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Abstract

A growing literature integrates theories of debt management into models of optimal fiscal policy. One promising theory argues that the composition of government debt should be chosen so that fluctuations in the market value of debt offset changes in expected future deficits. This complete market approach to debt management is valid even when the government only issues non-contingent bonds. A number of authors conclude from this approach that governments should issue long term debt and invest in short term assets.

We argue that the conclusions of this approach are too fragile to serve as a basis for policy recommendations. This is because bonds at different maturities have highly correlated returns, causing the determination of the optimal portfolio to be ill-conditioned. To make this point concrete we examine the implications of this approach to debt management in various models, both analytically and using numerical methods calibrated to the US economy. We find the complete market approach recommends asset positions which are huge multiples of GDP. Introducing persistent shocks or capital accumulation only worsens this problem. Increasing the volatility of interest rates through habits partly reduces the size of these positions but at the cost of introducing extreme volatility in asset holdings. Across these simulations we find no presumption that governments should issue long term debt - policy recommendations can be easily reversed through small perturbations in the specification of shocks or small variations in the maturity of bonds issued.

We further extend the literature by removing the assumption that governments every period costlessly repurchase all outstanding debt. This exacerbates the size of the required positions, worsens their volatility and in some cases produces instability in debt holdings.

We conclude that it is very difficult to insulate fiscal policy from shocks by using the complete markets approach to debt management. Given the limited variability of the yield curve using maturities is a poor way to substitute for state contingent debt. The result is the positions recommended by this approach conflict with a number of features that we believe are important in making bond markets incomplete e.g allowing for transaction costs, liquidity effects, etc..Until

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these features are all fully incorporated we remain in search of a theory of debt management capable of providing robust policy insights.

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1 Introduction

Traditionally the practise of debt management has focused on either minimizing the interest cost of borrowing, supporting short term interest rates set by monetary policy makers or assisting capital markets through providing appropriate amounts of risk free assets and liquidity at key maturities (see Missale (1999) for an excellent survey). However these practical considerations do not always have straightforward theoretical support. For instance, Nosbusch (2008) shows how cost minimization may be incompatible with optimal fiscal policy and the 1994 Mexican peso crisis painfully illustrates the macro costs of pursuing a cost minimizing debt issuance strategy.

More recently a growing literature has focused on an alternative approach to debt management, based on the idea that fiscal policy and debt structure should be *jointly* determined. This approach builds from the insight that a key influence on fiscal policy is the government's ability to offset unexpected fluctuations in government expenditure or revenue by managing the size, composition and value of debt.

This fiscal motivation for debt management raises two important research issues. The first is the extent to which bond markets are characterized by market incompleteness, that is, the extent to which governments are unable to issue state contingent claims. This issue is examined in Marcet and Scott (2005) and Scott (2007) who study the behavior of taxes, debt, deficits and output in OECD economies and conclude that fiscal policy is constrained by bond market incompleteness. The second research issue concerns what type of debt governments should issue and in what proportion. In a seminal contribution, Angeletos (2002) outlines what we refer to as a complete market approach to debt management. Under the assumption of a Ramsey planner, who seeks to minimize the deadweight loss arising from distortionary taxation, Angeletos shows i) even if a government only issues non-contingent bonds it can still exploit fluctuations in the yield curve and achieve the complete market outcome and ii) the optimal structure for government debt can be solved for by choosing the maturity structure that supports the complete market allocation for fiscal policy. Using this theory of debt management, and in the case of government expenditure shocks only, Angeletos shows that it is optimal for governments to issue long term debt and invest in short term assets.

Buera and Nicolini (2004) also analyze this complete market approach to deriving the optimal debt structure in the presence of bond market incompleteness. They show that the magnitude of the positions implied by this approach are extremely large multiples of GDP and at a level that seems implausible given the actual practice of government debt management. Angeletos (2002) notes this result but conjectures that it is a consequence of assuming an endowment economy and that allowing for capital accumulation should result in more plausible bond holdings[1]. Further, even if *quantitatively* the implications of this complete market approach appear wrong the *qualitative* recommendation to issue long term debt and hold short term assets receives additional support from Barro (1999) and (2003) and Nosbusch (2008).

We argue that the large positions found by Buera and Nicolini reflect a generic problem: since bonds at different maturities have very highly correlated returns, the determination of the optimal portfolio is in general ill-conditioned, and the optimal positions change very strongly with small changes in the environment. To make this point concrete we examine the optimal portfolio of maturities in a variety of models and study the robustness of the results. We introduce capital accumulation and consumption habits, the latter motivated by the need to match the observed volatility of interest rate spreads. We also extend the standard debt management model by introducing the realistic assumption that governments cannot buyback existing debt before maturity as opposed to the existing literature where governments costlessly every period buy back all existing debt and issue new securities[2]. We use simulations and calibrate model parameters to post war US data and then consider i) whether the quantitative problems noted by Buera and Nicolini are overcome ii) whether the qualitative recommendations of the complete market approach are robust across different model specifications. Solving some of the aspects of these simulations are non-trivial. Finding the debt management positions with capital and habits requires characterizing recursively these positions. The model solution for the model with habits is non-standard in a way that, to our knowledge, has not been recognized before. Our analysis of the no buyback case is, to our knowledge, new in the literature of optimal policy.

Our results reveal a large class of problems for the complete market approach to debt management. The size of debt positions actually increases with the introduction of capital accumulation, the recommendation to issue long term and invest short term can readily be overturned with small variations in parameters, the suggested portfolio positions are extremely sensitive to small variations in model structure and optimal debt issuance shows considerable volatility from period to period. Assuming that government holds all debt until maturity only worsens these problems and introduces additional channels of instability. The fact that quantitatively our predicted holdings differ in such a marked way from what is observed in practise is for us a source of concern with

[1] *"However, this disturbing result [of debt holdings exploding to plus and minus infinity] is mostly an artefact of an economy without capital"* Angeletos (2002)

[2] Their assumption is standard in the complete market literature, but it is not innocuous in determining the quantity of bonds that needs to be held in order to complete markets with debt management.

this complete market approach to debt management. However it can inevitably be argued that the discrepancy between our model and observed practice either reflects the sub-optimality of existing debt management practice or the simplified version of the models we examine. In this light the worry about the large positions we find may seem as a detail obscuring the big picture. But even in this light the inability of this approach to arrive at qualitatively stable and robust implications should be troublesome. Specifically, we find many cases where the sign of the positions is reversed and the government should issue short, and purchase long debt. Undermining our criticism of the complete market approach to debt management on the grounds of unrealistic model calibration is to misconstrue our point. If optimal debt management depends so strongly on the model being stimulated then the complete market approach provides few robust insights. Our point is merely reinforced by the claim that there exists an alternative model specification that produces even more different portfolio recommendations.

The numerical examples that we consider serve to demonstrate a deeper point about why the model fails quantitatively and qualitatively. Both the extreme nature of the portfolio positions and their instability and fragility are linked to the same fundamental reason - it is not very efficient to use different maturities to build a portfolio. It is well known in finance that in order to build an optimally diversified portfolio it is desirable to use a base of assets with returns as orthogonal as possible. In the limit, assets with perfectly mutually correlated returns are useless for insurance. For example, if the yield curve was perfectly flat in all periods it would be impossible to insure with a portfolio of maturities. Although the yield curve is not perfectly flat in the real world it does show very limited variability from one quarter to the next, therefore different maturities have payoffs that are very highly (though not perfectly) mutually correlated. This causes the equation that delivers the optimal portfolio to be ill-conditioned, causing the large instability and fragility of the results. This near-singularity can be seen in the examples we compute below in a clear way.

In summary, this complete market approach to debt management, with its focus on insulating optimal fiscal policy from unexpected shocks, yields implausible and unstable recommendations for optimal debt portfolios. We conclude that a theory of optimal debt management needs to retain the focus of providing insurance against fiscal shocks but has to supplement this with explicit recognition of the capital market imperfections, such as transaction costs, short selling constraints and liquidity effects, that lead to effective bond market incompleteness. In other words a successful theory of debt management needs to take the reasons for bond market incompleteness seriously if we are to produce an adequate theory of debt management that can provide reliable policy insights[3].

[3] Angeletos (2002) is careful to make this point in his conclusion too - " I do not suggest that we should abandon

The structure of the paper is as follows. In Section 2 we briefly outline the complete market approach to debt management of Angeletos (2002) and Buera and Nicolini (2004) and, using a model calibrated to US data, show the qualitative predictions emphasized by Angeletos and the quantitative problem raised by Buera and Nicolini. In Section 3 we introduce capital accumulation and consider the robustness of these quantitative and qualitative implications. Since fluctuations in the yield curve are the key influence in determining the optimal debt structure in Section 4 we introduce habits in order that our model can match the observed volatility of yield curves and consider once again the models robustness. In Section 5 we extend the literature by assuming the government never buys back its own bonds. A final section concludes.

2 Complete Market Approach to Debt Management

In this section we follow in the footsteps of Angeletos (2002) and Buera and Nicolini (2004), henceforth ABN, and outline the complete market approach to debt management. We consider the full commitment model of Lucas and Stokey (1983) augmented to include a productivity shock. We calibrate the model using US data and use this as the foundations for our subsequent analysis.

2.1 The Economy

The economy produces a single good that cannot be stored. The agent is endowed with one unit of time that it allocates between leisure and labour. Technology for every period t is given by:

$$c_t + g_t \leq \theta_t (1 - x_t), \tag{1}$$

where x_t, c_t and g_t represent leisure, private consumption and government expenditure respectively and θ_t represents a productivity shock. We shall refer to this version of our model, with some abuse of terminology, the endowment economy[4].

We assume $h_t \equiv (g_t, \theta_t)$ are stochastic and exogenous and represent the only sources of uncertainty in the model. In every period there is a finite number, N, of possible realizations of these shocks $\overline{h}_n \equiv (\overline{g}_n, \overline{\theta}_n)$, $n = 1, ..., N$. As usual, $h^t = (h_0, h_1, ..., h_t)$ represents the history of shocks up to and including period t. Governments and consumers have full information, that is, all variables

research on incomplete markets (which is) ... very important for understanding both the positive and normative aspects of fiscal policy". This paper is motivated by comments in response to Marcet and Scott (2007) that Angeletos (2002) shows the irrelevance of incomplete market approaches to debt management rather than any direct claim in Angeletos (2002).

[4] Strictly speaking this is an endowment economy augmented with work effort or a Robinson Crusoe economy.

dated t are restricted to be measurable with respect to h^t. As is standard, we will suppress the dependence of the endogenous variables on h^t whenever there is no confusion.

Preferences are given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U\left(c_t\right)+V\left(x_t\right)\right], \tag{2}$$

where $0 < \beta < 1$. For simplicity we assume U and V are strictly increasing and strictly concave in their respective arguments. The government has two instruments to finance government expenditure - a flat tax on labour or issue debt/lend to the consumer.

The case of complete markets using Arrow securities requires the government to issue N distinct contingent bonds at time t, each paying one unit of consumption contingent on $h_{t+1} = \overline{h}_n$ for $n = 1, ...N$. The quantity $b_t\left(h^t, \overline{h}_n\right)$ denotes the amount of government bonds issued in period t that pay one unit of consumption in period $t+1$ if $h_{t+1} = \overline{h}_n$ if realization h^t occurred.

The consumer's budget constraint is:

$$\begin{aligned} c_t\left(h^t\right)+ & \sum_{n=1}^{N} q_t\left(h^t, \overline{h}_n\right) \; b_t\left(h^t, \overline{h}_n\right) \\ \leq & \left(1-\tau_t^x\left(h^t\right)\right) w_t\left(h^t\right)\left(1-x_t\left(h^t\right)\right)+b_{t-1}\left(h^{t-1}, h_t\right), \end{aligned} \tag{3}$$

for all t and h^t, where $q_t\left(h^t, \overline{h}_n\right)$ is the price in terms of consumption of one bond $b_t\left(h^t, \overline{h}_n\right)$, $\tau_t^x\left(h^t\right)$ is the tax on labour and $w_t\left(h^t\right)$ is the wage earned by the consumer.

Finally, the government faces the constraint:

$$g_t\left(h^t\right)+b_{t-1}\left(h^{t-1}, h_t\right) \leq \tau_t^x\left(h^t\right) w_t\left(h^t\right)\left(1-x_t\left(h^t\right)\right)+\sum_{n=1}^{N} q_t\left(h^t, \overline{h}_n\right) b_t\left(h^t, \overline{h}_n\right). \tag{4}$$

Let c denote the sequence of all consumptions $\{c_0, c_1, ...\}$, and similarly for all other variables. A competitive equilibrium is defined as a feasible allocation (c, x, g), a price system (w, q) and a government policy (g, τ^x, b) such that, given the price system and government policy, (c, x) solves the firm's and consumer's maximization problem and also satisfies the sequence of government budget constraints (4).

The optimal Ramsey problem chooses policy by selecting the competitive equilibrium that maximizes (2). As shown, for example, in Chari and Kehoe (1999), this is equivalent to maximizing utility with (1) and the constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[c_t U_{c,t} - (1-x_t) V_{x,t}\right] = b_{-1} U_{c,0}, \tag{5}$$

where b_{-1} is the amount of liabilities inherited by the government in period 0, U_c is the marginal utility of consumption and V_x is the marginal utility of leisure.

