Centre for Economic Policy Research

BANCO DE ESPAÑA
Eurosistema

European Summer Symposium in International Macroeconomics (ESSIM) 2008

Hosted by
Banco de España

Tarragona, Spain; 20-25 May 2008

Expectations, Learning and Business Cycle Fluctuations

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We are grateful to the Banco de España for their financial and organizational support.

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May 14, 2008

Preliminary and incomplete

Abstract

Recently there has been a renewed interest in the notion that expectations are a source of business cycle fluctuation. This paper explores learning dynamics as a specific theory of shifting expectations, and assesses its implications for the amplification and propagation of technology shocks in real business cycle models. The benchmark model delivers volatility in output comparable to a rational expectations analysis with a standard deviation of technology shock that is 20 percent smaller, and has substantially more volatility in investment and hours. The model captures persistence in these series, unlike standard models. Inherited from real business cycle theory, the benchmark model suffers a comovement problem between consumption, hours, output and investment. An augmented model that is consistent with expectations driven business cycles, in the sense of Beaudry and Portier (2006), resolves these counterfactual predictions. This richer model produces additional amplification and propagation, requiring 30 percent smaller technology shocks than a rational expectations analysis while providing a superior characterization of other second order moments of observed data.

*The authors thank seminar participants at the Australian National University, Columbia University, Society of Economic Dynamics Prague 2007, Universitat Automoma and CREI-Universitat Pompeu Fabra and Northwestern University. We thank Albert Marcet and Jaume Ventura for useful conversations. The views expressed in the paper are those of the authors and are not necessarily reflective of views at the Federal Reserve Bank of New York or the Federal Reserve System. The usual caveat applies.

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1 Introduction

Recently there has been a renewed interest in shifting expectations as a source of business cycle fluctuations. A range of models have been explored that rely variously on multiple equilibria, exogenous news about future productivity and imperfect information — see, for example, Benhabib and Farmer (1994), Schmitt-Grohe (2000), Beaudry and Portier (2006), Jaimovich and Rebelo (2008) and Lorenzoni (2008). These frameworks seek not only to provide additional amplification and propagation of primitive shocks but also to resolve comovement problems that arise in standard real business cycle theory.

This paper proposes an alternative theory based on learning dynamics. In the context of an otherwise standard stochastic growth model, we consider an environment in which households and firms have an incomplete model of the macroeconomy, knowing only their own objectives, constraints and beliefs. Consequently, they do not have a model of how aggregate state variables, such as the capital stock, real wages and interest rates, are determined. They forecast exogenous variables relevant to their decision problems by extrapolating from historical patterns in observed data. This belief structure has the property that beliefs affect the true data generating process of the economy which in turn affects belief formation. The economy is therefore self-referential: shifts in beliefs about future returns to labor and capital affect current market clearing prices which in turn can reinforce beliefs. Current prices become less informative about future economic conditions generating fluctuations in real activity.

This kind of mechanism driving business cycle fluctuations can be found in early writings on macroeconomic dynamics. For example, Pigou (1926), on page 122, writes:

> "[...] a rise in prices, however brought about, by creating some actual and some counterfeit prosperity for business man, is liable to promote an error of optimism, and a fall in prices an error of pessimism, and this mutual stimulation of errors and price movements may continue in a vicious spiral until it is checked by some inference from outside."

Hence shifts in expectations, whether in part due to changes in fundamentals or in part due to error is a source of business cycle fluctuation that may be self-fulfilling. Our model is very much in the spirit of this quote. Learning breaks the tight link between fundamentals and, through expectations formation, equilibrium prices and allocations, giving rise to additional volatility relative to a rational expectations analysis of the model. Moreover, learning might be thought to improve the internal propagation mechanisms of the model, since beliefs are a function of historical data, introducing an additional state variable.

Calibrating the model to match properties of post war U.S. quarterly data, the central results of the paper are as follows. First, learning amplifies technology shocks. Relative to a rational expectations analysis of the model, a 20 percent smaller standard deviation of technology shocks is required to match the standard deviation of HP filtered output data. Moreover, model implications for the relative volatility of investment and hours to output are much improved, being some 40 and 25 percent greater than under rational expectations. Second, the persistence properties of our model bear much closer resemblance to observed data. The first order autocorrelation properties of output, hours and investment are well matched despite shocks being identically and independently distributed over time. These features of the data are typically problematic for real business cycle theory as documented by Cogley and Nason (1993) and Rotemberg and Woodford (1996). In general, the learning model provides a superior characterization of second order moments of observed data than does the model under rational expectations.

