TEMPORAL AGGREGATION, SYSTEMATIC SAMPLING, AND THE HODRICK-PRESCOTT FILTER

2007

Agustín Maravall and Ana del Río

Documentos de Trabajo
N.º 0728

BANCO DE **ESPAÑA**
Eurosistema

TEMPORAL AGGREGATION, SYSTEMATIC SAMPLING, AND THE HODRICK-PRESCOTT FILTER

TEMPORAL AGGREGATION, SYSTEMATIC SAMPLING, AND THE HODRICK-PRESCOTT FILTER

Agustín Maravall and Ana del Río

BANCO DE ESPAÑA

Documentos de Trabajo. N.º 0728
2007

ISSN: 0213-2710 (print)
ISSN: 1579-8666 (on line)
Depósito legal: M. 40217-2007
Unidad de Publicaciones, Banco de España

Abstract

Maravall and del Río (2001), analized the time aggregation properties of the Hodrick-Prescott (HP) filter, which decomposes a time series into trend and cycle, for the case of annual, quarterly, and monthly data, and showed that aggregation of the disaggregate component cannot be obtained as the exact result from direct application of an HP filter to the aggregate series. The present paper shows how, using several criteria, one can find HP decompositions for different levels of aggregation that provide similar results. We use as the main criterion for aggregation the preservation of the period associated with the frequency for which the filter gain is ½; this criterion is intuitive and easy to apply. It is shown that the Ravn and Uhlig (2002) empirical rule turns out to be a first-order approximation to our criterion, and that alternative —more complex— criteria yield similar results. Moreover, the values of the parameter λ of the HP filter, that provide results that are approximately consistent under aggregation, are considerably robust with respect to the ARIMA model of the series. Aggregation is seen to work better for the case of temporal aggregation than for systematic sampling. Still a word of caution is made concerning the desirability of exact aggregation consistency. The paper concludes with a clarification having to do with the questionable spuriousness of the cycles obtained with HP filter.

Keywords: Time series; Filtering and Smoothing; Time aggregation; Trend estimation; Business cycles; ARIMA models.

JEL classification: C22; C43; C82; E32; E66.

1 Introduction

The subjectiveness in the concept of business cycle has resulted in multiple methodologies for its identification [see, for example, Canova (1998)]. Yet, despite substantial academic criticism [see, for example, Cogley (2001), Cogley and Nason (1995), Harvey (1997), Harvey and Jaeger (1993), or King and Rebelo (1993)], the so-called Hodrick-Prescott (HP) filter [Hodrick and Prescott (1997)] has become central to the paradigm for business-cycle estimation at many economic institutions (examples are the IMF, the OECD, or the ECB). The HP filter decomposes a time series into two components: a long-term trend and a stationary cycle, and requires the prior specification of a parameter known as lambda (λ) that tunes the smoothness of the trend, and determines, for a given model for the series, the main period of the cycle that the filter will produce. Nevertheless, as pointed out by Wynne and Koo (1997), the parameter does not have an intuitive interpretation for the user, and its choice is considered an important weakness of the HP method [Dolado et al. (1993)].

The use of the same λ for series with different periodicity will (broadly) maintain the frequency associated with the cycle spectral peak, and hence will produce cycles that are inconsistent under time aggregation. For example, if the frequency is $\omega = \pi / 60$ radians, the monthly data will show a cycle concentrated around a period of 10 years; for annual data, the period becomes 120 years. Obviously, different periodicities require different values of λ.

For quarterly data (the frequency most often used for business-cycle analysis) there is an implicit consensus in employing the value of $\lambda = 1600$, originally proposed by Hodrick and Prescott, based on a somewhat mystifying reasoning ("…a 5% cyclical component is moderately large, as is a 1/8 of 1% change in the growth rate in a quarter…"). Still, the consensus around this value undoubtedly reflects the fact that analysts have found it useful. The consensus, however, disappears when other frequencies of observation are used. For example, for annual data, Baxter and King (1999) recommend the value $\lambda = 10$, Cooley and Ohanian (1991), Apel et al. (1996), and Dolado et al. (1993) employ $\lambda = 400$, while Backus and Kehoe (1992), Giorno et al. (1995) or European Central Bank (2000) use the value $\lambda = 100$, which is also the default value in the popular econometrics program EViews [EViews (2005)]. Concerning monthly data (a frequency seldom used), the default value in EViews is 14400, while the Dolado et al. reasoning would lead to $\lambda = 4800$.

None of the references mentioned addresses the issue of the relationship between the values of λ used for different observation frequencies. In particular, if λ_M is used for monthly data, how do the implied quarterly cycles compare with those obtained directly from the quarterly data with $\lambda_Q = 1600$? Also, what value of λ_A applied to annual observations yields cycles that are close to the ones obtained by aggregating the cycles obtained for quarterly data with $\lambda_Q = 1600$? Ravn and Uhlig (2002) use an empirical rule to obtain these "consistent under time aggregation" values of λ. Using as reference the value of λ_Q (for quarterly data), and letting λ_D denote the value for an alternative frequency of observation, they restrict attention to the relationship

$$\lambda_D = (k)^n \lambda_Q, \tag{1.1}$$

where k is the ratio of the number of observations per year for the alternative and quarterly frequencies respectively (thus $k = 3$ and $k = 1/4$ when the alternative frequencies are the monthly and annual ones) and n is a positive integer. Ravn and Uhlig (RU) present evidence that $n = 4$ appears to be the best choice. For $\lambda_Q = 1600$, this implies $\lambda_M = 129600$ and $\lambda_A = 6.25$.

Section 4 of the paper addresses the issue of consistency under temporal aggregation of the HP cycle from the perspective of preserving an important filter property, namely, the period associated with the frequency for which the filter gain is ½. Higher frequencies will belong mostly to the cycle; lower ones, to the trend. The criterion is easy to apply and yields results that are very close to those obtained by RU. In fact, it is shown how the RU rule turns out to be a first-order approximation to the criterion we consider. Section 5 considers criteria that preserve alternative characteristics of the HP filter and the results are found robust.

But the frequency domain properties of the cycle obtained will depend, not only on the filter, but also on the spectrum of the series at hand. This is analyzed in Section 6 and it is seen that, for an important class of models, the results are robust and remarkably close to those obtained with the simple criterion of Section 4. The closeness is stronger for the case of temporal aggregation than for the case of systematic sampling (in particular, when the model is not far from noninvertibility for the zero frequency). The robustness of the results is confirmed by a Least Squares exercise (Section 7). Finally, Section 8 discusses some limitations that should be taken into account when estimating and comparing cycles for different series periodicity.

Appendix A addresses a point having to do with the spuriousness of the HP filter. It is shown how, under very general conditions and for any linear process, the HP filter trend and cycle estimators can be given a perfectly sensible model-based interpretation that fully respects whatever model may have been identified for the series. Appendix B details how the autocovariances of the aggregate model can be obtained from those of the disaggregate model following the Wei and Stram procedure (extended to the systematic sample case).

The paper centers on monthly, quarterly, and annual frequencies of observation, and uses the widely accepted value $\lambda_Q = 1600$ as the pivotal value for the comparisons. The analysis, however, generalizes trivially to any other frequencies of observation and pivotal value for λ. The discussion is illustrated with some five macroeconomic series (the industrial production IPI series for the US, Japan, France, and Italy, and the US unemployment series) spanning the period January 1962 – December 2005 (528 monthly observations). The series are taken from the OECD database and are available at (www.bde.es $\rightarrow$ Professionals $\rightarrow$ Econometrics Software).

2 The Hodrick-Prescott Filter

Let B denote the lag operator, such that $B^j x_t = x_{t-j}$, and $\nabla = 1 - B$ denote the regular difference. For the rest of the paper, "w.n. (0, ν)" will denote a white noise (i.e., niid) variable with zero mean and variance ν. Suppose we are interested in decomposing x_t into a long-term trend m_t and a residual, c_t, to be called "cycle". From the time series realization $(x_1 \ldots x_T)$, the HP filter provides the sequences $(m_1 \ldots m_T)$ and $(c_1 \ldots c_T)$ such that

$$x_t = m_t + c_t \qquad t = 1, \ldots, T, \tag{2.1}$$

and the loss function

$$\sum_{t=1}^{T} c_t^2 + \lambda \sum_{t=3}^{T} \left(\nabla^2 m_t\right)^2 \tag{2.2}$$

is minimized. The first term in (2.2) penalizes large residuals (i.e., poor fit), while the second term penalizes lack of smoothness in the trend. The parameter λ regulates the trade-off between the two criteria: larger values of λ will produce smoother trends and increase the variance of the cycle. King and Rebelo (1993) showed that the filter could be given an unobserved component (UC) model derivation whereby x_t is the realization of a stochastic process consisting of (2.1), where

$$\nabla^2 m_t = a_{mt}, \quad a_{mt} \sim \text{w.n.}(0, \nu_m); \quad c_t \sim \text{w.n.}(0, \nu_c = \lambda \nu_m); \tag{2.3}$$

with a_{mt} orthogonal to c_t. Under these assumptions, the HP filter solution is equivalent to the minimum mean square error (MMSE) estimator of m_t and c_t obtained by the Kalman filter. Kaiser and Maravall (2001) show that the HP estimators can also be derived with an ARIMA-model-based (AMB) algorithm. We summarize this approach.

From (2.1) and (2.3) it follows that $\nabla^2 x_t = a_{mt} + \nabla^2 c_t$ and hence the reduced form for x_t is an IMA(2,2) process, say

$$\nabla^2 x_t = \theta_{HP}(B) b_t = (1 + \theta_1 B + \theta_2 B^2) b_t, \quad b_t \sim \text{w.n.}(0, \nu_b) \tag{2.4}$$

where the identity

$$\theta_{HP}(B)\, \theta_{HP}(F)\, \nu_b = \nu_m + \nabla^2 (1-F)^2\, \nu_c \tag{2.5}$$

determines the parameters θ_1, θ_2, and ν_b; see Section 4.4 in Kaiser and Maravall (2001) or Appendix A in Maravall and del Río (2001). For the pivotal value, $\lambda = 1600$, it is found that

$$\theta_{HP}(B) = 1 - 1.7771 B + .79944 B^2; \; V_b = 2001.4. \tag{2.6}$$

It should be stressed that the model-based interpretation (2.3) – (2.4) is simply meant to provide an algorithm, and not the model that could presumably be generating the series [see, for example, Pollock (2006)]. We shall refer to the model (2.3) – (2.4) as the "artificial" model. It will be most unlikely that the artificial model coincides with the model actually identified for the series (obviously, a white-noise business-cycle makes no sense) and the discrepancy between the artificial and identified model underlies the criticism made on occasion of the HP filter. This spuriousness issue is discussed in Appendix A where it is shown that, if properly interpreted, the trend and cycle estimators provided by the HP filter are MMSE of components with sensible trend and cycle models, that aggregate into whatever model might have been identified for the series.

The r.h.s. of (2.5) implies that $\nabla^2 x_t$ has a positive spectral minimum (equal to ν_m), and hence $\theta_{HP}(B)b_t$ is an invertible process; therefore, $\theta_{HP}(B)^{-1}$ will converge. The MMSE estimator of m_t and c_t obtained with the Wiener-Kolmogorov (WK) filter are the ones obtained with the HP filter, which can thus be expressed as

$$\vartheta_m(B,F) = \frac{\nu_m}{\nu_a}\frac{1}{\theta_{HP}(B)\theta_{HP}(F)}, \tag{2.7}$$

$$\vartheta_c(B,F) = \frac{\nu_c}{\nu_a}\frac{(1-B)^2(1-F)^2}{\theta_{HP}(B)\theta_{HP}(F)}, \tag{2.8}$$

where $F(=B^{-1})$ denotes the forward operator, such that $F^j x_t = x_{t+j}$. The estimators of m_t and c_t can be obtained through

$$\hat{m}_t = \vartheta_m(B,F)x_t, \quad \hat{c}_t = \vartheta_c(B,F)x_t. \tag{2.9}$$

The filters (2.7) and (2.8) are symmetric, centered, and convergent. From (2.3) and (2.5), the filter (2.7) can alternatively be expressed in terms of the HP parameter λ as:

$$\vartheta_m(B,F) = \frac{1}{1+\lambda(1-B)^2(1-F)^2}. \tag{2.10}$$

It will prove useful to look at the frequency domain representation of the filter (2.10). If $\omega\in[0,\pi]$ denotes the frequency measured in radians, replacing B by the complex number $e^{-i\omega}$, and using the identity $2\cos(j\omega) = e^{-ij\omega}+e^{ij\omega}$, gives the frequency response function (also the gain) of the trend estimation filter:

$$G_m(\omega,\lambda) = \frac{1}{1+4\lambda(1-\cos\omega)^2}. \tag{2.11}$$

The gain function of the filter that estimates the cycle is $G_c(\omega,\lambda) = 1-G_m(\omega,\lambda)$. Equating the pseudo-autocovariance functions (ACF) of the two sides of both equations in (2.9), and taking the Fourier transform (FT) yields

$$S_{\hat{m}}(\omega,\lambda) = [G_m(\omega,\lambda)]^2 S_x(\omega); \tag{2.12a}$$

$$S_{\hat{c}}(\omega,\lambda) = [G_c(\omega,\lambda)]^2 S_x(\omega), \tag{2.12b}$$

where $S_{\hat{m}}(\omega,\lambda)$, $S_{\hat{c}}(\omega,\lambda)$, and $S_x(\omega)$ are the spectra or pseudo-spectra (hereafter also denoted spectra) of $\hat{m}_t$, $\hat{c}_t$, and x_t. The squared gain of the filter indicates thus how much the frequencies of x_t will contribute to the variance of the estimators $\hat{m}_t$ and $\hat{c}_t$. Given that seasonal variation (or noise) should not contaminate the cyclical signal, the variable x_t in (2.9) and (2.12) will typically be a seasonally adjusted (SA) series or a trend-cycle component.

