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A NOTE ON MINIMUM MEAN SQUARED ERROR ESTIMATION OF SIGNALS WITH UNIT ROOTS

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Using an ARIMA parametrization, this note provides a very simple proof of how the Wiener–Kolmogorov–Whittle filter to estimate signals in time series can be extended to the nonstationary case. The proof is valid for any number and type of unit roots (not simply those implied by differencing) in both the signal and the overall model.

1. Introduction

The use of the Wiener–Kolmogorov filter, as developed in Whittle (1963), to estimate signals or unobserved components in ARIMA models became a powerful tool in applied time series work when the technique was extended to cover nonstationary series and signals. Key references are the pioneer work of Cleveland and Tiao (1976) and the comprehensive paper by Bell (1984). An important practical application, for example, has been the so-called ARIMA-model-based seasonal adjustment method; see Burman (1980) and Hillmer and Tiao (1982).

In this note we present an extremely simple alternative proof of how Whittle's result can be extended, in a rather natural way, to provide the minimum mean squared error estimator of the signal in the general nonstationary (unit roots) case.

2. General background

Consider a series x_t that can be decomposed as the sum of two independent unobserved components,

$$x_t = s_t + n_t, \tag{1}$$

which follow the ARIMA models

$$\phi_s(B)s_t = \theta_s(B)b_t, \tag{2a}$$

$$\phi_n(B)n_t = \theta_n(B)c_t, \tag{2b}$$

where b_t and c_t are independent Gaussian white noises with variances V_b and V_c, and ϕ_s, θ_s, ϕ_n and θ_n are finite polynomials in B, the lag operator. Let s_t represent the signal of interest. Since the roots of the autoregressive polynomials ϕ_s and ϕ_n imply peaks in the spectrum of x_t for certain frequencies, and since different components are associated with different peaks, it is assumed that ϕ_s and ϕ_n have no roots in common [for a more complete discussion, see Pierce (1979)]. Furthermore, although θ_s and θ_n may be noninvertible, it will be assumed that they do not share the same unit root, so that the overall model for x_t is invertible.

Eqs. (1)–(2) imply that the observed series x_t follows the ARIMA model

$$\phi(B)x_t = \theta(B)a_t, \tag{3}$$

where

$$\phi(B) = \phi_s(B)\phi_n(B), \tag{4}$$

and the polynomial $\theta(B)$ and the variance of a_t, V_a, can be obtained through the identity

$$\theta(B)a_t = \phi_n(B)\theta_s(B)b_t + \phi_s(B)\theta_n(B)c_t, \tag{5}$$

easily derived from expressions (1) to (4).

3. Estimation of the signal

Let all summation signs in this section extend from $-\infty$ to ∞ and consider the linear estimator of s_t,

$$\hat{s}_t = \nu(B)x_t, \tag{6}$$

where $\nu(B) = \sum \nu_j B^j$. In order to simplify notation, let a polynomial in B be denoted simply by a greek letter; an upper bar will denote the polynomial with B replaced by $F = B^{-1}$. Thus, for example, $\phi_s = \phi_s(B)$ and $\bar{\phi}_s = \phi_s(F)$.

The mean squared error (MSE) of $\hat{s}_t$ is

$$\begin{aligned} MSE &= \mathrm{E}(s_t - \hat{s}_t)^2 = \mathrm{E}\left[s_t - \nu(s_t + n_t)\right]^2 \\ &= \mathrm{E}\left[(1-\nu)\frac{\theta_s}{\phi_s}b_t - \nu\frac{\theta_n}{\phi_n}c_t\right]^2 \end{aligned} \tag{7}$$

and it will always be finite, even when the roots of ϕ_s and ϕ_n are nonstationary, when

$$1 - \nu = \phi_s \omega_1, \tag{8a}$$

$$\nu = \phi_n \omega_2, \tag{8b}$$

where ω_1 and ω_2 are polynomials in B such that $\sum_j \omega_{ij}^2 < \infty$ for $i = 1, 2$. From eqs. (8), the following identity is obtained:

$$1 = \phi_s \omega_1 + \phi_n \omega_2. \tag{9}$$

Equating the autocovariance generating functions on both sides of (5) yields

$$\theta \bar{\theta} V_a = \phi_n \bar{\phi}_n \theta_s \bar{\theta}_s V_b + \phi_s \bar{\phi}_s \theta_n \bar{\theta}_n V_c. \tag{10}$$

Dividing both sides of (10) by $\theta \bar{\theta} V_a$ and then adding and subtracting the polynomial $\phi_s \phi_n \omega$ in the right-hand side, where ω is any arbitrary polynomial in B, (10) can be rewritten

$$1 = \phi_s \left[\frac{\bar{\phi}_s \theta_n \bar{\theta}_n V_c}{\theta \bar{\theta} V_a} + \phi_n \omega \right] + \phi_n \left[\frac{\bar{\phi}_n \theta_s \bar{\theta}_s V_b}{\theta \bar{\theta} V_a} - \phi_s \omega \right]. \tag{11}$$

Comparing (9) and (11), it is seen that

$$\omega_1 = \frac{\bar{\phi}_s \theta_n \bar{\theta}_n V_c}{\theta \bar{\theta} V_a} + \phi_n \omega, \tag{12a}$$

$$\omega_2 = \frac{\bar{\phi}_n \theta_s \bar{\theta}_s V_b}{\theta \bar{\theta} V_a} - \phi_s \omega, \tag{12b}$$

and the filter ν, from (8b), becomes

$$\nu = \frac{\phi_n \bar{\phi}_n \theta_s \bar{\theta}_s V_b}{\theta \bar{\theta} V_a} - \phi_s \omega. \tag{13}$$

Let W be the set $W = \{\omega \,|\, \sum \omega_j^2 < \infty\}$, and denote by Ω the set of filters defined by (13) with $\omega \in W$. The *MSE* of $\hat{s}_t$ will always be finite if $\nu \in \Omega$. When the roots of ϕ are nonstationary, only filters $\nu \in \Omega$ will yield finite *MSE* estimators.