The improvement in fit can be traced to shifting beliefs acting as an endogenous news shock or demand shock. The only source of exogenous variation are technology shocks which have two effects. First, as in standard real business cycle theory, a temporary technology shock shifts the production frontier with well understood implications. Second, in subsequent periods, households revise their beliefs in response to changed market opportunities. In particular, households are more optimistic about the future path of returns to capital and more pessimistic about future returns to labor relative to rational expectations. The former leads to substitution of current consumption for future consumption and a high marginal utility of income, an effect reinforced by the lower projected wages. Combined, these expectations effects induce a larger fall in consumption and consequently a larger shift in labor supply and investment relative to rational expectations in the period after the shock. This amplification of standard substitution and income effects in response to a technology shock explains the increased volatility in these variables. The delayed adjustment in beliefs explains the persistence. Furthermore, these observations highlight our connection to Pigou (1926): expectations about returns to capital and wages are in part validated by the data. Shifts in expectations about future returns to labor and capital are for a given technological frontier and endogenous to the technology shock. In this sense they are endogenous news shocks. Moreover, they have similarity to demand shocks in so far as hours and consumption negatively comove in response to a revision in expectations.

As there is additional endogenous variation in the marginal utility of income for a given production frontier, the models suffers a comovement problem. Hours and consumption display negative correlation. The third result of the paper is to show that in a model of the kind proposed by Beaudry and Portier (2006) this comovement problem can be resolved. That paper explores primitive assumptions on technology and preferences that are consistent with so called expectations driven business cycles. That is, in response to an expectational shock, output, hours, investment and consumption display positive comovement. We propose a new pairing of assumptions that delivers this property. They are a small degree of increasing returns combined with non-separability in utility between consumption and hours. The first assumption provides an endogenous shift in the production frontier from external economies so that consumption does not crowd out investment, while the latter assists capturing the comovement between hours and consumption. Under these assumptions, which introduce no additional states variables, our model provides an even better characterization of observed data. Moreover, the interaction of learning with these model features provides additional amplification and propagation relative to a rational expectations model with these characteristics and our baseline model under learning. The modified model implies some 30 percent greater volatility in output for a given technology shock.

The model is demonstrated to be robust to a range of alternative assumptions. In particular, our analysis could be criticized on the grounds that it is well understood that real business cycle theory fails to account for various properties of observed data without augmenting the model with additional frictions such as variable capital utilization and investment adjustment costs. We show that our benchmark model performs well when compared to rational expectations models with these features.

Finally, we compare our analysis to earlier explorations of learning as a source of amplification and propagation. In particular, we revisit the analysis of Williams (2002) which also looked at learning in a standard real business cycle model. That paper concludes that learning based on extrapolating historical patterns in observed data, as considered here, is unlikely to help improve the performance of real business cycle models. We reproduce that analysis in the context of our model and show that this is indeed the case. The difference in conclusions stems from the failure in William's (2002) to model optimal decisions conditional on maintained beliefs following Marcet and Sargent (1989) and Preston (2005).

The paper proceeds as follows. Section 2 lays out a simple real business cycle model. Section 3 discusses the assumed beliefs structure in some detail. Section 4 details the data and calibration. Section 5 presents the core results under our benchmark assumptions. Section 6 gives results for a model consistent with expectations driven business cycles in the sense of Beaudry and Portier (2006). Section 7 provides some robustness exercises. Finally, Section 8 concludes.

2 A Simple Model

The following section details a simple stochastic growth model similar in spirit to Kydland and Prescott (1982), Prescott (1986) and King, Plosser, and Rebelo (1988). A continuum of households face a canonical consumption allocation problem and decide how much to consume of the economy's single available good, how much to invest, and how much labor to supply to firms in the production of the available good. A continuum of competitive firms produce goods using labor and capital as inputs. The major difference to this earlier literature is the incorporation of near-rational beliefs, delivering an anticipated utility model of the kind discussed by Kreps (1998) and Sargent (1999). The analysis follows Marcet and Sargent (1989) and Preston (2005), solving for optimal decisions conditional on current beliefs. Various mechanisms of persistence, such as investment adjustment costs and variable capital utilization are abstracted from. This provides sharp results regarding the ability of near-rational expectations to replicate salient features of the data. The sequel demonstrates that frictions of this kind tend to further amplify the effects identified in our benchmark analysis.