2.2 The Complete Markets Approach to Debt Management

Under the assumption of complete markets it is always possible to back out the optimal contingent bond holdings. For given c, x that satisfy (5) define a sequence of random variables z such that

$$z_t(h^{t-1}, h_t) \equiv E\left(\sum_{s=0}^{\infty} \frac{\beta^s}{U_{c,t}} \left[c_{t+s}\ U_{c,t+s} - (1 - x_{t+s})\ V_{x,t+s}\right] \middle| h^{t-1}, h_t \right) \tag{6}$$

then the government should issue in period $t-1$ an amount of debt/credit such that, for every n,

$$b_{t-1}\left(h^{t-1}, \overline{h}_n\right) = z_t\left(h^{t-1}, \overline{h}_n\right) \tag{7}$$

ABN show how to use $\{z\}$ to derive the optimal structure of government debt even when the government can only issue bonds that yield a non-contingent payoff at given different maturities. We call this the complete markets approach to debt management even though it is applied to the case of bonds with a non-contingent payoff.

Assume the government can only issue a sequence of bonds that yield a non-contingent payoff at different maturities. We assume throughout the paper that the number of maturities equals N (that is the number of possible realizations of the shocks). Let b_t^j denote the amount of government bonds issued that pay one unit of consumption with certainty in period $t+j$, and let p_t^j denote the market price of this bond in terms of consumption in period t, both p_t^j and b_t^j are a function of h^t. Assume for now that the maturities are consecutive, that is, there is a bond maturing for each $j = 1, ..., N$. Moreover, assume that in every period the government buys back the entire stock of outstanding debt, so that the budget constraint of the government is

$$g_t + \sum_{j=0}^{N-1} p_t^j\ b_{t-1}^j \leq \tau_t^x\ w_t\ \ (1 - x_t) + \sum_{j=1}^{N} p_t^j\ b_t^j \tag{8}$$

for all t and h^t, and symmetrically for the consumer, where $p_t^0 \equiv 1$. Equilibrium prices satisfy

$$p_t^j = \beta^j \frac{E_t\left(U_{c,t+j}\right)}{U_{c,t}} \tag{9}$$

ABN prove that if bond prices are sufficiently variable, then one can choose each period a portfolio of maturities $(b_t^1, ..., b_t^N)$ such that

$$\sum_{j=0}^{N-1} p_t^j\left(h^t\right)\ b_{t-1}^j\left(h^{t-1}\right) = z_t\left(h^t\right) \tag{10}$$

almost surely, for all t. This can be done because even though bonds issued in $t-1$ are not contingent on the realization of h_t, today's value of last period's debt $\sum_{j=0}^{N-1} p_t^j\left(h^t\right)\ b_{t-1}^j\left(h^{t-1}\right)$ is state contingent due to the fact that bond prices vary with the state of nature h_t.

Consider the special case in which productivity is constant $\theta_t=\bar{\theta}$ and government expenditure follows a two step Markov process taking values $\overline{g}_H>\overline{g}_L>0$ with probabilities of remaining in the same state π_{HH} and π_{LL}. If $b_{-1}^j=0$ for $j=1,2$ then it is well known that variables dated t in the Ramsey allocation depend only on the shock g_t. Therefore in the Ramsey equilibrium, consumption, prices, etc. take two values, one for each realization of the shock. Formally, $z_t\left(h^{t-1},\overline{g}_i\right)=\overline{z}^i$, $p_t^1\left(h^{t-1},\overline{g}_i\right)\equiv\overline{p}^i$ and so on for $i=H,L$ and for all t. Assuming in addition that $g_0=\overline{g}_H$ it turns out $\overline{z}^H=0<\overline{z}^L$. Under these conditions (10) becomes

$$b_{t-1}^1\left(h^{t-1}\right)+\overline{p}^i\ b_{t-1}^2\left(h^{t-1}\right)=\overline{z}^i\ \text{ for } i=H,L\ \forall t \tag{11}$$

The necessary and sufficient condition for this problem to have a unique solution is that $\overline{p}^L\neq\overline{p}^H$ such that

$$\begin{pmatrix} b_{t-1}^1(h^{t-1}) \\ b_{t-1}^2(h^{t-1}) \end{pmatrix}=\begin{pmatrix} 1 & \overline{p}^H \\ 1 & \overline{p}^L \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \overline{z}^L \end{pmatrix}=\begin{pmatrix} \frac{\overline{p}^H\overline{z}^L}{\overline{p}^H-\overline{p}^L} \\ \frac{-\overline{z}^L}{\overline{p}^H-\overline{p}^L} \end{pmatrix}\equiv\begin{pmatrix} B^1 \\ B^2 \end{pmatrix} \tag{12}$$

for all t. Therefore in this case the amount of debt issued at each maturity is time invariant and assuming standard utility functions, we have $\overline{p}^H<\overline{p}^L$ so that $B^2>0$ and $B^1<0$. In other words, the optimal debt management policy is for the government to hold short term assets and issue long term liabilities

2.3 Simulations

As stressed by ABN the one-period ahead variability of long rates $\left(\overline{p}^H-\overline{p}^L\right)$ is not large (both in canonical DSGE models and the real world) so that (12) implies large positions in B^2 are needed to achieve the complete market outcome and a matching but offsetting large position in B^1. To document this problem we calibrate our model to US data and perform simulations. We assume the utility function:

$$\frac{c_t^{1-\gamma_1}}{1-\gamma_1}+\eta\frac{x_t^{1-\gamma_2}}{1-\gamma_2}$$

and set $\beta=0.98$, $\gamma_1=1$[5] and $\gamma_1=2$. We set η such that the government's deficit equals zero in the non stochastic steady state and use the steady state condition to fix the fraction of

[5] Assuming utility to be logarithmic helps simplify our analysis when we allow for capital accumulation. In this case capital is taxed only in periods 0 and 1 and capital taxes are zero thereafter.

leisure at 30% of the time endowment. We assume $b_{-1} = 0$. We borrow Chari's et al. (1991) calibration of the government spending and the technological processes, which they chose to match average share of government spending, the variance and serial correlation of consumption growth in the US. Assuming a two state symmetric Markov process for government expenditure we have $\overline{g}_i = g^* (1 + \xi_i)$, $i = H, L$ and $\xi_H = 0.07 = -\xi_L$. g^* equals to 25% of GDP in the non stochastic steady state and the transition probabilities are assumed $\pi^g_{HH} = \pi^g_{LL} = 0.95$. For the technological process we assume $\overline{\theta}_i = \phi_i$, $i = H, L$ and $\phi_H = 0.04 = -\phi_L$. The transition probabilities of the symmetric Markov process are $\pi^\theta_{HH} = \pi^\theta_{LL} = 0.91$. In the simulations we show also the case in which the technological process is more persistent than government expenditure ($\pi^\theta_{HH} = \pi^\theta_{LL} = 0.98$).

To test the sensitivity of our debt management recommendations we consider a range of simulations including both productivity and expenditure shocks, when only productivity or expenditure is the source of uncertainty and also for different degrees of persistence for the shocks. We show results for transition probabilities $\begin{pmatrix} \pi_{HH} & \pi_{HL} \\ \pi_{LH} & \pi_{LL} \end{pmatrix} = \mu\Delta + (1-\mu) I$ where Δ are the calibrated probabilities chosen above and $I = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$. When $\mu = 1$ shocks have the persistence suggested by Chari's et al. (1991) calibration, when $\mu = 0$ is shocks are i.i.d. and for $\mu \in (0,1)$ we have intermediate levels of persistence. Critical to the size of the debt positions the complete market approach recommends is the volatility of the yield curve so as we change the persistence of the shocks we maintain the unconditional variance to the same calibrated level.

Table 1 reports our simulation results[6]. We quote the unconditional average of the ratio of the value of debt positions with total output (in other words 7.50 means a position of 750% of GDP on average). In the case with either only government expenditure or productivity shocks the economy is characterized by only two states of the world and so the complete market outcome is attained by issuing only two maturities. In the case where we have both productivity and expenditure shocks we have four possible states of the world and so the complete market approach requires issuing four different maturities. We follow Buera and Nicolini (2004) and choose the maturities issued by minimizing the absolute value of the debt positions.

INSERT TABLE 1

Focusing on only one source of uncertainty we find the qualitative recommendations of Angeletos (2002) hold - governments should issue long term debt and invest in short term assets. In the case of persistent government expenditure shocks the optimal positions are large multiples of GDP (the

[6] Appendix A provides detailed description of the computational methods used to produce the simulations.

long term debt issued is more than 7 times GDP). The required positions are large because with persistent productivity shocks fluctuations in the fiscal position (z) are large and, as shown in Table 1, fluctuations in the long term interest rate are small. In the case of i.i.d. expenditure shocks or only productivity shocks (whether they are i.i.d. or persistent) the optimal debt positions are much smaller (although still substantially larger than the debt positions we see for OECD economies). It is when we turn to the model that allows for both shocks that we see clearly the problems noted by Buera and Nicolini (2004). Firstly, the required positions are enormous - the government needs to issue debt at each maturity in amounts that vary between 400% and 16000% GDP. Secondly, although the model still recommends issuing long term debt and investing in short term securities the maturity structure is complex and varies dramatically with small changes in maturity. In the case of intermediate persistence in shocks ($\mu = 0.33$) the government should invest in one period bonds, issue 2 year bonds worth 5900% of GDP and invest in three year bonds worth 16000% GDP.

The final rows of Table 1 show simulation results for an economy with both shocks but where we modify the calibrated parameters to allow for a productivity shock that is more persistent than the government expenditure shock. We find two other areas in which the predictions of the complete market approach are volatile and non-robust. We reverse the recommendation that governments should issue long term debt and invest in short term assets. Changing the persistence of shocks affects the slope of the yield curve and flips around the size of the positions so that now the government should issue short term debt and invest in long term assets. Angeletos (2002) is careful to stress that his recommendation of issuing long term debt and holding short term assets is based on the assumption that adverse government expenditure shocks lead to an increase in interest rates. Under both assumptions on relative persistence we have adverse expenditure shocks leading to higher interest rates and yet when productivity shocks are more persistent we now have governments issuing short debt and investing in long term securities. The reason is that whilst interest rates still rise with adverse expenditure shocks the yield curve is now downward sloping e.g. short rates rise more than long rates. The second sign of non-robustness occurs when we remove the option of the government to change the maturities it issues. In particular, in the case of $\mu = 0.333$ the maturity structure that minimizes the absolute positions is 1,2,3 and 29 but if we restrict the government to issue maturities at 1,4,13 and 30 (the maturities that minimize the debt positions in the case of persistent shocks, $\mu = 1$) then the matrix of returns becomes singular up to machine precision and the optimal positions tend to plus and minus infinity (numbers for this case, therefore, are not reported in Table 1). Since we can hardly have full confidence on the set of calibrated parameter values it would be desirable that policy recommendations do not change much when parameters vary slightly.

Therefore in the case of an endowment economy calibrated to US data we find that the complete market approach to debt management i) recommends positions that are large multiples of GDP ii) the size of debt positions varies sharply with small changes in maturity and involves simultaneously both issuing and investing in bonds of adjacent maturities iii) is extremely sensitive to small changes in parameter specifications with no presumption that it is always optimal for the government to issue long term debt and invest in short term bonds[7].

3 Introducing Capital Accumulation

The endowment economy is a useful workhorse model but the magnitude and sensitivity of the debt positions we outlined in the previous section could be an artefact of its simplicity. Therefore in this section we use the complete market optimal tax model of Chari et al (1994) to consider Angeletos' (2002) claim that capital mitigates these problems.

3.1 Complete Markets

Assume there are two factors of production: labour $(1-x)$ and capital k, with output produced through a Cobb Douglas function such that the economy's resource constraint is :

$$c_t + g_t + k_t - (1-\delta)k_{t-1} \leq \theta_t \; k_{t-1}^{\alpha}(1-x_t)^{1-\alpha} = \theta_t \; F\left(k_{t-1}, x_t\right) \tag{13}$$

where δ is the depreciation rate. As before, the exogenous shocks are $h = (g, \theta)$. The government now has three policy instruments to finance g: taxes on labour τ^x, taxes on capital τ^k and debt/credit.

For this problem to be of interest we need to restrict capital taxes in two ways. First we need to bound the initial period capital tax to prevent the planner from achieving the first best through a capital levy. We therefore add the constraint $\tau_0^k \leq \overline{\tau}^k$ for a fixed constant $\overline{\tau}^k$. We also need to assume that capital taxes are decided one period in advance (see also Farhi (2005)) otherwise debt and taxes in equilibrium would be undetermined and the role of debt management could

[7] We have focused purely on the properties of the debt structure implied by the complete market approach. However, another source of mismatch with the data comes from the second order properties of deficit and debt. As shown in Marcet and Scott (2005) the complete market Ramsey outcome implies that debt should show i) *less* persistence than other variables and ii) a negative co-movement of deficit and debt. These findings are replicated in our simulations here. However in practice debt shows *greater* persistence than other variables and a positive co-movement of deficit and debt, suggesting at the very least that this complete market approach has not influenced the practice of debt management.

be supplanted by state contingent capital taxation.[8] Note that as a result of this assumption we denote by τ_t^k the tax that is applied to capital income in period t even though this tax is set with information on h^{t-1}.

