**TEMPORAL AGGREGATION, SYSTEMATIC SAMPLING,**

**AND THE HODRICK-PRESCOTT FILTER**

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**September 2006**

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## Abstract

The time aggregation properties of the Hodrick-Prescott (HP) filter, to decompose a time series into trend and cycle, are analyzed for the case of annual, quarterly, and monthly data. It is seen that aggregation of the disaggregate component estimators cannot be obtained as the exact result from applying an HP filter to the aggregate series (and viceversa). Nevertheless, using several criteria, one can find HP decompositions for different levels of aggregation that provide similar results. (The approximation works better for the case of temporal aggregation than for systematic sampling.) Moreover, the values of the parameter $\lambda$ of the HP filter, that provide results that are approximately consistent under aggregation, are considerably robust with respect to the ARIMA model of the series, and close to the ones that are obtained with the empirical rule of Ravn and Uhlig (2002). The paper concludes with a word of caution concerning the desirability of exact aggregation consistency.

**Keywords:** Time series; Filtering and Smoothing; Time aggregation; Business cycles; ARIMA models.

**JEL classification:** C22; C43; C82; E32; E66.

## 1. Introduction

The subjectiveness in the concept of business cycle has derived in multiple methodologies for its identification (see, for example, Canova (1998)). Be that as it may, and despite substantial academic criticism (see, for example, Cogley and Nason (1995), Harvey (1997), or King and Rebelo (1993)), the so-called Hodrick-Prescott (HP) filter has become the core of the paradigm for business-cycle estimation at policy making institutions (some examples are the IMF, the OECD, or the ECB). The HP filter decomposes a time series into two components: a long-term trend component and a stationary cycle (see Hodrick and Prescott (1980), Kydland and Prescott (1990), and Prescott (1986)); it is a linear filter that requires previous specification of a parameter known as lambda, $\lambda$. This parameter tunes the smoothness of the trend, and should depend on the periodicity of the data and on the main period of the cycle that is of interest to the analyst. Nevertheless, as pointed out by Wyne and Koo (1997), the parameter does not have an intuitive interpretation for the user, and its choice is considered an important weakness of the HP method (Dolado et al. (1993)).

For quarterly data (the frequency most often used for business-cycle analysis), there is an implicit consensus in employing the value of $\lambda = 1600$, originally proposed by Hodrick and Prescott (1980) based on a somewhat mystifying reasoning ("…a 5% cyclical component is moderately large, as is a 1/8 of 1% change in the growth rate in a quarter…"). Still, the consensus around this value undoubtedly reflects the fact that analysts have found it useful. The consensus, however, disappears when other frequencies of observation are used. For example, for annual data, Baxter and King (1999) recommend the value $\lambda = 10$, Cooley and Ohanian (1991), Apel et al. (1996), and Dolado et al. (1993) employ $\lambda = 400$, while Backus and Kehoe (1992), Giorno et al. (1995) or European Central Bank (2000) use the value $\lambda = 100$, which is also the default value in EViews (see EViews (2005)). Concerning monthly data (a frequency seldom used), the popular econometrics program EViews uses the default value 14400, while the Dolado et al. reasoning would lead to $\lambda = 4800$.

None of the references mentioned addresses the issue of the relationship between the values of $\lambda$ used for different observation frequencies. In particular, if $\lambda_M$ is used for monthly data, how do the implied quarterly cycles compare with the ones obtained directly on the quarterly data with $\lambda_Q = 1600$? Also, what value of $\lambda_A$ applied to annual observations yields cycles that are close to the ones obtained by aggregating the cycles obtained for quarterly data with $\lambda_Q = 1600$? Ravn and Uhlig (2002) use an empirical criterion to obtain these "consistent under time aggregation" values of $\lambda$. Using as reference the value of $\lambda_Q$ (for quarterly data), and letting $\lambda_D$ denote the value for an alternative frequency of observation, they restrict attention to the criterion

$$\lambda_D = (k)^n \lambda_Q \quad , \tag{1.1}$$

where k is the ratio of the number of observations per year for the alternative and quarterly frequencies (thus $k = 3$ and $k = 1/4$ when the alternative frequencies are the monthly and annual ones, respectively,) and n is a positive integer. (It is worth noticing that the values of $\lambda_A = 400$, 100, and $\lambda_M = 4800$, 14400 mentioned above are obtained by applying the previous rule for $n = 1$ and $n = 2$.) Ravn and Uhlig (RU) present evidence that $n = 4$ appears to be the best choice. For $\lambda_Q = 1600$, this implies $\lambda_M = 129600$ and $\lambda_A = 6.25$ .

The present paper addresses the issue of consistency under temporal aggregation of the HP filter from another perspective. First, the aggregation properties of the filter are analized: even though temporal aggregation of an HP cycle does not yield a cycle that can be obtained with an HP filter directly applied to the aggregate data, by choosing an appropriate value of $\lambda$, the differences can be made small. A relatively simple criterion for that choice, based on the preservation of desirable filter properties, is presented. The results are very close to those obtained by RU, and it is shown how their criterion turns out to be a first-order approximation to the one we propose.

Some alternative criteria that take into consideration properties of the series model and structure are also discussed. It is seen that the results are remarkably close to those obtained with the previous (simpler) criteria, although the closeness is generally stronger for the case of temporal aggregation (or averaging) than for the case of systematic sampling. The last section discusses some limitations that should be taken into account when searching for consistency under time aggregation.

The paper centers on monthly, quarterly, and annual frequencies of observation, and uses the widely accepted value $\lambda_Q = 1600$ as the pivotal value for the comparisons. The discussion, however, generalizes trivially to any other frequencies of observation and pivotal value for $\lambda$. The discussion is illustrated with some examples based on five macroeconomic series (the IPI series for the US, Japan, France, and Italy, and the US unemployment series). All of them span the period January 1962 – December 2005, and contain 528 monthly observations; the series are taken from the OECD data base.

## 2. The Hodrick-Prescott Filter

Assume that the observed series follows the ARIMA(p,d,q) model

$$(1+\phi_1 B+...+\phi_p B^p)\nabla^d x_t = (1+\theta_1 B+...+\theta_q B^q) a_t ,$$

where B is the lag operator, such that $B^j x_t = x_{t-j}$, and $\nabla = 1-B$ is the regular difference. For the rest of the paper, "w.n. (0, $\nu$)" will denote a white noise (i.e., niid) variable with zero mean and variance $\nu$. We further assume that $a_t$ is a w.n. (0, $\nu_a$) innovation. The polynomials in B $\phi(B)$ and $\theta(B)$ are assumed stationary and invertible, respectively (see Box and Jenkins (1970)). When p=0, the model name will be abbreviated to IMA(d,q).

Assume we are interested in decomposing $x_t$ into a long-term trend $m_t$ and a residual, $c_t$, to be called "cycle". For the time series realization $(x_1 \ldots x_T)$, the HP filter provides the sequences $(m_1 \ldots m_T)$ and $(c_1 \ldots c_T)$ such that

$$x_t = m_t + c_t \qquad\qquad t = 1,\ldots,T, \tag{2.1}$$

and the loss function

$$\sum_{t=1}^{T} c_t^2 + \lambda \sum_{t=3}^{T} \left(\nabla^2 m_t\right)^2 \tag{2.2}$$

is minimized. The first term in (2.2) penalizes large residuals (i.e., poor fit), while the second term penalizes lack of smoothness in the trend. The parameter $\lambda$ regulates the trade-off between the two criteria: larger values of $\lambda$ will produce smoother trends and increase the variance of the cycle. The solution is given by the expression $\hat{m} = A^{-1}x$, with $A = I + \lambda KK'$, where $\hat{m}$ is the estimated value of the vector $(m_1 \ldots m_T)'$, x is the vector $(x_1 \ldots x_T)'$, $K$ is a (T-2)xT matrix with elements $K_{ij} = 1$ if $i = j$ or $i = j+2$, $K_{ij} = -2$ if $i = j+1$, and $K_{ij} = 0$ otherwise (see Danthine and Girardin (1989)). King and Rebelo (1993) showed that the filter could be given an unobserved component (UC) model derivation whereby x is the realization of a stochastic process consisting of (2.1), where

$$\nabla^2 m_t = a_{mt}, \qquad\qquad a_{mt} \sim \text{w.n.}(0, \nu_m)\ ; \tag{2.3a}$$

$$c_t \sim \text{w.n.}(0, \nu_c)\ ; \tag{2.3b}$$

with $a_{mt}$ orthogonal to $c_t$. Under these assumptions, the HP filter solution is equivalent to finding the minimum mean square error (MMSE) estimator of $m_t$ and $c_t$ with the Kalman filter. Kaiser and Maravall (2001) show that the HP estimators can also be derived with a different model-based algorithm. We summarize this approach.

From (2.1) and (2.3) it follows that $\nabla^2 x_t = a_{mt} + \nabla^2 c_t$ and hence the reduced form for $x_t$ is an IMA(2,2) process, say

$$\nabla^2 x_t = \theta_{HP}(B) a_t = (1 + \theta_1 B + \theta_2 B^2) a_t, \qquad a_t \sim \text{w.n.}(0, \nu_a), \tag{2.4}$$

where the identity

$$\theta_{HP}(B)a_t = a_{mt} + \nabla^2 c_t \tag{2.5}$$

determines the parameters $\theta_1$, $\theta_2$, and $\nu_a$ (see Appendix A). The fact that the r.h.s. of (2.5) is the sum of two orthogonal components, one of them white noise, implies that $\nabla^2 x_t$ has a strictly positive spectral minimum, and hence $\theta_{HP}(B)a_t$ is an invertible process; therefore, $\theta_{HP}(B)^{-1}$ will converge. The MMSE estimator of $m_t$ and $c_t$ can be obtained with the Wiener-Kolmogorov (WK) filter. When

$$\lambda = \nu_c / \nu_m \tag{2.6}$$

these estimators are the ones obtained with the HP filter, which can thus be expressed as (see Kaiser and Maravall (2001))

$$\vartheta_m(B,F) = \frac{v_m}{v_a} \frac{1}{\theta_{HP}(B)\theta_{HP}(F)} \quad , \tag{2.7}$$

$$\vartheta_c(B,F) = \frac{v_c}{v_a} \frac{(1-B)^2(1-F)^2}{\theta_{HP}(B)\theta_{HP}(F)} \quad , \tag{2.8}$$

where $F(=B^{-1})$ denotes the forward operator, such that $F^j x_t = x_{t+j}$. The estimators of $m_t$ and $c_t$ will be obtained through

$$\hat{m}_t = \vartheta_m(B,F)x_t \ , \quad \hat{c}_t = \vartheta_c(B,F)x_t \quad . \tag{2.9}$$

The filters (2.7) and (2.8) are symmetric, centered, and convergent. Since (2.5) implies

$$\theta_{HP}(B)\ \theta_{HP}(F)\nu_a = \nu_m + (1-B)^2(1-F)^2\ \nu_c \quad , \tag{2.10}$$

considering (2.6), the filter (2.7) can alternatively be expressed in terms of the HP parameter $\lambda$ as:

$$\vartheta_m(B,F) = \frac{1}{1+\lambda(1-B)^2(1-F)^2} \quad . \tag{2.11}$$

It will prove useful to look at the frequency domain representation of the filter (2.11), given by its Fourier transform (FT). If $\omega \in [0, \pi]$ denotes the frequency measured in radians, replacing B by the complex number $e^{-i\omega}$, and using the identity $2\cos(j\omega) = e^{-ij\omega} + e^{ij\omega}$, the FT of (2.11) is given by

$$G_m(\omega,\lambda) = \frac{1}{1+4\lambda(1-\cos\omega)^2} \qquad (2.12)$$

and $G_c(\omega,\lambda) = 1 - G_m(\omega,\lambda)$. Expression (2.12) is the gain function of the trend filter. Equating the pseudo-autocovariance functions (ACF) of the two sides of the second equation in (2.9), and taking the FT yields

$$S_{\hat{m}}(\omega,\lambda) = [G_m(\omega,\lambda)]^2\, S_x(\omega); \qquad S_{\hat{c}}(\omega,\lambda) = [G_c(\omega,\lambda)]^2\, S_x(\omega), \qquad (2.13)$$

where $S_{\hat{m}}(\omega,\lambda)$, $S_{\hat{c}}(\omega,\lambda)$, and $S_x(\omega)$ are the pseudo-spectra of $\hat{m}_t$, $\hat{c}_t$, and $x_t$. Given that the pseudo-spectrum provides a description of how different frequencies contribute to the variation of $x_t$, expression (2.13) shows that the squared gain of the filter indicates which frequencies are selected from $x_t$ to provide the estimators $\hat{m}_t$ and $\hat{c}_t$. Given that seasonal variation (or noise) should not contaminate the cyclical signal, the variable $x_t$ in (2.9) and (2.13) will typically be a seasonally adjusted series or a trend-cycle component.

The WK filters (2.7) and (2.8) extend from $-\infty$ to $\infty$. Their convergence, however, would allow us to use a finite truncation. But, as characterizes all 2-sided filters, estimation of the component at both ends of a finite series requires future observations, still unknown, and observations prior to the first one available. As shown by Cleveland and Tiao (1976) the optimal (MMSE) estimator for end points can be obtained by extending the series at both ends with forecasts and backcasts, so that expression (2.9) remains valid with $x_t$ replaced by the extended series. There is no need however to truncate the filter: using the approach in Burman (1980), Kaiser and Maravall (2001) present the algorithm for the HP filter case, and show how the effect of the infinite extensions can be exactly captured with only 4 forecasts and backcasts. The WK application of the HP filter is computationally efficient and analytically convenient.