2.1 Microfoundations

Households. Households maximize their intertemporal utility derived from consumption and leisure

$$\hat{E}_t^j \sum_{T=t}^{\infty} \beta^{T-t} \left[\ln C_T^j + \nu \left(1 - H_t^j\right)\right] \tag{1}$$

subject to the flow budget constraint

$$C_t^j + K_{t+1}^j = R_t^K K_t^j + W_t H_t^j + (1-\delta) K_t^j \tag{2}$$

where C_t^j denotes household $j's$ consumption, K_t^j the holdings of the aggregate capital stock available at the beginning of period t, with $K_0^j > 0$ given; and H_t^j represents the fraction of the available time (normalized to one unit per period) spent on non-leisure activities.[1] The function $v(\cdot)$ is concave. The functional forms are chosen to be consistent with a balance growth path — see King, Plosser and Rebelo (1988). Households supply labor and capital in perfectly competitive markets. R_t^K is the rental rate of capital and W_t is the real wage. The household's discount factor and the depreciation rate of capital satisfy $0 < \beta, \delta < 1$.

The expectation operator $\hat{E}_t^j$ denotes agent j's subjective beliefs. In forming expectations, households and firms observe only their own objectives, constraints and realizations of aggregate variables that are exogenous to their decision problems and beyond their control. The agent's problem is to choose sequences of consumption, hours worked, and capital in order to maximize (1) subject to (2), taking as given prices and the capital stock available at the beginning of the period. The precise specification of beliefs is described in the next section.

Household optimization yields the conditions

$$W_t = C_t^j v_H \left(1 - H_t^j\right) \tag{3}$$

from equating the marginal rate of substitution between an additional unit of consumption

[1] Here, following large part of the literature, it is assumed that households own the capital stock.

and additional unit labor supply to their relative prices and

$$\beta \hat{E}_t^j \left[\frac{C_t^j}{C_{t+1}^j} \left(R_{t+1}^K + (1-\delta) \right) \right] = 1 \tag{4}$$

the Euler equation from equating the marginal rate of substitution between consumption today and tomorrow to the real interest rate.

The paper's primary goal is the quantitative evaluation of the model. Following Kydland and Prescott (1982), it is useful to employ a log-linear approximation of the model around a balanced growth path. For any variable G_t define $g_t = G_t/X_t$ as the corresponding variable in efficiency units, where X_t is the level of technology in period t described further below. The model is then studied in log deviation from a non-stochastic steady state in these normalized variables so that $\hat{g}_t = \ln(g_t/\bar{g})$, with $\bar{g}$ denoting the steady state value of g_t. Details of the steady state and the log-linear approximation are confined to the appendix. Here it suffices to note that consumption, investment, output, the capital stock and real wages grow at the rate technological progress in the balanced growth path so that

$$y_t = \frac{Y_t}{X_t}; \; c_t = \frac{C_t}{X_t}; \; i_t = \frac{I_t}{X_t}; \; w_t = \frac{W_t}{X_t} \; \text{ and } k_t = \frac{K_t}{X_{t-1}}$$

are stationary. Hours and the rental rate of capital are stationary on the balanced growth path. Studying the approximated model also facilitates economic interpretation of later results.

Log-linearizing, solving the flow budget constraint forward, imposing the no-Ponzi condition and substituting for hours using a log-linear approximation to (3) gives the intertemporal budget constraint

$$\epsilon_c \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \hat{c}_T^j = \beta^{-1} \hat{k}_t^j + \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[\epsilon_w \hat{w}_T + \bar{R} \hat{R}_T^K - \beta^{-1} \hat{\gamma}_T \right].$$

The coefficients ϵ_c and ϵ_w are constants that are composites of model primitives, $\bar{R} > 0$ the gross rental rate, and $\hat{\gamma}_t = \ln(X_t/(X_{t-1}\bar{\gamma}))$, the log deviation of the growth rate in technological progress relative to steady state. This relation states the expected present value of consumption must be equal to the capital stock available at the beginning of the period plus the expected present value of the wage and rental income. These latter variables are outside the control of the single agent, given the assumption of competitive markets.