The WK filters (2.7) and (2.8) extend from $-\infty$ to ∞. Their convergence, however, would allow us to use a finite truncation. But, as characterizes all 2-sided filters, estimation of the component at both ends of a finite series requires future observations, still unknown, and observations prior to the first one available. The optimal (MMSE) estimator for end points can be obtained by extending the series with forecasts and backcasts, so that expression (2.9) remains valid with x_t replaced by the extended series. There is no need however to truncate the filter: using the approach in Burman (1980), Kaiser and Maravall (2001) present the algorithm for the HP filter case, and show how the effect of the infinite extensions can be exactly captured with only four forecasts and backcasts. The WK application of the HP filter is computationally efficient and analytically convenient.

3 Temporal Aggregation of the Hodrick-Prescott Filter

We shall consider two types of aggregation. In the first one ("temporal aggregation") the aggregate variable is the sum (or average) of the disaggregate variable; in the second one ("systematic sampling") the aggregate variable is obtained by periodically sampling one observation from the disaggregate variable.

Given that different values of λ have to be used for different series periodicity and that the HP filter is only linear for fixed λ, aggregation of an HP cycle will not yield the cycle that would result of a direct application of an HP filter to the aggregate series. [This point is discussed in detail in Maravall and del Río (2001).] As mentioned in Section 1, a variety of (seemingly arbitrary) values of λ have been used for different frequencies of observation. The first question that comes to mind is: how relevant can be the lack of aggregation consistency between the different values of λ? Figures 1, 2, and 3 display the cycles estimated for the USA Industrial Production Index during the period 1962-2005 for different values of λ and frequencies of observation. Figure 1 compares the estimates for the last 200 months using the RU rule ($\lambda = 130000$) and the EViews default value ($\lambda = 14400$). Figure 2 compares, for the case of systematic sampling, the cycles for the last 50 quarters obtained directly with the consensus value $\lambda = 1600$ and indirectly by aggregating the monthly cycles using the EViews default value. Finally, Figure 3 compares, for the case of temporal aggregation, the annual cycles for the full period obtained directly with the value $\lambda = 400$ and indirectly with the same EViews monthly value.

Direct inspection of the figures shows that, although the most salient features of the cycles may roughly be robust to variations in λ, the differences are nevertheless important and increase with the level of aggregation. The next question is whether one can derive λ values for different frequencies of observation such that consistency under time aggregation is approximately preserved. Specifically, given the HP decomposition of the quarterly series with λ_Q as parameter,

(a) can we obtain a value λ_A that provides a direct HP decomposition of the annual series with components that are close to the ones obtained by aggregating the quarterly components?

(b) can we obtain a value of λ_M that provides monthly components that, when aggregated, are close to the components of the direct quarterly decomposition?

In summary, we seek values of λ —say λ_M, λ_Q, and λ_A— such that direct application of the HP filter to the monthly, quarterly, and annual series yields cycles that are very approximately consistent. We shall consider first criteria based on the preservation of some feature of the filter.

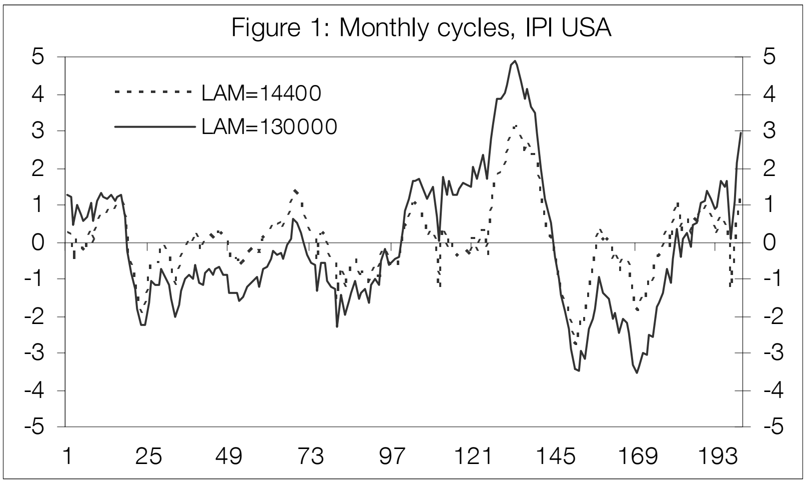


Figure 1: Monthly cycles, IPI USA
LAM=14400
LAM=130000
5
4
3
2
1
0
-1
-2
-3
-4
-5
1
25
49
73
97
121
145
169
193

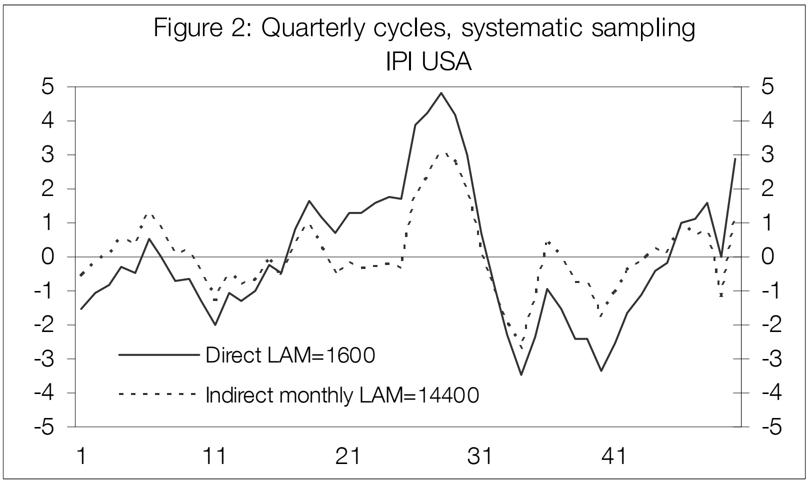


Figure 2: Quarterly cycles, systematic sampling
IPI USA
Direct LAM=1600
Indirect monthly LAM=14400
5
4
3
2
1
0
-1
-2
-3
-4
-5
1
11
21
31
41

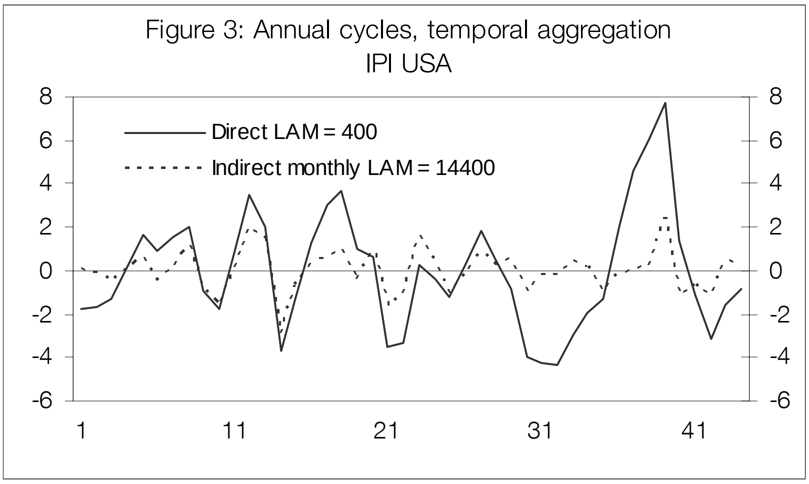


Figure 3: Annual cycles, temporal aggregation
IPI USA
Direct LAM = 400
Indirect monthly LAM = 14400
8
6
4
2
0
-2
-4
-6
1
11
21
31
41

4 Aggregation Criteria Based on the Preservation of Filter Characteristics; the Ravn and Uhlig Rule

4.1 Aggregation by Fixing the Period for which the Gain is One Half (the Cycle of Reference)

In the engineering literature, a well-known family of filters designed to remove (or estimate) the low-frequency component of a series is the Butterworth family [see, for example, Pollock (1997, 2003), or Gómez (2001)]. The filter is described by its gain function which, for the two-sided expression and the sine-type subfamily, can be expressed as

$$G_m(\omega) = \left[1 + \left(\frac{\sin(\omega/2)}{\sin(\omega_0/2)}\right)^{2d}\right]^{-1}, \quad 0 \le \omega \le \pi, \tag{4.1}$$

and depends on two parameters, d and ω_0. Given that $G(\omega_0) = .5$, the parameter ω_0 is the frequency for which 50% of the filter gain has been achieved (Figure 4). Thus frequencies lower than ω_0 will go mostly to the trend, while frequencies higher than ω_0 will be assigned mainly to the cycle. We shall refer to the cycle associated with that frequency as the "cycle of reference". Setting d=2 and $\beta = \left[\sin^4(\omega_0/2)\right]^{-1}$, the gain can also be expressed as

$$G_m(\omega) = \left[1 + \beta \sin^4(\omega/2)\right]^{-1}. \tag{4.2}$$

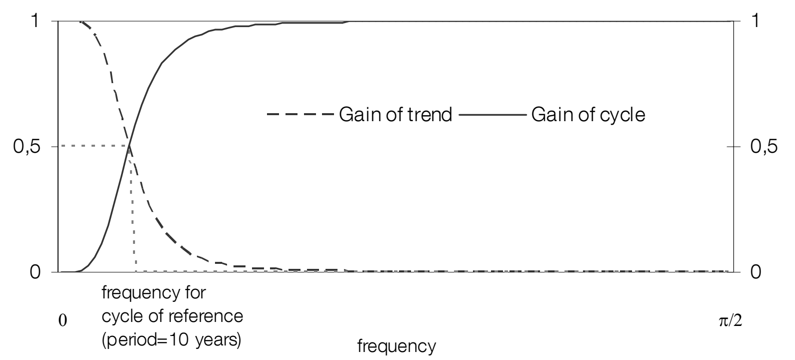
From the identity $2\sin^2(\omega/2) = 1 - \cos(\omega)$, (4.2) can be rewritten as

$$G_m(\omega) = \left[1 + (\beta/4)(1 - \cos\omega)^2\right]^{-1},$$

which, considering (2.11), shows that the filter is precisely the HP filter, with $\lambda = \beta/16$. Corresponding to $\omega = \omega_0$, one finds

$$\lambda_0 = \left[4(1 - \cos\omega_0)^2\right]^{-1}. \tag{4.3}$$

Figure 4: Gain of the HP filter (LAM=1600)



Therefore, knowing the parameter ω_0, the HP filter parameter λ can be easily obtained, and vice versa. If τ denotes the period of $\cos\omega$, τ is related to ω through

$$\tau = 2\pi / \omega . \tag{4.4}$$

Using (4.3) and (4.4), we can express the period τ directly as a function of λ, as

$$\tau = 2\pi / a\cos\left(1 - \frac{1}{2\sqrt{\lambda}}\right). \tag{4.5}$$

Equations (4.3)-(4.5) allow us to move from period to frequency, and then to λ (and vice versa) in a simple way. The frequency ω_0 —or its associated period τ_0— provide a more intuitive characterization of the cycle than the HP parameter λ. For example, from (4.5) the consensus value $\lambda_Q = 1600$ implies an associated period of (very approximately) 10 years. The choice of a 10-year period cutting point (between periods that will be mostly assigned to the trend and those that will be mostly assigned to the cycle) seems easier to interpret than the choice of a value for λ. The preservation of the period of the cycle of reference provides an attractive criterion for finding values of λ that yield relatively consistent results under aggregation.

Our procedure amounts to the following. Starting with λ_0 for some frequency of observation (for example, quarterly), the associated period τ_0 (in quarters) is found through (4.5). Consider another frequency of observation (for example, monthly or annual). Expressed in this frequency, τ_0 implies the period

$$\tau_D = k\ \tau_0 , \tag{4.6}$$

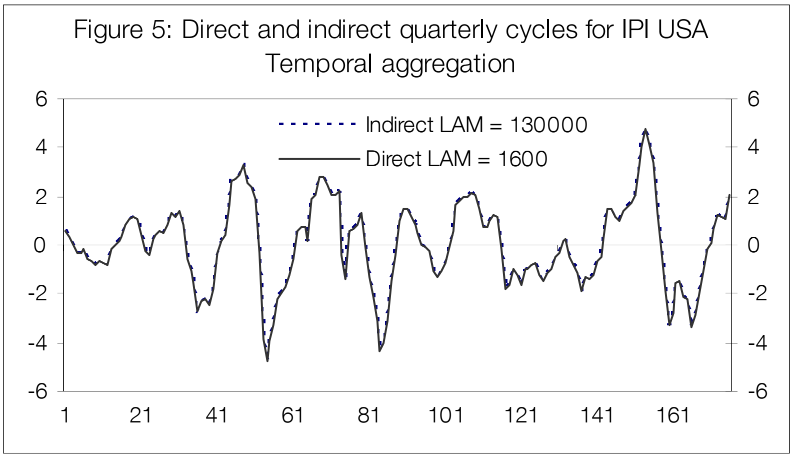
where k is as in (1.1). From (4.4) and (4.6), $\omega_D = \omega_0 / k$, so that λ_D can be obtained with (4.3) with ω_0 replaced by ω_D. The procedure is simple to apply, and can be used for aggregating or disaggregating series with any frequency of observation.