## 3. Temporal Aggregation of the Hodrick-Prescott Filter

We shall consider two types of aggregation. In the first one ("temporal aggregation") the aggregate variable is the sum (or average) of the disaggregate variable; in the second one ("systematic sampling") the aggregate variable is obtained by periodically sampling one observation from the disaggregate variable. In order to analyze the aggregation properties of the HP filter, it will prove helpful to use the UC representation. We consider first the case of temporal aggregation. Let the aggregate series consist of non-overlapping sums of k consecutive values of the disaggregate series. We adopt the following notation:

- Aggregate series will be represented by capital letters; disaggregate series by small letters.

- T and t denote the time sub-index for the aggregate and disaggregate series respectively, so that, if T and t are aligned $X_{T-j} = x_{t-jk} + x_{t-jk-1} + \ldots + x_{t-(j+1)k+1}$ , j=0, 1, 2, …., and similarly for the disaggregate and aggregate components, $m_t$, and $M_T$, and $c_t$ and $C_T$, respectively.

- b denotes the backward operator and $\nabla$ the difference operator for the disaggregate series. Thus $\nabla_j x_t = (1-b^j)x_t = x_t - x_{t-j}$.

- $S_k$ is the aggregation operator for the disaggregate series: $S_k = 1 + b + \ldots + b^{k-1}$. Thus

$$X_{T-j} = S_k x_{t-jk} \; ; \tag{3.1}$$

$$\nabla_k = \nabla S_k \; . \tag{3.2}$$

- B denotes the backward operator and *D* the difference operator for the aggregate series. Thus $DX_T = (1-B)X_T = X_T - X_{T-1}$.

- We shall denote an "HP-type decomposition" the decomposition of an IMA(2,2) model into the sum of an IMA(2,0) model plus orthogonal white noise.

Starting with a disaggregate HP-type decomposition, we aggregate the filters, and see if they are consistent with a direct HP-type decomposition on the aggregate series. As shown by Wei and Stram (1986), aggregation preserves the order of integration of the series, thus, from (2.4), we consider $D^2 X_T$. From the identity $D^2 X_T = X_T - 2X_{T-1} + X_{T-2}$ it is easily found using (3.1) that $D^2 X_T = \nabla_k^2 S_k x_t$, or using (2.4) and (3.2),

$$D^2 X_T = S_k^3 \theta_{HP}(b) a_t = \alpha(b) a_t \; , \tag{3.3}$$

where $\alpha(b)$ is a polynomial in b of order 3k-1. In terms of the aggregate time units, it follows that $D^2 X_T$ follows an IMA(2,2) process. Proceeding in a similar way with the aggregate trend component, it is found that $D^2 M_T = \nabla_k^2 S_k m_t = S_k^3 a_{mt}$, where $\alpha_m(b) = S_k^3$ is a polynomial in b of order 3k-3, and hence $D^2 M_T$ is also an IMA(2,2) model. As for the cycle, it is immediately seen that $C_T$ is a white-noise variable, with variance $\nu_C = k\nu_c$. The trend specification implies that the aggregate components cannot be seen as the result of a direct HP-type decomposition of the aggregate series: the HP-filter does not preserve itself under temporal aggregation.

When the aggregate variable is obtained by systematic sampling every k periods, the aggregate series is obtained as $X_{T-j} = x_{t-jk}$, j=0, 1, 2... , and, from

$D^2 X_T = X_T - 2X_{T-1} + X_{T-2} = (1 - 2b^k + b^{2k}) x_t = \nabla_k^2 x_t$ , using (2.4) and (3.2) it is found that $D^2 X_T = S_k^2 \theta_{HP}(b) a_t = \alpha(b) a_t$ where $\alpha(b)$ is of order 2k. It follows that, in terms of the aggregate time units, $D^2 X_T$ has nonzero covariances only for lags 0, ±1 and ±2. Similarly to the temporal aggregation case, the aggregate variable $X_T$ follows an IMA(2,2) model. Using a similar derivation for the (systematically sampled) aggregate trend component, it is obtained that $D^2 M_T = S_k^2 a_{mt}$ where $S_k^2$ is a polynomial in b of order 2(k-1). This implies that $M_T$ is an IMA(2,1) process. As for the aggregate cycle, $C_T$ $(= c_t)$ is a white noise variable, with variance $\nu_C = \nu_c$. Therefore, although systematic sampling implies a model for the aggregate trend filter different than the one implied by temporal aggregation (an IMA(2,1) versus an IMA(2,2) model), it still remains true that the decomposition for the aggregate obtained by aggregating the filters for the disaggregate components is not of the HP type.

Thus, on the one hand, aggregation of an HP cycle does not yield an aggregate cycle that can be seen as the result of a direct application of an HP-type cycle to the aggregate series. On the other hand, as mentioned in Section 1, a variety of (mostly arbitrary) values of $\lambda$ have been used for different frequencies of observation. The first question that comes to mind is: how relevant can be the lack of aggregation consistency between the different values of $\lambda$? Figures 1, 2, and 3 display the cycles estimated for the USA Industrial Production Index during the period 1962-2005 for different values of $\lambda$ and frequencies of observation. Figure 1 compares the estimates for the last 200 months using the RU rule ($\lambda = 130000$) and the EViews default value ($\lambda = 14400$). Figure 2 compares, for the case of systematic sampling, the cycles for the last 50 quarters obtained directly with the consensus value $\lambda = 1600$ and indirectly by aggregating the monthly cycles for the EViews default value. Finally, Figure 3 compares, for the case of temporal aggregation, the annual cycles for the full period obtained directly with the value $\lambda = 400$ and indirectly from the EViews monthly default value.

Figure 1

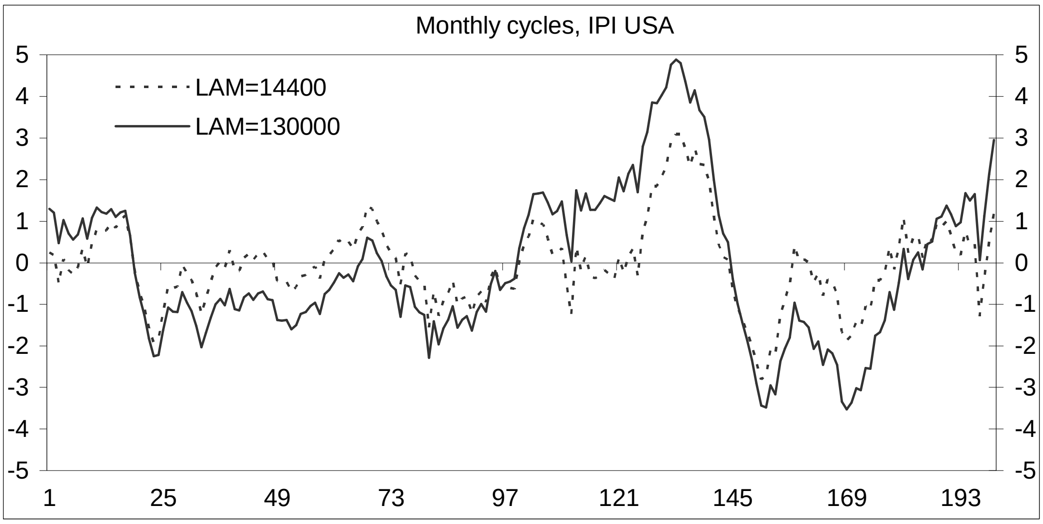



Figure 2

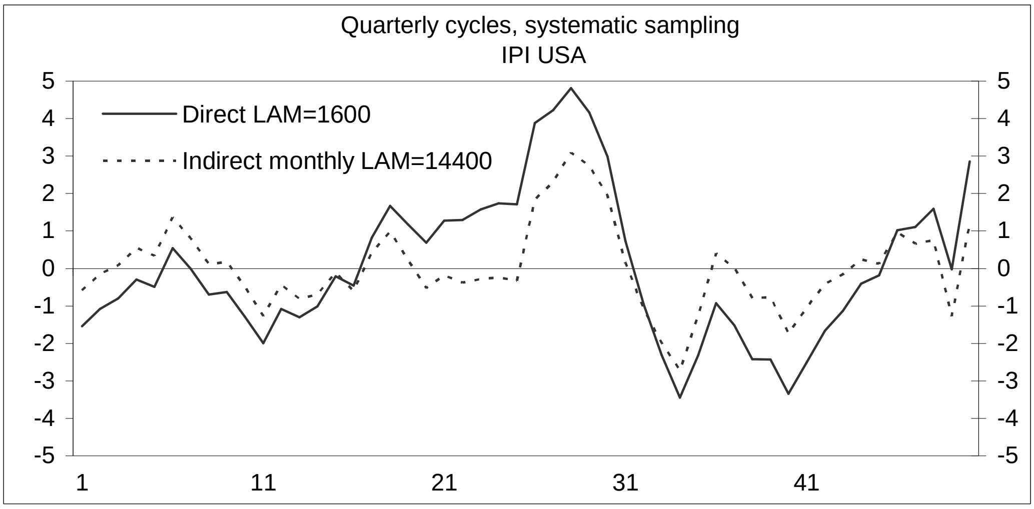
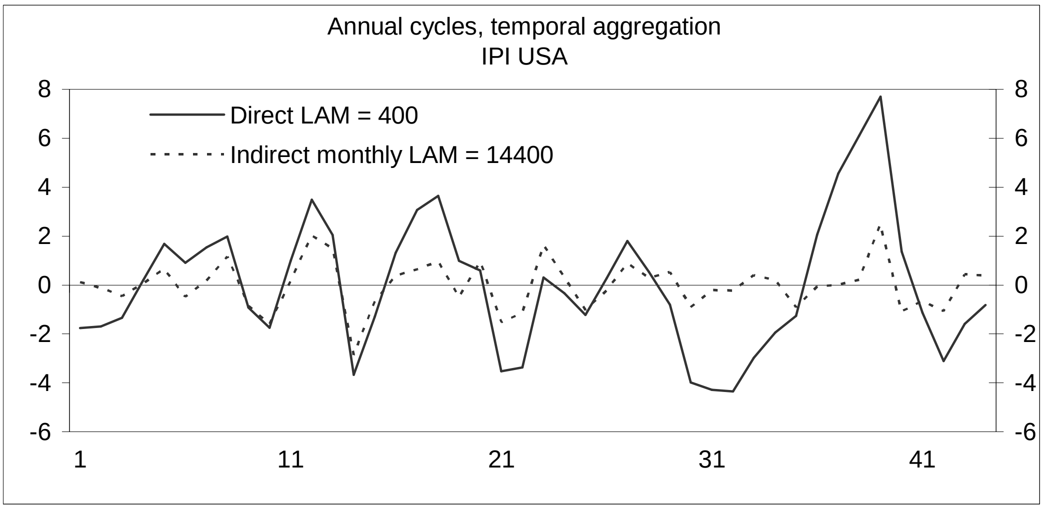



Figure 3



Direct inspection of the figures shows that, although the main features of the cycles (periods of sustained expansion or recession) are broadly robust to variations in $\lambda$, the differences are considerable and increase with the level of aggregation. The next question is whether one can, from the structure of the filter, derive $\lambda$ values for different frequencies of observation such that consistency under time aggregation is approximately preserved. Specifically, given the HP decomposition of the quarterly series,

a) can we obtain a value $\lambda_A$ that provides a direct HP decomposition of the annual series with components that are close to the ones obtained by aggregating the quarterly components?

b) can we obtain a value of $\lambda_M$ that provides an HP-decomposition of the monthly series with components that, when aggregated, are close to the components of the direct quarterly decomposition?

To get a first insight, we look at the aggregation properties of the model-based version of the HP filter. From (2.5), (3.2) and (3.3), $D^2 X_T = S_k^3 (a_{mt} + \nabla^2 c_t) = S_k^3 a_{mt} + S_k \nabla_k^2 c_t$, and the ACF of the r.h.s. of this equation is given by

$$S_k^3 \overline{S}_k^3 \nu_m + S_k \nabla_k^2 \overline{S}_k \overline{\nabla}_k^2 \nu_c \quad , \tag{3.4}$$

where an upper bar denotes the corresponding polynomial with B replaced by F (i.e. $\overline{\nabla}_k = 1 - F^k$). Given that, from (3.3), $D^2 X_{T-j} = S_k^3 \theta_{HP}(b) a_{t-jk}$, j = 0, 1, 2, the variance and lags 1 and 2 autocovariances of $D^2 X_{T-j}$, to be denoted $\Gamma_0$, $\Gamma_1$, and $\Gamma_2$, measured in the aggregate time units, are equal to the variance and lags k and 2k autocovariances of (3.4), to be denoted $\gamma_0$, $\gamma_{1k}$, and $\gamma_{2k}$, measured in the disaggregate time units. If the direct HP decomposition of the aggregate series $X_T$ is given by

$$X_T = M_T + C_T$$

$$D^2 M_T = A_{MT} \quad , \qquad A_{MT} \sim \text{w.n.}\ (0, \nu_M) \quad ,$$

with $C_T$ white noise (0, $\nu_C$) orthogonal to $A_{MT}$, setting $\Gamma_j = \gamma_{jk}$ (j=0,1,2), a system of three equations is obtained, of the type:

$$\begin{aligned} \nu_M + 6\nu_C &= a_{10}\nu_m + a_{20}\nu_c \\ -4\nu_C &= a_{11}\nu_m + a_{21}\nu_c \quad . \\ \nu_C &= a_{12}\nu_m + a_{22}\nu_c \end{aligned} \tag{3.5}$$

Given $\nu_m$ and $\nu_c$, the system of three equations with two unknowns will not have a solution for $\nu_M$ and $\nu_C$, i.e., for the HP-parameter of the aggregate decomposition $\lambda_A = \nu_C / \nu_M$. Inversely, given $\nu_M$ and $\nu_C$, the system will not have a solution for $\nu_m$ and $\nu_c$. (That is, given $\lambda_A$, no disaggregated value $\lambda_d = \nu_c / \nu_m$ can be found.)