To determine optimal consumption decisions, combine the intertemporal budget constraint with a log-linear approximation to (4) to yield

$$\begin{aligned}\hat{c}_t^j &= \frac{1-\beta}{\epsilon_c}\left[\beta^{-1}\left(\hat{k}_t^j - \hat{\gamma}_t\right) + \bar{R}\hat{R}_t^K + \epsilon_w \hat{w}_t\right] \\ &\quad + \hat{E}_t^j \sum_{T=t}^{\infty} \beta^{T-t}\left[\frac{(1-\beta)}{\epsilon_c} - \beta\right]\beta\bar{R}\hat{R}_{T+1}^K + \\ &\quad \hat{E}_t^j \sum_{T=t}^{\infty} \beta^{T-t}\frac{(1-\beta)}{\epsilon_c}\beta\epsilon_w\hat{w}_{T+1}. \qquad (5)\end{aligned}$$

The consumption decision rule comprises three terms. The first shows the impact that the current level of the capital stock and current prices have on consumption. The second and third terms show how expected variations in permanent income affect current consumption. The former has two parts corresponding to the positive income effect and the negative substitution effect of higher returns to capital on current consumption. The latter only one part as the income and substitution effects of a wage increase both increase current consumption.

Firms. There is a continuum of identical competitive firms of mass one. Each produces the economy's only good using capital and labor as inputs according to the production function

$$Y_t^i = \left(K_t^i\right)^{\alpha}\left(X_t H_t^i\right)^{1-\alpha}$$

where $0 < \alpha < 1$. Labor augmenting technological progress, X_t, satisfies the stochastic process

$$\ln\left(X_{t+1}/X_t\right) = \ln\bar{\gamma} + z_{t+1}$$

where z_t is an independent, identically distributed random variable with zero mean and standard deviation σ_z. $\bar{\gamma} > 0$ is the steady state rate of technology growth. This aggregate disturbance is the only source of exogenous variation in the model. Each firm chooses labor and capital inputs to maximize profits

$$\Pi_t^i = Y_t^i - R_t^K K_t^j - W_t H_t^j$$

taking factor prices as given. The first order conditions to a firm's optimization problem provides

$$\begin{aligned}W_t &= (1-\alpha)\left(K_t^i\right)^{\alpha}\left(X_t\right)^{1-\alpha}\left(H_t^i\right)^{-\alpha} \\ R_t^K &= \alpha\left(K_t^i\right)^{\alpha-1}\left(X_t H_t^i\right)^{1-\alpha}\end{aligned}$$

which equate factor prices with their real marginal products.

2.2 Market clearing and aggregate dynamics

We are interested in studying the behavior of macroeconomic aggregates. As households have the same preferences and constraints; firms the same technology; and beliefs are assumed homogenous across all agents (although they are assumed not to be aware of that) the analysis considers a symmetric equilibrium in which $\hat{k}_t^i = \hat{k}_t^j = \hat{k}_t$; $\hat{H}_t^j = \hat{H}_t^i = \hat{H}_t$; $\hat{\imath}_t^i = \hat{\imath}_t^j = \hat{\imath}_t$ for all i, j, t.

Integrating over the continuum provides aggregate consumption demand

$$\begin{aligned} \hat{c}_t &= \frac{1-\beta}{\epsilon_c}\left[\beta^{-1}\hat{k}_t + \bar{R}\hat{R}_t^K - \beta^{-1}\hat{\gamma}_t + \epsilon_w \hat{w}_t\right] \\ & \quad + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\left[\frac{(1-\beta)}{\epsilon_c} - \beta\right]\beta\bar{R}\hat{R}_{T+1}^K + \\ & \quad \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t}\frac{(1-\beta)}{\epsilon_c}\beta\epsilon_w\hat{w}_{T+1} \end{aligned} \tag{6}$$

where $\int \hat{E}_t^j dj = \hat{E}_t$ denotes average expectations in the population. The aggregate consumption dynamics inherit the properties of individual decision rules. This is the only model equation that depends on expectations in our benchmark model, and therefore of central focus. If near-rational expectations are to be a source of amplification and propagation of primitive shocks, the effects must originate here.