For the consensus value $\lambda_Q = 1600$, (4.5) implies a period of 39.7 quarters. Thus, for annual data, the period of the cycle of reference is, according to (4.6), $\tau = 9.9$ years. From (4.4), $\omega_A = 2\pi / 9.9$, and finally (4.3) yields $\lambda_A = 6.65$. On the other hand, in terms of monthly observations, the period of 39.7 quarters is equal to 119.1 months. Using (4.4) and (4.3), it is found that the equivalent value for monthly data is $\lambda_M = 129119$. Thus, using this criterion, values of λ that are consistent under aggregation are

$$\lambda_M = 129119 \ ; \quad \lambda_Q = 1600 \ ; \quad \lambda_A = 6.65 . \tag{4.7}$$

These values are very close the ones that result from the RU rule. An example can illustrate the difference with respect to other proposed values. In Giorno et al. (1995) the method used by the OECD for the estimation of the output gap is described: it uses the HP filter with $\lambda_Q = 1600$ and $\lambda_A = 100$. These values are referred to as "de facto industry standards"; they are also used by the European Central Bank (2000) and default values in EViews. Using $\lambda_A = 100$ for annual data, from (4.5), the period of the cycle of reference is $\tau_A = 19.8$ years which, in terms of quarterly data, becomes $\tau_Q = 79.2$ quarters, very different from the 39.7 quarters associated with the consensus λ_Q value.

For the cases of temporal aggregation and systematic sampling, figures 5, 6, 7, and 8 compare the direct and indirect cycles obtained with (4.7) for the USA IPI example: the two cycles are seen to be virtually indistinguishable for the case of temporal aggregation, and very close for the case of systematic sampling.



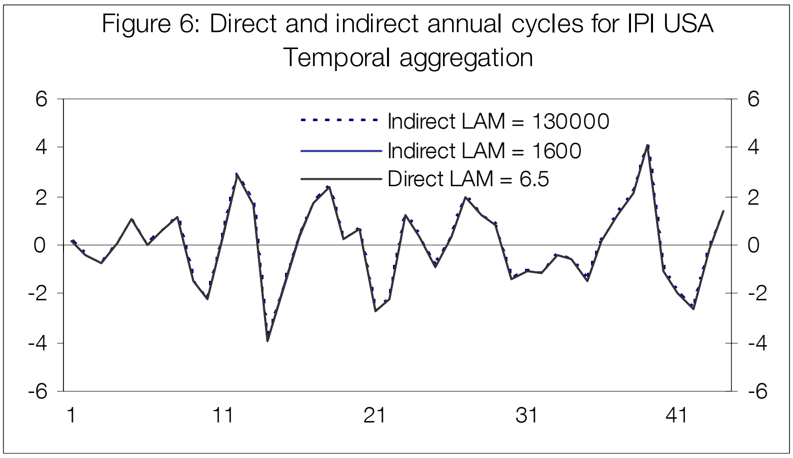


Figure 7: Direct and indirect quarterly cycles for IPI USA
Systematic sampling

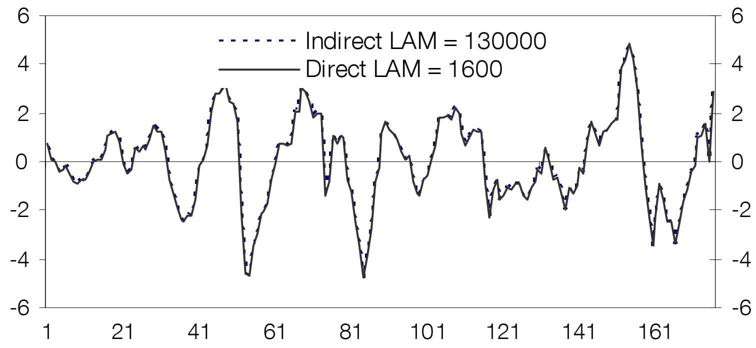
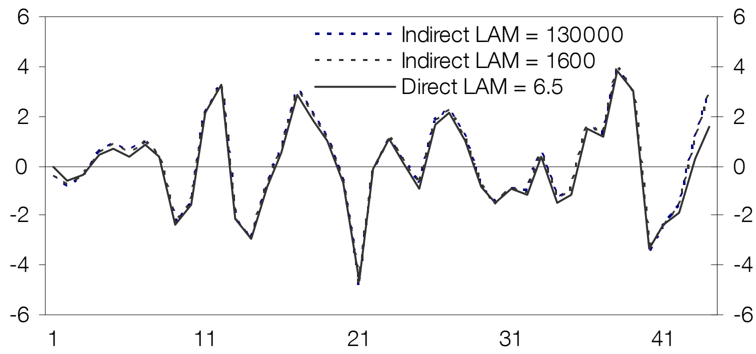


Figure 8: Direct and indirect annual cycles for IPI USA
Systematic sampling



The convenience of using λ values that are consistent under time aggregation is illustrated with the following example. Figure 9 shows the cycles estimated for the quarterly USA IPI and unemployment series during the period 1962-2005 using the consensus value $\lambda_Q = 1600$ for both. The figure reveals a very stable inverse relationship between the two cycles throughout the entire period. Recessions in industrial production are associated with expansions in unemployment, and viceversa, with the association moving in close to perfect phase.

Figures 10 and 11 show the monthly and annual cycles for the two series using the λ-values obtained with the criterion of maintaining the period associated with the 50% gain. It is seen how the relationship between the two series is preserved, so that inferences concerning the relationship between the cycles are robust with respect to the measurement time units.

Figure 9: Quarterly cycle, temporal aggregation (LAM=1600)

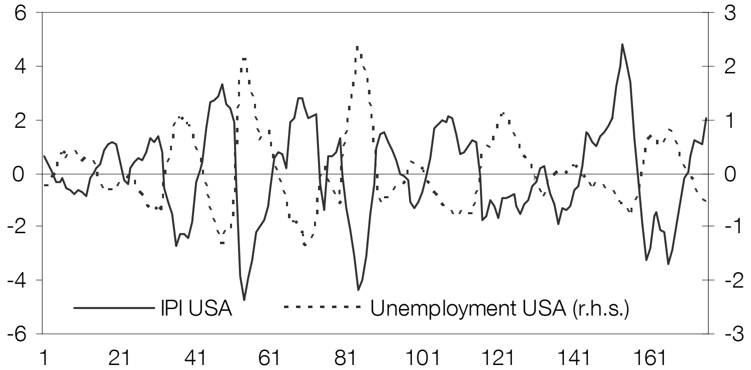


Figure 10: Monthly cycle (LAM=130000)

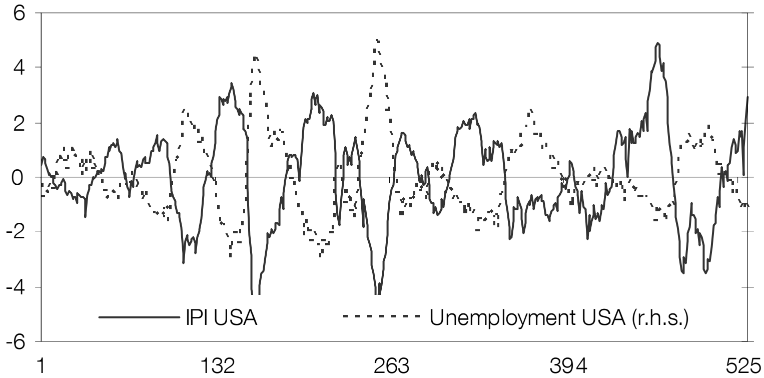


Figure 11: Annual cycle, temporal aggregation (LAM=6.5)

4.2 *Relationship with the Ravn and Uhlig Rule*

As mentioned in the introduction, Ravn and Uhlig (2002) provide a simple rule to compute values of λ for different frequencies of observation that appear to be approximately consistent under aggregation. If λ_Q is the reference value for quarterly data, and λ_D denotes the value for an alternative frequency of observation, they look at relationships of the type (1.1) and present evidence that good results are obtained for $j=4$. If $\lambda_Q = 1600$, this rule yields the monthly and annual values $\lambda_M = 129600$ and $\lambda_A = 6.25$, close to the ones obtained in (4.7). This closeness can be explained as follows.

Let λ_0 be the HP parameter for a given periodicity of observation, and let ω_0 and τ_0 be the frequency and period associated with $G(\omega_0) = .5$. We wish to obtain the equivalent value for λ_0, say λ_D, for another observation periodicity, using the criterion of preserving the period τ_0. Let ω_D and τ_D be the frequency and period associated to $G(\omega_D) = .5$. Then, preservation of the period implies that $\tau_D = k\ \tau_0$ or, equivalently, $\omega_D = \omega_0 / k$, so that, according to (4.3),

$$\lambda_D = \frac{1}{4\left(1-\cos\left(\omega_0 / k\right)\right)^2}. \tag{4.8}$$

Further, from (4.3),

$$\cos\omega_0 = 1 - 1/(2\sqrt{\lambda_0}). \tag{4.9}$$

Considering the power series expansion

$$\cos x = 1 - x^2/2 + \text{higher order terms}, \tag{4.10}$$

letting $x = \omega_0$ and comparing (4.9) and (4.10), after simplification,

$$\omega_0 \cong \lambda_0^{-1/4}. \tag{4.11}$$

Letting $x = \omega_0 / k$ in (4.10), $\cos\left(\omega_0 / k\right) \cong 1 - \omega_0^2 / 2k^2$, so that, considering (4.11), after simplification (4.8) becomes

$$\lambda_D \cong k^4\ \lambda_0. \tag{4.12}$$

Expression (4.12) shows that the RU rule turns out to be a first-order approximation to the criterion of preserving the period of the cycle for which the gain of the filter is 1/2. The approximation will work better for larger values of λ, as shown in Table 1. (Note: in the table, the value of τ for RU is the period associated with the condition that Gain = .5 when the RU value of λ is employed.)

Table 1: Performance of approximation

Frequency of observation		G = .5 criterion	RU criterion
Every month	λ	129120	129600
	τ (months)	119.1	119.2
Every 2 months	λ	8081	8100
	τ (2 months)	59.55	59.58
Every 3 months	λ	1600	1600
	τ (quarter)	39.70	39.70
Every 4 months	λ	508	506
	τ (4 months)	29.77	29.75
Every 6 months	λ	101.3	100
	τ (6 months)	19.85	19.79
Once a year	λ	6.65	6.25
	τ (years)	9.92	9.76

From the table, starting from the quarterly value of $\lambda_Q = 1600$, the period of the cycle associated with ω such that $G(\omega) = .5$ is $\tau = 119.1$ months. Let λ denote the value for another frequency of observation, obtained with the same criterion, and let λ^{RU} denote the value obtained with the RU rule. If τ and τ^{RU} are the period of the cycles associated with Gain = .5 when λ and λ^{RU}, respectively, are used, for monthly data: $\tau_M - \tau_M^{RU} = .1$ months; for annual data: $\tau_A - \tau_A^{RU} = 1.9$ months; for data recorded every two years: $\tau_{2Y} - \tau_{2Y}^{RU} = 9$ months. Thus the annual frequency seems to provide a rough limit for the validity of the approximation. The criterion of preserving the period of the cycle that represents the cutting point between "mostly trend" and "mostly cycle" periods provides a sensible rationale to the empirical rule of RU.

5 Criteria Based on Alternative Filter Characteristics

5.1 Replacing the Gain by the Squared Gain

Section 4 used as aggregation criterion the preservation of the period associated with the frequency for which the filter gain is ½. This period was referred to as the cutting point between trend and cycle in the series. But, in view of (2.12), one could consider the way variances are filtered, and use perhaps as criteria the preservation of the period associated with the frequency for which the squared gain of the cycle filter equals ½.

From $G_c(\omega,\lambda) = 1 - G_m(\omega,\lambda)$, it is found that if ω_0 denotes the frequency for which $\left[G_c(\omega_0)\right]^2 = .5$, then $G_m(\omega_0) = 1 - \sqrt{.5}$ and, from (2.11), the associated value of λ, say λ_0 is:

$$\lambda_0 = \frac{c_1}{(1 - \cos\omega_0)^2}, \tag{5.1}$$

where $c_1 = \left[1/(1 - \sqrt{.5}) - 1\right]/4$. Therefore, the relationship between λ_0 and the period associated with ω_0, say τ_0, is given by:

$$\tau_0 = \frac{2\pi}{a\cos(1 - \sqrt{c_1/\lambda})}. \tag{5.2}$$

Replacing equations (4.3) and (4.5) with (5.1) and (5.2), one can proceed as in Section 4.1. to obtain values of λ for different frequencies of observations that yield consistent results. For the pivotal value of $\lambda_Q = 1600$ it is obtained that $\lambda_M = 128854$ and $\lambda_A = 6.89$.

5.2 Preserving the period associated with the roots of $\theta_{HP}(B)$

As seen in (2.7) and (2.8), the model-based algorithm depends on the polynomial $\theta_{HP}(B) = 1 + \theta_1 B + \theta_2 B^2$, fully determined from the λ parameter. Appendix A shows that this polynomial in B will show up as part of the AR polynomial in the model for the cycle (this model is implied by the convolution of the HP filter and the ARIMA model for the series). The roots of $\theta_{HP}(B)$ will be a pair of complex conjugate roots associated with a cyclical frequency [McElroy (2006)]. Thus another criterion for aggregation could be the preservation of the period that corresponds to that frequency.

McElroy shows that the dependence of the roots frequency on λ is given by

$$\omega = a\tan\left[(2q + 2\sqrt{q}\sqrt{q+16})^{1/2}/4\right]. \tag{5.3}$$

where $q = 1/\lambda$. Proceeding as in Section 4.1, starting with a value λ_0, we obtain ω_0 with (5.3), then (4.6) transforms this frequency into the equivalent one (say ω_D) for the different periodicity of observations. Solving (5.3) for λ, one obtains the associated value

$$\lambda_D = \frac{1 + (\tan\omega_D)^2}{4(\tan\omega_D)^4}. \tag{5.4}$$

For the pivotal value of $\lambda_Q = 1600$, the roots of $\theta_{HP}(B)$ —given by (2.6)— have frequency 0.1117 and the associated period is 14 years. The values of λ that provide consistent cycles for the monthly, quarterly and annual periodicities are found to be:

$\lambda_M = 130082$; $\lambda_Q = 1600$; $\lambda_A = 5.84$.