Focusing on the relationship between monthly, quarterly, and annual aggregation the coefficients $a_{ij}$ in (3.5) are obtained from (3.4), yielding the matrices

$$A_{MQ} = \begin{bmatrix} 141 & 18 \\ 50 & -12 \\ 1 & 3 \end{bmatrix} , \qquad A_{QA} = \begin{bmatrix} 580 & 24 \\ 216 & -16 \\ 6 & 4 \end{bmatrix} ,$$

where the subindices refer to monthly to quarterly and to quarterly to annual aggregation. For the quarterly value $\lambda_Q = 1600$, setting $\nu_m = 1$ and $\nu_c = 1600$, it is easily seen that no equivalent annual value $\lambda_A$ can be found because the system has no solution for $\nu_M$ and $\nu_C$. Likewise, setting $\nu_M = 1$ and $\nu_C = 1600$ no equivalent monthly value $\lambda_M$ can be obtained because the system has no solution for $\nu_m$ and $\nu_c$. However, solving the system (3.5) in a least-square sense yields

$$\lambda_M = 114013 \quad ; \qquad \lambda_A = 7.19 \; , \tag{3.6}$$

values that can be seen to be fairly consistent with $\lambda_Q = 1600$. In summary, although there will be no set of $\lambda$'s that will exactly satisfy the aggregation constraints, we can find values that may provide close approximations. The criterion just used -based on a "fictitious" model-based interpretation of the filter- is not very appealing. More relevantly, in the next section we consider a simple criterion to obtain these values, based on the preservation of an important feature of the filter.

## 4. Aggregation by Fixing the Period for which the Gain is One Half (the Cycle of Reference).

In the engineering literature, a well-known family of filters designed to remove (or estimate) the low-frequency component of a series is the Butterworth family (see, for example, Pollock, 1997, or Gómez, 2001). The filter is described by its gain function which, for the two-sided expression and the sine-type subfamily, can be expressed as ($\omega$ measured in radians)

$$G_m(\omega) = \left[ 1 + \left( \frac{\sin(\omega/2)}{\sin(\omega_0/2)} \right)^{2d} \right]^{-1}, \qquad 0 \le \omega \le \pi , \tag{4.1}$$

and depends on two parameters, d and $\omega_0$. Given that $G(\omega_0) = .5$, the parameter $\omega_0$ is the frequency for which 50% of the filter gain has been achieved (see Figure 4). We shall refer to the cycle associated with that frequency as the "cycle of reference". Setting d=2 and $\beta = \left[\sin^4(\omega_0/2)\right]^{-1}$, the gain can also be expressed as

$$G_m(\omega) = \left[1 + \beta \sin^4(\omega/2)\right]^{-1} . \tag{4.2}$$

From the identity $2\sin^2(\omega/2) = 1 - \cos(\omega)$, (4.2) can be rewritten as

$$G_m(\omega) = \left[1 + (\beta/4)(1-\cos\omega)^2\right]^{-1} ,$$

which, considering (2.12), shows that the filter is precisely the HP filter, with $\lambda = \beta/16$, or

$$\lambda_0 = \left[4(1-\cos\omega_0)^2\right]^{-1} . \tag{4.3}$$

Therefore, knowing the parameter $\omega_0$, the HP filter parameter $\lambda$ can be easily obtained, and viceversa. If $\tau$ denotes the period of $\cos\omega$, $\tau$ is related to $\omega$ through

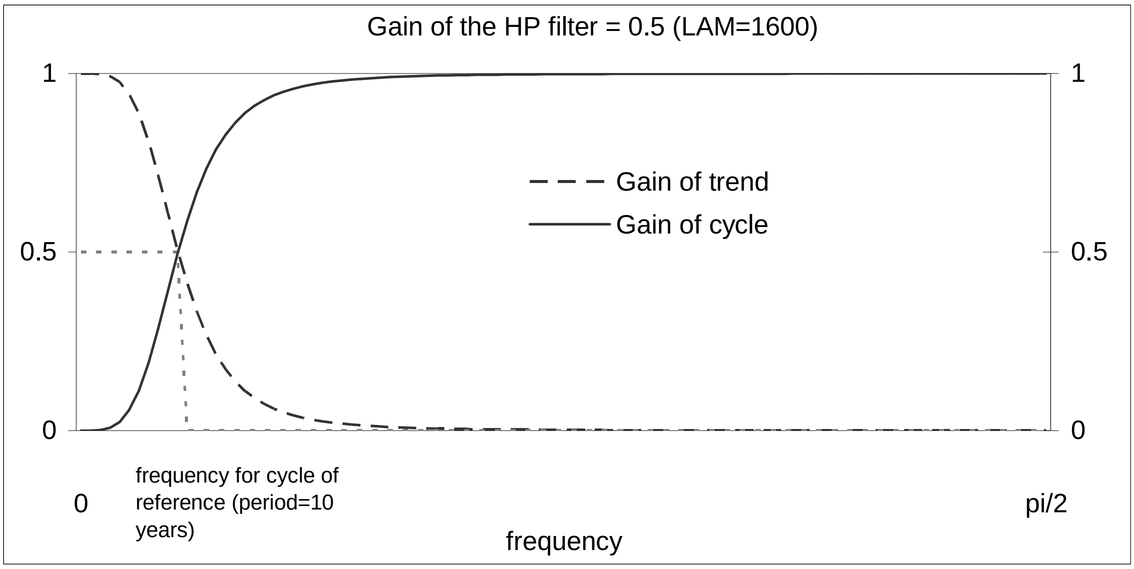
$$\tau = 2\pi/\omega . \tag{4.4}$$

Using (4.3) and (4.4), we can express the period $\tau$ directly as a function of $\lambda$, as

$$\tau = 2\pi / a\cos\left(1 - \frac{1}{2\sqrt{\lambda}}\right) . \tag{4.5}$$

Equations (4.3)-(4.5) allow us to move from period to frequency, and then to $\lambda$ (and viceversa) in a simple way. The frequency $\omega_0$ -or its associated period $\tau_0$ - provide a more intuitive and alternative characterization of the cycle than the HP parameter $\lambda$: $\tau_0$ represents a cutting point, in the sense that cycles with periods $\tau > \tau_0$ will be mostly assigned to the trend, while those with $\tau < \tau_0$ will be mostly assigned to the cycle. For example, from (4.5) the consensus value $\lambda_Q = 1600$ implies an associated period of (very approximately) 10 years. The choice of a 10-year period cutting point seems easier to interpret than the Hodrick and Prescott rationalization quoted in Section 1. The preservation of the period of the cycle of reference provides an intuitively attractive criterion for finding values of $\lambda$ that yield relatively consistent results under aggregation. (Within the context of Butterworth highpass filters, Pollock (2003) addresses the related problem of transforming the filter cut-off frequency. His procedure relies on replacing the B operator (viewed as a complex variable) by an appropriately chosen function that preserves some properties of the filter. The transformed filter, however, will not be of the HP-type.)

Figure 4




Our procedure amounts to the following. Starting with $\lambda_0$ for some frequency of observation (for example, quarterly), the associated period $\tau_0$ (in quarters) is found through (4.5). Consider another frequency of observation (for example, monthly or annual). Expressed in this frequency, $\tau_0$ implies the period

$$\tau_A = k \ \tau_0 \tag{4.6}$$

where k is as in (1.1). From (4.4), $\omega_A = \omega_0 / k$, and $\lambda_A$ is obtained from (4.3) with $\omega_0$ replaced by $\omega_A$. The procedure of finding $\lambda$'s that are approximately consistent under aggregation by means of fixing the period of the cycle for which the gain of the filter is .5 is simple to apply, and can be used for aggregating or disaggregating series with any frequency of observation.

For the consensus value $\lambda_Q = 1600$ for quarterly series, (4.5) implies a period of 39.7 quarters. Thus, for annual data, the period of the cycle of reference is, according to (4.6), $\tau = 9.9$ years. From (4.4), $\omega_A = 2\pi / 9.9$, and (4.3) yields $\lambda_A = 6.65$. On the other hand, in terms of monthly observations, the period of 39.7 quarters is equal to 119.1 months. Using (4.4) and (4.3), it is found that the equivalent value for monthly data is $\lambda_M = 129119$. Thus, using this criterion, values of $\lambda$ that are consistent for the monthly and annual frequencies of observation are

$$\lambda_M = 129119 \ ; \qquad \lambda_A = 6.65 \ , \tag{4.7}$$

These values are very close the ones that result from the RU rule, and not far from the values in (3.6). An example can illustrate the difference with respect other proposed values. In Giorno et al. (1995) the method used by the OECD for the estimation of the output gap is described: it uses the HP filter with $\lambda_Q = 1600$ and $\lambda_A = 100$. These values are referred to as "de facto industry standards"; they are also used by the European Central Bank (2000) and the default values in EViews. Using $\lambda_A = 100$ for annual data, from (4.5), the period of the cycle of reference is $\tau_A = 19.8$ years or, very approximately, 20 years, which in terms of quarterly data becomes $\tau_Q = 79.2$ quarters. From (4.3) and (4.4), this would imply the use of $\lambda_Q = 25199$, considerably higher than the "de facto" value of 1600.

For the cases of temporal aggregation and systematic sampling, figures 5, 6, 7, and 8 compare the direct and indirect cycles obtained with (4.7) for the USA IPI example: the two cycles are seen to be virtually indistinguishable for the case of temporal aggregation, and rather close for the case of systematic sampling.

Figure 5

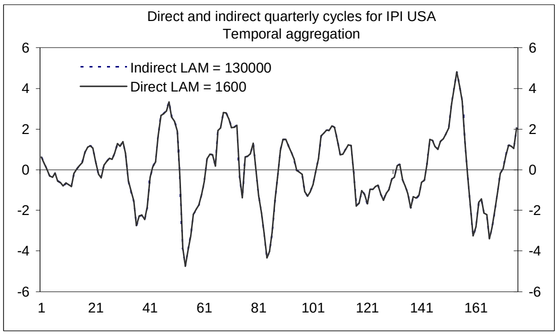



Figure 6

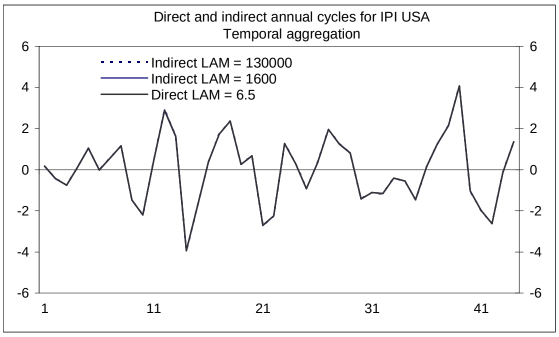



Figure 7

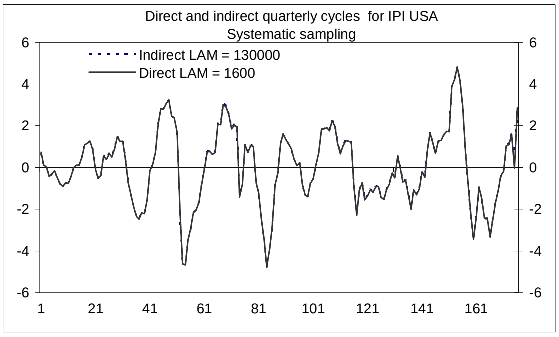
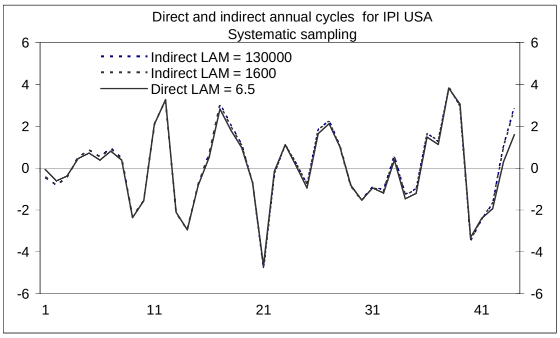



Figure 8




The convenience of using $\lambda$ values that are consistent under time aggregation is illustrated with the following example. Figure 9 shows the cycles estimated for the quarterly USA IPI and unemployment series during the period 1962-2005 using the consensus value $\lambda_Q = 1600$ for both. The figure reveals a very stable inverse relationship between the two cycles throughout the entire period. Recessions in industrial production are associated with expansions in unemployment, and viceversa, with the association moving in nearly perfect phase. (The zeroes for both series are nearly identical and the lag-zero crosscorrelation is -.88.)

Figures 10 and 11 show the monthly and annual cycles for the two series using the $\lambda$-values obtained with the criterion of maintaining the period associated with the 50% gain. It is seen how the relationship between the two series is perfectly preserved, so that inferences concerning the relationship between the cycles are robust with respect to the measurement time units.