As before, we start with the case of complete markets where the government has full access to a complete set of contingent Arrow securities. The consumer's budget constraint is:

$$\begin{aligned} c_t\left(h^t\right)+k_t\left(h^t\right)+\sum_{n=1}^{N} q_t\left(h^t,\overline{h}_n\right)\ b_t\left(h^t,\overline{h}_n\right) \leq \left[\left(1-\tau_t^k\left(h^{t-1}\right)\right) r_t\left(h^t\right)+1-\delta\right] k_{t-1}\left(h^{t-1}\right) \\ +\left(1-\tau_t^x\left(h^t\right)\right) w_t\left(h^t\right)\left(1-x_t\left(h^t\right)\right)+b_{t-1}\left(h^{t-1},h_t\right) \end{aligned}$$

and the government's:

$$\begin{aligned} g_t\left(h^t\right)+b_{t-1}\left(h^{t-1},h_t\right) \leq \tau_t^k\left(h^{t-1}\right)\ r_t\left(h^t\right)\ k_{t-1}\left(h^{t-1}\right) \\ +\tau_t^x\left(h^t\right)\ w_t\left(h^t\right)\ \left(1-x_t\left(h^t\right)\right)+\sum_{n=1}^{N} q_t\left(h^t,\overline{h}_n\right)\ b_t\left(h^t,\overline{h}_n\right) \end{aligned}$$

where r_t denotes the rental price of capital.

The Ramsey problem is now augmented with the consumer's Euler equation with respect to capital, viz.,

$$U_{c,t}=\beta E_t\left\{U_{c,t+1}\left[\left(1-\tau_{t+1}^k\right) r_{t+1}+1-\delta\right]\right\}. \tag{14}$$

Firms' maximization implies $r_t=F_{k,t}$, $w_t=F_{l,t}$.

The results of Chari and Kehoe (1999) guarantee that the implementability constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t\left[c_t U_{c,t}-(1-x_t) V_{x,t}\right]=U_{c,0}\left[\left(\left(1-\tau_0^k\right) F_{k,0}+1-\delta\right) k_{-1}+b_{-1}\right]$$

plus the feasibility constraint (13) are necessary and sufficient conditions for a competitive equilibrium.

For given sequences c, k, x that satisfy these conditions we can build the expected discounted sum of future surpluses of the government in each period z:

$$\begin{aligned} z_t^k\left(h^{t-1},h_t\right) \equiv E\left(\sum_{s=0}^{\infty} \beta^s\left[c_{t+s}\frac{U_{c,t+s}}{U_{c,t}}-(1-x_{t+s})\frac{V_{x,t+s}}{U_{c,t}}\right]\middle| h^t\right) \\ -\left[\left(1-\tau_t^k\left(h^{t-1}\right)\right) F_{k,t}\left(h^t\right)+1-\delta\right]\ k_{t-1}\left(h^{t-1}\right). \end{aligned} \tag{15}$$

[8] See Chari and Kehoe (1999) for a detailed discussion of this issue.

Once more we can solve for the optimal portfolio by using (7) for each feasible c, k, x. However, unlike the case for the endowment economy the optimal bond positions are no longer constant. Chari et al. (1994) show that the Ramsey solution to this problem satisfies the recursive structure:

$$\left[k_t, c_t, x_t, \tau_t^x, \tau_t^k\right]' = G(h_t, k_{t-1})$$

for all $t \geq 1$ for some time-invariant function G. Using Proposition 1A in Marcet and Scott (2005) this implies the existence of a time-invariant function $D : R^3 \to R$ such that

$$D(k_{t-1}, \overline{h}_n) = z_t^k(h^{t-1}, \overline{h}_n)$$

for all $t \geq 1$, all h^t all n and the z_t^k obtained when plugging the optimal solution in (15). In other words, even though $z_t^k(h^{t-1}, \overline{h}_n)$ potentially depends on all past shocks these are effectively summarized by the previous period capital stock in the optimal solution. Therefore using (7) we have that the Ramsey optimum for debt under complete markets is $b_{t-1}(h^{t-1}, \overline{h}_n) = D(k_{t-1}, \overline{h}_n)$. The result of adding capital is therefore that the bond positions that complete the market are no longer constant but are a function of the capital stock.

3.2 The Complete Market Approach to Debt Management

We now turn to the standard debt management case where the government issues debt that pays a fixed amount at the time of maturity. We assume the government issues N consecutive maturities. The government can effectively complete the markets if it can find bond holdings b_{t-1}^j for each maturity such that

$$\sum_{j=0}^{N-1} p_t^j(h^{t-1}, \overline{h}_n)\, b_{t-1}^j\left(h^{t-1}\right) = D(k_{t-1}(h^{t-1}), \overline{h}_n) \tag{16}$$

for all t, all h^t and all n. Since the recursive structure of the Ramsey solution implies $P^n\left(k_{t-1}(h^t), h_t\right) = p_t^n(h^t)$ for N time-invariant functions P^n, for all $t \geq 1$, all h^{t-1} and all n, this gives N equations to solve for the unknowns $(b_{t-1}^1\left(h^{t-1}\right), ..., b_{t-1}^N\left(h^{t-1}\right))$ in each period. More precisely, letting $\Pi : R_+ \to R^{N \times N}$ be defined as

$$\Pi(k) \equiv \begin{bmatrix} 1 & P^1\left(k, \overline{h}_1\right) & ... & P^{N-1}\left(k, \overline{h}_1\right) \\ \vdots & & & \vdots \\ 1 & P^1\left(k, \overline{h}_n\right) & ... & P^{N-1}\left(k, \overline{h}_n\right) \end{bmatrix}$$

and assuming $\Pi(k_t)$ is invertible with probability one,[9] then the time-invariant function $B : R_+ \to R^N$ given by

$$\begin{bmatrix} b^1_{t-1} \\ \vdots \\ b^N_{t-1} \end{bmatrix} = [\Pi(k_{t-1})]^{-1} \begin{bmatrix} D(k_{t-1}, \overline{h}^1) \\ \vdots \\ D(k_{t-1}, \overline{h}_n) \end{bmatrix} \equiv B(k_{t-1}) \tag{17}$$

gives the portfolio that effectively completes the markets for all $t \geq 1$, all h^t.[10] To show this use simple algebra to check that these bond holding satisfy (16).

Therefore, with capital accumulation the amount issued of maturity j at time t is not constant as in the ABN case but is now a time-invariant function of current capital. Note that, perhaps surprisingly, the quantity of bonds $(b^1_t, ..., b^1_t)$ does not depend on the values of g_t, θ_t beyond their effect on k_t. The function B can be readily found by plugging the equilibrium functions D and P^j of the Ramsey solution in the above formula.

3.3 Simulations

Table 2 summarizes the results for simulations of the model with capital accumulation. The details of how we obtain the simulations for this and all the models solved in this paper are given in appendix A. This appendix gives a step-by-step account of how each policy function is computed and how the discounted budget constraints are insured.

We take standard values and we set $\alpha = 0.4$, the depreciation rate $\delta = 0.05$, assume that the initial value of government debt is always zero and set the initial capital stock equal to its deterministic steady state value in the Ramsey allocation[11]. As the bond holdings issued in each period are no longer constant we report both the average structure of the value of debt and also the average of the 5% lowest and 5% highest positions for each maturity, so as to indicate the volatility of the positions.

INSERT TABLE 2

The results show that adding capital accumulation only exacerbates the magnitude of the positions. In particular the positions in the case where the economy is subject to only one source of

[9] The "probability" statement is with respect to the distribution on k_t induced by the Ramsey solution.

[10] Notice that the Ramsey solution is only fully recursive for $t \geq 1$, because variables such as consumption or capital are only time-invariant functions after period 1, but the portfolio that completes the markets turns out to be time-invariant for $t \geq 0$.

[11] More precisely, we consider the deterministic steady state when g_t, θ_t are equal to the constants g^*,θ^*, there are no capital taxes and labour taxes are constant.

uncertainty are substantially higher than before. The size of the positions is once more influenced by the slope of the yield curve and how it fluctuates over time. When we allow for capital accumulation we allow another margin through which agents can smooth consumption and so interest rates and bond prices are less volatile requiring larger positions to achieve the complete market outcome. As we explained above capital accumulation also makes the optimal debt positions time varying and Table 2 shows that the required variation can be substantial. For instance, in the case of persistent productivity and expenditure shocks although on average the government issues long term debt worth 33600% of GDP in 5% of the periods it issues long term debt worth on average around 12800% GDP and at the other extreme in 5% of periods issuance averages around 89000%. So as well as finding the positions to be very volatile in response to small changes in maturity (the issuance pattern still shows a zig zag pattern of oscillating signs across maturities) the complete market approach also requires enormous and variable levels of debt issuance at each maturity. It is not anymore the case that varying the relative persistence of the two shocks (making productivity more persistent than government expenditure) leads to a reversal of the qualitative recommendation to issue long term debt and invest in short term bonds.

However now we find a yet another dimension in which the qualitative recommendation of the complete markets approach to debt management is undermined. Let us go back to the case where the economy is perturbed only by a persistent productivity shock and then use the complete market approach to solve for the optimal debt positions when the government issues a one period bond and a j-period bond, $j = 2, ..., 30$. Any one of these j's is sufficient to complete the markets. The optimal bond positions are shown in Figure 1 as a function of each possible j. For $j < 18$ the government should issue short term debt and invest in long term bonds. However when the government is constrained to issue long term bonds of maturity 18 or greater than the result flips around and now issuing long term debt and investing short term becomes optimal. The notion that optimal portfolio structure can change so dramatically depending on whether the government issues a 17 or 18 year bond seems both an undesirable property and worrying from a policy perspective. Further evidence of the sensitivity of the model to small changes in specification is shown in Figure 2 which plots the policy function for the case where the government issues either a 1 period or 16 period bond. Figure 2 shows that if the level of capital is below $k^* = 1291$ the government should issue short term debt and invest in long term assets. At this level of capital the matrix of returns is non-invertible and the sign of the bond holding at each maturity switches. This singularity is well within the support of capital stocks: we find in our simulations that the probability of capital being less than k^* in the steady state distribution is 49.8%. Therefore the level of capital shows an enormous shift in debt positions with long term debt going from large negative values to large positive ones when the capital stock is close to the median value of its distribution. This means that as the capital stock becomes close to its median the asset position may change dramatically from one period to the next.

INSERT FIGURE 1 AND FIGURE 2

4 Habits and Term Structure Volatility

We have so far documented that the complete market approach to solving for optimal debt policy when bond markets are incomplete produces debt holdings which are enormous multiples of GDP, show substantial volatility and considerable sensitivity to small perturbations in assumptions. One potential criticism of this finding is that it is based around models which produce very little volatility in long bond prices or the slope of the yield curve. The counterfactual properties of debt management we have documented may arise from this counterfactually low yield curve volatility which requires magnification by huge and volatile debt positions.

To understand this point consider again the simple model of section 2 when g can take two possible values $\overline{g}_H, \overline{g}_L$, but modify that model by assuming the government issues a short bond that matures in S periods ($S < M$). In this case markets would be effectively completed by a portfolio b_t^S, b_t^M satisfying

$$\overline{p}_t^{S-1,i} b_{t-1}^S + \overline{p}_t^{M-1,i} \; b_{t-1}^M = \overline{z}_t^i \;\; \text{for } i = H, L \;\; \text{for all } h^{t-1}$$

where $\overline{p}_t^{S-1,H}$ is shorthand for $\overline{p}_t^{S-1}(h^{t-1}, \overline{g}_H)$ and so on. Using the complete market methodology this implies

$$b_{t-1}^M = \frac{\overline{p}_t^{S-1,H} \; \overline{z}_t^L - \overline{p}_t^{S-1,L} \; \overline{z}_t^H}{\overline{p}_t^{S-1,H} \; \overline{p}_t^{M-1,L} - \overline{p}_t^{S-1,L} \; \overline{p}_t^{M-1,H}}$$

The closer to zero is the denominator $[\overline{p}_t^{S-1,H} \; \overline{p}_t^{M-1,L} - \overline{p}_t^{S-1,L} \; \overline{p}_t^{M-1,H}]$ then *ceteris paribus* the larger is the absolute value of b_{t-1}^M. Log-linearizing this denominator around 1 and rearranging yields:

$$(S-1)(spr_t^H - spr_t^L) + (M-S)(r_t^{M-1,H} - r_t^{M-1,L})$$

where $spr_t^i \equiv r_t^{M-1,i} - r_t^{S-1,i}$ is the interest rate spread between long and short bonds for realizations $i = H, L$, and r is the one-period net interest rate at each maturity for each given realization of the shock. Clearly $\left(spr_t^H - spr_t^L\right)$ is closely related to the variability of the spread conditional on information up to $t-1$,[12] so this expression shows that the greater the volatility of the spread conditional on past information, and the larger the one period ahead volatility of the return on the M period bond, the smaller the required optimal position to complete the market.

To see how this may have biased our results against the complete market model we first construct an empirical analog of the conditional volatility of the spread, then we present a model with habits that can potentially match this volatility and use our empirical results to calibrate the model and then examine the size and stability of the optimal positions.