A log-linear approximation to the model yields the remaining model equations as follows. Aggregate capital dynamics are given by the usual accumulation equation

$$\hat{k}_{t+1} = \left[\alpha\frac{\bar{y}}{\bar{k}} + \frac{(1-\delta)}{\bar{\gamma}}\right]\hat{k}_t + (1-\alpha)\frac{\bar{y}}{\bar{k}}\hat{H}_t - \frac{\bar{c}}{\bar{k}}\hat{c}_t - \frac{(1-\delta)}{\bar{\gamma}}\hat{\gamma}_t. \tag{7}$$

The labor-leisure choice determines aggregate labor supply as

$$\epsilon_L \frac{\bar{H}}{1-\bar{H}}\hat{H}_t = -\hat{c}_t + \hat{w}_t \tag{8}$$

where ϵ_L is the elasticity of labor supply and $\bar{H} > 0$ denotes the number of hours supplied in steady state. Factor prices are expressed as

$$\hat{w}_t = \alpha\hat{k}_t - \alpha\hat{H}_t \tag{9}$$

$$\hat{R}_t^K = \hat{\gamma}_t + (\alpha - 1)\hat{k}_t + (1-\alpha)\hat{H}_t. \tag{10}$$

And the resource constraint provides

$$\alpha \hat{k}_t + (1-\alpha)\,\hat{H}_t = \frac{\bar{c}}{\bar{y}}\hat{c}_t + \frac{\bar{\imath}}{\bar{y}}\hat{\imath}_t. \tag{11}$$

Given our assumption about technological progress, equations (6) - (11) together with the expectations formation mechanism specified in the next section completely determine the aggregate dynamics of the economy.

3 Beliefs

Optimal decisions require households to forecast the evolution of future wages and returns to capital. They are assumed to use a simple econometric model, relating wages and the capital rental rate to the aggregate stock of capital. That is

$$\hat{R}^K_t = \omega^r_0 + \omega^r_1 \hat{k}_t + e^r_t, \tag{12}$$

$$\hat{w}_t = \omega^w_0 + \omega^w_1 \hat{k}_t + e^w_t \tag{13}$$

and

$$\hat{k}_{t+1} = \omega^k_0 + \omega^k_1 \hat{k}_t + e^k_t \tag{14}$$

where e^r_t, $e^r_t$and e^r_t are i.i.d. shocks.[2] The beliefs therefore contain the same variables that appear in the minimum state variable rational expectations solution to the model. And, while the rational expectations solution does not contain a constant, it has the natural interpretation under learning of capturing uncertainty about the steady state (equivalently the level of technology).

Rational Expectations. The model solution under rational expectations implies (to a first order approximation) that labor and capital prices and the next period capital stock are linearly related to aggregate capital, with *time-invariant* coefficients $\omega^r_0 = \omega^w_0 = \omega^k_0 = 0$ and $\omega^r_1 = \bar{\omega}^r_1$, $\omega^w_1 = \bar{\omega}^w_1$, $\omega^k_1 = \bar{\omega}^k_1$. Hence, the agents' forecasting model nests beliefs that would be observed in a rational expectations equilibrium. Furthermore, under rational expectations $e^r_t = \bar{\omega}^r_3 \hat{\gamma}_t$, $e^w_t = \bar{\omega}^w_3 \hat{\gamma}_t$ and $e^k_t = \bar{\omega}^k_3 \hat{\gamma}_t$.

[2]The variables in agents' model are written in deviations from steady state. In alternative the model could be specified in log levels [MORE].

Perpetual learning. Agents estimate equations (12) – (14), updating their coefficient estimates every period as new data becomes available. Following recent literature, household's update their estimates using a discounted least-squares estimator, assigning lower weight to older observations, in order to protect against structural change.[3] Letting $\omega' = (\omega_0, \omega_1)$, $z_t = \left(\hat{R}_t^K, \hat{w}_t, \hat{k}_{t+1}\right)$ and $q_{t-1} = \left(1, \hat{k}_t\right)$, the algorithm can be written in recursive terms as

$$\hat{\omega}_t = \hat{\omega}_{t-1} + gR_t^{-1}q_{t-1}\left(z_t - \hat{\omega}'_{t-1}q_{t-1}\right)' \tag{15}$$

$$R_t = R_{t-1} + g\left(q_{t-1}q'_{t-1} - R_{t-1}\right) \tag{16}$$

where $\hat{\omega}_t$ denotes the current period's coefficient estimate and $g \in (0,1)$ denotes the constant gain, determining the rate at which older observations are discounted. The constant gain assumption delivers perpetual learning, as market participants 'forget' the past. However, the model has the property that if beliefs were instead given by a recursive least square algorithm, defined by $g = t^{-1}$, the learning process would converge to the rational expectations coefficients.[4] Under the constant gain algorithm, agents' estimates converge to a distribution. Evans and Honkapohja (2001) show that for a gain sufficiently close to zero the distribution is normal and centered around the time-invariant coefficients of the rational expectations coefficients.