Figure 9

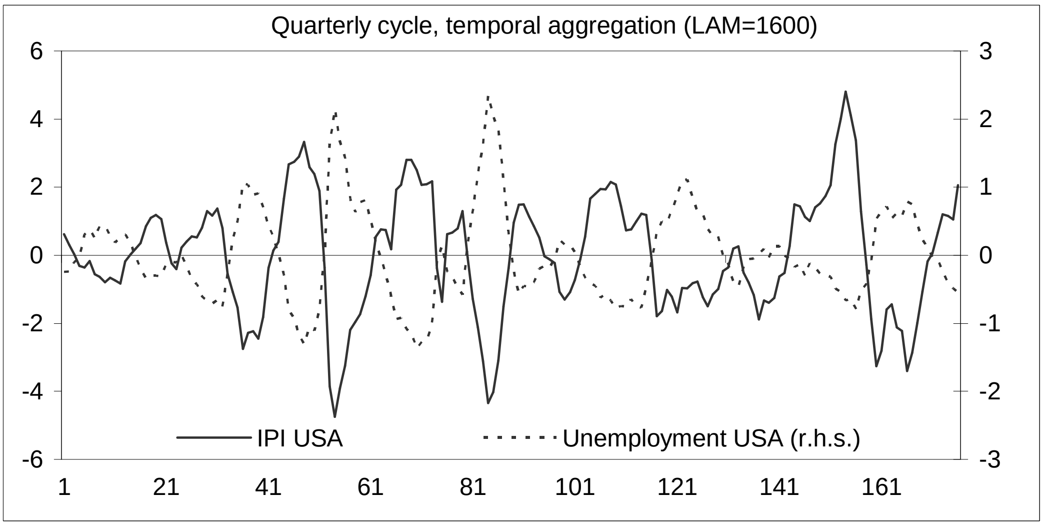



Figure 10

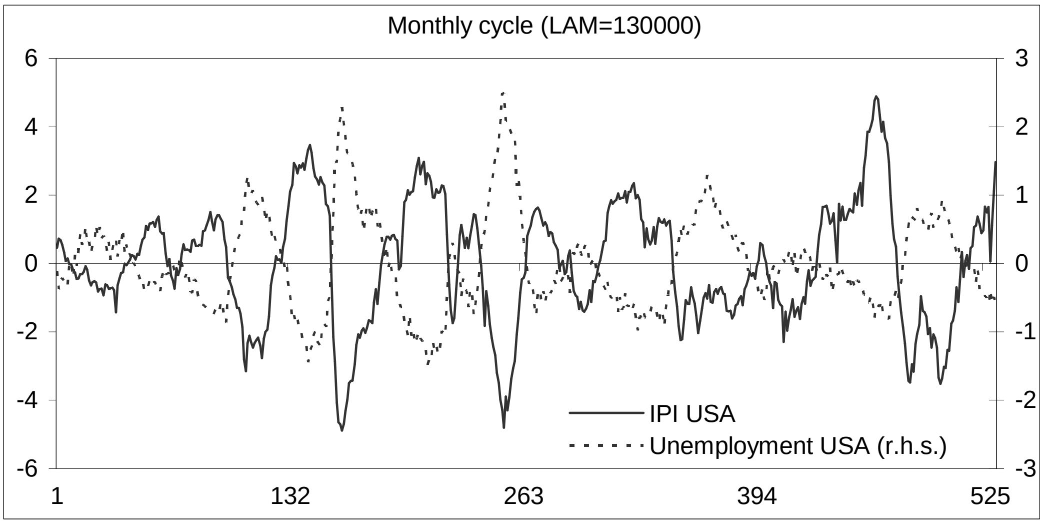
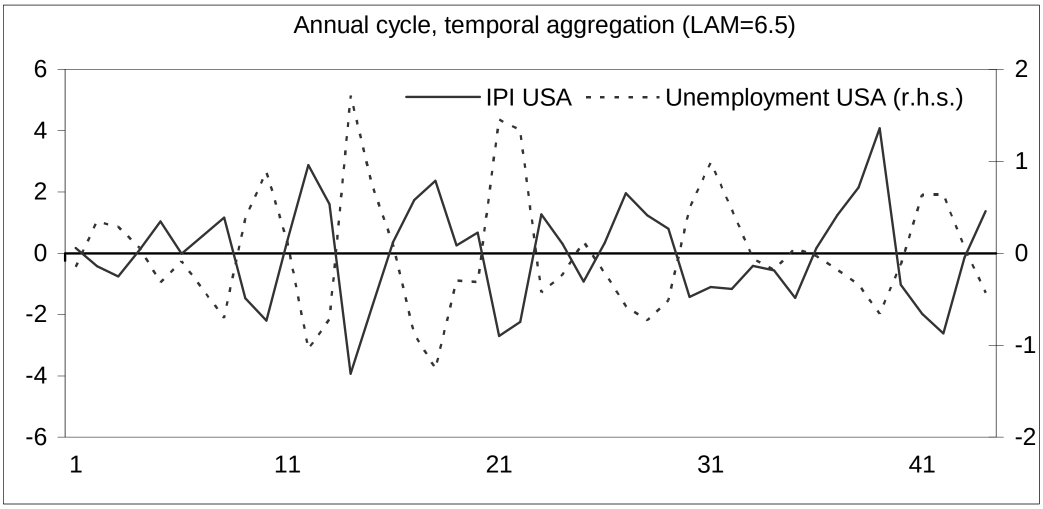



Figure 11



## 5. Relationship with the Ravn and Uhlig Rule

As mentioned in the introduction, Ravn and Uhlig (2002) provide a simple rule to compute values of $\lambda$ for different frequencies of observation that appear to be approximately consistent under aggregation. If $\lambda_Q$ is the reference value for quarterly data, and $\lambda_D$ denotes the value for an alternative frequency of observation, they look at relationships of the type (1.1) and present evidence that good results are obtained for $j=4$. If $\lambda_Q = 1600$, this rule yields the monthly and annual values $\lambda_M = 129600$, $\lambda_A = 6.25$, close to the ones obtained in (4.7). This closeness can easily be explained.

Let $\lambda_0$ be the HP parameter for a given periodicity of observation, and let $\omega_0$ and $\tau_0$ be the frequency and period associated with $G(\omega_0) = .5$. We wish to obtain the equivalent value for $\lambda_0$, say $\lambda_A$, for another observation periodicity, using the criterion of preserving the period $\tau_0$. Let $\omega_A$ and $\tau_A$ be the frequency and period associated to $G(\omega_A) = .5$. Then, preservation of the period implies that $\tau_A = k\,\tau_0$ or, equivalently, $\omega_A = \omega_0 / k$, so that, according to (4.3),

$$\lambda_A = \frac{1}{4\left(1-\cos\left(\omega_0 / k\right)\right)^2} \quad . \tag{5.1}$$

Further, from (4.3),

$$\cos\omega_0 = 1 - 1/(2\sqrt{\lambda_0}\,) \quad , \tag{5.2}$$

and assume $\lambda_0$ is relatively large. Considering the power series expansion

$$\cos x = 1 - x^2/2 + \text{higher order terms} , \tag{5.3}$$

letting $x = \omega_0$ and comparing (5.2) and (5.3), after simplification,

$$\omega_0 \cong \lambda_0^{-1/4} \quad . \tag{5.4}$$

[A remark on the approximation implied by (5.4): For the consensus value $\lambda_0 = 1600$, (5.2) yields $\omega_0 = .15828$, or $\tau_0 = 39.70$ quarters, while through (5.3) one obtains $\omega_0 = .15811$, or $\tau_0 = 39.74$ quarters, that is, the difference is approximately one tenth of a month.]

Letting $x = \omega_0 / k$ in (5.3), $\cos(\omega_0 / k) \cong 1 - \omega_0^2 / 2k^2$, so that, considering (5.4), (5.1) becomes

$$\lambda_A \cong k^4 \ \lambda_0 \ . \tag{5.5}$$

Expression (5.5) shows that the RU rule turns out to be a first-order approximation to the criterion of preserving the period of the cycle for which the gain of the filter is .5. The approximation will work better for larger values of $\lambda$, as shown in Table 1. (Note: $\tau$ for RU is the period associated with Gain = .5 when the RU value of $\lambda$ is employed.)

Table 1: Performance of approximation

| Frequency of observation | | G = .5 criterion | RU criterion |
|---|---|---|---|
| Every month | $\lambda$ | 129120 | 129600 |
| | $\tau$ (months) | 119.1 | 119.2 |
| Every 2 months | $\lambda$ | 8081 | 8100 |
| | $\tau$ (2 months) | 59.55 | 59.58 |
| Every 3 months | $\lambda$ | 1600 | --- |
| | $\tau$ (quarter) | 39.70 | --- |
| Every 4 months | $\lambda$ | 508 | 506 |
| | $\tau$ (4 months) | 29.77 | 29.75 |
| Every 6 months | $\lambda$ | 101.3 | 100 |
| | $\tau$ (6 months) | 19.85 | 19.79 |
| Once a year | $\lambda$ | 6.65 | 6.25 |
| | $\tau$ (years) | 9.92 | 9.76 |

From the table, starting from the quarterly value of $\lambda_Q = 1600$, the period of the cycle associated with $\omega$ such that $G(\omega) = .5$ is $\tau = 119.1$ months. Let $\lambda$ denote the value for another frequency of observation, obtained with the same criterion, and let $\lambda^{RU}$ denote the value obtained with the RU criterion. If $\tau^{RU}$ is the period of the cycle associated with Gain = .5 when $\lambda^{RU}$ is used, for monthly data: $\tau_M - \tau_M^{RU} = .1$ months; for annual data: $\tau_A - \tau_A^{RU} = 1.9$ months; for data recorded every two years: $\tau_{2Y} - \tau_{2Y}^{RU} = 9$ months. Thus the annual frequency seems to provide a rough limit for the validity of the approximation, and for observations recorded with a once-a-year or higher frequency, the two criteria give similar results. The criterion of preserving the period of the cycle that represents the cutting point between “mostly trend” and “mostly cycle” periods provides a sensible rationale to the empirical rule of RU.

## 6. Aggregation by Fixing the Period Associated with the Maximum of the Cycle Spectrum

The previous criterion of preserving the cycle of reference is based solely on properties of the HP filter. But ultimately, the properties of the cycle obtained are a combination of two factors: the characteristics of the filter and the stochastic properties of the series in question. We consider now their interaction. To describe the cycle we consider its spectrum, which can be computed through expression (2.13). (In Kaiser and Maravall (2005) the general model for the cycle is derived, and it is seen to be a stationary ARMA model, with the polynomial $\theta_{HP}(B)$ of (2.4) appearing in the autoregressive part.) Series with different stochastic structures will imply different spectra for the cycle even when the same HP filter is used.

As an example, consider two series that follow a standard and a second-order random-walk model, namely

$$\nabla x_{1t} = a_t \quad , \tag{6.1a}$$

$$\nabla^2 x_{2t} = a_t \quad . \tag{6.1b}$$

Expressions (2.8) and (6.1) show that the estimators of the cycle can be expressed in terms of the innovations $a_t$ as

$$\hat{c}_{1t} = k_c \frac{(1-B)(1-F)^2}{\theta_{HP}(B)\,\theta_{HP}(F)} a_t \quad , \tag{6.2a}$$

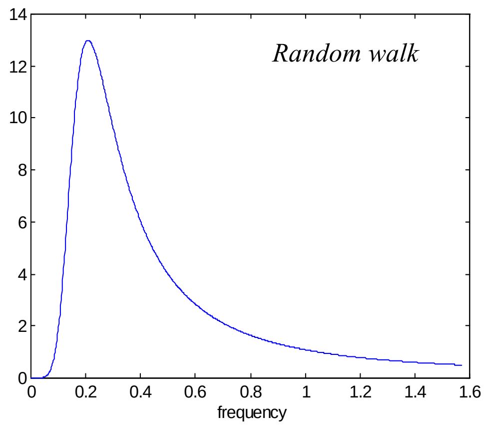
$$\hat{c}_{2t} = k_c \frac{(1-F)^2}{\theta_{HP}(B)\,\theta_{HP}(F)} a_t \quad , \tag{6.2b}$$

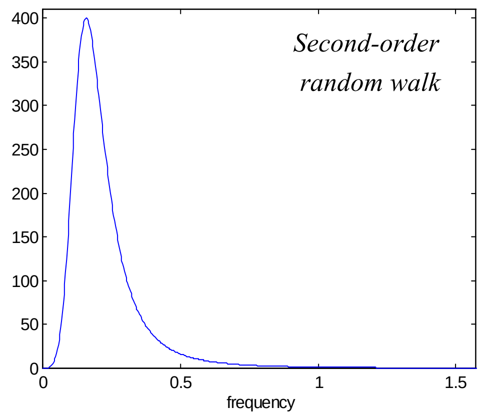
where $k_c = \nu_c / \nu_a$. The FT of the ACF of (6.2a) and (6.2b) yield the two spectra, namely,

$$S_{c1}(\omega) = \frac{8\lambda^2 (1-\cos\omega)^3}{\left[1+4\lambda(1-\cos\omega)^2\right]^2} V_a \ ,$$

$$S_{c2}(\omega) = \frac{4\lambda^2 (1-\cos\omega)^2}{\left[1+4\lambda(1-\cos\omega)^2\right]^2} V_a \ .$$

Figure 12: Spectra of the cycle component of first and second-order random walk (λ=1600)




The two spectra are displayed in Figure 12 for $\lambda = 1600$. Both have the shape of a stochastic cyclical component spectrum, with the variance concentrated around the spectral peak. The cycle associated with that peak will be denoted the "cycle of dominance". A natural criterion for aggregation could be preservation of the cycle of dominance. (This is similar to approximating spectral densities by preserving the mode, see Durbin and Koopman (2000)). An advantage of this approach is that it combines the characteristics of the filter with the specific features of the series, it has the disadvantage that no general rule to find equivalent values of $\lambda$ can be obtained, since the equivalence depends on the ARIMA model for the series. Nevertheless, two issues are of interest. First, what is the equivalence for some of the most relevant ARIMA models? Second, if the simpler criterion of fixing the period associated with the cycle of reference of Section 4 (or the RU rule) is used, are the results likely to be much different from the ones obtained with the criterion of fixing the period associated with the cycle of dominance?