First we estimate the one step ahead forecast error variance of the spread between the ten and one year US bond over the period 1949 to 2004. Applying standard model selection criteria on a VAR of lags and a set of related macroeconomic variables (e.g. GDP growth, interest rates, primary deficit, inflation) yields an equation of the form

$$spr_t = \alpha_1 + \alpha_2 spr_{t-1} + \alpha_3 \frac{def_{t-2}}{gdp_{t-2}} + \alpha_4 r_{t-2} + \varepsilon_t$$

where $\frac{def_t}{gdp_t}$ is the primary deficit/GDP ratio and r_t is the one year real interest rate. We interpret the residual ε as the one-step-ahead forecast error of the spread, and the variance of ε is our measure of $var_{t-1}(spr_t)$ which leads to an estimate of $\frac{\sqrt{E(var_{t-1}(\varepsilon_t))}}{Er_t}$ equal to 0.341. Comparing this to our model simulations confirms how poorly they perform in terms of producing volatility in the yield curve. For instance, in the model without capital the conditional volatility, defined as $\left(\frac{\sqrt{E(var_{t-1}(spr_t))}}{Er_t}\right)$,where the spread is defined between one and 10 period bonds, is equal to only 0.0033 for the model with government spending shocks, 0.0436 for the model with technology shocks and only 0.163 even if we allow for both shocks. Our simulations therefore produce insufficient volatility in the yield curve relative to the data and so require large positions to be held at each maturity.

To evaluate the extent of this problem we extend our model by introducing habits into the utility function. This approach has been widely used as a means of matching asset market puzzles in the literature e.g. Constantinides (1990), Campbell and Cochrane (1999). In essence it makes interest rates a function of consumption growth and the slope of the yield curve depends on the rate of change of consumption growth and so raises the volatility of both.

With habits in consumption the utility function of the consumer becomes:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U\left(c_t, c_{t-1}\right) + V\left(x_t\right)\right], \tag{18}$$

[12] More precisely, in the case that $\Pr ob_{t-1}(g_t = g^i) = .5$ it is easy to check that $E_{t-1}\left|spr_t\right| = spr_t^H - spr_t^L$.

The resource and budget constraints are the same as in the endowment model developed in Section 2. Equilibrium prices are given as before by (9) where marginal utility of consumption is now given by:

$$U_{c,t} \equiv \frac{\partial\, U\left(c_t, c_{t-1}\right)}{\partial c_t} + \beta E_t \left[\frac{\partial\, U\left(c_{t+1}, c_t\right)}{\partial c_t}\right] \tag{19}$$

Notice that this marginal utility depends on both past, current and expected future levels of consumption because of the presence of internal habits. The implementability condition is (5) as in section 2 but with $U_{c,t}$ given by the above formula. The presence of future variables in $U_{c,t}$ introduces some technical difficulties and non-standard aspects in the optimal policy: unlike in the case of section 2 the first order conditions of the Ramsey policy include some intertemporal terms since c_t now appears in $U_{c,t}, U_{c,t-1}$ and $U_{c,t+1}$. Also, in order to write recursively the problem we need to operate on the implementability constraint until we can express the Lagrangian as a recursive sum from period 1 onwards. The policy function is time-invariant $c_t = G(h_t, c_{t-1})$ for $t \geq 1$, but it is a different function in period zero.[13] These issues are all carefully addressed in the appendix.

In the appendix we discuss how to compute the policy function G (different from the function in the previous section) and all h^{t-1} and the solution for period 0. Using a similar argument as in the last section, we conclude

$$\left[b_t^1, ..., b_t^N\right]' = B(c_t) \tag{20}$$

for some time-invariant function B and all h^{t-1}. Thus, the level and composition of debt that effectively complete the markets now varies with current consumption.

In our simulations we assume the functional form

$$U\left(c_t, c_{t-1}\right) + V\left(x_t\right) \equiv \frac{\left(c_t - \chi c_{t-1}\right)^{1-\gamma_1}}{1-\gamma_1} + \eta \frac{x_t^{1-\gamma_2}}{1-\gamma_2},$$

and we calibrate the degree of habit persistence, χ, so as to match the conditional variance of the forecast error of the spread in the US data that we estimated above.

INSERT TABLE 3

Table 3 summarizes the results from simulations of our various cases. In order to maximize the volatility of interest rates we only show results for persistent shocks. We were unable to find a

[13] Some of these technical difficulties would be avoided by assuming external habits. There is a literature that studies how optimal fiscal policy can be used to treat the externality that arises from the spillovers of others' consumption caused by the external habits. See, for example, Ljungqvist and Uhlig (2000), Alonso-Carrera, Caballé and Raurich (2004) and references therein. We chose internal habits to avoid dealing with issues of externalities.

value for χ that matches observed volatility in the case where government expenditure is the only source of uncertainty. We therefore exclude this case from our table of results. The introduction of habits does raise the volatility of long term interests and the interest rate spread and so does lower the magnitude of the debt positions. However, although the magnitude of the positions is reduced they remain large (for instance in the case with both shocks the government has to issue 22 year bonds to the value of 11.48 times GDP and invest in 10 year bonds worth 18.23 times GDP). Whilst allowing for habits attenuates the size of the required positions it also creates a substantial additional problem. Increasing the volatility of interest rates and the term spread reduces the average size of the positions but at the expense of substantially increasing their volatility. For instance, if we focus on the higher 5% realizations of the long bond issuance they are on average 99.10 times GDP, while if we focus on the lowest 5% realizations we find that the government invests in 22 year bonds to the value of 62.69 GDP on average in this interval. Therefore the complete market approach recommends hugely volatile positions and, once again, the simple qualitative recommendation of issuing long term bonds and investing in short term assets is easily overturned since the government invests heavily in long maturities in many periods.

The reason behind these results is shown in Figure 3 which reports the policy functions for the value of the bond positions as a function of consumption.[14] The policy functions for bonds of 10, 15 and 22 period maturity show a spike at the same level of consumption. At this level of consumption the matrix of returns is non-invertible and at this point the sign of the bond holding switches. Therefore for only small changes in consumption we see an enormous shift in debt positions with long term debt going from large negative values to large positive ones. This reversal of optimal debt management occurs despite the fact that interest rates do rise in response to adverse expenditure shocks - a combination that Angeletos (2002) and others stress as important for making it optimal for governments to issue long term debt.

INSERT FIGURE 3

[14]Four lines appear in each graph of Figure 3, each line for each current realization of the shocks. This may seem odd given our previous observation that the position of the bonds is determined by current consumption. But what is reported in the Figure is the *value* of the bond, which is multiplied by the price and, therefore, contingent on today's realization of the shocks.

5 Ruling out Debt Buyback

In this section we examine the implications for our analysis of removing the counterfactual assumption, prevalent throughout the literature, that governments costlessly buy back every period the entire stock of debt and then reissue again whatever quantity and with whatever composition they desire. Ruling out debt buyback has an obvious empirical motivation. Many governments rarely intervene in secondary bond markets and instead only issue new debt or retire maturing debt. As a consequence the amount of new issues in any one period is normally a small proportion of the stock of outstanding debt. We will take this observation as a motivation for ruling out buyback and consider the case of a government that never retires debt before it matures. Presumably this important feature of debt management is based around considerations of transactions costs but our focus here is on understanding the implications of this practice rather than deriving the behavior endogenously.

From the point of view of the complete market literature it might seem at first glance that whether the government buys back the whole debt or not is innocuous. It is well known that under no transaction costs, no arbitrage and full commitment an agent can achieve the same allocations whether the debt is bought back every period or all debt is held until maturity, so it would seem that introducing debt buyback in the analysis is a matter of convenience. But it is clear that the quantity of bonds that needs to be held is different whether or not debt is bought back every period. Also, as we shall show, the evolution across time of each case can be quite different[15].

Consider once more the endowment model of Section 2 (with no habits and no capital) and assume only g varies and takes two values - either $\overline{g}_L$ or $\overline{g}_H$, with initial conditions $g_0 = \overline{g}_H$. Each period the government can issue two types of bonds: a one-period and an M-period bond (denoted $\mathbf{b}^1$ and $\mathbf{b}^M$ respectively).[16] The government is now assumed to never buy (sell) back the bonds it has issued (purchased) before they mature. This means that, letting $\mathbf{b}_t^1$ and $\mathbf{b}_t^M$ denote the new bonds issued by the government at t, the proceeds that the consumer receives at time t from bonds purchased in the past are given by $\mathbf{b}_{t-1}^1 + \mathbf{b}_{t-M}^M$, that is, from the the one period bond issued last year ($\mathbf{b}_{t-1}^1$) and the M period bond issued M periods ago ($\mathbf{b}_{t-M}^M$).

The period t budget constraint of the consumer is now:

$$c_t - (1 - \tau_t^x) w_t (1 - x_t) + p_t^1 \mathbf{b}_t^1 + p_t^M \mathbf{b}_t^M = \mathbf{b}_{t-1}^1 + \mathbf{b}_{t-M}^M. \tag{21}$$

[15] A similar point is made in Bronner, Martín and Ventura (2007), showing that the presence or absence of secondary bond markets makes a big difference in the presence of default.

[16] Note that we use boldface for the bond positions when the government holds bonds until maturity.

where as before $\mathbf{b}_t^1$ and $\mathbf{b}_t^M$ are functions of h^t.

Although the government only ever issues two types of bonds there are of course $M-1$ maturities available at any given point in time: in addition to the bonds $(\mathbf{b}_{t-1}^1, \mathbf{b}_{t-M}^M)$ that mature and produce income at t, there are also long bonds that have not yet matured: namely, $\mathbf{b}_{t-M+1}^M$,...,$\mathbf{b}_{t-1}^M$. Therefore the total value of debt outstanding that has not yet matured is given by $\sum_{i=1}^{M-1} \overline{p}^{M-i-1}\ \mathbf{b}_{t-i}^M$. Even though these non-maturing bonds do not show up in the government's and consumer's budget constraint at t they may nonetheless affect the actions of the government since they influence the income that will be available in the future.

In Appendix B we show that a version of the standard implementability constraint can be found and that, for zero initial debt, the optimal allocations c, x are the same as in section 2. We also show that in order to complete the markets the government will issue bonds ensuring that the total value of debt each period equals the discounted sum of surpluses z:

$$\begin{aligned} \mathbf{b}_t^1 + \sum_{i=0}^{M-1} \overline{p}^{M-i-1,H}\ \mathbf{b}_{t-i}^M &= 0 \\ \mathbf{b}_t^1 + \sum_{i=0}^{M-1} \overline{p}^{M-i-1,L}\ \mathbf{b}_{t-i}^M &= \overline{z}^L \end{aligned} \tag{22}$$

for all $t \geq 0$ (recall $\overline{p}^0 \equiv 1$). This equation is analogous to (10) in section 2 since it states that total wealth of the consumer, including unmatured bonds, has to equal the discounted sum of future deficits.

Given initial conditions b_{-1}^1, b_{-i}^M for $i = 1, ..., M-1$ this equation describes the whole evolution of b_t^1, b_t^M. To see the implications of this equation we start by focussing on the steady state.

5.1 Bonds at Steady State

Let us go back to the case of section 2, where the government buys back the entire stock of debt every period. A generalization of formula (12) for any long maturity M yields

$$\begin{pmatrix} b_t^1 \\ b_t^M \end{pmatrix} = \begin{pmatrix} B^1 \\ B^M \end{pmatrix} \equiv \begin{pmatrix} \frac{\overline{p}^{M-1,H}\ \overline{z}^L}{\overline{p}^{M-1,H} - \overline{p}^{M-1,L}} \\ \frac{-\overline{z}^L}{\overline{p}^{M-1,H} - \overline{p}^{M-1,L}} \end{pmatrix} \tag{23}$$

for all $t \geq 0$. Therefore, B^1, B^M give the steady state of bonds with buybacks.

To find the steady state without buybacks that we consider in this section, we set $\mathbf{b}_t^M = \mathbf{B}_{ss}^M$ and $\mathbf{b}_t^1 = \mathbf{B}_{ss}^1$ for all t in (22) which gives

$$\begin{pmatrix} 1\ , & \sum_{i=0}^{M-1} \overline{p}^{i,H} \\ 1\ , & \sum_{i=0}^{M-1} \overline{p}^{i,L} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{ss}^1 \\ \mathbf{B}_{ss}^M \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{z}^L \end{pmatrix}. \tag{24}$$

yielding

$$\mathbf{B}_{ss}^{M}=\frac{-z_L}{\sum_{i=1}^{M-1}\left(\overline{p}^{M-i,H}-\overline{p}^{M-i,L}\right)}$$

For standard utility functions it holds that $\overline{p}^{i,H}<\overline{p}^{i,L}$ for all $i=1,...,M$ so that, as before, $\mathbf{B}_{ss}^{M}>0$ showing that in the endowment economy with only expenditure shocks if we are at steady state the government should issue long bonds, as in Section 2, in order to mop up the variations in z_t.

Let us now compare these steady state positions with the quantities of ABN given by (23). In the case of $M>2$ we can expect $\left|\sum_{i=1}^{M-1}\left(\overline{p}^{i,H}-\overline{p}^{i,L}\right)\right|>\left|\overline{p}^{M-1,H}-\overline{p}^{M-1,L}\right|$ implying $B_{ss}^{M}<B_{ss}^{M}$. In other words, in steady state the government will issue lower amounts of long term bonds every period if it holds the bonds until maturity. This is not surprising, any long bond now constitutes debt for the following M periods, so that less debt needs to be issued every period.