5.3 Summary Remark

The three criteria yield similar results, similar also to the ones obtained with the RU rule. The value of λ for monthly data consistent with $\lambda = 1600$ for quarterly data is always very close to $\lambda_M = 130000$. For annual data, λ_A ranges between roughly 6 and 7, a small range compared to the range of values that have been proposed in the literature (between 6 and 400). In fact, graphical comparison of the cycles obtained with criteria 4.1, 4.2, 5.1, or 5.2 would practically reproduce Figures 5 to 8; the differences would be indistinguishable.

The criterion of Section 5.1, based on the Squared Gain, does not really provide a "cutting-point" interpretation given that a 50% assignment of the variance to the cycle does not imply that the remaining 50% is assigned to the trend. There is a loss due to the appearance of a covariance between the trend and cycle estimators. Concerning the criterion of preserving the period associated with the roots of $\theta_{HP}(B)$, its main justification can be found in the time domain: for long enough lag and horizon, the eventual autocorrelation and forecast functions will contain a cyclical component with that same period.

Altogether, of the criteria we have considered that are based on the preservation of the characteristics of the filter, the first one (Section 4.1) seems the most intuitive and attractive. It provides moreover a nice rationale to the simple RU rule.

6 Aggregation by Fixing the Period Associated with the Maximum of the Cycle Spectrum

The previous criteria are based solely on properties of the HP filter. But ultimately, the properties of the resulting cycle are a combination of two factors: the characteristics of the filter and the stochastic properties of the series in question. We consider now their interaction. To describe the cycle we consider its spectrum, which can be computed through expression (2.12) and will always be expressed in units of 2π. Series with different stochastic structures will imply different spectra for the cycle even when the same HP filter is used.

As an example, consider two series that follow a standard and a second-order random-walk model, as in

$$\nabla x_{1t} = a_t \,, \quad \nabla^2 x_{2t} = a_t \,. \qquad (6.1)$$

Expressions (2.8) and (6.1) show that the estimators of the cycle can be expressed in terms of the innovations a_t as

$$\hat{c}_{1t} = k_c \frac{(1-B)(1-F)^2}{\theta_{HP}(B)\,\theta_{HP}(F)} a_t \,, \qquad (6.2a)$$

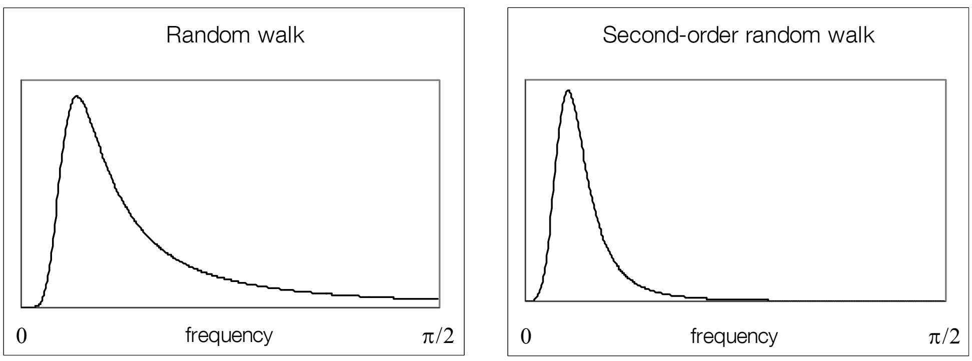
$$\hat{c}_{2t} = k_c \frac{(1-F)^2}{\theta_{HP}(B)\,\theta_{HP}(F)} a_t \,, \qquad (6.2b)$$

where $k_c = \nu_c / \nu_a$. The FT of the ACF of (6.2a) and (6.2b) yield the two spectra, namely,

$$S_{c1}(\omega) = \frac{8\lambda^2 (1-\cos\omega)^3}{\left[1+4\lambda(1-\cos\omega)^2\right]^2} V_a \,,$$

$$S_{c2}(\omega) = \frac{4\lambda^2 (1-\cos\omega)^2}{\left[1+4\lambda(1-\cos\omega)^2\right]^2} V_a \,.$$

Figure 12: Spectra of the cycle component of first and second-order random walk (λ=1600)



The two spectra are displayed in Figure 12 for $\lambda = 1600$. Both have the shape of a stochastic cyclical component spectrum, with the variance concentrated around the spectral peak. The cycle associated with that peak will be denoted the "cycle of dominance". A natural criterion for aggregation could be preservation of the cycle of dominance. [This is similar to approximating spectral densities by preserving the mode, see Durbin and Koopman (2000).] An advantage of this approach is that it combines the characteristics of the filter with the specific features of the series; it has the disadvantage that no general rule for finding equivalent values of λ can be obtained, since the equivalence depends on the model for the series. Nevertheless, two issues are of interest. First, what is the equivalence for some of the most relevant ARIMA models? Second, if the simpler criterion of fixing the period associated with the cycle of reference of section 4 (or the RU rule) is used, are the results likely to be much different from those obtained with the criterion of fixing the period associated with the cycle of dominance?

We consider IMA(1,1) and IMA(2,2) models, both of which are consistent under temporal aggregation and systematic sampling [Brewer (1973)]. The IMA(d,d) formulation is also attractive because it is the limiting model for time aggregates of ARIMA(p,d,q) models [Tiao (1972)]. We encompass both cases under the specification

$$\nabla^d x_t = \theta(B) a_t = (1 + \theta_1 B + \theta_2 B^2) a_t, \tag{6.3a}$$

where $d = 1$ and $\theta_2 = 0$ for the IMA(1,1) case, and $d = 2$ for the IMA(2,2) one. For an alternative (aggregate) frequency of observation (6.3a) becomes

$$\nabla^d X_T = \Theta(B) A_T = (1 + \Theta_1 B + \Theta_2 B^2) A_T. \tag{6.3b}$$

Let $\theta_Q = (\theta_1, \theta_2)$ and $\theta_D = (\Theta_1, \Theta_2)$ denote the vectors with the MA parameters of the quarterly model and of the model for the alternative frequency of observation (annual or monthly). Likewise, let $S_Q(\omega | \theta_Q, \lambda_Q)$ and $S_D(\omega | \theta_D, \lambda_D)$ denote the corresponding spectra. The procedure for obtaining the equivalent values of λ for the transformed series can be summarized as follows:

1. Given θ_Q and λ_Q, obtain the frequency ω_Q ($\omega \in [0, \pi]$) such that $S_Q(\omega | \theta_Q, \lambda_Q)$ is maximized, as well as the associated period τ_Q.
2. Transform τ_Q into τ_D and obtain the associated frequency ω_D.
3. Use the relationship between the variance and covariances of the disaggregate and aggregate series to find θ_D given θ_Q.
4. Find $\tilde{\omega}$ such that $S_D(\omega | \theta_D, \lambda_D)$ is maximized, and λ_D such that $\tilde{\omega} = \omega_D$.

Although the procedure is general, in our application we fix $\lambda_Q = 1600$ for quarterly data and derive the values λ_M and λ_A that preserve the period associated with the cycle spectral peak. Step 3 requires the derivation of the model for the annual or monthly series, given the model for the quarterly one. If (6.3a) is the model for the more disaggregate series, the model for the aggregate series will be of the type (6.3b). In order to obtain the Θ and V_A parameters, we follow the Wei and Stram (1986) approach, extended to cover also the case of systematic sampling, as detailed in Appendix B. In brief, if Γ and γ denote the vector of autocovariances of $\nabla^d X_T$ and $\nabla^d x_t$, a matrix M is computed such that

$$\Gamma = M \gamma. \tag{6.4}$$

Expressing Γ and γ as functions of the model parameters, the parameters for the alternative frequency can be obtained as functions of the quarterly parameters.

For the IMA(1,1), and IMA(2,2) models, the matrices M in (6.4) that relate annual to quarterly, and quarterly to monthly, covariances are given in Table 2, for both the temporal aggregation and systematic sampling cases.

Table 2: Matrices M that relate aggregate and disaggregate covariances

Model	Frequencies	Temporal Aggregation	Systematic Sampling
IMA(1,1)	Quarterly to Annual Aggreg.	$\begin{bmatrix} 44 & 80 \\ 10 & 24 \end{bmatrix}$	$\begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$
	Monthly to Quarterly Aggreg.	$\begin{bmatrix} 19 & 32 \\ 4 & 11 \end{bmatrix}$	$\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$
IMA(2,2)	Quarterly to Annual Aggreg.	$\begin{bmatrix} 580 & 1092 & 912 \\ 216 & 456 & 512 \\ 6 & 22 & 56 \end{bmatrix}$	$\begin{bmatrix} 44 & 80 & 62 \\ 10 & 24 & 32 \\ 0 & 0 & 1 \end{bmatrix}$
	Monthly to Quarterly Aggreg.	$\begin{bmatrix} 141 & 252 & 180 \\ 50 & 111 & 132 \\ 1 & 6 & 21 \end{bmatrix}$	$\begin{bmatrix} 19 & 32 & 20 \\ 4 & 11 & 16 \\ 0 & 0 & 1 \end{bmatrix}$

6.1 IMA(1,1) Model

Combining (6.3a) –with $d=1$ and $\theta_2=0$– with (2.8), it is seen that the cycle estimator follows the model

$$\hat{c}_t = \frac{\nu_c}{\nu_a} \frac{(1-B)(1-F)^2}{\theta_{HP}(B)\,\theta_{HP}(F)} (1+\theta B)\, a_t ,$$

so that, considering (2.12), its spectrum is given by

$$S_c(\omega|\theta,\lambda) = \frac{8\lambda^2 (1-\cos\omega)^3}{\left[1+4\lambda(1-\cos\omega)^2\right]^2}(1+\theta^2+2\theta\cos\omega)V_a , \tag{6.5}$$

and maximizing (6.5) with respect to ω yields

$$\tilde{\omega} = a\cos\left[1+\frac{\theta}{\lambda(1+\theta)^2} - \sqrt{\frac{3}{4\lambda}+\frac{\theta^2}{\lambda^2(1+\theta)^4}}\right]. \tag{6.6}$$

Solving (6.6) for λ, it is obtained that

$$\tilde{\lambda} = \frac{3}{4\left(1-\cos\tilde{\omega}\right)^2} - \frac{2\theta}{(1+\theta)^2\left(1-\cos\tilde{\omega}\right)} . \tag{6.7}$$

(a) From Quarterly to Annual Data

Steps (1) and (2) above are a direct application of (6.6) and (4.4) with $\theta = \theta_Q$ and $\lambda = \lambda_Q$, and of (4.6) with $k = 1/4$. From (6.3a) and (6.3b), $\gamma = (\gamma_0, \gamma_1) = [(1+\theta_Q^2)V_a, \theta_Q V_a]$, $\Gamma = (\Gamma_0, \Gamma_1) = [(1+\Theta_1^2)V_A, \Theta_1 V_A]$, or, considering (6.4) with the appropriate M matrix from Table 2, the system of covariance equation is

$$\Gamma_0 = (44 + 80\theta_Q + 44\theta_Q^2)V_a,$$

$$\Gamma_1 = (10 + 24\theta_Q + 10\theta_Q^2)V_a.$$

Therefore, $(1+\Theta_1^2)/\Theta_1 = c$, where $c = (44 + 80\theta_Q + 44\theta_Q^2)/(10 + 24\theta_Q + 10\theta_Q^2)$. The MA parameter of the annual IMA(1,1) model is given by the invertible solution of equation

$$z^2 - cz + 1 = 0. \tag{6.8}$$

For the case of systematic sampling, using the appropriate matrix from Table 2, the system of covariance equations becomes

$$\Gamma_0 = (4 + 6\theta_Q + 4\theta_Q^2)V_a, \quad \Gamma_1 = \theta_Q V_a.$$

Defining $c = (4 + 6\theta_Q + 4\theta_Q^2)/\theta_Q$, the MA parameter for the IMA(1,1) annual model is again the invertible solution of (6.8). Having obtained Θ_1, setting $\theta = \Theta_1$, and $\tilde{\omega} = \omega_D$ in (6.7), the equivalent value of λ for annual series, λ_A, is obtained. The period associated with the cycle spectral maximum will be identical for the quarterly and annual series.

(b) From Quarterly to Monthly Data

Step (1) and (2) are as in the previous case, except that now, $\tau_M = 3\tau_Q$, the aggregate series is the quarterly one, and hence $\Gamma_0 = (1+\theta_Q^2)V_a$, $\Gamma_1 = \theta_Q V_a$, $\gamma_0 = (1+\theta_M^2)V_a$, and $\gamma_1 = \theta_M V_a$. Using the appropriate matrix M from Table 2, the system of covariance equations is given by

$$(1+\theta_Q^2)V_A = (19 + 32\theta_M + 19\theta_M^2)V_a,$$

$$\theta_Q V_A = (4 + 11\theta_M + 4\theta_M^2)V_a.$$

Letting $c_1 = (1+\theta_Q^2)/\theta_Q$, it is found that θ_M is the invertible solution of (6.8), with $c_2 = (32 - 11c_1)/(19 - 4c_1)$. The equation has complex solutions when $\theta_Q \geq 0.3$ so that IMA(1,1) monthly models aggregate into IMA(1,1) quarterly models with the MA parameter restricted to the range $-1 < \theta_Q < 0.3$.