We consider IMA(1,1) and IMA(2,2) models, both of which are consistent under temporal aggregation and systematic sampling (Brewer (1973)). (The IMA(d,d) formulation is also attractive because it is the limiting model for time aggregates of ARIMA(p,d,q) models; Tiao (1972).) We encompass both cases under the specification

$$\nabla^d x_t = \theta(B) a_t = (1 + \theta_1 B + \theta_2 B^2) a_t , \tag{6.3a}$$

where $d = 1$ and $\theta_2 = 0$ for the IMA(1,1) case, and $d = 2$ for the IMA(2,2) one. For an alternative (aggregate) frequency of observation (6.3a) becomes

$$\nabla^d X_t = \Theta(B) A_t = (1 + \Theta_1 B + \Theta_2 B^2) A_t . \tag{6.3b}$$

Let $\theta_Q = (\theta_1, \theta_2)$ and $\theta_D = (\Theta_1, \Theta_2)$ denote the vectors with the MA parameters of the quarterly model and of the model for the alternative frequency of observation (annual or monthly). Likewise, let $S_Q(\omega | \theta_Q, \lambda_Q)$ and $S_D(\omega | \theta_D, \lambda_D)$ denote the corresponding spectra. The procedure to obtain the equivalent values of $\lambda$ for the transformed series can be summarized as follows.

1. Given $\theta_Q$ and $\lambda_Q$, obtain the frequency $\omega_Q$ ($\omega \in [0, \pi]$) such that $S_Q(\omega | \theta_Q, \lambda_Q)$ is maximized and the associated period $\tau_Q$.
2. Transform $\tau_Q$ into $\tau_D$ and obtain the associated frequency $\omega_D$.
3. Use the relationship between the variance and covariances of the disaggregate and aggregate series to find $\theta_D$ given $\theta_Q$.
4. Find $\tilde{\omega}$ such that $S_D(\omega | \theta_D, \lambda_D)$ is maximized, and $\lambda_D$ such that $\tilde{\omega} = \omega_D$.

Although the procedure is general, in our application we fix $\lambda_Q = 1600$ for quarterly data and derive the values $\lambda_M$ and $\lambda_A$ that preserve the period associated with the cycle spectral peak. Step 3 requires the derivation of the model for the annual or monthly series, given the model for the quarterly one. If (6.3a) is the model for the more disaggregate series, the model for the aggregate series will be of the type (6.3b). In order to obtain the $\Theta$ and $V_A$ parameters, we follow the Wei and Stram (1986) approach, extended to cover also the case of systematic sampling, as detailed in Appendix B. In brief, if $\Gamma$ and $\gamma$ denote the vector of autocovariances of $\nabla^d X_T$ and $\nabla^d x_t$, a matrix M is computed such that

$$\Gamma = M \gamma . \tag{6.4}$$

Expressing $\Gamma$ and $\gamma$ as functions of the model parameters, the parameters for the alternative frequency can be obtained as functions of the quarterly ones.

For the IMA(1,1), and IMA(2,2) models, the matrices M in (6.4) that relate annual to quarterly, and quarterly to monthly, covariances are given in Table 2, for both the temporal aggregation and systematic sampling cases. (Appendix B details some examples.)

Table 2: Matrices M that relate aggregate and disaggregate covariances.

| Model | Frequencies | Temporal Aggregation | Systematic Sampling |
|---|---|---|---|
| IMA(1,1) | Quarterly to Annual Aggreg. | $\begin{bmatrix} 44 & 80 \\ 10 & 24 \end{bmatrix}$ | $\begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$ |
| | Monthly to Quarterly Aggreg. | $\begin{bmatrix} 19 & 32 \\ 4 & 11 \end{bmatrix}$ | $\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$ |
| IMA(2,2) | Quarterly to Annual Aggreg. | $\begin{bmatrix} 580 & 1092 & 912 \\ 216 & 456 & 512 \\ 6 & 22 & 56 \end{bmatrix}$ | $\begin{bmatrix} 44 & 80 & 62 \\ 10 & 24 & 32 \\ 0 & 0 & 1 \end{bmatrix}$ |
| | Monthly to Quarterly Aggreg. | $\begin{bmatrix} 141 & 252 & 180 \\ 50 & 111 & 132 \\ 1 & 6 & 21 \end{bmatrix}$ | $\begin{bmatrix} 19 & 32 & 20 \\ 4 & 11 & 16 \\ 0 & 0 & 1 \end{bmatrix}$ |

## 6.1. IMA(1,1) Model

When $x_t$ follows an IMA(1,1) model, combining (6.3a) -with $d=1$ and $\theta_2 = 0$- with (2.8), it is seen that the cycle estimator follows the model

$$\hat{c}_t = \frac{\nu_c}{\nu_a} \frac{(1-B)(1-F)^2}{\theta_{HP}(B)\,\theta_{HP}(F)} (1+\theta B)\, a_t \ ,$$

so that, considering (2.10), its spectrum is given by

$$S_c(\omega|\theta,\lambda) = \frac{8\lambda^2 (1-\cos\omega)^3}{\left[1+4\lambda(1-\cos\omega)^2\right]^2}(1+\theta^2+2\theta\cos\omega)V_a \ , \tag{6.5}$$

and maximizing (6.5) with respect to $\omega$ yields

$$\tilde{\omega} = a\cos\left[1+\frac{\theta}{\lambda(1+\theta)^2} - \sqrt{\frac{3}{4\lambda}+\frac{\theta^2}{\lambda^2(1+\theta)^4}}\right] . \tag{6.6}$$

Solving (6.6) for $\lambda$, it is obtained that

$$\tilde{\lambda} = \frac{3}{4\left(1-\cos\tilde{\omega}\right)^2} - \frac{2\theta}{(1+\theta)^2\left(1-\cos\tilde{\omega}\right)} \quad . \tag{6.7}$$

(a) From Quarterly to Annual Data

Steps (1) and (2) above are a direct application of (6.6) and (4.4) with $\theta = \theta_Q$ and $\lambda = \lambda_Q$, and of (4.6) with k = ¼ . From (6.3a) and (6.3b),

$$\gamma = (\gamma_0, \gamma_1) = [(1+\theta_Q^2)V_a, \theta_Q V_a]$$

$$\Gamma = (\Gamma_0, \Gamma_1) = [(1+\Theta_1^2)V_A, \Theta_1 V_A] \;,$$

or, considering (6.4), with the appropriate M matrix from Table 2,

$$\Gamma_0 = (44 + 80\,\theta_Q + 44\,\theta_Q^2)V_a \quad ,$$

$$\Gamma_1 = (10 + 24\,\theta_Q + 10\,\theta_Q^2)V_a \quad .$$

Therefore,

$$\frac{1+\Theta_1^2}{\Theta_1} = \frac{44 + 80\,\theta_Q + 44\,\theta_Q^2}{10 + 24\,\theta_Q + 10\,\theta_Q^2} \quad ,$$

and, letting $c = (44 + 80\,\theta_Q + 44\,\theta_Q^2)/(10 + 24\,\theta_Q + 10\,\theta_Q^2)$, the MA parameter of the annual IMA(1,1) model is given by the invertible solution of equation

$$\Theta_1^2 - c\,\Theta_1 + 1 = 0 \quad . \tag{6.8}$$

For the case of systematic sampling, using the appropriate matrix from Table 2 and proceeding in a similar way,

$$\Gamma_0 = (4 + 6\,\theta_Q + 4\,\theta_Q^2)V_a \quad ,$$

$$\Gamma_1 = \theta_Q V_a \quad .$$

Defining $c = (4 + 6\,\theta_Q + 4\,\theta_Q^2)/\theta_Q$, the MA parameter for the IMA(1,1) annual model is again the invertible solution of (6.8). Having obtained $\Theta_1$, setting $\theta = \Theta_1$, and $\tilde{\omega} = \omega_D$ in (6.7), the equivalent value of $\lambda$ for annual series, $\lambda_A$, is obtained. The period associated with the cycle spectral maximum will be identical for the quarterly and annual series.

(b) From Quarterly to Monthly Data

Step (1) and (2) are as in the previous case, except that now, $\tau_M = 3\tau_Q$, the aggregate series is the quarterly one, and hence $\Gamma_0 = (1+\theta_Q^2)V_a$, $\Gamma_1 = \theta_Q V_a$, $\gamma_0 = (1+\theta_M^2)V_a$, and $\gamma_1 = \theta_M V_a$. Using the appropriate matrix M from Table 2, the system of covariance equations is given by

$$(1+\theta_Q^2)V_A = (19+32\theta_M + 19\theta_M^2)V_a$$

$$\theta_Q V_A = (4+11\theta_M + 4\theta_M^2)V_a$$

Letting $c_1 = (1+\theta_Q^2)/\theta_Q$, it is obtained that $\theta_M$ is the invertible solution of the equation $x^2 + c_2 x + 1 = 0$, where $c_2 = (32 - 11c_1)/(19 - 4c_1)$. The equation has complex solutions when $\theta_Q \geq 0.3$ so that IMA(1,1) monthly models aggregate into IMA(1,1) quarterly models with the MA parameter restricted to the range $-1 < \theta_Q < 0.3$.

For the case of systematic sampling and using the appropriate M matrix from Table 2, the system of covariance equations is replaced by:

$$(1+\theta_Q^2)V_A = (3+4\theta_M + 3\theta_M^2)V_a$$

$$\theta_Q V_A = \theta_M V_a \quad ,$$

so that, if $c_1 = (1+\theta_Q^2)/\theta_Q$ and $c_2 = (4 - c_1)/3$, the value of $\theta_M$ is the invertible solution of the equation $x^2 + c_2 x + 1 = 0$. The system yields complex solutions when $\theta_Q > 0.33$ and hence systematic sampling of monthly IMA(1,1) models yields quarterly IMA(1,1) models with the MA parameter restricted to the range $-1 < \theta_Q < 0.33$.

With the quarterly value set at $\lambda_Q = 1600$, Table 3 displays the equivalent monthly and annual values of $\lambda$, obtained with the criterion of preserving the period associated with the cycle spectral peak, when the series follows an IMA(1,1) process, and for different values of the MA parameter $\theta_Q$. It is seen that when $\theta_Q$ is not close to -1, the period associated with the cycle spectral peak takes a value between roughly 7 and 7.5 years. Thus the model parameter has a moderate effect on the period of the cycle of dominance.

Table 3: IMA(1,1): monthly and annual $\lambda$ values that preserve the period of the cycle spectral peak for $\lambda_Q$=1600.

| $\theta_Q$ | Period of the cycle of dominance (in years) | Equivalent values of $\lambda$ | | | |
|---|---|---|---|---|---|
| | | temporal aggregation | | systematic sampling | |
| | | annual ($\lambda_A$) | monthly ($\lambda_M$ in $10^3$) | annual ($\lambda_A$) | monthly ($\lambda_M$ in $10^3$) |
| -0.8 | 5.72 | 6.53 | 129.3 | 20.86 | 71.4 |
| -0.6 | 7.14 | 6.12 | 129.8 | 10.85 | 112.0 |
| -0.4 | 7.41 | 6.05 | 129.8 | 8.21 | 123.4 |
| -0.2 | 7.50 | 6.03 | 129.9 | 7.33 | 127.2 |
| 0.0 | 7.53 | 6.02 | 129.9 | 6.97 | 128.8 |
| 0.2 | 7.55 | 6.02 | 129.9 | 6.81 | 129.8 |
| 0.4 | 7.56 | 6.01 | - (*) | 6.74 | - (*) |
| 0.6 | 7.56 | 6.01 | - | 6.70 | - |
| 0.8 | 7.56 | 6.01 | - | 6.69 | - |

(*) Values of $\theta_Q$ for the lines marked "-" cannot be obtained by aggregation of monthly IMA(1,1) models.

When aggregation of the series is made through temporal aggregation the results are seen to be very stable. The monthly equivalent values $\lambda_M$ are always close to 130000, and the annual equivalent value $\lambda_A$ lies between 6 and 6.5 . These values are close to the ones obtained with the RU rule or the criterion of preserving the period of the cycle of reference, given by (4.7), and far from the "de facto industry standards." When aggregation is achieved through systematic sampling, the results are less stable, in particular in the neighborhood of $\theta_Q = -1$.