In order to compare the debt positions when buyback is ruled out we shall focus on the ratio of the *value* of total long debt ($RVLD$)

$$RVLD^{j}\equiv\frac{\overline{p}^{M-1,j}\ B^{M}}{\left(\sum_{i=1}^{M-1}\overline{p}^{i,j}\right)\mathbf{B}_{ss}^{M}}\quad .$$

This compares the value of total government long debt outstanding in the model of section 2 (in the numerator) with the value of total long debt outstanding in the model of this section. Even though bonds are constant at steady state, this ratio depends on the realization $j=H,L$ due to the fact that prices change with the current realization j.

To gain some insight on likely values of this ratio we assume g iid and β close to 1. Under some approximations, appendix B shows that

$$E(RVLD)\approx 1-\frac{1}{M}\tag{25}$$

Therefore, according to this approximation, ruling out buyback unambiguously increases the value of total long debt held by the government, since $1-\frac{1}{M}<1$. Also, it is clear that $1-\frac{1}{M}$ is increasing in M and it goes from .5 to 1 as M varies from 2 to ∞. Therefore the value of government debt outstanding is about twice as large as in the model of section 2 with buybacks when the long bond has maturity $M=2$, or it can be equal to the model of section 2 as we consider M arbitrarily large.

We now turn to the analysis of steady state for the short bond. We have

$$\frac{B^{1}}{\mathbf{B}_{ss}^{1}}=RVLD^{H}$$

where the equality uses the first equations in (24) and in (23) and the definition of $RVLD^H$. Therefore, except for the discrepancy between $RVLD^H$ and $E\,(RVLD)$, which is likely to be small, we can claim that for the case of iid expenditure shocks and β close to 1 we have

$$\frac{B^1}{\mathbf{B}^1_{ss}} \approx 1 - \frac{1}{M}$$

Therefore ruling out buyback leads to larger positions in short bonds as well, going from twice as large to equal as M grows.

Therefore, if we focus in the steady state, introducing buyback does not alter the endowment models' qualitative implication that governments should issue long bonds and invest in short assets. It does however reduce the size of long bonds issued every period, whilst increasing the amount of one period bonds held. The total value of long government bonds outstanding is larger than under buyback.

5.2 Stability of the steady state

We now turn to consider the whole dynamics of bond positions. We show how when we rule out the buyback assumption not only does the complete market approach to debt management produce volatile asset positions it may even produce unstable fluctuations.

Consider the not unreasonable case when the government's initial debt position is not equal to its steady state value. In the ABN case where buyback is allowed convergence to the steady state is in one period. However, with the assumption of no buy back convergence to the steady state is less obvious. Simple algebra gives, from (22) that

$$\mathbf{b}^M_t = A + \sum_{i=1}^{M-2} \rho_i \ \mathbf{b}^M_{t-i} \tag{26}$$

for all $t \geq 0$, where the constants A, ρ_i are given by

$$\begin{aligned} A &\equiv \frac{z_L}{\overline{p}^{L,M-1} - \overline{p}^{H,M-1}} \\ \rho_i &\equiv \frac{\overline{p}^{H,M-i-1} - \overline{p}^{L,M-i-1}}{\overline{p}^{L,M-1} - \overline{p}^{H,M-1}} \qquad i = 1, ..., M-2 \end{aligned}$$

Equation (26) shows that $\mathbf{b}^M_t$ follows a linear, deterministic, difference equation of order $M-2$. Given initial conditions $\mathbf{b}^M_{-i}$, $i = 1, ..., M-2$ equation (26) gives the whole solution for long bonds. As is well known, this difference equation converges to the steady state if and only if all the roots of the polynomial $1 + \sum_{i=1}^{M-2} \rho_i \ L^i$ occur for values L that are larger than one in modulus.

It is clear that this difference equation could easily be unstable. For example, in the case $M = 3$ (26) gives

$$\mathbf{b}_t^3 = A + \rho_1 \ \mathbf{b}_{t-1}^3.$$

Using the prices for i.i.d shocks derived in Appendix B, the coefficient that multiplies $\mathbf{b}_{t-1}^3$ is:

$$\rho_1 = \frac{\overline{p}^{H,1} - \overline{p}^{L,1}}{\overline{p}^{L,2} - \overline{p}^{H,2}} = -\frac{1}{\beta} < -1$$

and so as a result the optimal debt structure does not converge but instead shows increasing oscillations, jumping from positive to negative values of increasing absolute value which in the limit tend to infinity. In this case, therefore, it is not true that the government would want to issue long bonds in all periods.

To see if instability is likely to arise in the relevant parameter values and to explore the size of the positions at steady state and check the validity of the approximations we have made we now turn to numerical analysis.

5.3 Simulations

For comparison purposes we calibrate the no buyback model using the same parameters as the endowment economy of Section 2. Table 4 reports results for the steady state where we assume the only source of uncertainty is government expenditure which follows a persistent two state Markov process[17].

INSERT TABLE 4

The analysis of the previous sub-section, based on the case of i.i.d shocks and certain approximations, suggested that in the case of no buyback the government would issue fewer bonds per period but the stock of bonds (i.e., the value of total outstanding unmatured long bonds) would be larger in steady state. The amount of short bonds held by the government would also be larger. These results are confirmed in our simulations for both persistent and i.i.d shocks. New issues of longer maturities each period are dramatically smaller in the case of buyback (and the ratio of new issues to outstanding stock broadly consistent with actual debt issuance patterns in the OECD), but the value of the stock of unmatured bonds and the holdings of short bonds are higher without buyback. For instance, whereas under buyback the government issues long term debt worth 716%

[17] In the case of i.i.d expenditure shocks we were never able to arrive at the steady state outcome if we started with non-steady state initial conditions. In the case of persistent expenditure shocks the model showed stability.

times GDP and invests in short term securities worth 704% of GDP when we rule out buyback these become 1073% and 716%. Therefore extending our complete market model to allow for more plausible restrictions on new debt issuance only exacerbates the problems documented previously.

It turns out that the calibrated case of Table 4 gives a stable difference equation (26), so that the bond positions would converge to the reported steady state. But this is not at all a robust feature of the model: if we lower slightly the persistence of the shocks we find instability. In particular, if we take $\pi^g_{HH} = .92$ we find that bond holdings are unstable. Furthermore, we have computed a number of case for iid shocks with various utility parameters and always found instability. Therefore for parameter values close to those in the calibrated case we find oscillatory behavior of bonds that do not converge to the steady state.

6 Conclusion

Macroeconomists have become increasingly interested in trying to embed policy recommendations for debt management into theories of optimal fiscal policy This literature has produced an appealing theory which we call the complete market approach to debt management. By exploiting variations in the yield curve the government can structure its debt so as to minimize the distortionary costs to taxation. Bond price movements help maintain the government's intertemporal budget constraint that requires equating the market value of government debt to the net present value of future primary fiscal surpluses. Successful debt management enables this to happen whilst minimizing changes to taxes. The great strength of this insight is that it can be applied even in the case when bond markets are incomplete in the sense that the government can not issue state-contingent debt. Further a number of authors have argued that this complete market approach offers a robust qualitative recommendation to debt managers - governments should issue long term debt and invest in short term bonds.

In this paper we have extensively reviewed the insights and implications of this complete market approach to debt management and identify a number of areas where this methodology is problematic:

i) As in Buera and Nicolini (2004) we find that the magnitude of the debt positions the government is required to hold are implausibly large multiples of GDP. We extend Buera and Nicolini's results by calibrating the model to US data and considering a range of extensions including capital accumulation, habits and ruling out the possibility of government buybacks. Although the magnitude of the positions does change substantially across these model specifications they remain throughout extremely large compared with observed practice.

ii) We identify an additional problem when we extend the model to allow for capital accumulation and habits. The required positions also show extremely large volatility. In particular increasing the volatility of interest rates only partly alleviates the size of positions but introduces a problem of extreme volatility. In some cases this volatility is so large that optimal positions for long term debt fluctuate between large negative and positive positions from one period to the next.

iii) It could be argued that these defects are a result of using inevitably stylized models, that the quantitative implications of the theory should not be taken too seriously, but that the qualitative features are robust. However we find that this complete market approach is also extremely sensitive to relatively small variations in parameters. Both the size and sign of positions can change dramatically with small changes in relative persistence of shocks or slight changes in the maturity of bonds that governments can issue. Therefore the qualitative implications of the model are not at all robust. The fragility of the model's implications are a major drawback to using this approach to provide insights to policymakers.

iv) We find that the recommendation that governments should issue long term debt and go short at shorter maturities is not robust. Under plausible parameterizations the model can easily produce the opposite recommendation. There may be good reasons why governments in the real world should issue long term debt, but the complete market methodology is not what produces this recommendation.

v) The debt management literature to date has tended to assume that every period the government can buyback the entire stock of debt and reissue the debt in whatever amount or proportion it desires. We show that this assumption is not without loss of generality. Ruling out buyback leads to larger positions having to be taken by the government and also raises concerns about the stability of debt management out of the steady state.

We have used various standard models to make these points. But the fundamental problem is that the yield curve moves very slowly. Then, a portfolio of bonds at different maturities can only provide as much fiscal insurance as complete markets if the size of the positions is so large that it magnifies many times over the scant variability of the yield curve. This problem is likely to arise in any model with a reasonable variability in the yield curve. In essence, the limited volatility of the yield curve makes maturities a poor substitute for state contingent debt. Therefore in order to exploit the maturity structure of debt the complete market approach requires large positions which conflict with other potential considerations of debt managers.

We conclude from this analysis that the recommendations of the complete market approach to debt management deviate too much from observed practice and are too fragile in response to small changes in model specification. It could of course be argued that the discrepancy between practice and theory merely confirms the sub-optimality of existing debt management practice. Under this interpretation our complete market result offers a direction in which to improve policy.

However, we believe these empirical discrepancies and fragilities require a more critical evaluation of the complete markets approach to debt management. For instance, if governments were to try and implement these policy recommendations they would have to buy and sell enormous amounts of bonds each period. This would entail all kinds of transaction costs, refinancing risks, and it would force some private agents in the economy to hold the opposite of the huge positions the government decided to take, possibly facing credit constraints. The government would have to hold very large amounts of private debt which could be defaulted upon. By explicitly ignoring these features of market incompleteness we believe the complete market approach is potentially misleading. The great strength of the complete market approach is it recognizes the importance of debt management in providing insurance against fiscal shocks. However the weakness with the complete market approach is it only focuses on fiscal insurance and abstracts from fundamental features of market incompleteness.

A successful theory of debt management will need to balance the insights of fiscal insurance with the constraints that incomplete markets provide. Whilst the complete market approach offers many insights we do not think it can be used to justify debt management policies or recommendations. We remain in search of a plausible theory of debt management.

APPENDIX A - Solution Details

Here we present the equations determining the equilibrium and we describe in detail the numerical computations for each model analyzed. In all cases it was assumed that initial government debt was zero, so we describe the solution procedure for this case, guaranteeing that there is no difference between the policy function of period zero and all other periods.

Numerical solution of the endowment economy (Section 2)

When $b_{-1} = 0$ the Lagrangian of the Ramsey problem of the endowment economy is:

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t) + V(x_t) + \lambda \left[U_{c,t} c_t - V_{x,t} (1 - x_t) \right] \right. \\ & \left. +\nu_t \left[\theta_t (1 - x_t) - c_t - g_t \right] \right\} \end{aligned} \tag{27}$$

The first order conditions of the problem for all t are

$$\begin{aligned} U_{c,t} + \lambda \left(U_{cc,t} c_t + U_{c,t} \right) - \nu_t &= 0 \\ V_{x,t} - \lambda \left(V_{xx,t} (1 - x_t) - V_{x,t} \right) + \nu_t &= 0 \\ \theta_t (1 - x_t) - c_t - g_t &= 0 \end{aligned} \tag{28}$$

plus the implementability constraint (5).

Moreover assume that the shocks follow a Markov process of N^2 states, $\{\overline{g}_i, \overline{\theta}_j\}$ with $i, j = 1, ..., N$.

For a given value of λ solving the model is trivial: for each period and each realization equations (28) give three equations to find the three unknowns c_t, x_t, ν_t as a (time-invariant) function of the exogenous shocks θ, g. In order to find the equilibrium λ we perform the following steps:

1. given the initial condition and the transition probabilities of the states, draw S series of T periods each of the shocks θ, g using a random number generator. Denote this realization $\left\{ \{ g_t^i, \theta_t^i \}_{t=1}^{T} \right\}_{i=1}^{S}$. The number of series S should be large enough for a certain expectation that we specify below to be computed accurately. T should be large enough for a certain discounted sum that we describe below to be computed accurately.

2. guess a value for λ. Solve system (28) for every state $\{\overline{g}_i, \overline{\theta}_j\}$ to get N^2 values of c, x and the corresponding surplus;

3. given the values from 2., for a given realization $\left\{g_t^i, \theta_t^i\right\}_{t=1}^T$ we could approximate the discounted with the sum $\sum_{t=0}^{T} \beta^t \left(c_t^i \frac{U_{c,t}^i}{U_{c,0}^i} - (1 - x_t^i) \frac{V_{x,t}^i}{U_{c,0}^i} \right)$. This amounts to setting all surpluses for $t > T$ equal to zero, and we can do better than that. We can reduce the error from truncating the sum by setting the surplus for $t > T$ to the average value of the deficit for each λ. So, to the truncated sum we add $\frac{1-\beta^T}{1-\beta} \left(\overline{c} \frac{\overline{U}_c}{U_{c,0}} - (1 - \overline{x}) \frac{\overline{V}_x}{U_{c,0}} \right)$ where $\overline{U}_c, \overline{c}, \overline{x}$ and $\overline{V}_x$ are computed at the mean of the shocks.