For the case of systematic sampling and using the appropriate M matrix from Table 2, the system of covariance equations is replaced by:

$$(1+\theta_Q^2)V_A = (3+4\theta_M + 3\theta_M^2)V_a,$$

$$\theta_Q V_A = \theta_M V_a,$$

so that, if $c_1 = (1+\theta_Q^2)/\theta_Q$ and $c = (4\text{-}c_1)/3$, the value of θ_M is the invertible solution of (6.8). The system yields complex solutions when $\theta_Q > 0.33$ and hence systematic sampling of monthly IMA(1,1) models yields quarterly IMA(1,1) models with the MA parameter restricted to the range $-1 < \theta_Q < 0.33$.

With the quarterly value set at $\lambda_Q = 1600$, Table 3 displays the equivalent monthly and annual values of λ, obtained with the criterion of preserving the period associated with the cycle spectral peak, when the series follows an IMA(1,1) process, and for different values of the MA parameter θ_Q. It is seen that the model parameter has a moderate effect on the period of the cycle of dominance.

Table 3: IMA(1,1): Monthly and annual λ values that preserve the period of the cycle spectral peak for $\lambda_Q = 1600$

θ_Q	Period of the cycle of dominance (in years)	Equivalent values of λ			
		temporal aggregation		systematic sampling	
		annual (λ_A)	monthly (λ_M in 10^3)	annual (λ_A)	monthly (λ_M in 10^3)
-0.8	5.72	6.53	129.3	20.86	71.4
-0.6	7.14	6.12	129.8	10.85	112.0
-0.4	7.41	6.05	129.8	8.21	123.4
-0.2	7.50	6.03	129.9	7.33	127.2
0.0	7.53	6.02	129.9	6.97	128.8
0.2	7.55	6.02	129.9	6.81	129.8
0.4	7.56	6.01	- (*)	6.74	- (*)
0.6	7.56	6.01	-	6.70	-
0.8	7.56	6.01	-	6.69	-

(*) Values of θ_Q for the lines marked "-" cannot be obtained by aggregation of monthly IMA(1,1) models.

For the case of temporal aggregation the results are seen to be very stable. The monthly equivalent values λ_M are always close to 130000, and the annual equivalent value λ_A lies between 6 and 6.5. These values are close to the ones obtained in Sections 4 and 5. When aggregation is achieved through systematic sampling, the results are less stable, in particular as θ_Q approaches -1.

6.2 IMA(2,2) Model

When z_t follows the IMA(2,2) model given by (6.3a), from (2.8) and (2.12) it is found that the HP cycle follows the model

$$\hat{c}_t = \frac{\lambda(1-F)^2 (1+\theta_1 B+\theta_2 B^2)}{1+\lambda(1-B)^2 (1-F)^2} a_t,$$

with spectrum

$$S_c(\omega|\lambda,\theta_1,\theta_2) = \frac{4\lambda^2 (1-\cos\omega)^2 \left[1+\theta_1^2+\theta_2^2+2\theta_1(1+\theta_2)\cos\omega+2\theta_2\cos 2\omega\right]}{\left[1+4\lambda(1-\cos\omega)^2\right]^2} V_a.$$

The maximum with respect to ω is achieved for the real and positive solution of a third degree polynomial in $\cos\omega$. Let this solution be

$$\tilde{\omega} = \tilde{\omega}(\lambda,\theta_1,\theta_2). \tag{6.9}$$

or, solving for λ,

$$\tilde{\lambda} = \frac{1}{4(1-\cos\tilde{\omega})} - \frac{\theta_1+\theta_1\theta_2+4\theta_2\cos\tilde{\omega}}{2(1-\cos\tilde{\omega})\left[1+\theta_1(\theta_1+1+\cos\tilde{\omega})+\theta_2(\theta_2-2)+(2+\theta_1)\cos\tilde{\omega})\right]} \tag{6.10}$$

Proceeding as in the previous section, given $\theta_Q = (\theta_1,\theta_2)$ and λ_Q for the quarterly model, we use (6.9) to compute the frequency for which the spectrum of the quarterly cycle reaches a maximum, and obtain the associated period. Expressing this period in terms of annual and monthly data, we obtain the annual and monthly associated frequencies. Once we know the parameters θ_1 and θ_2 of the annual and monthly model, (6.10) provides the values of λ_A and λ_M. The monthly and annual series also follow IMA(2,2) models and, in order to derive the parameters, we follow as before the Wei-Stram procedure.

Let x_t, (θ_1,θ_2), and V_a denote the disaggregate series, the MA parameters of its model, and its innovation variance, respectively. Likewise, let X_T, (Θ_1,Θ_2), and V_A denote the aggregate series, the MA parameters of its model, and its innovation variance. If $(\gamma_0,\gamma_1,\gamma_2)$ and $(\Gamma_0,\Gamma_1,\Gamma_2)$ represent the variance, lag-1, and lag-2 autocovariances of $\nabla^2 x_t$ and $\nabla^2 X_T$, respectively, we have

$$\gamma_0 = (1+\theta_1^2+\theta_2^2)V_a, \tag{6.11a}$$

$$\gamma_1 = \theta_1(1+\theta_2)V_a, \tag{6.11b}$$

$$\gamma_2 = \theta_2 V_a, \tag{6.11c}$$

and, replacing (θ_1,θ_2) and V_a by (Θ_1,Θ_2) and V_A, similar expressions hold for Γ_0, Γ_1 and Γ_2. If γ and Γ denote the vectors $\gamma = (\gamma_0,\gamma_1,\gamma_2)'$ and $\Gamma = (\Gamma_0,\Gamma_1,\Gamma_2)'$, the relevant M matrices in (6.4) are given in Table 2.

Given γ, one can obtain Γ and, using the inverse relationship $\gamma = M^{-1}\Gamma$, given Γ, one can obtain γ. The aggregate/disaggregate MA parameters are found by factorizing the ACF obtained, as in Maravall and Mathis (1994, Appendix A).

Table 4, which is analogous to Table 3 for the IMA(2,2) case, displays the monthly and annual λ values that are consistent with the quarterly value $\lambda_Q = 1600$, under the criterion of preserving the period associated the cycle spectral peak (the MA values θ_1 and θ_2 are restricted to lie in the invertible region). Compared to the IMA(1,1) case, the IMA(2,2) model increases the length of the period of the cycle of dominance, and the θ parameters are seen to have a small effect on λ_A and λ_M. The results are again close to those obtained with the criteria of Sections 4 and 5.

Table 4: IMA(2,2): monthly and annual λ values that preserve the period of dominance for λ_Q=1600

<table>
<tr><th rowspan="3">$\theta_{Q,1}$</th><th rowspan="3">$\theta_{Q,2}$</th><th rowspan="3">Period of the cycle of dominance (years)</th><th colspan="4">Equivalent values of λ</th></tr>
<tr><th colspan="2">temporal aggregation</th><th colspan="2">systematic sampling</th></tr>
<tr><th>annual (λ_A)</th><th>monthly (λ_M in 10^3)</th><th>annual (λ_A)</th><th>monthly (λ_M in 10^3)</th></tr>
<tr><td>0.2</td><td>0.0</td><td>9.9</td><td>6.02</td><td>131.8</td><td>6.24</td><td>129.6</td></tr>
<tr><td>0.0</td><td>0.0</td><td>9.9</td><td>6.03</td><td>128.6</td><td>6.24</td><td>129.6</td></tr>
<tr><td>0.0</td><td>0.2</td><td>10.0</td><td>6.01</td><td>131.2</td><td>6.23</td><td>131.8</td></tr>
<tr><td>-0.2</td><td>0.0</td><td>9.9</td><td>6.03</td><td>127.7</td><td>6.24</td><td>129.6</td></tr>
<tr><td>-0.2</td><td>0.2</td><td>10.0</td><td>6.01</td><td>131.1</td><td>6.23</td><td>131.4</td></tr>
<tr><td>-0.4</td><td>0.0</td><td>9.9</td><td>6.04</td><td>125.2</td><td>6.26</td><td>129.6</td></tr>
<tr><td>-0.4</td><td>0.2</td><td>10.0</td><td>6.01</td><td>130.8</td><td>6.23</td><td>130.9</td></tr>
<tr><td>-0.6</td><td>0.0</td><td>9.7</td><td>6.05</td><td>117.7</td><td>6.29</td><td>129.5</td></tr>
<tr><td>-0.6</td><td>0.2</td><td>9.9</td><td>6.02</td><td>129.7</td><td>6.24</td><td>129.8</td></tr>
<tr><td>-0.8</td><td>0.2</td><td>9.9</td><td>6.04</td><td>125.7</td><td>6.28</td><td>129.6</td></tr>
<tr><td>-0.8</td><td>0.4</td><td>10.0</td><td>5.98</td><td>133.5</td><td>6.22</td><td>133.9</td></tr>
<tr><td>-1.0</td><td>0.2</td><td>9.4</td><td>5.99</td><td>102.9</td><td>6.59</td><td>102.7</td></tr>
<tr><td>-1.0</td><td>0.4</td><td>10.0</td><td>5.98</td><td>132.9</td><td>6.24</td><td>133.0</td></tr>
<tr><td>-1.4</td><td>0.6</td><td>10.1</td><td>5.74</td><td>140.8</td><td>6.23</td><td>140.9</td></tr>
</table>

7 Least squares minimization of the distance between direct and indirect cycle

For a particular application, it is always possible to compute close-to-equivalent values of λ through least-squares minimization of the distance between the direct and indirect aggregate cycles. If λ_0 is the value of λ applied to the disaggregate series, the value λ_d to use for direct adjustment is given by

$$\hat{\lambda}_d = \arg\min \sum_t \left[\hat{C}_{i,t}(\lambda_0) - \hat{C}_{d,t}(\lambda_d)\right]^2 \qquad (7.1)$$

where $\hat{C}_{i,t}(\lambda_0)$ and $\hat{C}_{d,t}(\lambda_d)$ denote the estimated indirect and direct aggregate cycle, respectively. This procedure is relatively cumbersome, depends on the particular realization, and may produce variability in the values of λ that could induce inconsistencies for the different levels of aggregation. It is nevertheless of interest to ascertain whether the solution is likely to yield values of λ that may strongly depart from the values obtained with the previous criteria.

We looked at the case of aggregating quarterly series into annual ones (using $\lambda_Q = 1600$ for direct estimation of the quarterly cycle), under temporal aggregation and systematic sampling, and for the IMA(1,1) and IMA(2,2) models for different values of the parameters. For each of the cases, only 100 simulations were made; the results seemed stable given our level of precision (first decimal point in λ_A). For each simulation, expression (7.1) was solved and $\hat{\lambda}_d$ estimated; then the mean and standard deviation of the $\hat{\lambda}_d$'s obtained were computed.

As before, except for the case of systematic sampling a model with an MA root close to -1, the values of λ_A are relatively stable and close to those obtained with the previous criteria. Notice that the value $\lambda_A = 6.65$, obtained according to the criteria of preserving the cycle of reference, is not significantly different from any of the values in Tables 5 and 6.

Table 5: Least square minimization: IMA(1,1) models

θ_Q	temporal aggregation λ_A		systematic sampling λ_A	
	mean	std. dev	mean	std. dev
-0.8	6.9	0.7	(*)	(*)
-0.6	6.8	0.4	15.1	11.4
-0.4	6.6	0.2	10.8	13.3
-0.2	6.7	0.2	8.4	4.1
0.0	6.6	0.2	7.4	1.7
0.2	6.7	0.2	7.1	1.2
0.4	6.7	0.2	7.1	1.2
0.6	6.6	0.1	7.1	1.2
0.8	6.6	0.1	7.0	1.2

(*) Numerical problems because of the flat surface of the objective function around the minimum.

Table 6: Least square minimization: IMA(2,2) models

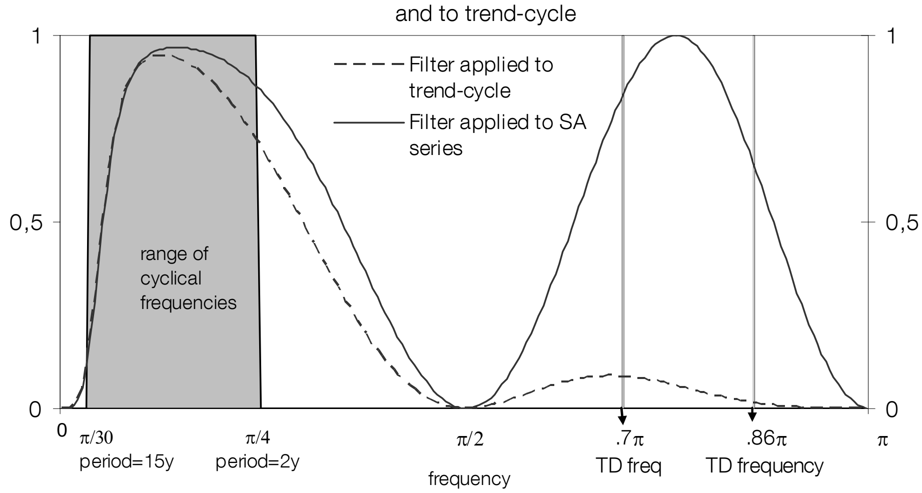
$\theta_{Q,1}$	$\theta_{Q,2}$	temporal aggregation		systematic sampling	
		λ_A		λ_A	
		mean	std. dev	mean	std. dev
-0.8	0	6.5	0.1	6.7	0.9
-0.6	0	6.5	0.1	6.7	0.7
-0.4	0	6.5	0.1	6.7	0.6
-0.2	0	6.5	0.1	6.6	0.6
0.0	0	6.5	0.1	6.5	0.7
0.2	0	6.5	0.1	6.4	0.6
0.4	0	6.5	0.1	6.6	0.6
0.6	0	6.5	0.1	6.6	0.6
0.8	0	6.5	0.1	6.5	0.6
-0.6	-0.3	6.6	0.1	6.9	1.1
-0.6	0.3	6.5	0.1	6.5	0.7
-0.5	-0.3	6.5	0.1	6.6	0.9
-0.5	0.3	6.5	0.1	6.6	0.6
-0.4	-0.3	6.5	0.1	6.6	0.9
-0.4	0.3	6.5	0.1	6.6	0.6
0.4	-0.3	6.5	0.1	6.6	0.6
0.4	0.3	6.5	0.1	6.5	0.6
0.5	-0.3	6.5	0.1	6.6	0.5
0.5	0.3	6.5	0.1	6.5	0.5
0.6	-0.3	6.5	0.1	6.6	0.6
0.6	0.2	6.5	0.1	6.5	0.5

8 Limitations of Consistency under Aggregation

Our objective has been to obtain values of λ for which direct and indirect estimation under time aggregation yield cycles that are consistent. Yet there are a number of reasons that can justify departures from aggregation consistency. For example, it can be argued that when monitoring a series observed once a year or once every two years, short- or medium-term analysis should not focus on the same frequencies as when the series is observed weekly or monthly. Evidently, a 3-year cycle may be of interest when monitoring a monthly series, but would hardly be helpful if the series is observed once every 2 years. Thus the analyst may not be interested in preserving as cycle of reference one designed for quarterly data, and the choice of λ may differ depending on the frequency of observation. Cycles used for different frequencies may not display good aggregation properties, yet they might be of more use to the analyst.