## 6.2. IMA(2,2) Model

When $z_t$ follows the IMA(2,2) model given by (6.3a), from (2.8) and (2.13) it is found that the HP cycle follows the model

$$\hat{c}_t = \frac{\lambda(1-F)^2 (1+\theta_1 B+\theta_2 B^2)}{1+\lambda(1-B)^2 (1-F)^2} a_t \quad ,$$

with spectrum

$$S_c(\omega|\lambda,\theta_1,\theta_2)=\frac{4\lambda^2(1-\cos\omega)^2\left[1+\theta_1^2+\theta_2^2+2\theta_1(1+\theta_2)\cos\omega+2\theta_2\cos 2\omega\right]}{\left[1+4\lambda(1-\cos\omega)^2\right]^2}V_a$$

The maximum with respect to $\omega$ is achieved for the real and positive solution of a third degree polynomial in $\cos\omega$. Let this solution be

$$\tilde{\omega}=\tilde{\omega}(\lambda,\theta_1,\theta_2) \quad . \tag{6.9}$$

or, solving for $\lambda$,

$$\tilde{\lambda}=\frac{1}{4(1-\cos\tilde{\omega})}-\frac{\theta_1+\theta_1\theta_2+4\theta_2\cos\tilde{\omega}}{2(1-\cos\tilde{\omega})\left[1+\theta_1(\theta_1+1+\cos\tilde{\omega})+\theta_2(\theta_2-2)+(2+\theta_1)\cos\tilde{\omega})\right]} \quad . \tag{6.10}$$

Proceeding as in the previous section, given $\theta_Q=(\theta_1,\theta_2)$ for the quarterly model and $\lambda_Q$, we use (6.9) to compute the frequency for which the spectrum of the quarterly cycle reaches a maximum, and the associated period. Expressing this period in terms of annual and monthly data, we obtain the annual and monthly associated frequencies. Once we know the parameters $\theta_1$ and $\theta_2$ of the annual and monthly model, (6.10) provides the values of $\lambda_A$ and $\lambda_M$. The monthly and annual series also follow IMA(2,2) models and, in order to derive the parameters, we follow as before the Wei-Stram procedure.

Let $x_t$, $(\theta_1,\theta_2)$, and $V_a$ denote the disaggregate series, the MA parameters of its model, and its innovation variance, respectively. Likewise, let $X_T$, $(\Theta_1,\Theta_2)$, and $V_A$ denote the aggregate series, the MA parameters of its model, and its innovation variance. If $(\gamma_0,\gamma_1,\gamma_2)$ and $(\Gamma_0,\Gamma_1,\Gamma_2)$ represent the variance, lag-1, and lag-2 autocovariances of $\nabla^2 x_t$ and $\nabla^2 X_T$, respectively, we have

$$\gamma_0=(1+\theta_1^2+\theta_2^2)V_a \ , \tag{6.11a}$$

$$\gamma_1=\theta_1(1+\theta_2)V_a \ , \tag{6.11b}$$

$$\gamma_2=\theta_2 V_a \quad , \tag{6.11c}$$

and, replacing $(\theta_1,\theta_2)$, and $V_a$ by $(\Theta_1,\Theta_2)$, and $V_A$, similar expressions hold for $\Gamma_0$, $\Gamma_1$ and $\Gamma_2$. If $\gamma$ and $\Gamma$ denote the vectors $\gamma=(\gamma_0,\gamma_1,\gamma_2)'$ and $\Gamma=(\Gamma_0,\Gamma_1,\Gamma_2)'$, the relevant M matrices in (6.4) are given in Table 2.

Given $\gamma$, one can obtain $\Gamma$ and, using the inverse relationship $\gamma = M^{-1}\Gamma$, given $\Gamma$, one can obtain $\gamma$. The aggregate/disaggregate MA parameters are found by factorizing the ACF obtained, as explained in the Appendix A.

Table 4 is analogous to Table 3 for the IMA(2,2) case, and displays the monthly and annual $\lambda$ values that are consistent with the quarterly value $\lambda_Q = 1600$, under the criterion of preserving the period associated the cycle spectral peak (the MA values $\theta_1$ and $\theta_2$ are restricted to lie in the invertible region). Altogether, the $\theta$ parameters have a moderate effect on $\lambda_A$ and $\lambda_M$, and the results are similar to those obtained for the IMA(1,1) case. As before, they are also close to those obtained with the criterion of preserving the cycle of reference or the RU rule.

Table 4: IMA(2,2): monthly and annual $\lambda$ values that preserve the period of dominance for $\lambda_Q$=1600.

| $\theta_{Q,1}$ | $\theta_{Q,2}$ | Period of the cycle of dominance (years) | Equivalent values of $\lambda$ | | | |
|---|---|---|---|---|---|---|
| | | | temporal aggregation | | systematic sampling | |
| | | | annual ($\lambda_A$) | monthly ($\lambda_M$ in $10^3$) | annual ($\lambda_A$) | monthly ($\lambda_M$ in $10^3$) |
| 0.2 | 0.0 | 9.9 | 6.02 | 131.8 | 6.24 | 129.6 |
| 0.0 | 0.0 | 9.9 | 6.03 | 128.6 | 6.24 | 129.6 |
| 0.0 | 0.2 | 10.0 | 6.01 | 131.2 | 6.23 | 131.8 |
| -0.2 | 0.0 | 9.9 | 6.03 | 127.7 | 6.24 | 129.6 |
| -0.2 | 0.2 | 10.0 | 6.01 | 131.1 | 6.23 | 131.4 |
| -0.4 | 0.0 | 9.9 | 6.04 | 125.2 | 6.26 | 129.6 |
| -0.4 | 0.2 | 10.0 | 6.01 | 130.8 | 6.23 | 130.9 |
| -0.6 | 0.0 | 9.7 | 6.05 | 117.7 | 6.29 | 129.5 |
| -0.6 | 0.2 | 9.9 | 6.02 | 129.7 | 6.24 | 129.8 |
| -0.8 | 0.2 | 9.9 | 6.04 | 125.7 | 6.28 | 129.6 |
| -0.8 | 0.4 | 10.0 | 5.98 | 133.5 | 6.22 | 133.9 |
| -1.0 | 0.2 | 9.4 | 5.99 | 102.9 | 6.59 | 102.7 |
| -1.0 | 0.4 | 10.0 | 5.98 | 132.9 | 6.24 | 133.0 |
| -1.4 | 0.6 | 10.1 | 5.74 | 140.8 | 6.23 | 140.9 |

## 7. Least squares minimization of the distance between direct and indirect cycle.

For a particular application, it is always possible to compute close-to-equivalent values of $\lambda$ through least-squares minimization of the distance between the direct and indirect aggregate cycles. If $\lambda_0$ is the value of $\lambda$ applied the disaggregate series, the value $\lambda_d$ to use for direct adjustment is given by

$$\hat{\lambda}_d = \arg\min \sum_t \left[\hat{C}_{i,t}(\lambda_0) - \hat{C}_{d,t}(\lambda_d)\right]^2 \qquad (7.1)$$

where $\hat{C}_{i,t}(\lambda_0)$ and $\hat{C}_{d,t}(\lambda_d)$ denote the estimated indirect and direct aggregate cycle, respectively. This procedure is relatively cumbersome, depends on the particular realization, and may produce variability in the values of $\lambda$ that could induce inconsistencies for the different levels of aggregation. It is nevertheless of interest to look at whether the solution is likely to yield values of $\lambda$ that may strongly depart from the values obtained with the previous criteria.

We looked at the case of aggregating quarterly series into annual ones (using $\lambda_Q = 1600$ for direct estimation of the quarterly cycle), under temporal aggregation and systematic sampling, and for the IMA(1,1) and IMA(2,2) models for different values of the parameters. For each one of the cases, only 100 simulations were made; the results seemed stable given our level of precision (first decimal point in $\lambda_A$). For each simulation, expression (7.1) was solved and $\hat{\lambda}_d$ estimated; then the mean and standard deviation of the $\hat{\lambda}_d$'s obtained were computed.

Again, except for the case of systematic sampling an IMA(1,1) model with a large and negative value of its MA parameter, the values of $\lambda_A$ are relatively stable and close to the ones obtained with the previous criteria. Notice that the value $\lambda_A = 6.65$, obtained with the criteria of preserving the cycle of reference, is, in none of the cases, significantly different from the values in Tables 5 and 6.

Table 5: Least square minimization: IMA(1,1) models.

| $\theta_Q$ | temporal aggregation | | systematic sampling | |
|---|---|---|---|---|
| | $\lambda_A$ | | $\lambda_A$ | |
| | mean | std. dev | mean | std. dev |
| -0.8 | 6.9 | 0.7 | (*) | (*) |
| -0.6 | 6.8 | 0.4 | 15.1 | 11.4 |
| -0.4 | 6.6 | 0.2 | 10.8 | 13.3 |
| -0.2 | 6.7 | 0.2 | 8.4 | 4.1 |
| 0.0 | 6.6 | 0.2 | 7.4 | 1.7 |
| 0.2 | 6.7 | 0.2 | 7.1 | 1.2 |
| 0.4 | 6.7 | 0.2 | 7.1 | 1.2 |
| 0.6 | 6.6 | 0.1 | 7.1 | 1.2 |
| 0.8 | 6.6 | 0.1 | 7.0 | 1.2 |

(*) Numerical problems because of the flat surface of the objective function around the minimum.

Table 6: Least square minimization: IMA(2,2) models.

| $\theta_{Q,1}$ | $\theta_{Q,2}$ | temporal aggregation | | systematic sampling | |
|---|---|---|---|---|---|
| | | $\lambda_A$ | | $\lambda_A$ | |
| | | mean | std. dev | mean | std. dev |
| -0.8 | 0 | 6.5 | 0.1 | 6.7 | 0.9 |
| -0.6 | 0 | 6.5 | 0.1 | 6.7 | 0.7 |
| -0.4 | 0 | 6.5 | 0.1 | 6.7 | 0.6 |
| -0.2 | 0 | 6.5 | 0.1 | 6.6 | 0.6 |
| 0.0 | 0 | 6.5 | 0.1 | 6.5 | 0.7 |
| 0.2 | 0 | 6.5 | 0.1 | 6.4 | 0.6 |
| 0.4 | 0 | 6.5 | 0.1 | 6.6 | 0.6 |
| 0.6 | 0 | 6.5 | 0.1 | 6.6 | 0.6 |
| 0.8 | 0 | 6.5 | 0.1 | 6.5 | 0.6 |
| -0.6 | -0.3 | 6.6 | 0.1 | 6.9 | 1.1 |
| -0.6 | 0.3 | 6.5 | 0.1 | 6.5 | 0.7 |
| -0.5 | -0.3 | 6.5 | 0.1 | 6.6 | 0.9 |
| -0.5 | 0.3 | 6.5 | 0.1 | 6.6 | 0.6 |
| -0.4 | -0.3 | 6.5 | 0.1 | 6.6 | 0.9 |
| -0.4 | 0.3 | 6.5 | 0.1 | 6.6 | 0.6 |
| 0.4 | -0.3 | 6.5 | 0.1 | 6.6 | 0.6 |
| 0.4 | 0.3 | 6.5 | 0.1 | 6.5 | 0.6 |
| 0.5 | -0.3 | 6.5 | 0.1 | 6.6 | 0.5 |
| 0.5 | 0.3 | 6.5 | 0.1 | 6.5 | 0.5 |
| 0.6 | -0.3 | 6.5 | 0.1 | 6.6 | 0.6 |
| 0.6 | 0.2 | 6.5 | 0.1 | 6.5 | 0.5 |

## 8. Limitations of Consistency under Aggregation

Although, as evidenced by figures 1-3, the main features of the cycle obtained are relatively robust to variations in $\lambda$, the use of values that are inconsistent under time aggregation may confuse the analyst. Yet there are a number of reasons that can justify departures from (approximately) exact consistency under aggregation. For example, it seems reasonable that when monitoring a series observed once a year or once every two years, short- or medium-term analysis should not focus on the same frequencies as when the series is observed weekly or monthly. Obviously, a 3-year cycle may be of interest when monitoring a monthly series, but would hardly be of help if the series is observed once every 2 years. Thus the analyst may not be interested in preserving as cycle of reference one designed for quarterly data, and the choice of $\lambda$ may differ depending on the frequency of observation. Cycles used for different frequencies may not display good aggregation properties, yet can be of more use to the analyst.

Additionally, there are also methodological reasons that can justify departures from aggregation consistency. In order to avoid contamination with seasonal frequencies, the HP filter is applied to seasonally adjusted (SA) data. Yet seasonal adjustment is a non-linear transformation (Ghysels et al. (1996); Maravall (2006)) and hence linear constraints –such as those implied by time aggregation- cannot be expected to be preserved. Further, the SA series is contaminated with noise and possibly with outliers, and this contamination may distort estimation of the cyclical signal. It is thus preferable to use as input to the HP filter the trend-cycle component (Kaiser and Maravall (2001)). This is illustrated in Figures 13 and 14, which plot the cycles estimated on the SA series and on the trend-cycle component of the monthly Italian and French IPIs (Jan 1962-Dec 2005). The rough similarities between the two cycles are more clearly discernible when the trend-cycle component is employed.