The Whittle filter, given by

$$\nu_0 = \frac{V_b}{V_a} \frac{\psi_s \bar{\psi}_s}{\psi \bar{\psi}},$$

where $\psi_s = \theta_s/\phi_s$ and $\psi = \theta/\phi$, becomes, in our ARIMA notation,

$$\nu_0 = \frac{V_b}{V_a} \frac{\phi_n \bar{\phi}_n \theta_s \bar{\theta}_s}{\theta \bar{\theta}}, \tag{14}$$

and it obviously belongs to Ω since then $\omega = 0$. When the roots of ϕ are stationary, ν_0 produces the minimum MSE estimator of s_t. Our aim is to show that (14) still yields the minimum MSE estimator of s_t in the nonstationary case.

4. The result

Let the polynomials $\theta_s, \theta_n, \phi_n$ and the variances V_b and V_c be fixed, assume $\phi_s = 1 + \phi_1 B + \cdots + \phi_p B^p$ and consider the roots of the equation $z^p + \phi_1 z^{p-1} + \cdots + \phi_p = 0$. The coefficient ϕ_j $(j = 1, \ldots, p)$ is given by $\phi_j = (-1)^j S_j$, where S_j is the symmetric function consisting of the sum of all combinations of the products of j roots; thus ϕ_j is a continuous function of the roots. Assume there are m real roots and $2n$ complex roots $(m + 2n = p)$. Express the complex conjugate roots in terms of the modulus, r_i $(r_i > 0)$, and the frequency, f_i $(f_i \in [0, 2\pi])$. Let the frequencies be fixed and construct the vector r with elements the real positive roots, the negative of the real negative roots, and the n moduli r_i of the complex roots. The vector r belongs to R_+^{m+n}. Define the vector u of the same dimension as r and with all elements equal to one. Let $F = (\phi_1, \ldots, \phi_p)$; since u is an interior point of R_+^{m+n}, F is a continuous function of r at u.

Expression (10) implies a system of covariance equations in the θ and V_a parameters, where all equations are polynomials. The Jacobian of the system evaluated at $F(u)$ will be therefore nonzero (except possibly on a zero measure set on the space of the ϕ_n, θ_s, θ_n, V_b and V_c parameters). Thus the θ and V_a parameters will be (a.e.) continuous in F at $F(u)$. It is then immediate, from (12) and (13), that the coefficients of ω_1, ω_2 and ν are also continuous functions of F at $F(u)$. Let v denote the vector with elements the coefficients of ν. For ν given by (13), $v = v[F(r), \omega]$ is a continuous function of r at u. Define the polynomials $\lambda = \omega_1(r, \omega)\theta_s$ and $\delta = -\omega_2(r, \omega)\theta_n$, with ω_1 and ω_2 given by (12); from (7) and (8), for $\nu \in \Omega$, the MSE of $\hat{s}_t$ can be expressed as

$$\begin{aligned} MSE\ \{F(r), v[F(r), \omega]\} &= \mathrm{E}[\lambda b_t + \delta c_t]^2 \\ &= V_b \sum \lambda_j^2 + V_c \sum \delta_j^2, \end{aligned}$$

and hence MSE is a continuous function of r at u.

In our notation, ν_0 given by (14), for the nonstationary case can be written as $v[F(u),0]$. Its minimum MSE property is established in the following theorem.

Theorem. $MSE\{F(u), v[F(u),0]\} \leq MSE\{F(u), v[F(u),\omega]\}$ *for any* $\omega \neq 0$.

Proof. Assume the theorem is false. Then $\exists\omega^*$, $\omega^* \in \mathrm{W}$ and $\omega^* \neq 0$, such that

$$MSE\{F(u), v[F(u),0]\} > MSE\{F(u), v[F(u),\omega^*]\}. \tag{15}$$

Then, by continuity, for a vector $\varepsilon > 0$, of the same dimension as r and with small enough positive elements, (15) implies

$$\begin{aligned} &MSE\{F(u-\varepsilon), v[F(u-\varepsilon),0]\} \\ &> MSE\{F(u-\varepsilon), v[F(u-\varepsilon),\omega^*]\}. \end{aligned} \tag{16}$$

Since for $F(u-\varepsilon)$ we are in the stationary case, $v[F(u-\varepsilon),0]$ minimizes the MSE of $\hat{s}_t$, and hence (16) cannot be true, proving the theorem.

Notice that the theorem assumes that all roots of ϕ_s are nonstationary. The case in which only some of them are nonstationary can be proved in an identical way simply by fixing the stationary roots and removing them from the vector r. The proof is thus general in that it can handle any number and type of unit roots (not simply those implied by differencing) in both ϕ_s and ϕ_n.

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