There are also methodological reasons that justify departures from aggregation consistency. In order to avoid contamination with seasonal frequencies, the HP filter is applied to SA data. Yet seasonal adjustment is a non-linear transformation [Ghysels et al. (1996); Maravall (2006)] and hence one cannot expect to preserve linear constraints-such as those implied by time aggregation. Further, the SA series is contaminated with noise and possibly with outliers or trading day effects, and this contamination may distort estimation of the cyclical signal. Figure 13 compares the gains of the convolution of the HP filter ($\lambda = 1600$) with the filter that provides the estimators of the SA series and of the trend-cycle for the model $\nabla\nabla_4 x_t = a_t$. The use of the trend-cycle improves the band-pass features of the cyclical filter in the sense that it performs a more drastic removal of frequencies that do not belong to the range of cyclical frequencies (in the figure, the periods between 2 and 15 years). It is thus preferable to use as input to the HP filter the trend-cycle component. This is illustrated in Figures 14 and 15, which plot the cycles estimated on the SA series and on the trend-cycle component of the monthly Italian and French IPIs (Jan 1962-Dec 2005). The similarities between the two cycles are more clearly discernible when the trend-cycle component is employed.

Figure 13: Gain of the HP filter applied to seasonally adjusted series and to trend-cycle



In general, filtering or pretreatment of a series prior to application of the HP filter may already affect aggregation. Outliers detected in a monthly series may well be different from those detected in an annual one. Trading-day and/or Easter effects may be significant for the monthly series, but not for the quarterly one. The ARIMA models used to extend the series, or to obtain the SA series or trend-cycle component, will hardly ever be exact aggregates when identified for different levels of aggregation. As a consequence, departures from aggregation consistency should be expected. Figures 16-19 illustrate these departures for the Italian and French IPI cycles obtained with the equivalent values of λ given by (4.7). As is usually the case, direct estimation provides a smoother series and, although the overall effect is moderate, it cannot be regarded as trivial. Given that little can be done to solve in a convincing manner this discrepancy problem, perhaps the best solution is to compute both the direct and indirect estimators, and this may serve as a reminder of our (many) measurement limitations.

Figure 14: Monthly cycles (LAM=130000)
Measured on SA series

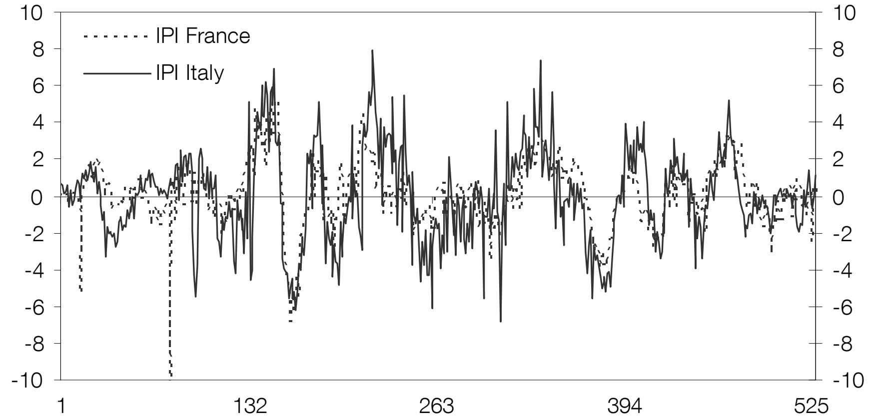
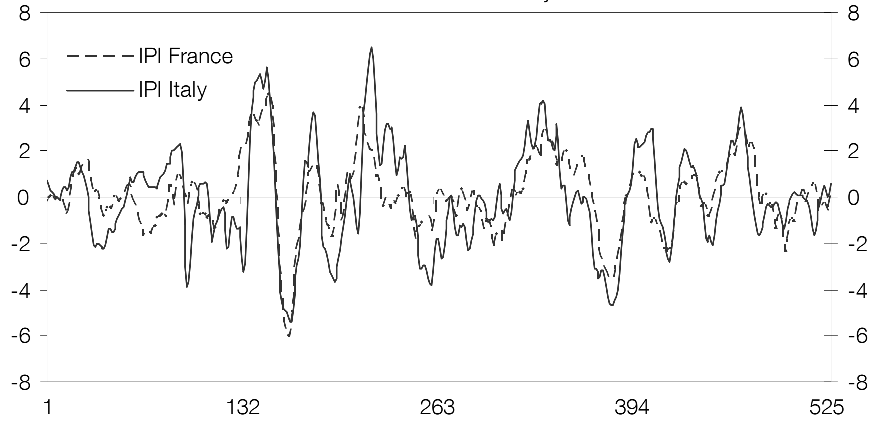


Figure 15: Monthly cycles (LAM=130000)
Measured on trend-cycle



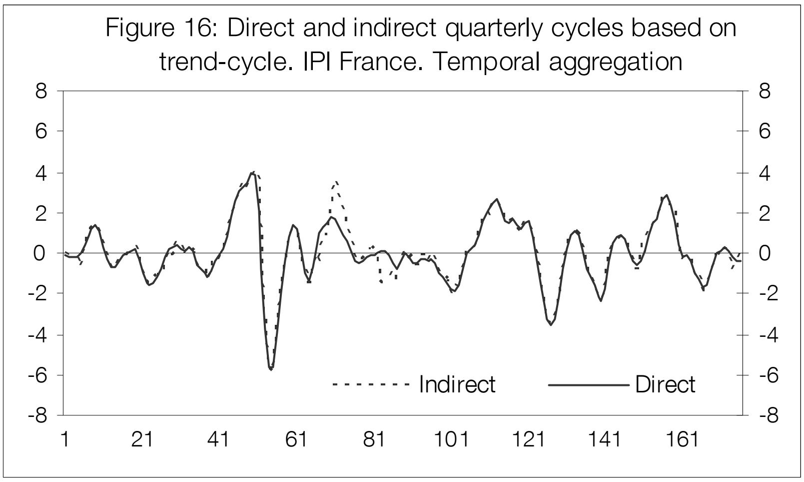


Figure 16: Direct and indirect quarterly cycles based on trend-cycle. IPI France. Temporal aggregation
8
6
4
2
0
-2
-4
-6
-8
1
21
41
61
81
101
121
141
161
Indirect
Direct

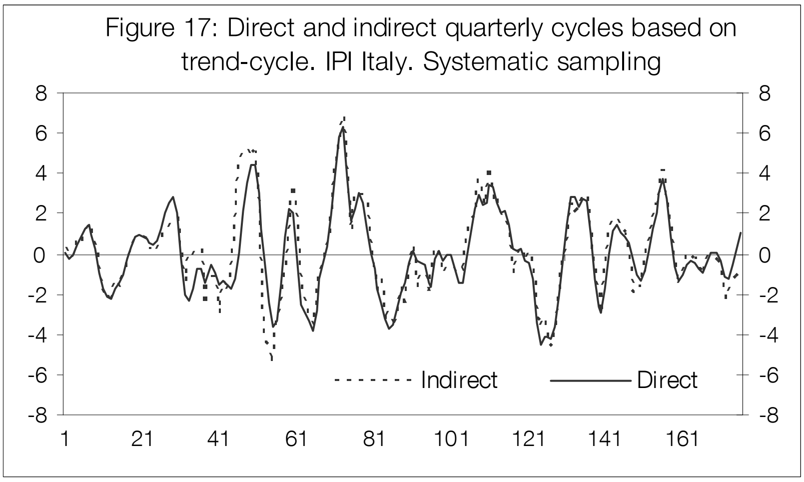


Figure 17: Direct and indirect quarterly cycles based on trend-cycle. IPI Italy. Systematic sampling
8
6
4
2
0
-2
-4
-6
-8
1
21
41
61
81
101
121
141
161
Indirect
Direct

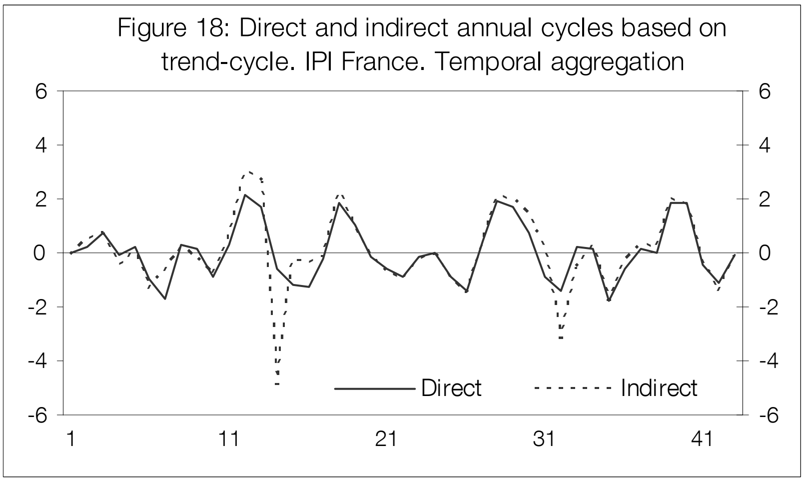
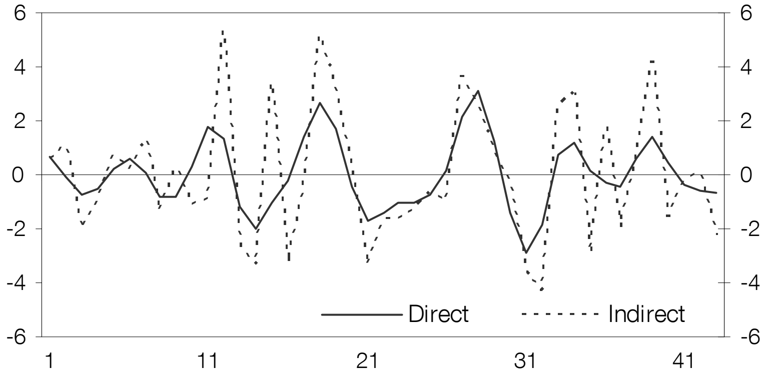


Figure 18: Direct and indirect annual cycles based on trend-cycle. IPI France. Temporal aggregation
6
4
2
0
-2
-4
-6
1
11
21
31
41
Direct
Indirect

Figure 19: Direct and indirect annual cycles based on trend-cycle. IPI Italy. Systematic sampling



9 Conclusions

We have analyzed the time aggregation properties of the Hodrick-Prescott (HP) filter, focusing on monthly, quarterly, and annual observations. Two types of aggregation have been considered: Temporal Aggregation, whereby the aggregate series consists of (non-overlapping) sums (or averages) of disaggregate values, and Systematic Sampling, whereby the aggregate series is equal to a value of the disaggregate series sampled at periodic intervals. The main results can be summarized as follows.

For the two types of aggregation, the HP filter does not preserve itself under aggregation in the following sense. Cycles estimated by aggregating cycles obtained for disaggregate data with an HP filter (indirect estimation) cannot be seen as the exact result of an HP filter applied to the aggregate data (direct estimation). Direct and indirect estimation of cycles computed with HP-type filters cannot yield identical results. In practice, this lack of aggregation consistency has led to an arbitrary choice of inconsistent λ 's for different levels of aggregation.

Several statistically-based criteria that provide values of λ that yield almost equivalent results have been considered. The first criterion, considered in Section 4.1, is to preserve, for different levels of aggregation, the period of the cycle associated with the frequency for which the gain of the HP filter is .5. Given that this frequency represents the cutting point between frequencies that will be mostly assigned to the cycle and those that will be mostly assigned to the trend, the criterion is intuitively attractive and simple to apply. Section 4.2 shows that the empirical rule suggested by Ravn and Uhlig (2002) turns out to be a first-order approximation to the previous criterion. Sections 5.1 and 5.2 consider criteria based on preserving different filter characteristics and it is seen that the results remain roughly unchanged.