Figure 13

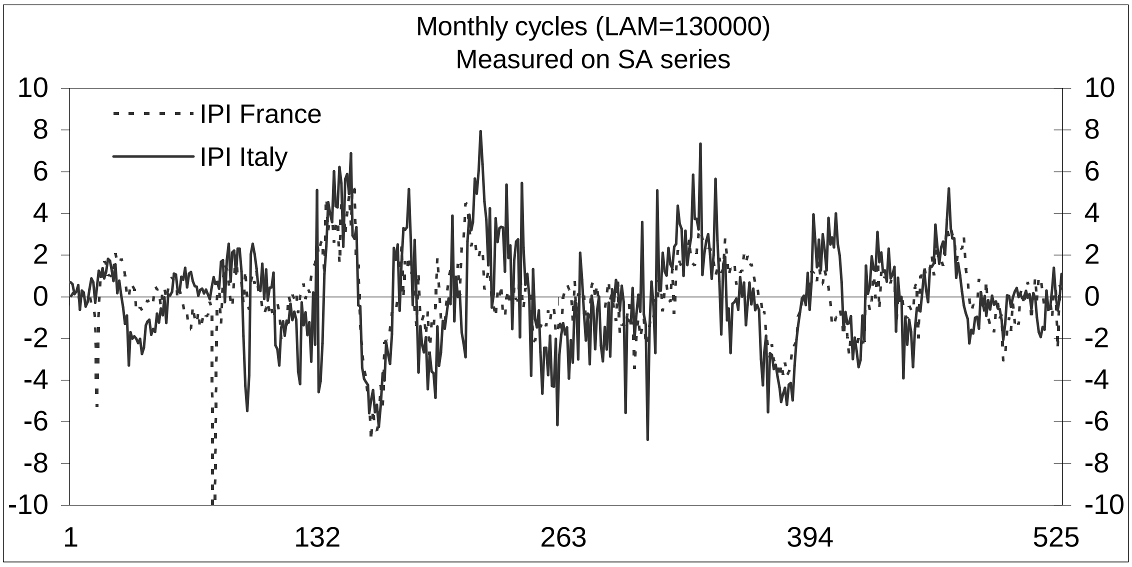



Figure 14

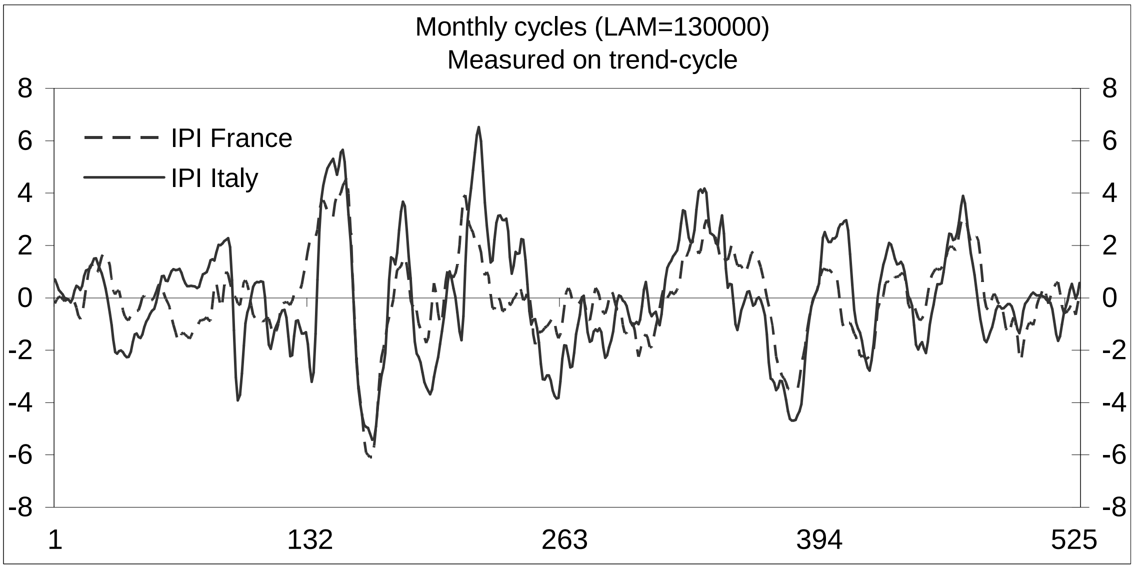
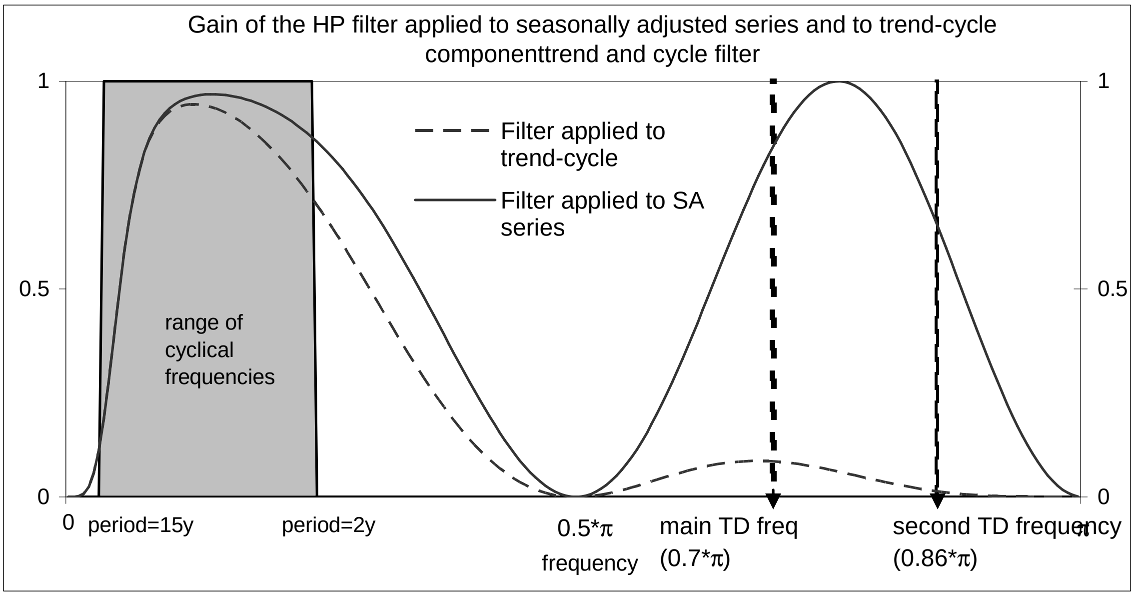



Figure 15 compares the gains of the convolution of the HP filter ($\lambda = 1600$) with the filter that provides the estimators of the SA series and of the trend-cycle estimator for the model $\nabla\nabla_4 x_t = a_t$. The use of the trend-cycle is seen to improve the band-pass features of the cyclical filter in the sense that it performs a more drastic removal of frequencies that do not belong to the range of cyclical frequencies (in the figure, this range cover the cycles with a period between 2 and 15 years). These non-cyclical frequencies can, on occasion, be relevant. Figure 16 exhibits, as an example, the cycle component obtained with the HP filter ($\lambda = 130000$) applied to the SA Japan IPI monthly series. Figure 17 presents its spectrum and the spectrum of the rate of growth of the SA series. It is clearly seen that the Trading Day (TD) effect has not been fully removed from the SA series and, although the associated frequencies do not belong to the cyclical range, they are nevertheless passed on to the cyclical component. (The spectrum estimate and the TD frequencies are as in Soukup and Findley (1999)) Figure 18 displays the smoother cycle obtained when this TD effect has been removed; the effect represents undesirable short-term variation.

Figure 15




In general, filtering or pretreatment of a series prior to application of the HP filter may already imply features that affect aggregation. Outliers detected in a monthly series may well be different from those detected in an annual one. Trading-day and/or Easter effects may be significant for the monthly series, but not for the quarterly one. The ARIMA models used to extend the series, or to obtain the SA series or trend-cycle component, will hardly ever be exact aggregates for different levels of aggregation. As a consequence, departures from aggregation consistency should be expected. Figures 19-22 illustrate these departures for the Italian and French IPI cycles (obtained with equivalent $\lambda$ values). Although the overall effect is moderate, it cannot be considered trivial. (Incidentally, a similar reflection applies to forecasting.) Given that not much can be done to solve in a convincing manner this discrepancy problem, perhaps the best solution is to compute both the direct and indirect estimators, and this may serve as a reminder of our (many) measurement limitations.

