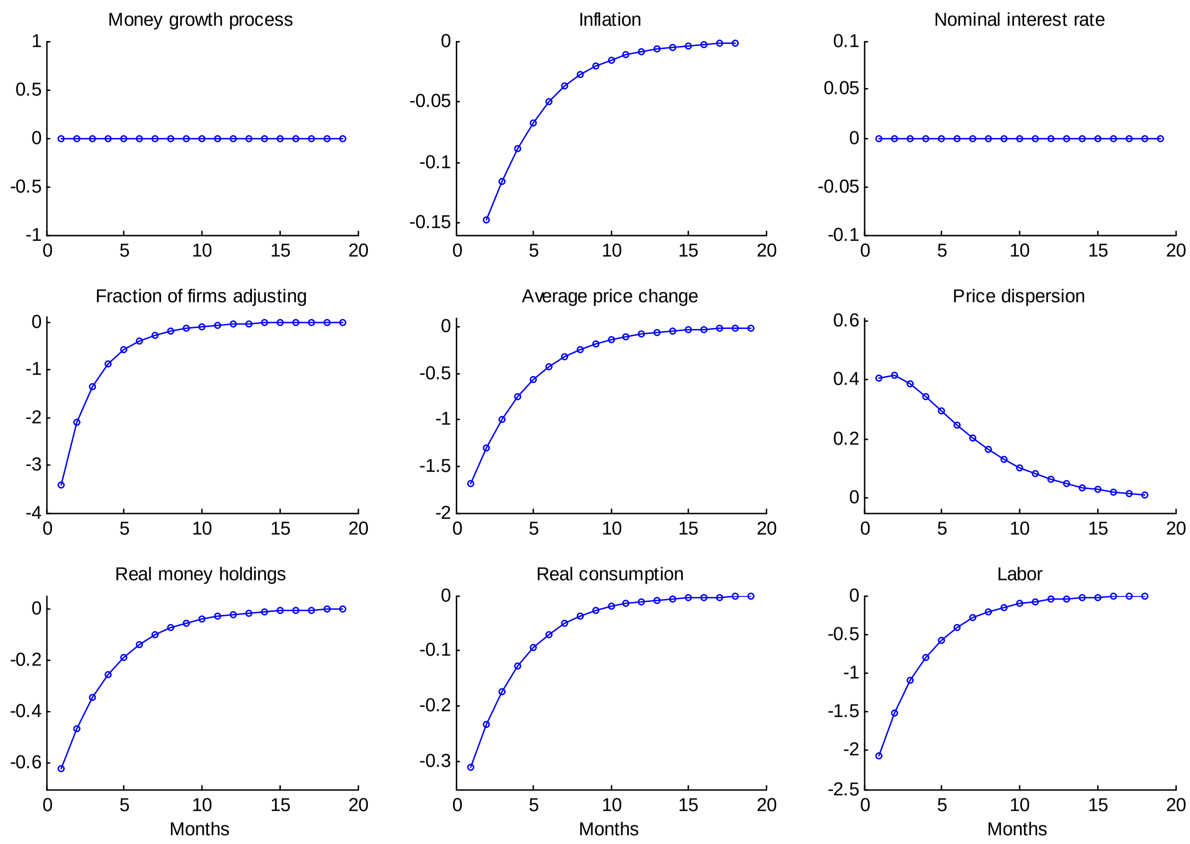


Figure 5.6 Transitional dynamics from shifted distribution: all firms from optimal sticky price.
Panels 1-5: difference from steady state in percentage points. Panels 6-9: percent deviation from steady state.

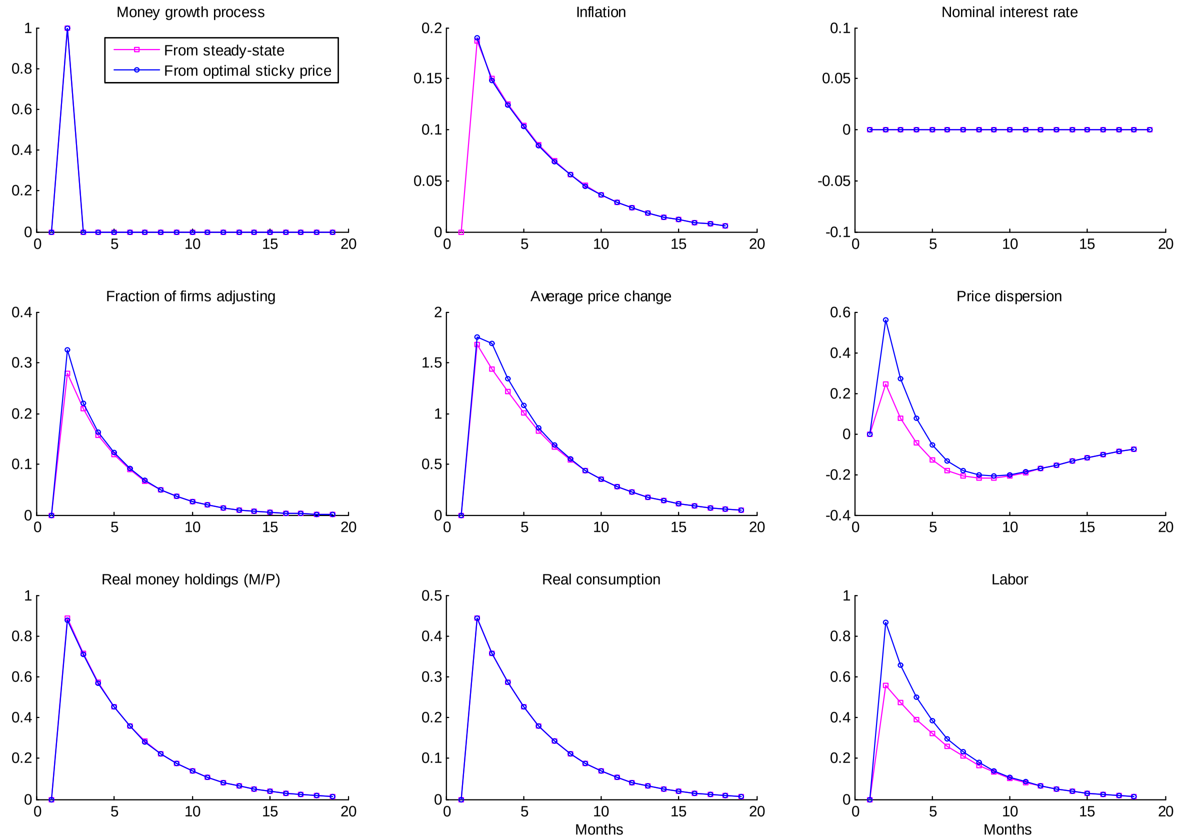


Figure 6. Impulse-responses from different initial distributions.

Panels 1-5: difference from steady state in percentage points. Panels 6-9: percent deviation from steady state.