But the properties of the estimated cycle will depend, not only on the filter, but also on the characteristics of the series at hand. We represent the latter with an ARIMA model and this allows us to derive the spectrum of the cycle estimator. In Section 6 equivalent values of λ for different levels of aggregation are derived, for IMA (1,1) and IMA (2,2) models, under the criterion of preserving the period associated with the frequency for which the cycle spectrum reaches a maximum. It is seen that, except for the case of systematic sampling of models that are close to non invertibility, the results are robust with respect to the model parameters, and very close to those obtained with the previous criteria. Finally, Section 7 uses as criterion least-square minimization of the distance between direct and indirect cycles. With the same exception as before, the results obtained are again very close.

For the quarterly consensus value $\lambda_Q = 1600$, the previous results yield monthly and annual equivalent values in the intervals

$$125000 < \lambda_M < 130000 , \quad 6 < \lambda_A < 7 ,$$

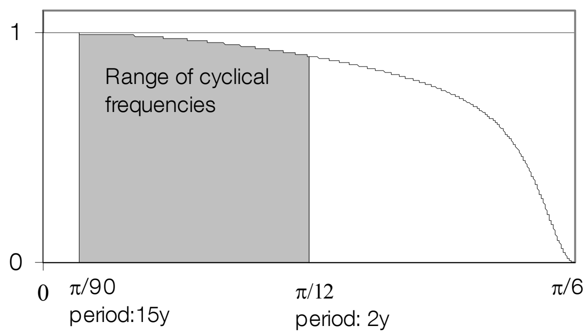
with the λ values more in the vicinity of the upper bounds. It follows that the RU rule can be safely used, with perhaps a slightly larger value of λ_A (such as $\lambda_A = 6.5$ or 6.75). For the case of systematic sampling of close-to-noninvertible models a smaller value for λ_M and a larger one for λ_A may perform better. Nevertheless, although consistency under aggregation is a desirable property, the final section explains and illustrates why optimal procedures are likely to induce inconsistencies between direct and indirect estimation, even when consistent values of λ are employed.

APPENDIX A: Identification of the Cycle and Spurious Results

Criticism of the HP filter has focused on two methodological points: It has been argued that the HP parameter λ should be estimated directly in a structural time series model (STSM) approach [see Harvey (1997)] and concern has been repeatedly expressed over the danger of violating the series structure by imposing a spurious cycle. A closer look will show that these two criticisms are not justified.

It is well known that the differencing operator ∇ has a strong effect on the low frequencies of x_t, including the range of cyclical frequencies. As an example, the gain of the trend extraction filter in the ARIMA-model-based decomposition of the Airline model popularized by Box and Jenkins (1970) and given by $\nabla\nabla_{12} x_t = (1 - .4B)(1 - .6B^{12})a_t$, is displayed in Figure A.1 for the range of frequencies between 0 and the first seasonal harmonic. It is seen how the trend filter picks up most of the variation for cyclical frequencies, so that the component should be more properly called "trend-cycle" component. (This feature also characterizes trends produced by the standard STSM approach or by the Henderson filters in X11.)

Figure A.1: Gain of Trend filter in Airline model



For the usual series length, regular differencing, as a rule, does not permit identification of business cycles through a "let the data speak" approach. A way to overcome this limitation is to use ad-hoc band-pass filters that extract the series variation for some range of frequencies, while respecting the information that the "let the data speak" approach provides (namely, that the identified ARIMA model transforms the series into white noise).

Consider a series that follows the general model

$$\nabla^d x_t = \psi(B)a_t, \quad (d < 3), \tag{A.1}$$

where $\psi(B)a_t$ is a stationary ARMA process, and the HP decomposition of x_t into trend $(\hat{m}_t)$ and cycle $(\hat{c}_t)$ given by (2.9), so that $x_t = \hat{m}_t + \hat{c}_t$. In general, the concept of spuriousness is questionable in the context of ad-hoc filters: the filter simply yields what it is designed to yield, without reference to a model. For the case of the HP filter, perhaps the reason for asserting its spuriousness can be found in its model-based interpretation given by King and Rebelo (the "artificial" model) which implies an IMA(2,2) structure for the series x_t, with the MA polynomial — $\theta_{HP}(B)$ — determined from λ. In so far as it is highly unlikely that x_t follows this model, it is argued that the filter is spurious [in particular for I(1) series].

But the argument is fallacious. The HP filter can be given another, perfectly sensible, model-based interpretation. The series x_t, given by (A.1), can be expressed as the sum of orthogonal trend (m_t) and cycle (c_t) components, with models given by

$$\theta_{HP}(B)\nabla^d m_t = \psi(B)\, a_{mt}, \quad a_{mt} \sim wn(0, V_m), \tag{A.2}$$

$$\theta_{HP}(B)\, c_t = \psi(B)\nabla^{2-d} a_{ct}, \quad a_{ct} \sim wn(0, V_c), \tag{A.3}$$

and $V_c / V_m = \lambda$. It is straightforward to check that the HP filter estimators in (2.7) – (2.9) are the MMSE estimators of m_t and c_t. Let $\gamma_{dm}(B,F)$ and $\gamma_{dc}(B,F)$ denote the ACFs of $\nabla^d m_t$ and $\nabla^d c_t$, respectively. From (A.2) and (A.3),

$$\gamma_{dm}(B,F) = k_m\, \psi(B)\psi(F) / \theta_{HP}(B)\theta_{HP}(F),$$

$$\gamma_{dc}(B,F) = k_c\, \psi(B)\nabla^2 \psi(F)(1-F)^2 / \theta_{HP}(B)\theta_{HP}(F),$$

where $k_m = V_m / V_a$ and $k_c = V_c / V_a$. The ACF of $\nabla^d (m_t + c_t)$ is equal to

$$V_a\, \psi(B)\psi(F)\left[k_m + k_c \nabla^2 (1-F)^2\right] / \theta_{HP}(B)\theta_{HP}(F) = \psi(B)\psi(F)V_a,$$

where use has been made of (2.5); it is thus equal to the ACF of $\nabla^d x_t$. It follows that, under our assumptions, model (A.1) and the model consisting of $x_t = m_t + c_t$, (A.2), and (A.3), are observationally equivalent. The ACFs implied by the two models are equal and so will be the spectra and the implied joint distribution of the observed series. The models will provide the same diagnostics, the same likelihood, and the same forecast function. [A related discussion can be found in Maravall and Kaiser (2005).]

One may disagree with the specification of the UC component model, but the results cannot be properly called spurious. The spectral shape of m_t will be that of a smooth trend (the small value of k_m will produce a narrow peak for the zero frequency) and the spectral shape of c_t will be that of a stochastic cycle, with the peak determined from the AR(2) polynomial $\theta_{HP}(B)$, which will have complex roots associated with a cyclical frequency [McElroy (2006)]. This frequency will be determined by the analyst choice of λ. It should be pointed out that, given (A.1), the UC model (A.2)-(A.3) is, in general, underidentified and hence the parameter $\lambda = k_c / k_m$ cannot be consistently estimated from the data. Identification can be achieved in a variety of ways. For example, in the STSM approach no model for the observed series is identified; the component models are directly specified and identification is achieved by a-priori restrictions on the orders of the MA polynomials in those models. In our approach, the condition that the component models be consistent with the model for the observed series is imposed (thereby avoiding spuriousness) and identification is achieved by a-priori selecting λ (so to speak, by choosing the band-pass features of the cycle filter).

In the STSM approach, the parameter λ is estimated as the ratio of the trend and cycle innovation variances. But in order to separate the trend from the cycle frequencies this ratio needs to be very small and, unless the series is abnormally long, the estimator of the ratio will not be significant; hence no cycle can be detected. This lack of resolution is more a limitation of the approach than a proof that no cycle information can be found in the series.

As an example, we consider a quarterly series that follows the random walk model (6.1a). Setting $\lambda = 1600$, the WK implementation of the HP filter implies estimation of c_t in the artificial model (2.3) and (2.4), with $\theta_{HP}(B)$ and V_b given by (2.6). Thus $k_m = 1/V_b = .0005$ and $k_c = 1600/V_b = .8$, and x_{1t} can be decomposed into orthogonal trend (m_t) and cycle (c_t) components that follow the models

$$\theta_{HP}(B)\,\nabla\, m_t = a_{mt}, \tag{A.4}$$

$$\theta_{HP}(B)\, c_t = \nabla a_{ct}, \tag{A.5}$$

with $Var(a_{mt}) = k_m$ and $Var(a_{ct}) = k_c$. It is straightforward to check that the filters that yield the MMSE of m_t and c_t in the above model are the HP filters (2.7) – (2.8), and that the sum of the spectra of m_t and c_t yields the spectrum of (6.1a). The spectrum of m_t is shown in figure A.2: it consists of a monotonically decreasing narrow peak around $\omega = 0$. Figure A.3 displays the spectrum of the cyclical component c_t: it has the standard shape of a stationary stochastic cycle, with the variance concentrated around a peak associated with a cycle period of 10 years. The two figures portray sensible (smooth) trend and cycle components, and the sum of their spectra yields exactly the spectrum of the random walk.

Figure A.2: Spectrum of Trend

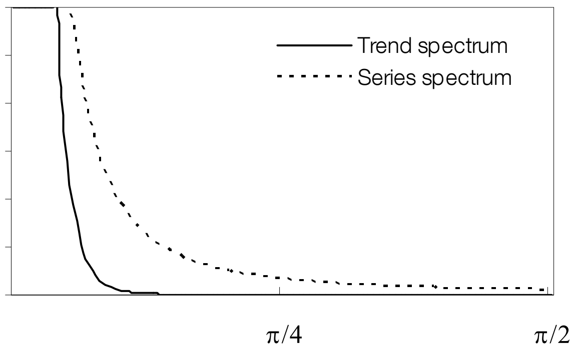
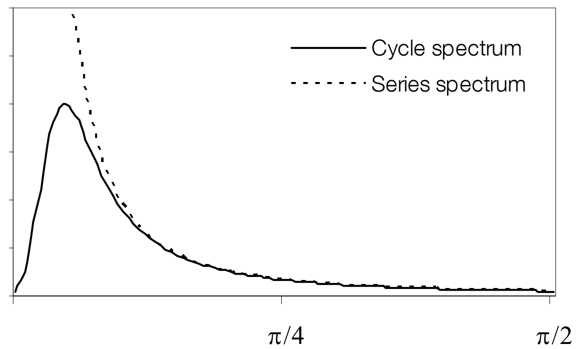


Figure A.3: Spectrum of cycle



Notice that attempts to estimate an innovation variance in the order of $0.0005\, V_a$ by means of the STSM approach, for a quarterly series with 100 or 200 observations, would be futile. The STSM obtained would say that the series simply consists of a random walk trend. This result would reflect the limits of the approach; it would not imply that the trend plus cycle decomposition produces a spurious result, induced by some model misspecification. If the random walk model is not rejected by the data, the UC model (A.4) – (A.5) will not be rejected either.

APPENDIX B: Construction of the Wei-Stram Aggregation Matrix

The Wei-Stram aggregation matrix, M, relates the covariances of the stationary transformation of the aggregate and disaggregate series. We consider IMA(d,d) models, with $d = 1$ and 2, so that the stationary tranformation of the series x_t is $\nabla^d x_t$. Let k be the order of aggregation ($k = 3$ and 4 when aggregating monthly into quarterly and quarterly into annual frequencies, respectively) and define $n = (d+1)$ for temporal aggregation and n=d for systematic sampling. Let γ_i and Γ_i be the autocovariance of order i for the stationary transformation of the disaggregate and aggregate series respectively. Wei and Stram (1986) prove the following relationship for the case of temporal aggregation:

$$\Gamma_i = S^{2n} \gamma_{(ki+n(k-1))} \quad i=0,1,\ldots \tag{B.1}$$

where $S = (1+B+B^2+\ldots+B^{k-1})$ is the aggregation operator. The systematic sampling case is not consider by them but, proceeding in a similar manner it is straightforward to find that (B.1) also holds (although the value of n will differ).

If x_t follows a IMA(d,q) model, then the aggregate series X_T follows a IMA(d,Q) process. When, as in our case, q=d, then Q=d also. If γ and Γ denote the column vectors with the i-th element equal to γ_i and Γ_i, respectively, the Wei-Stram procedure permits us to obtain the relationship $\Gamma = M\gamma$, where M is constructed as follows. Let c be a $1x(2n(k-1)+1)$ row vector with elements (c_i) the coefficients of B^i in the polynomial S^{2n}. Define the matrix A as the $(Q+1) \times (kQ + 2n(k-1)+1)$ matrix:

$$A = \begin{bmatrix} c & 0_k & 0_{(Q-1)k} \\ 0_k & c & 0_{(Q-1)k} \\ 0_{2k} & c & 0_{(Q-2)k} \\ \ldots & \ldots & \ldots \\ 0_{(Q-1)k} & c & 0_k \\ 0_{(Q-1)k} & 0_k & c \end{bmatrix}$$

where 0_j is a (1xj) row vector of zeros. Adding the column (n(k-1)+1-j) of matrix A to the column (n(k-1)+1+j) of the same matrix, for j=1 to n(k-1), and then deleting the first n(k-1) columns, we obtain a new matrix A*. The matrix M consists of the first q+1 columns of A*.

Consider as a first example systematic sampling of a quarterly IMA(1,1) model which is aggregated to annual frequency. In this case n=d=1, the vector c contains the coefficients of $S^{2n} = (1+B+B^2+B^3)^2$, that is $c = (1\ 2\ 3\ 4\ 3\ 2\ 1)$, and A is the following (2x11) matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{bmatrix}.$$

Then, $A^* = \begin{bmatrix} 4 & 6 & 4 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{bmatrix}$, and the matrix M is $\begin{bmatrix} 4 & 6 \\ 0 & 1 \end{bmatrix}$.