Figure 16

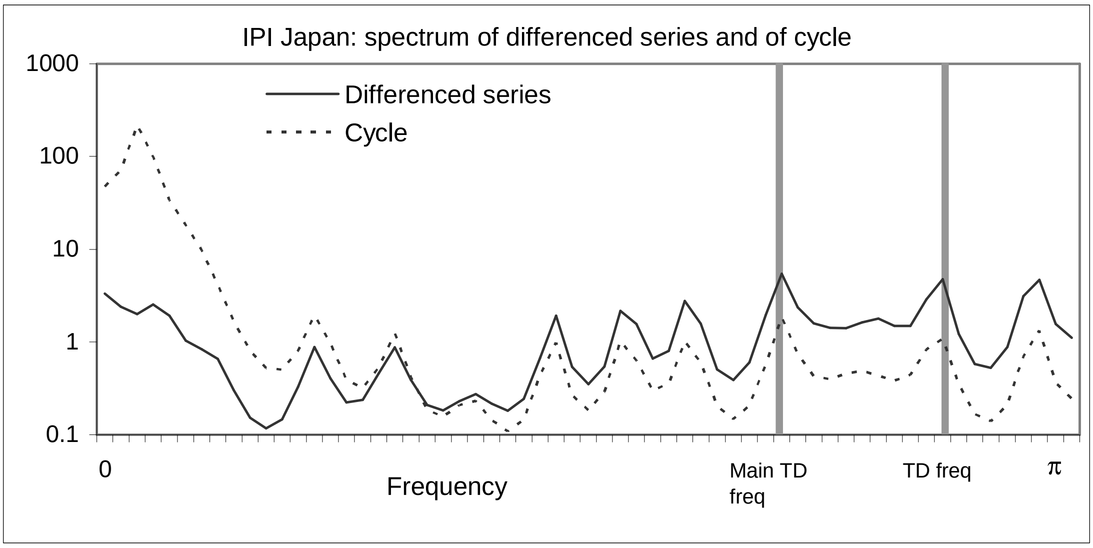



Figure 17

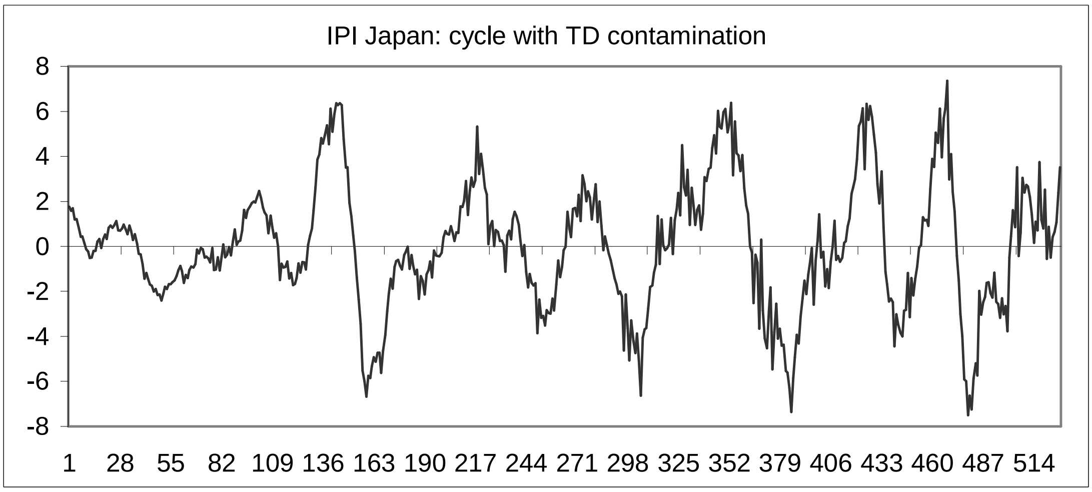



Figure 18

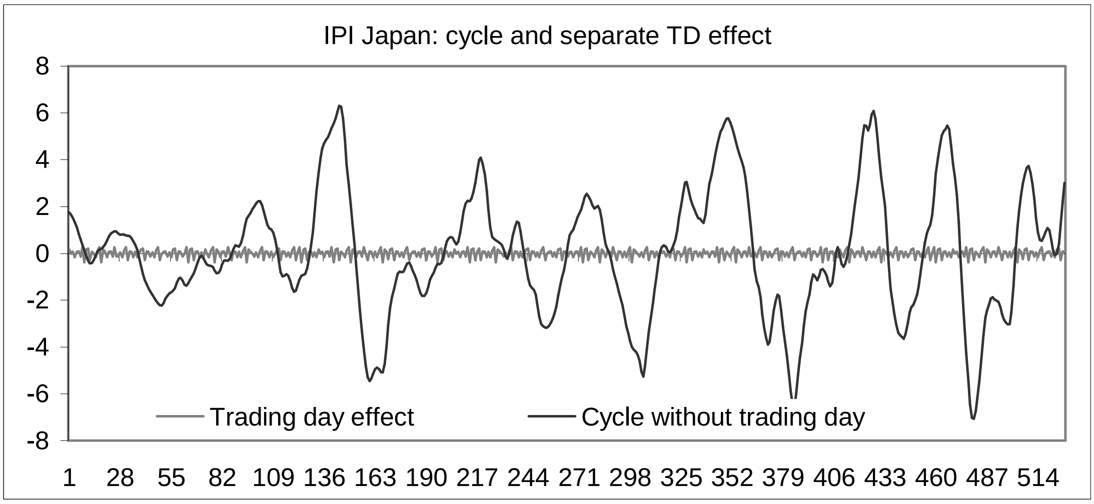


Figure 19

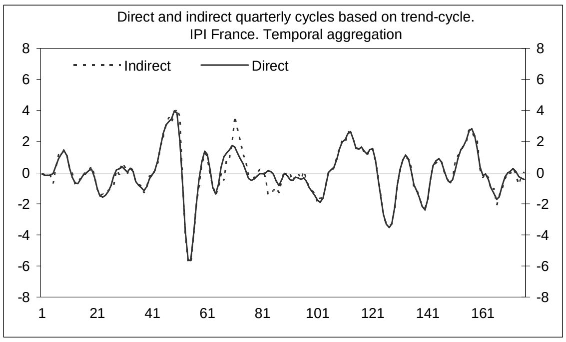



Figure 20

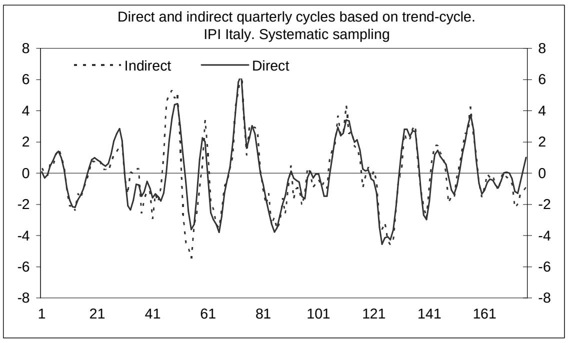



Figure 21

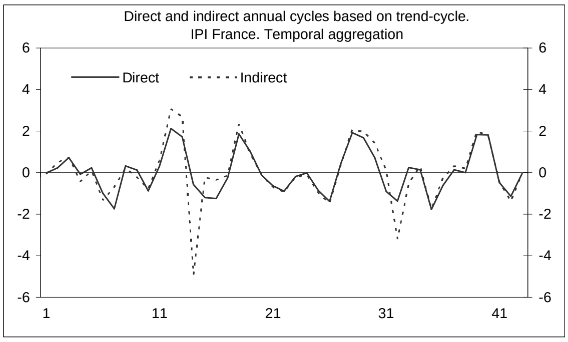
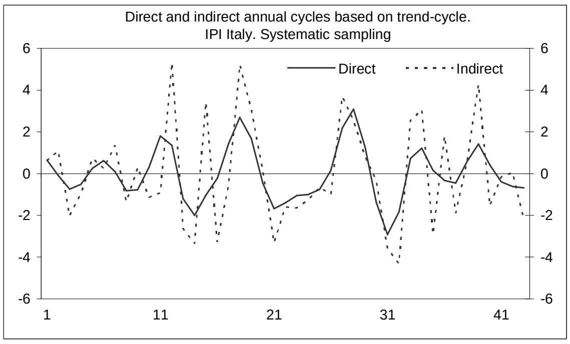



Figure 22




## 9. Conclusions

We have analyzed the time aggregation properties of the Hodrick-Prescott (HP) filter, focusing on monthly, quarterly, and annual observations. Two types of aggregation have been considered: Temporal Aggregation, whereby the aggregate series consists of (non-overlapping) sums (or averages) of disaggregate values, and Systematic Sampling, whereby the aggregate series is equal to a value of the disaggregate series sampled at periodic intervals. The main results can be summarized as follows.

For the two types of aggregation, the HP filter does not preserve itself under aggregation in the following sense. Cycles estimated by aggregating cycles obtained for disaggregate data with an HP filter (indirect estimation) cannot be seen as the exact result of an HP filter applied to the aggregate data (direct estimation). Direct and indirect estimation of cycles computed with HP-type filters cannot yield identical results. In practice, this lack of aggregation consistency, plus the fact that the broad features of the computed cycle are roughly robust to the choice of the HP parameter $\lambda$, has led to an arbitrary choice of inconsistent $\lambda$'s for different levels of aggregation. Yet, as seen in Section 3, careful analysis of the filter indicates that an adequate choice may provide direct and indirect estimators that are very close.

Several statistically-based criteria that provide close-to equivalent $\lambda$ values are considered. The first criterion, considered in Section 4, is to preserve, for different levels of aggregation, the period of the cycle associated with the frequency for which the gain of the HP filter is .5.

Given that this frequency represents the cutting point between frequencies that will be mostly assigned to the cycle and those that will be mostly assigned to the trend, the criterion is intuitively attractive. Further, it is simple to apply and general in the sense that, once the period for the cutting point (or, equivalently, the $\lambda$ parameter) is chosen, the equivalent values for other levels of aggregation are uniquely determined. Section 5 shows that the empirical criterion suggested by Ravn and Uhlig (2002) turns out to be a first-order approximation to the criterion of preserving the period associated with a filter gain of .5 .

The properties of the estimated cycle will depend, not only on the filter, but also on the characteristics of the series at hand. We represent the latter with an ARIMA model and this allows us to derive the spectrum of the cycle. In Section 6 equivalent values of $\lambda$ for different levels of aggregation are derived under the criterion of preserving the period associated with the frequency for which the cycle spectrum reaches a maximum, for IMA(1,1) and IMA(2,2) models. It is seen that, except for the case of systematic sampling of nearly non-invertible models, the results are robust with respect to the model parameters, and very close to the ones obtained with the first criterion. Finally, Section 7 uses as criterion least-square minimization of the distance between direct and indirect cycles. With the same exception as before, the results obtained are again similar to those obtained with the first criterion.

For the quarterly consensus value $\lambda_Q = 1600$, the previous results yield monthly and annual equivalent values in the intervals

$$115000 < \lambda_M < 130000 \quad , \qquad 6 < \lambda_A < 7 \quad ,$$

with the $\lambda$ values more in the vicinity of the upper bounds. It follows that the RU rule can be safely used (with perhaps a slightly larger value of $\lambda_A$, such as $\lambda_A = 6.5$ or 6.75); this empirical rule can be rationalized as providing values that approximately preserve the period associated with a filter gain of .5, and the period associated with the peak of the cycle spectrum. The values for the case of systematic sampling of close-to-noninvertible models lie outside the previous limits, indicating a smaller value for $\lambda_M$ and a larger one for $\lambda_A$, but the distance is still moderate. Nevertheless, although consistency under aggregation is a desirable property, the final section (Section 8) explains and illustrates why, because of applied and – under the present “state of the art”- methodological considerations, optimal procedures are likely to induce some inconsistencies between direct and indirect estimation.

## Appendix A: Factorization of an MA(2) process

Given the variance, $\gamma_0$, and the lag-1 and lag-2 autocovariances $\gamma_1$ and $\gamma_2$, of an MA(2) model, the MA parameters can be obtained by solving the nonlinear system of equations (6.11). We provide a simpler procedure, based on the general algorithm in Maravall and Mathis (1994, appendix A).

Compute $b = \gamma_1 / \gamma_2$ and $c = \gamma_0 / \gamma_2 - 2$, and let $y_1$ and $y_2$ be the two solutions of the equation $y^2 + by + c = 0$. Consider first the case in which the two roots are real. Solve the two equations $z^2 - y_j z + 1 = 0$, j=1,2, and select in each case the root $z_j$ such that $|z_j| \geq 1$. If the two roots selected are $z_1$ and $z_2$, the MA(2) polynomial is given by $\theta(B) = (1\text{-}z_1B)\ (1\text{-}z_2B)$.

When the roots $y_1$ and $y_2$ are complex, let $y_1 = a + bi$ and $y_2 = a - bi$, and define $k = a^2 - b^2 - 4$, $m = 2ab$, $h = +\left[\left(|k| + \left(k^2 + m^2\right)^{1/2}\right)/2\right]^{1/2}$. If $k \geq 0$, let $c = h$, $d = m/2h$; if $k < 0$, let $d = [\text{sign}(m)]\,h$, $c = m/2d$. Consider the two complex numbers $z_1 = z_1^r + z_1^i i$ and $z_2 = z_2^r + z_2^i i$, where $z_1^r = (-a + c)/2$, $z_1^i = (-b + d)/2$, $z_2^r = (-a - c)/2$, and $z_2^i = (-b - d)/2$, and denote by $z_j$ the one with the smallest modulus. Then $\theta_1 = 2z_j^r$, $\theta_2 = \left(z_j^r\right)^2 + \left(z_j^i\right)^2$, and the MA(2) polynomial is given by $\theta(B) = 1 + \theta_1 B + \theta_2 B^2$. The innovation variance can be obtained through $V_a = \gamma_0 /(1 + \theta_1^2 + \theta_2^2)$.

## Appendix B: Construction of the Stram-Wei Aggregation Matrix

The Stram-Wei aggregation matrix, M, relates the covariances of the stationary transformation of the aggregate and disaggregate series. We consider IMA(d,d) models, with $d = 1$ and 2, so that the stationary tranformation of the series $x_t$ is $\nabla^d x_t$.

Let k be the order of aggregation ($k = 3$ and 4 when aggregating monthly into quarterly and quarterly into annual frequencies, respectively,) and define $n = (d + 1)$ for temporal aggregation and n=d for systematic sampling. Let $\gamma_i$ and $\Gamma_i$ be the autocovariance of order i for the stationary transformation of the disaggregate and aggregate series respectively. Stram and Wei (1986) prove the following relationship for the case of temporal aggregation:

$$\Gamma_i = S^{2n}\gamma_{(ki+n(k-1))} \qquad i=0,1,\dots \qquad (B.1)$$

where $S=(1+B+B^2+\dots+B^{k-1})$ is the aggregation operator. The systematic sampling case is not consider by them but, proceeding in a similar manner it is straightforward to find that (B.1) also holds (although the value of $n$ will differ).

If $x_t$ follows an IMA(d,d) model, then the aggregate series $X_T$ follows an IMA(d,Q) process with $Q=d$. If $\gamma$ and $\Gamma$ denote the column vectors with the i-th element equal to $\gamma_i$ and $\Gamma_i$, respectively, the Stram-Wei procedure permits us to express the relationship between $\gamma$ and $\Gamma$ as $\Gamma = M\gamma$ , where M is constructed as follows. Let $c$ be a $1x(2n(k-1)+1)$ row vector with elements $(c_i)$ the coefficients of $B^i$ in the polynomial $S^{2n}$. Define the matrix A as the following $(Q+1)\,x\,(kQ+2n(k-1)+1)$ matrix:

$$A=\begin{bmatrix} c & 0_k & 0_{(Q-1)k} \\ 0_k & c & 0_{(Q-1)k} \\ 0_{2k} & c & 0_{(Q-2)k} \\ \dots & \dots & \dots \\ 0_{(Q-1)k} & c & 0_k \\ 0_{(Q-1)k} & 0_k & c \end{bmatrix}$$

where $0_j$ is a (1xj) row vector of zeros. Adding the column (n(k-1)+1-j) of matrix A to the column (n(k-1)+1+j) of the same matrix, for j=1 to n(k-1), and then deleting the first n(k-1) columns, we obtain a new matrix A*. The matrix M consists of the first q+1 columns of A*.

Consider as a first example a quarterly IMA(1,1) model which is aggregated to annual frequency. In this case we have $k=4$, and $d=q=Q=1$. For temporal aggregation n=(d+1)=2 and considering the coefficients of $S^{2n}=(1+B+B^2+B^3)^4$, given by $c=(1\ 4\ 10\ 20\ 31\ 40\ 44\ 40\ 31\ 20\ 10\ 4\ 1)$, it is found that A is the following (2x17) matrix:

$$A=\begin{bmatrix} 1 & 4 & 10 & 20 & 31 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 10 & 20 & 31 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 \end{bmatrix},$$

and hence $A^*=\begin{bmatrix} 44 & 80 & 62 & 40 & 20 & 8 & 2 & 0 & 0 & 0 & 0 \\ 10 & 24 & 32 & 40 & 44 & 40 & 31 & 20 & 10 & 4 & 1 \end{bmatrix}$. Thus $M=\begin{bmatrix} 44 & 80 \\ 10 & 24 \end{bmatrix}$.

For the case of systematic sampling n=d=1, the vector c contains the coefficients of $S^{2n}=(1+B+B^2+B^3)^2$, that is $c=(1\ 2\ 3\ 4\ 3\ 2\ 1)$, and A is the following (2x11) matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{bmatrix} .$$

Then, $A^* = \begin{bmatrix} 4 & 6 & 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{bmatrix}$, and the matrix M is $\begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$.

As a last example, consider a monthly IMA(2,2) model and its quarterly temporal aggregate. In this case, k = 3, d = 2, q = 2, and n = d+1 = 3. The vector c, with elements the coefficients of $(1+B+B^2)^2$, is equal to $c = (1\ 6\ 21\ 50\ 90\ 126\ 141\ 126\ 90\ 50\ 21\ 6\ 1)$, and hence A is the (3x19) matrix:

$$A = \begin{bmatrix} 1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1 \end{bmatrix},$$

A* is the (3x13) matrix

$$A^* = \begin{bmatrix} 141 & 252 & 180 & 100 & 42 & 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 50 & 111 & 132 & 142 & 126 & 90 & 50 & 21 & 6 & 1 & 0 & 0 & 0 \\ 1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1 \end{bmatrix},$$

and the matrix M is given by

$$M = \begin{bmatrix} 141 & 252 & 180 \\ 50 & 111 & 132 \\ 1 & 6 & 21 \end{bmatrix} .$$

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