   Finally, compute the average of discounted sums

$$\frac{1}{S} \sum_{i=1}^{S} \left[\sum_{t=0}^{T} \beta^t \left(c_t \frac{U_{c,t}}{U_{c,0}} - (1 - x_t) \frac{V_{x,t}}{U_{c,0}} \right) \right] + \frac{1 - \beta^{T+1}}{1 - \beta} \left(c_T \frac{\overline{U}_{c,T}}{U_{c,0}} - (1 - x_T) \frac{\overline{V}_{x,T}}{U_{c,0}} \right) \tag{29}$$

   which should be a good approximation to

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(c_t \frac{U_{c,t}}{U_{c,0}} - (1 - x_t) \frac{V_{x,t}}{U_{c,0}} \right)$$

   for S, T sufficiently large. Notice that (29) is a function of λ.

4. Iterate on λ until (29) is close to zero. The result is the equilibrium λ

Given this equilibrium λ and values of c, x and of the surpluses, we compute the prices in all the states of the bonds with different maturities, computing the expectation on marginal utilities as a simple sum over all possible future states.

We also can compute the z's by a regression of the realized discounted sum on the exogenous variables.

For every maturity we can calculate the value of the matrix of returns and compute:

$$b = Pz$$

where $\underset{(N^2 \times 1)}{b}$ is the vector of bonds, $\underset{(N^2 \times N^2)}{P}$ is the matrix of the returns and $\underset{(N^2 \times 1)}{z}$ is the vector of conditional expected discounted surpluses.

Numerical solution of the economy with capital (Section 3)

Even with zero initial debt the model with capital has a different policy function in period zero from the following periods. The reason is that the return to capital is in the right side of the implementability constraint.

Assuming $\tau_0^k = 0$, the Lagrangian of the Ramsey problem is:

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \beta^t \{ U(c_t) + V(x_t) + \lambda \left[U_{c,t} c_t - V_{x,t} (1 - x_t) \right] \\ &\quad + \nu_t \left[F(k_{t-1}, 1 - x_t, \theta_t) + (1-\delta) k_{t-1} - c_t - g_t - k_t \right] \\ &\quad - \lambda \left[b_{-1} + (F_{k,0} + 1 - \delta) k_{-1} \right] U_{c,0} \} \end{aligned}$$

and the first order conditions are:

for $t > 0$:

$$\begin{aligned} U_{c,t} + \lambda (U_{cc,t} c_t + U_{c,t}) - \nu_t &= 0 \\ V_{x,t} - \lambda (V_{xx,t} (1 - x_t) - V_{x,t}) + \nu_t F_{x,t} &= 0 \\ \nu_t - \beta E_t \left[\nu_{t+1} (F_{k,t+1} + 1 - \delta) \right] &= 0 \\ F(k_{t-1}, 1 - x_t, \theta_t) + (1-\delta) k_{t-1} - c_t - g_t - k_t &= 0 \end{aligned} \tag{30}$$

for $t = 0$:

$$\begin{aligned} U_{c,0} + \lambda (U_{cc,0} c_0 + U_{c,0}) - \nu_0 - \lambda \left[b_{-1} + (F_{k,0} + 1 - \delta) k_{-1} \right] U_{cc,0} &= 0 \\ V_{x,0} - \lambda (V_{xx,0} (1 - x_0) - V_{x,0}) + \nu_0 F_{x,0} - \lambda F_{kx,0} k_{-1} &= 0 \\ \nu_0 - \beta E_0 \left[\nu_1 (F_{k,1} + 1 - \delta) \right] &= 0 \\ U_{c,0} - \beta E_0 \left[U_{c,1} \left(\tau_1^k F_{k,1} + 1 - \delta \right) \right] &= 0 \\ F(k_{-1}, 1 - x_0, \theta_0) + (1-\delta) k_{-1} - c_0 - g_0 - k_0 &= 0 \end{aligned} \tag{31}$$

We assume log utility and $b_{-1} = 0$.

The numerical procedure that we follow has step 1) as above. The following steps are now a bit more involved:

1. guess a value for λ. Given results in Chari, Christiano and Kehoe (1994) the solution after period 1 is given by a time-invariant function of the state variables k_{t-1}, g_t, θ_t, so we parameterize the function

$$E_t \left[U_{c,t+1} (F_{k,t+1} + 1 - \delta) \right] = \Phi(\beta; k_{t-1}, g_t, \theta_t), \text{ for } t \geq 1$$

where Φ is a polynomial with parameters β.

2. Given the assumption of log utility the first equation in (30) gives $U_{c,t} = \nu_t$ and the third equation in (30) gives

$$U_{c,t} = \Phi(\beta; k_{t-1}, g_t, \theta_t)$$

Given Φ and a conjecture for β we draw a long realization (10000 periods) of the shocks and we use system (30) to generate long run simulations for all variables and iterate on β with PEA (den Haan and Marcet (1990)) to find the fixed point β_f. In this way we find an approximation to the policy function for $t > 0$ consistent with λ.

3. period 0 is different from the other periods. Now $U_{c,0} \neq \nu_0$. To find the optimal choice for period 0, guess a value for k_0. For every value of g_1, θ_1 solve period 1 variables using system (30) replacing E_1 by the approximate function found in the previous step. Averaging over all states for g_1, θ_1 compute $E_0[\nu_1(F_{k,1} + 1 - \delta)]$, $E_0(U_{c,1}F_{k,1})$, and $E_0(U_{c,1})$ consistent with each k_0. Finally, solve the system (31) for the period $t = 0$ variables, setting $\tau_1^k = \left(1 - \frac{U_{c,0}}{\beta E_0(U_{c,1}F_{k,1})} + (1-\delta)\frac{E_0(U_{c,1})}{E_0(U_{c,1}F_{k,1})}\right)$, the level of capital tax that satisfies the first order conditions of the consumer.

4. perform a long simulation (100000 periods) of the model given k_0 found in step 3. and using $\Phi(\beta_f; k_{t-1}, g_t, \theta_t)$ for the remaining periods, given the realization for (c_t, x_t, k_t) from point 4), we approximate the infinite sum of the surpluses as a function of the states:

$$E_t \sum_{j=t+1}^{\infty} \beta^{j-t} \{U_{c,j}c_j + V_{x,j}(1 - x_j)\} = \Omega\left(\widetilde{\beta}_f; k_{t-1}, g_t, \theta_t\right);$$

by constructing the infinite sums in the expectation and running one regression of that infinite sum on $\Omega\left(\widetilde{\beta}_f; k_{t-1}, g_t, \theta_t\right)$. This is used to reduce the error in truncating the infinite sum as in step 3 of the previous model.

5. short simulation: we draw 10000 realizations of the shocks for the first 50 periods. We solve (30) given k_0 and we compute the infinite sum of the expected surplus in period 0 as an average of the infinite sums using the short simulations for the first 50 periods and $\Omega\left(\widetilde{\beta}_f; k_{t-1}, g_t, \theta_t\right)$ for $t = 51$; to approximate the left side of the budget constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t [U_{c,t}c_t - V_{x,t}(1 - x_t)] = [b_{-1} + (F_{k,0} + 1 - \delta)k_{-1}]U_{c,0}$$

and we iterate on λ in a similar way as before, until this constraint is approximately satisfied.

Given the equilibrium λ, Φ, β_f and the realizations of (c_t, x_t, k_t) of the long simulation of point 4), in order to get the bond prices at different maturities (from 1 to 30 years), approximate the expectations of future marginal utilities as a function of the current states of the economy.

Select 10000 consecutive periods of the long simulation. The vector of D's in (17) is given by $D(k_{t-1}(h^{t-1}), \overline{g}_i, \overline{\theta}_j) = \Omega\left(\widetilde{\beta}_f; k_{t-1}(h^{t-1}), \overline{g}_i, \overline{\theta}_j\right) + \left[U_{c,t}c_t + V_{x,t}(1-x_t)\right](h^{t-1}, \overline{g}_i, \overline{\theta}_j)$, the prices Π are computed in a similar way, and (17) gives the equilibrium maturities for the one period bond and different maturities of the longer bonds.

Numerical solution of the economy with consumption habits (Section 4)

In the case of habits the Lagrangian of the Ramsey problem with zero initial debt is exactly as in (27) replacing $U(c_t)$ by $U(c_t, c_{t-1})$ and with $U_{c,t}$ given by (19). This means that in the discounted sum of the Lagrangian future consumptions appear in the term dated t, so that in order to formulate the model recursively we have to rearrange the Lagrangian

For this purpose, use the notation

$$\begin{aligned} U_{1,t} &= \frac{\partial\, U(c_t, c_{t-1})}{\partial c_t} \qquad U_{2,t} = \frac{\partial\, U(c_t, c_{t-1})}{\partial c_{t-1}} \\ U_{11,t} &= \frac{\partial^2\, U(c_t, c_{t-1})}{\partial (c_t)^2} \qquad U_{22,t} = \frac{\partial^2\, U(c_t, c_{t-1})}{\partial (c_{t-1})^2} \qquad U_{12,t} = \frac{\partial^2\, U(c_t, c_{t-1})}{\partial c_{t-1} \partial c_t} \end{aligned} \tag{32}$$

With this notation $U_{c,t} = U_{1,t} + \beta\, E_t U_{2,t+1}$ and the Lagrangian can be written as

$$\begin{aligned} L &= E_0 \sum_{t=0}^{\infty} \beta^t\ \{U(c_t, c_{t-1}) + V(x_t) + \lambda\ [(U_{1,t} + \beta\, U_{2,t+1})\ c_t - V_{x,t}(1-x_t)] \\ &\quad + \nu_t [\theta_t (1-x_t) - c_t - g_t]\} \\ &= U(c_0, c_{-1}) + V(x_0) + \lambda\ [U_{1,0}\ c_0 - V_{x,0}(1-x_0)]\ + \nu_0 [\theta_0 (1-x_0) - c_0 - g_0] \\ &\quad + E_0 \sum_{t=1}^{\infty} \beta^t\ \{U(c_t, c_{t-1}) + V(x_t) + \lambda\ [U_{1,t}\ c_t + U_{2,t}\ c_{t-1} - V_{x,t}(1-x_t)] \\ &\quad + \nu_t [\theta_t (1-x_t) - c_t - g_t]\} \end{aligned}$$

where we used the notation in (32) and the the law of iterated expectations in the first equality, and for the last equality we set aside the $t = 0$ term and reorder.

Notice that the term multiplying λ is different in period $t = 0$ as in future periods so that the solution is only recursive after $t > 0$. This means that even with zero initial debt the policy function is different in the first period, unlike the endowment model of section 2, but similar to the case with capital. Notice also that c_{t-1} is the only variable from the past that appears in the terms dated $t \geq 1$ so that this is a sufficient state variable. This implies that the optimal solution can be written recursively as $c_t = G(h_t, c_{t-1})$ for $t > 0$ but a different decision function applies at time zero.

The FOC for $t > 0$ are given by the following expression

$$(1+\lambda)\left(U_{1,t} + \beta E_t U_{2,t+1}\right) + \lambda\left[U_{11,t}c_t + U_{12,t}\ c_{t-1} + \beta E_t\left(U_{12,t+1}c_{t+1}\right) + \beta\ c_t\ E_t U_{21,t+1}\right] = \nu_t$$

but the term $U_{12,t}\ c_{t-1}$ is absent for the FOC at t=0.

To take care of the different first order condition for period zero we compute c_0 from an analog to step 3 in the algorithm described for the model of section 3.

Notice that the above FOC imply that the expectations $E_t U_{2,t+1}$, $E_t\left(U_{12,t+1}c_{t+1}\right)$ and $E_t U_{21,t+1}$ need to be approximated in order to use this FOC to solve for c_t. Given the functional form for U used in the simulations we have

$$\begin{aligned}
E_t U_{2,t+1} &= -\chi E_t\left[(c_{t+1} - \chi c_t)^{-\gamma_1}\right] \\
E_t U_{12,t+1} &= \chi\gamma_1\ E_t\left[(c_{t+1} - \chi c_t)^{-\gamma_1 - 1}\right] \\
E_t\left(U_{12,t+1}c_{t+1}\right) &= \chi\gamma_1\ E_t\left[c_{t+1}(c_{t+1} - \gamma c_t)^{-\gamma_1 - 1}\right]
\end{aligned}$$

We proceed by parameterizing three expectations

$$\begin{aligned}
E_t\left[(c_{t+1} - \gamma c_t)^{-\gamma_1}\right] &= \Phi_1\left(\beta^1; c_{t-1}, g_t, \theta_t\right) \\
E_t\left[(c_{t+1} - \gamma c_t)^{-\gamma_1 - 1}\right] &= \Phi_2\left(\beta^2; c_{t-1}, g_t, \theta_t\right) \\
E_t\left[c_{t+1}(c_{t+1} - \gamma c_t)^{-\gamma_1 - 1}\right] &= \Phi_3\left(\beta^3; c_{t-1}, g_t, \theta_t\right)
\end{aligned}$$

Then we solve for a rational expectations equilibrium given λ, we compute the initial consumption separately, check the value of the implementability constraint for each λ, and iterate on λ until the implementability constraint is satisfied.