As a second example, consider a monthly IMA(2,2) model and its quarterly temporal aggregate. In this case, k = 3, d = q = 2, and n = d+1 = 3. The vector c, with elements the coefficients of $(1+B+B^2)^6$, is equal to $c=(1\ 6\ 21\ 50\ 90\ 126\ 141\ 126\ 90\ 50\ 21\ 6\ 1)$, and hence A is the (3x19) matrix:

$$A=\begin{bmatrix} 1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1 \end{bmatrix},$$

A* is the (3x13) matrix

$$A^*=\begin{bmatrix} 141 & 252 & 180 & 100 & 42 & 12 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 50 & 111 & 132 & 142 & 126 & 90 & 50 & 21 & 6 & 1 & 0 & 0 & 0 \\ 1 & 6 & 21 & 50 & 90 & 126 & 141 & 126 & 90 & 50 & 21 & 6 & 1 \end{bmatrix},$$

and the matrix M is given by

$$M=\begin{bmatrix} 141 & 252 & 180 \\ 50 & 111 & 132 \\ 1 & 6 & 21 \end{bmatrix}.$$

REFERENCES

APEL, M., J. HANSEN and H. LINDBERG (1996). "Potential Output and Output Gap", *Quarterly Review of the Bank of Sweden,* 3, pp. 24-35.

BACKUS, D. K., and P. J. KEHOE (1992). "International Evidence on the Historical Properties of Business Cycles", *The American Economic Review*, 82, pp. 864-888.

BAXTER M., and R. G. KING (1999). "Measuring Business Cycles: Approximate Band-Pass Filters for Economic Time Series", *Review of Economics and Statistics,* 81, pp. 575-593.

BOX, G. E. P., and G. M. JENKINS (1970). *Time Series Analysis: Forecasting and Control*, San Francisco: Holden-Day.

BREWER, K. R. W. (1973). "Some Consequences of Temporal Aggregation and Systematic Sampling for ARMA and ARMAX Models", *Journal of Econometrics,* 1, pp. 133-154.

BURMAN, J. P. (1980). "Seasonal Adjustment by Signal Extraction", *Journal of the Royal Society A,* 143, pp. 321-337.

CANOVA, F. (1998). "Detrending and Business Cycle Facts", *Journal of Monetary Economics,* 41, pp. 475-512.

COGLEY, T. (2001). "Alternative Definitions of the Business Cycle and Their Implications for Business Cycle Models: A Reply", *Journal of Economic Dynamics and Control,* 25, pp. 1103-1107.

COGLEY, T., and J. M. NASON (1995). "Effects of the Hodrick-Prescott Filter on Trend and Difference Stationary Time Series: Implications for Business Cycle Research", *Journal of Economic Dynamics and Control,* 19, pp. 253-278.

COOLEY, T. F., and L. E. OHANIAN (1991). "The Cyclical Behavior of Prices", *Journal of Monetary Economics,* 28, pp. 25-60.

DOLADO, J. J., M. SEBASTIÁN and J. VALLÉS (1993). "Cyclical Patterns of the Spanish Economy", *Investigaciones Económicas,* XVII, pp. 445-473.

DURBIN, J., and S. J. KOOPMAN (2000). "Time Series Analysis of Non-gaussian Observations Based on State Space Models from both Classical and Bayesian Perspectives", *Journal of the Royal Statistical Society B,* 62, pp. 3-56.

EUROPEAN CENTRAL BANK (2000). *Monthly Bulletin,* October.

EVIEWS (2005). *Version 5.1: Documentation,* March.

GHYSELS, E., C. W. J. GRANGER and P. L. SIKLOS (1996). "Is Seasonal Adjustment a Linear or Nonlinear Data-Filtering Process?", *Journal of Business and Economic Statistics,* 14, pp. 374-397.

GIORNO, C., P. RICHARDSON, D. ROSEVEARE and P. VAN DEN NOORD (1995). *Estimating Potential Output, Output Gaps and Structural Budget Balances,* OECD Economics Department Working Papers, 152.

GÓMEZ, V. (2001). "The Use of Butterworth Filters for Trend and Cycle Estimation in Economic Time Series", *Journal of Business and Economic Statistics,* 19, pp. 365-373.

HARVEY, A. C. (1997). "Trends, Cycles and Autoregressions", *Economic Journal,* 107, pp. 192-201.

HARVEY, A. C., and A. JAEGER (1993). "Detrending, Stylized Facts and the Business Cycle", *Journal of Applied Econometrics,* 8, pp. 231-247.

HODRICK, R., and E. C. PRESCOTT (1997). "Post War U.S. Business Cycles: An Empirical Investigation", *Journal of Money, Credit, and Banking,* 29, pp. 1-16.

KAISER, R., and A. MARAVALL (2001). "Measuring Business Cycles in Economic Statistics", *Lecture Notes in Statistics,* 154, New York: Springer-Verlag.

— (2005). "Combining Filter Design with Model-based Filtering: An Application to Business-cycle Estimation", *International Journal of Forecasting,* 21, pp. 691-710.

KING, R. G., and S. T. REBELO (1993). "Low Frequency Filtering and Real Business Cycles", *Journal of Economic Dynamics and Control,* 17, pp. 207-233.

MARAVALL, A. (2006). "An Application of the TRAMO-SEATS Automatic Procedure; Direct versus Indirect Adjustment", *Computational Statistics and Data Analysis,* 50, pp. 2167-2190.

MARAVALL, A., and A. MATHIS (1994). "Encompassing Univariate Models in Multivariate Time Series", *Journal of Econometrics,* 61, pp. 197-233.

MARAVALL, A., and A. DEL RÍO (2001). *Time Aggregation and The Hodrick-Prescott Filter,* Working Paper 0108, Research Department, Banco de España.

MCELROY, T. S. (2006). *Exact Formulas for the Hodrick-Prescott Filter,* mimeo, US Bureau of the Census.

POLLOCK, D. S. G. (1997). "Data Transformations and Detrending in Econometrics", in C. Heij, H. Schumacher, B. Hanzon and K. Praagman (eds.), *System Dynamics in Economic and Financial Models,* Chichester: Wiley, (Chapter 11).

— (2003). "Improved Frequency Selective Filters", *Computational Statistics & Data Analysis,* 42, pp. 279-297.

— (2006). "Econometric methods of signal extraction", *Computational Statistics & Data Analysis,* 50, pp. 2268-2292.

PROIETTI, T. (2005). "New algorithms for dating the business cycle", *Computational Statistics & Data Analysis,* 49, pp. 447-498.

RAVN, M. O., and H. UHLIG (2002). "On Adjusting the Hodrick-Prescott Filter for the Frequency of Observations, *Review of Economics and Statistics,* 84, pp. 371-376.

TIAO, G. C. (1972). "Asymptotic Behavior of Time Series Aggregates", *Biometrika,* 59, pp. 525-531.

WEI, W. S. W., and D. O. STRAM (1986). "Temporal Aggregation in the Arima Process", *Journal of Time Series Analysis,* 7, pp. 279-292.

WYNNE, M. A., and J. KOO (1997). *Business Cycles Under Monetary Union EU and US Business Cycles Compared*, Federal Reserve Bank of Dallas, Working paper 97-07.

BANCO DE ESPAÑA PUBLICATIONS

WORKING PAPERS[1]

0626 CRISTINA BARCELÓ: A Q-model of labour demand.
0627 JOSEP M. VILARRUBIA: Neighborhood effects in economic growth.
0628 NUNO MARTINS AND ERNESTO VILLANUEVA: Does limited access to mortgage debt explain why young adults live with their parents?
0629 LUIS J. ÁLVAREZ AND IGNACIO HERNANDO: Competition and price adjustment in the euro area.
0630 FRANCISCO ALONSO, ROBERTO BLANCO AND GONZALO RUBIO: Option-implied preferences adjustments, density forecasts, and the equity risk premium.
0631 JAVIER ANDRÉS, PABLO BURRIEL AND ÁNGEL ESTRADA: BEMOD: A dsge model for the Spanish economy and the rest of the Euro area.
0632 JAMES COSTAIN AND MARCEL JANSEN: Employment fluctuations with downward wage rigidity: The role of moral hazard.
0633 RUBÉN SEGURA-CAYUELA: Inefficient policies, inefficient institutions and trade.
0634 RICARDO GIMENO AND JUAN M. NAVE: Genetic algorithm estimation of interest rate term structure.
0635 JOSÉ MANUEL CAMPA, JOSÉ M. GONZÁLEZ-MÍNGUEZ AND MARÍA SEBASTIÁ-BARRIEL: Non-linear adjustment of import prices in the European Union.
0636 AITOR ERCE-DOMÍNGUEZ: Using standstills to manage sovereign debt crises.
0637 ANTON NAKOV: Optimal and simple monetary policy rules with zero floor on the nominal interest rate.
0638 JOSÉ MANUEL CAMPA AND ÁNGEL GAVILÁN: Current accounts in the euro area: An intertemporal approach.
0639 FRANCISCO ALONSO, SANTIAGO FORTE AND JOSÉ MANUEL MARQUÉS: Implied default barrier in credit default swap premia. (The Spanish original of this publication has the same number.)
0701 PRAVEEN KUJAL AND JUAN RUIZ: Cost effectiveness of R&D and strategic trade policy.
0702 MARÍA J. NIETO AND LARRY D. WALL: Preconditions for a successful implementation of supervisors' prompt corrective action: Is there a case for a banking standard in the EU?
0703 PHILIP VERMEULEN, DANIEL DIAS, MAARTEN DOSSCHE, ERWAN GAUTIER, IGNACIO HERNANDO, ROBERTO SABBATINI AND HARALD STAHL: Price setting in the euro area: Some stylised facts from individual producer price data.
0704 ROBERTO BLANCO AND FERNANDO RESTOY: Have real interest rates really fallen that much in Spain?
0705 OLYMPIA BOVER AND JUAN F. JIMENO: House prices and employment reallocation: International evidence.
0706 ENRIQUE ALBEROLA AND JOSÉ M.ª SERENA: Global financial integration, monetary policy and reserve accumulation. Assessing the limits in emerging economies.
0707 ÁNGEL LEÓN, JAVIER MENCÍA AND ENRIQUE SENTANA: Parametric properties of semi-nonparametric distributions, with applications to option valuation.
0708 ENRIQUE ALBEROLA AND DANIEL NAVIA: Equilibrium exchange rates in the new EU members: external imbalances vs. real convergence.
0709 GABRIEL JIMÉNEZ AND JAVIER MENCÍA: Modelling the distribution of credit losses with observable and latent factors.
0710 JAVIER ANDRÉS, RAFAEL DOMÉNECH AND ANTONIO FATÁS: The stabilizing role of government size.
0711 ALFREDO MARTÍN-OLIVER, VICENTE SALAS-FUMÁS AND JESÚS SAURINA: Measurement of capital stock and input services of Spanish banks.
0712 JESÚS SAURINA AND CARLOS TRUCHARTE: An assessment of Basel II procyclicality in mortgage portfolios.
0713 JOSÉ MANUEL CAMPA AND IGNACIO HERNANDO: The reaction by industry insiders to M&As in the European financial industry.
0714 MARIO IZQUIERDO, JUAN F. JIMENO AND JUAN A. ROJAS: On the aggregate effects of immigration in Spain.
0715 FABIO CANOVA AND LUCA SALA: Back to square one: identification issues in DSGE models.
0716 FERNANDO NIETO: The determinants of household credit in Spain.
0717 EVA ORTEGA, PABLO BURRIEL, JOSÉ LUIS FERNÁNDEZ, EVA FERRAZ AND SAMUEL HURTADO: Actualización del modelo trimestral del Banco de España.
0718 JAVIER ANDRÉS AND FERNANDO RESTOY: Macroeconomic modelling in EMU: how relevant is the change in regime?

---

1. Previously published Working Papers are listed in the Banco de España publications catalogue.

0719 FABIO CANOVA, DAVID LÓPEZ-SALIDO AND CLAUDIO MICHELACCI: The labor market effects of technology shocks.

0720 JUAN M. RUIZ AND JOSEP M. VILARRUBIA: The wise use of dummies in gravity models: Export potentials in the Euromed region.

0721 CLAUDIA CANALS, XAVIER GABAIX, JOSEP M. VILARRUBIA AND DAVID WEINSTEIN: Trade patterns, trade balances and idiosyncratic shocks.

0722 MARTÍN VALLCORBA AND JAVIER DELGADO: Determinantes de la morosidad bancaria en una economía dolarizada. El caso uruguayo.

0723 ANTON NAKOV AND ANDREA PESCATORI: Inflation-output gap trade-off with a dominant oil supplier.

0724 JUAN AYUSO, JUAN F. JIMENO AND ERNESTO VILLANUEVA: The effects of the introduction of tax incentives on retirement savings.

0725 DONATO MASCIANDARO, MARÍA J. NIETO AND HENRIETTE PRAST: Financial governance of banking supervision.

0726 LUIS GUTIÉRREZ DE ROZAS: Testing for competition in the Spanish banking industry: The Panzar-Rosse approach revisited.

0727 LUCÍA CUADRO, MARCEL FRATZSCHER AND CHRISTIAN THIMANN: The transmission of emerging market shocks to global equity markets.

0728 AGUSTÍN MARAVALL AND ANA DEL RÍO: Temporal aggregation, systematic sampling, and the Hodrick-Prescott filter.

BANCO DE ESPAÑA Eurosistema	Unidad de Publicaciones Alcalá, 522; 28027 Madrid Telephone +34 91 338 6363. Fax +34 91 338 6488 e-mail: publicaciones@bde.es www.bde.es