We build the elements of the system of equations that give the bonds at each maturity by approximating the corresponding functions of future discounted deficits and prices, for each value of the state variable c_t and for each possible future realization, now these functions have to depend on past consumption.

APPENDIX B - No Buy Back

Here we show the analysis for the model of section 5. This case has not been looked at in detail in the optimal policy literature. We show that an implementability condition analogous to the one used in the case of Arrow securities or in the ABN case can be found. We also prove that under standard conditions the equilibrium debt portfolio in each period is given by (22). Finally we give the formulae for the approximations that have been used in the main text in order to characterize the size and the stability of the bond positions.

Implementability constraint

We first show that, analogous to the case with Arrow securities or ABN, all equilibrium conditions can be summarized in a period-0 discounted budget constraint. Given any feasible sequence c, x we define the corresponding price for a bond that matures in i periods p_t^i as

$$p_t^i \equiv \beta^i E_t \left(\frac{U_{c,t+i}}{U_{c,t}} \right) \qquad i = 0, 1, ..., M \tag{33}$$

so $p_t^0 \equiv 1$. Assuming there is a secondary market for unmatured bonds, even if the government does not participate in this market, insures that these are equilibrium secondary market prices for unmatured bonds of each maturity that would implement the competitive equilibrium c, x.

For sufficiency we need to assume for each t and for almost all histories h^{t-1} that the corresponding bond prices satisfy

$$p_t^{M-1}(h^{t-1}, \overline{g}_H) \neq p_t^{M-1}(h^{t-1}, \overline{g}_L) \quad . \tag{34}$$

This condition fails, for example, in the the risk neutral case when U is linear. In this case it is not possible to complete the markets by bonds of different maturities. But in general if U has some curvature this condition holds generically. The same condition is needed to complete the markets in section 2.

We now show that the implementability condition

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[c_t U_{c,t} - (1 - x_t) V_{x,t} \right] = \left(\mathbf{b}_{-1}^1 + \sum_{i=1}^{M} p_0^{M-i} \ \mathbf{b}_{-i}^M \right) U_{c,0} \quad . \tag{35}$$

is necessary and sufficient for a competitive equilibrium

Result 1

Consider the asset economy of section 5. Take initial conditions on bonds $\mathbf{b}_{-1}^1, \mathbf{b}_{-1}^M, ..., \mathbf{b}_{-(M-1)}^M$ *as given.*

If a feasible sequence c, x is a competitive equilibrium it satisfies (35).

Furthermore, if a feasible sequence c, x satisfies (35) and (34), then c, x is a competitive equilibrium.

Proof

We first prove that (35) is necessary. If c, x is a competitive equilibrium it means that all period-t budget constraints (21) are satisfied for a no-Ponzi portfolio $\mathbf{b}^1, \mathbf{b}^M$, when bond prices are given by (33) and taxes

$$\frac{V_{x,t}}{U_{c,t}} = \theta_t(1 - \tau_t^x) \tag{36}$$

Denote the consumer deficit by $d_t \equiv c_t - (1 - x_t)\frac{V_{x,t}}{U_{c,t}}$. Add and subtract to both sides of (21) the value of old unmatured bonds held by the government to obtain that an equilibrium $c, x, \mathbf{b}^1, \mathbf{b}^M$ must satisfy:

$$d_t + p_t^1 \mathbf{b}_t^1 + \sum_{i=0}^{M-1} p_t^{M-i} \mathbf{b}_{t-i}^M = \mathbf{b}_{t-1}^1 + \sum_{i=1}^{M} p_t^{M-i} \mathbf{b}_{t-i}^M \tag{37}$$

The value of the portfolio held at the end of the period satisfies

$$\begin{aligned} p_t^1 \mathbf{b}_t^1 + \sum_{i=0}^{M-1} p_t^{M-i} \mathbf{b}_{t-i}^M &= \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \left(\mathbf{b}_t^1 + \sum_{i=0}^{M-1} \beta^{M-i-1} \frac{U_{c,t+M-i}}{U_{c,t+1}} \mathbf{b}_{t-i}^M \right) \right] \\ &= \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \left(\mathbf{b}_t^1 + \sum_{i=0}^{M-1} \beta^{M-i-1} E_{t+1} \left(\frac{U_{c,t+M-i}}{U_{c,t+1}} \right) \mathbf{b}_{t-i}^M \right) \right] \\ &= \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \left(\mathbf{b}_t^1 + \sum_{i=1}^{M} p_{t+1}^{M-i} \mathbf{b}_{t+1-i}^M \right) \right], \end{aligned} \tag{38}$$

where the first equality follows from (33) and simple algebra, the second equality follows from the law of iterated expectations, and the third from simple algebra, the pricing equation for p_t^i and the definition of $p_t^0 = 1$.

Letting total consumer wealth be $W_t \equiv \mathbf{b}_{t-1}^1 + \sum_{i=1}^{M} p_t^{M-i} \mathbf{b}_{t-i}^M$ have

$$d_t + \beta E_t \left(\frac{U_{c,t+1}}{U_{c,t}} W_{t+1} \right) = W_t$$

Iterating forward and assuming no Ponzi games on total wealth yields

$$E_t \sum_{j=0}^{\infty} \beta^j \frac{U_{c,t+j}}{U_{c,t}} d_{t+j} = \mathbf{b}_{t-1}^1 + \sum_{i=1}^{M} p_t^{M-i} \mathbf{b}_{t-i}^M \tag{39}$$

for all $t = 0, 1, \ldots$. This holds, in particular, for $t = 0$, and it proves that (35) holds for an equilibrium c, x. This shows necessity.

To prove sufficiency we need to find a portfolio $\widetilde{\mathbf{b}}^1, \widetilde{\mathbf{b}}^M$ and tax sequence that is a competitive equilibrium together with c, x. Obviously taxes that satisfy (36) can always be found to ensure consumer's maximization. If prices are given by (33) and the consumer budget constraint is satisfied this insures that agents choose $\widetilde{\mathbf{b}}^1, \widetilde{\mathbf{b}}^M$ optimally. Then the government budget constraint is certain to hold by Walras' law. Hence all that remains to be checked is period-t budget constraints of the consumer and the corresponding no-Ponzi condition hold for some $\widetilde{\mathbf{b}}^1, \widetilde{\mathbf{b}}^M$. Let us now find this portfolio.

Given a feasible c, x that satisfies (35) consider the corresponding discounted values z defined by (6) and the prices defined by (33) associated with these sequences. For all $t > 0$ and h^t define the portfolios $\widetilde{\mathbf{b}}$ as

$$\begin{bmatrix} \widetilde{\mathbf{b}}^1_{t-1}(h^{t-1}) \\ \widetilde{\mathbf{b}}^M_{t-1}(h^{t-1}) \end{bmatrix} = \begin{bmatrix} 1 & , & p_t^{M-1}(h^{t-1}, \overline{g}_H) \\ 1 & , & p_t^{M-1}(h^{t-1}, \overline{g}_L) \end{bmatrix}^{-1} \begin{bmatrix} z_t(h^{t-1}, \overline{g}_H) - \sum_{i=2}^M p_t^{M-i}(h^{t-1}, \overline{g}_H)\, \widetilde{\mathbf{b}}^M_{t-i}(h^{t-i}) \\ z_t(h^{t-1}, \overline{g}_L) - \sum_{i=2}^M p_t^{M-i}(h^{t-1}, \overline{g}_L)\, \widetilde{\mathbf{b}}^M_{t-i}(h^{t-i}) \end{bmatrix} \tag{40}$$

given initial conditions $\widetilde{\mathbf{b}}^M_{-i} = \mathbf{b}^M_{-i}$ $i = 1, \ldots, M$. Assumption (34) insures that the inverse matrix in the right side exists almost surely. Notice that the sequence c, x considered (and the corresponding z, p) and past $\mathbf{b}^M$'s fully determine the right side of this equation, so this defines $\widetilde{\mathbf{b}}^1, \widetilde{\mathbf{b}}^M$ for given initial conditions, for almost all realizations and all periods. We now show that precisely these portfolios satisfy the period t budget constraints. Clearly, the last equation implies

$$\widetilde{\mathbf{b}}^1_{t-1}(h^{t-1}) + \sum_{i=1}^M p_t^{M-i}(h^{t-1}, \overline{g}_j)\widetilde{\mathbf{b}}^M_{t-i}(h^{t-i}) = z_t(h^{t-1}, \overline{g}_j)$$

for $j = H, L$ all $t > 0$ a.s. so that

$$\widetilde{\mathbf{b}}^1_{t-1} + \sum_{i=1}^M p_t^{M-i}\widetilde{\mathbf{b}}^M_{t-i} = E_t \sum_{j=0}^\infty \beta^j \frac{U_{c,t+j}}{U_{c,t}} d_{t+j} \tag{41}$$

for all $t > 0$ a.s. by definition of z. This shows, first, that the portfolio $\widetilde{\mathbf{b}}^1, \widetilde{\mathbf{b}}^M$ thus constructed satisfies the Ponzi conditions. Also, this equation implies that

$$\begin{aligned}
\widetilde{\mathbf{b}}_{t-1}^{1}+\sum_{i=1}^{M} p_t^{M-i}\widetilde{\mathbf{b}}_{t-i}^{M}-d_t &= \beta E_t\left(\frac{U_{c,t+1}}{U_{c,t}}\sum_{j=1}^{\infty}\beta^j\frac{U_{c,t+j}}{U_{c,t+1}}d_{t+j}\right)\\
&= \beta E_t\left[\frac{U_{c,t+1}}{U_{c,t}}E_{t+1}\left(\sum_{j=0}^{\infty}\beta^j\frac{U_{c,t+1+j}}{U_{c,t+1}}d_{t+1+j}\right)\right]\\
&= \beta E_t\left[\frac{U_{c,t+1}}{U_{c,t}}\left(\widetilde{\mathbf{b}}_t^1+\sum_{i=1}^{M}p_{t+1}^{M-i}\widetilde{\mathbf{b}}_{t+1-i}^{M}\right)\right]=p_t^1\widetilde{\mathbf{b}}_t^1+\sum_{i=0}^{M-1}p_t^{M-i}\widetilde{\mathbf{b}}_{t-i}^{M}
\end{aligned}$$

where the first equality follows from simple algebra, the second from the law of iterated expectations, the third uses (41) again, and the last equality is because (38) holds for $\widetilde{\mathbf{b}}$ also. Subtracting $\sum_{i=1}^{M-1} p_t^{M-i}\widetilde{\mathbf{b}}_{t-i}^{M}$ from both the first and last expression of this string of equalities we see the budget constraint in period t is satisfied and therefore that c, x is a competitive equilibrium.■

This implies that the Ramsey optimum with no buybacks is found, as in the complete markets case, by considering (35) as the only restriction on the choice of c, x.

Equilibrium bonds

In the text we consider the case where initial debt in all maturities is zero $\left(\mathbf{b}_{-1}^1, \mathbf{b}_{-1}^M, ..., \mathbf{b}_{-(M-1)}^M\right) = 0$. In this case this implementability condition is exactly equal to the condition of section 2 so that the equilibrium c, x in sections 2 and 5 coincide. If initial debt would not be zero the first order conditions for the first M periods differ from those in the later periods. We prefer to leave this case aside for comparability with section 2.

Given a feasible c, x it is clear from the argument in the proof of result 1 that equilibrium bonds are given by (40) and, therefore, that equation (26) that we use in the main text gives the whole path for equilibrium long bonds. Short bonds are then given by (41).

Some formulae for ratio of bonds with and without buybacks in iid case

Under some additional assumptions we can characterize the size of the positions at steady state and the stability of the difference equation (26).

Assume g is iid and initial debt holdings are zero. Since the solution for c, x is the same as in section 2 we are confident that c_t is serially independent so that $K \equiv E_t U_{c,t+M}$ is a constant, independent of h^t, t and M. Then bond prices are $\overline{p}^{M-i,j} = \beta^{M-i} \frac{K}{\overline{U}_c^j}$ for $j = H, L$, where $\overline{U}_c^j$ is the marginal utility today if the shock $j = H, L$ is realized.

It follows that

$$\frac{B^M}{\mathbf{B}_{ss}^M} = \frac{\sum_{i=1}^{M-1} \beta^i \left(\frac{1}{\overline{U}_c^H} - \frac{1}{\overline{U}_c^L} \right)}{\beta^{M-1} \left(\frac{1}{\overline{U}_c^H} - \frac{1}{\overline{U}_c^L} \right)} = \frac{\beta - \beta^M}{\beta^{M-1}(1-\beta)} \tag{42}$$

Using the approximation

$$E\left(p_t^i\right) = \beta^i E\left(\frac{K}{U_{c,t}}\right) \approx \beta^i \frac{K}{E\left(U_{c,t}\right)} = \beta^i$$

and (42) we have

$$E(RVLD) \approx \frac{\beta^{M-1}}{\left(\sum_{i=0}^{M-1} \beta^i\right)} \frac{\beta - \beta^M}{\beta^{M-1}(1-\beta)} = \frac{\beta - \beta^M}{1 - \beta^M}$$

Simple algebra shows that this ratio is increasing in M, going from $\frac{\beta-\beta^2}{1-\beta^2}$ for the shortest maturity $M = 2$ to β as $M \to \infty$. Therefore,

$$\frac{\beta - \beta^2}{1 - \beta^2} < \frac{\beta - \beta^M}{1 - \beta^M} < \beta$$

Furthermore, if we consider the relevant case of large β we can use l'Hôpital rule to conclude

$$E(RVLD) \approx \lim_{\beta \to 1} \frac{\beta - \beta^M}{1 - \beta^M} = 1 - \frac{1}{M} \tag{43}$$

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Table 1 : Simulation Results - Endowment economy

Shocks							Interest rates			
g							**H**	**L**		
		B_1	B_{30}			R_1	2.23	1.85		
	$\mu = 1$	-7.04	7.16			R_{30}	2.10	1.98		
		B_1	B_{30}			R_1	3.95	0.13		
	$\mu = 0$	-0.79	0.81			R_{30}	2.28	1.80		
$\boldsymbol{\theta}$							**H**	**L**		
		B_1	B_{30}			R_1	1.07	2.93		
	$\mu = 1$	-0.85	0.90			R_{30}	1.85	2.21		
		B_1	B_{30}			R_1	-3.13	7.21		
	$\mu = 0$	-0.17	0.18			R_{30}	1.86	2.21		
$\mathbf{g}, \boldsymbol{\theta}$							**HH**	**HL**	**LH**	**LL**
		B_1	B_4	B_{13}	B_{30}	R_1	1.23	3.25	0.90	2.71
	$\mu = 1$	-16.15	41.32	-86.71	57.66	R_{30}	1.92	2.28	1.79	2.15
$\pi^g_{HH} = 0.95$		B_1	B_2	B_3	B_{29}	R_1	-5.75	7.21	-2.98	4.16
$\pi^\theta_{HH} = 0.91$	$\mu = 1/3$	-4.22	58.48	-161.22	106.37	R_{29}	1.92	2.28	1.79	2.14
		B_1	B_5	B_{18}	B_{30}	R_1	2.00	2.45	1.64	2.71
	$\mu = 1$	63.82	-140.94	163.15	-75.64	R_{30}	1.97	2.22	1.85	2.09
$\pi^g_{HH} = 0.95$		B_1	B_2	B_3	B_{29}	R_1	-3.34	6.96	-2.74	4.02
$\pi^\theta_{HH} = 0.98$	$\mu = 1/3$	5.77	-85.8	210.19	-129.51	R_{29}	1.91	2.28	1.79	2.14

Table 2 : Simulation Results - Capital Accumulation

Shocks						Interest rates				
							H	**L**		
g		B_1	B_{30}							
	$\mu = 1$	-14.49	12.36			R_1	2.08	1.98		
	$E_{+5\%}$	-18.29	9.41			R_{30}	2.07	2.00		
	$E_{-5\%}$	-11.65	16.3							
		B_1	B_{30}							
	$\mu = 0$	-9.23	7.19			R_1	2.06	1.99		
	$E_{+5\%}$	-9.50	6.90			R_{30}	2.04	2.03		
	$E_{-5\%}$	-8.94	7.46							
$\boldsymbol{\theta}$							**H**	**L**		
		B_1	B_{30}							
	$\mu = 1$	-8.49	6.26			R_1	2.26	1.85		
	$E_{+5\%}$	-12.5	3.56			R_{30}	2.01	2.07		
	$E_{-5\%}$	-5.62	10.10							
		B_1	B_{30}							
	$\mu = 0$	-3.49	1.47			R_1	2.01	2.07		
	$E_{+5\%}$	-3.93	1.19			R_{30}	2.02	2.06		
	$E_{-5\%}$	-3.12	1.82							
$\mathbf{g}, \boldsymbol{\theta}$							**HH**	**HL**	**LH**	**LL**
		B_1	B_4	B_{16}	B_{30}					
	$\mu = 1$	-30.10	42.54	-48.18	33.44	R_1	2.46	1.67	2.26	1.48
	$E_{+5\%}$	-34.33	26.14	-97.58	15.94	R_{30}	2.03	2.07	1.94	1.98
	$E_{-5\%}$	-26.30	63.28	-16.46	66.29					
$\pi_H^g = 0.95$		B_1	B_9	B_{13}	B_{29}					
$\pi_H^\theta = 0.91$	$\mu = 1/3$	-14.38	32.62	-30.74	10.42	R_1	2.04	1.97	2.00	1.92
	$E_{+5\%}$	-18.80	26.24	-36.75	8.16	R_{29}	2.01	2.05	2.00	2.03
	$E_{-5\%}$	-11.00	41.44	-25.37	11.91					
		B_1	B_5	B_{18}	B_{30}					
	$\mu = 1$	-77.85	153.10	-207.77	130.19	R_1	2.55	1.63	2.42	1.50
	$E_{+5\%}$	-109.15	138.74	-226.37	106.12	R_{30}	2.09	2.05	2.02	1.99
	$E_{-5\%}$	-55.63	167.34	-189.63	161.17					
$\pi_H^g = 0.95$		B_1	B_9	B_{14}	B_{29}					
$\pi_H^\theta = 0.98$	$\mu = 1/3$	-12.58	21.44	-23.13	12.20	R_1	2.07	1.94	2.03	1.90
	$E_{+5\%}$	-34.93	13.46	-54.90	8.63	R_{29}	2.03	2.00	2.05	2.01
	$E_{-5\%}$	-5.48	70.24	-18.56	17.44					

Note: The positions and the interest rates are obtained as average of 10000 period simulation. $E_{\pm 5\%}$ denote the average conditional on the realization being among the highest or lowest 5% values of the bonds.

Table 3: Simulation Results - with consumption habits

Shocks	Habits							Interest rates			
$\boldsymbol{\theta}$								**H**	**L**		
	$\chi = 0$		B_1	B_{10}			R_1	1.07	2.93		
		$\mu = 1$	-1.03	1.07			R_{10}	1.58	2.47		
	$\chi = 0.28$		B_1	B_{10}							
		$\mu = 1$	-0.621	0.617			R_1	-0.64	5.16		
		$E_{+5\%}$	-0.669	0.577			R_{10}	1.37	2.73		
		$E_{-5\%}$	-0.570	0.656							
$\mathbf{g}, \boldsymbol{\theta}$								**HH**	**HL**	**LH**	**LL**
	$\chi = 0$		B_1	B_{10}	B_{16}	B_{30}	R_1	1.23	3.15	0.90	2.71
		$\mu = 1$	-4.60	71.74	-159.02	101.39	R_{30}	1.92	2.28	1.79	2.15
$\pi^g_{HH} = 0.95$											
$\pi^\theta_{HH} = 0.91$	$\chi = 0.25$		B_1	B_{10}	B_{15}	B_{22}					
		$\mu = 1$	-0.48	-18.23	7.01	11.48	R_1	0.09	5.39	-0.77	3.50
		$E_{+5\%}$	-0.50	-27.45	-91.14	-62.69	R_{22}	1.82	2.44	1.62	2.21
		$E_{-5\%}$	-0.45	-7.24	90.36	99.10					

Note: idem as previous table.

Table 4: Debt Management with No Buyback - Steady State values

		Stock	*Flows*
$\mu = 1$	B_1	B_{30}	B_{30}
No Buy Back	-10.60	10.73	0.26
Buy Back	-7.04	7.16	7.16
$\mu = 0$	B_1	B_{30}	B_{30}
No Buy Back	-0.83	0.84	0.03
Buy Back	-0.79	0.81	0.81

Figure 1: Sensitivity of Portfolio Structure - Capital Accumulation and Persistent Technology Shocks

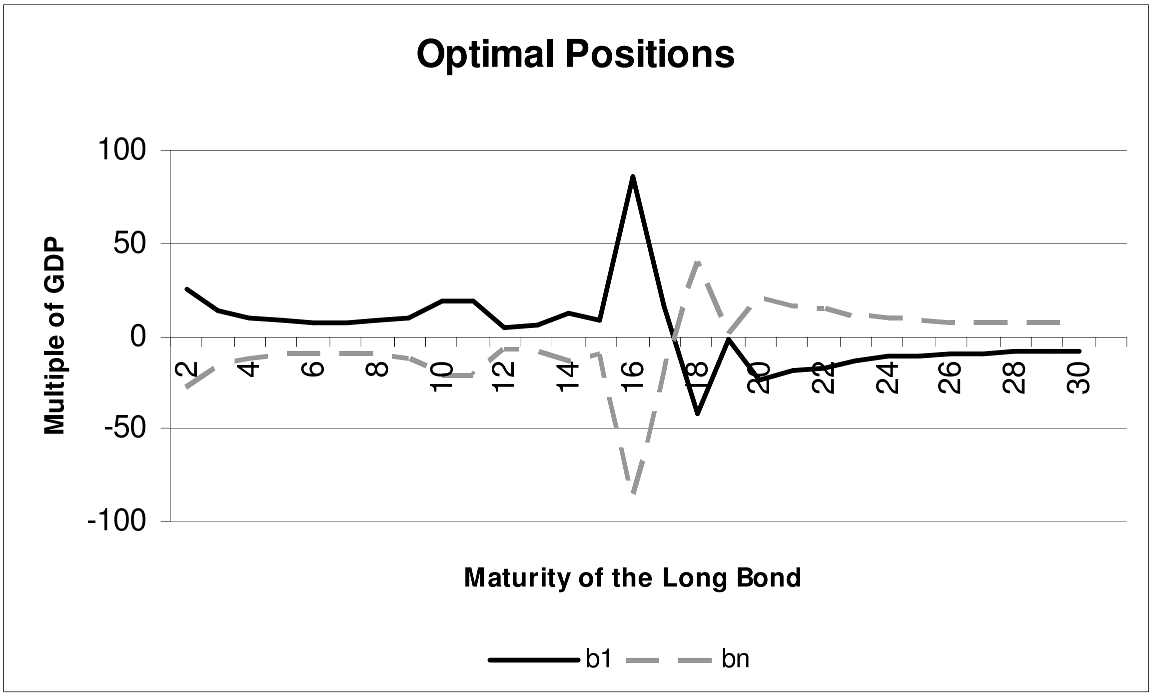
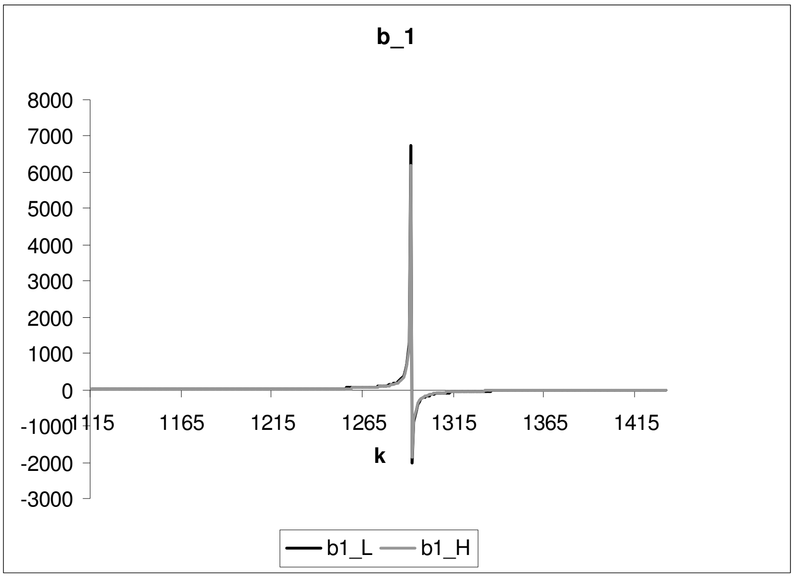


Figure 2: Policy Functions for Debt Issuance - Capital Accumulation and Persistent Technology Shocks



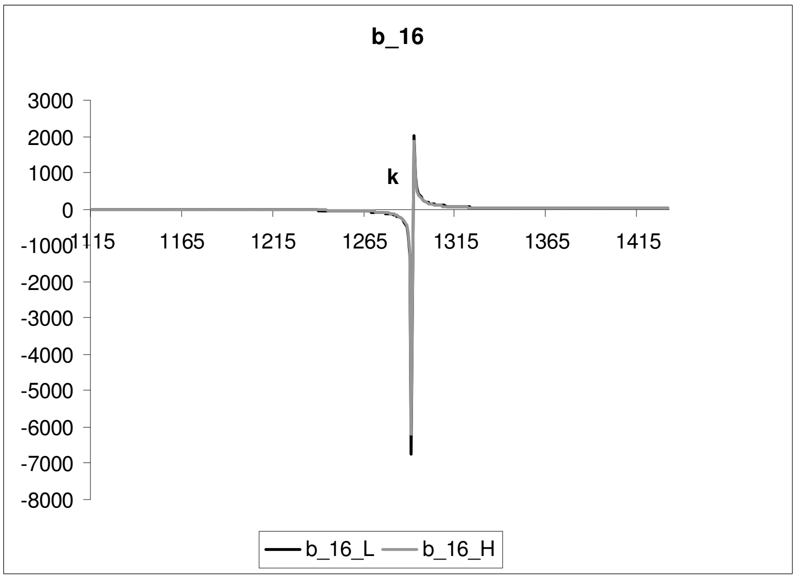


Figure 3 : Policy Functions for Debt Issuance - Habits and Both Shocks