Timing and information. We assume that agents' update their estimates at the end of the period, after making consumption and labor supply decisions. This avoids simultaneous determination of the parameters defining agents' forecast functions and current prices and quantities. However, to compare the model under learning with the predictions under rational expectations, we assume that agents' expectations are determined simultaneously with consumption and labor supply decisions, so that agents observe all variables that are determined at time t, including $\hat{k}_{t+1}$. For example, the one-period-ahead forecast for $\hat{R}_t^K$ is

$$\hat{E}_t\hat{R}_{t+1}^K = \hat{\omega}_{0,t-1}^r + \hat{\omega}_{1t-1}^r\hat{k}_{t+1}$$

where $\hat{\omega}_{0,t-1}^r$ and $\hat{\omega}_{1t-1}^r$ are the previous period's estimates of belief parameters that define the period t forecast function. Hence, they observe the same variables that a 'rational' agent would

[3] Of course we consider a stationary model environment with a single shock so as to clearly isolate the role of expectations in generating business cycle fluctuations.

[4] In the limit, agents have an infinite amount of data.

observe. The only difference is that they are attempting to learn the 'correct' coefficients that characterize optimal forecasts. An alternative approach may be to assume expectations are formed before taking decisions, but this would render comparison of the learning model to the benchmark real business cycle model difficult as rational expectations would not be a special case of the assumed belief structure.

That beliefs are updated a period after new data arrives is a key component of learning as a friction. Like models of sticky information — see, for example, Reis (XXXX) — where only some firms can update information about the state of the economy, we assume that all agents can revise their beliefs in response to new data, but only with a one period lag subject to the constraint of the constant gain learning technology. More generally, beliefs are a state variable that are sluggish by assumption, much like habit formation, price indexation, investment adjustment costs and labor market search (where it is assume that new matches are only productive in future periods).

Concerning agents' information about $\hat{\gamma}_t$, we assume that it is unobserved in equations (12) – (14).[5] If it were observed, agents would not face an inference problem and therefore learn immediately, given that the only disturbance in the model is the technology shock. This particular assumption could be relaxed if an additional shock were introduced in the model. We refrain from doing this to ensure comparability with the standard real business cycle framework.

Finally in forecasting over the decision horizon agents do not take into account that they update their beliefs in subsequent periods. The model is therefore one of anticipated utility — see Sargent (1999).

True Data Generating Process. Using (12) – (14) to substitute for expectations in (6) and solving, delivers the actual data generating process

$$
\begin{aligned}
z_t &= T_1\left(\hat{\omega}_{t-1}\right) q_{t-1} + T_2\left(\hat{\omega}_{t-1}\right) \hat{\gamma}_t && (17)\\
\hat{\omega}_t &= \hat{\omega}_{t-1} + g R_t^{-1} q_{t-1} \left(\left[\left(T_1\left(\hat{\omega}_{t-1}\right) - \hat{\omega}'_{t-1} \right) q_{t-1} + T_2\left(\hat{\omega}_{t-1}\right) \hat{\gamma}_t \right] \right)' && (18)\\
R_t &= R_{t-1} + g\left(q_{t-1} q'_{t-1} - R_{t-1} \right) && (19)
\end{aligned}
$$

[5] Given that variables in equations (12)-(14) are expressed in efficiency units, agents could extract $\hat{\gamma}_t$ from observing X_t.

and

$$\begin{bmatrix} \hat{c}_t \\ \hat{\imath}_t \\ \hat{H}_t \end{bmatrix} = \Psi z_t, \tag{20}$$

where $T_1(\hat{\omega})$ and $T_2(\hat{\omega})$ are nonlinear functions of the previous period's estimates of beliefs and Ψ is a matrix comprised of composites of primitive model parameters. The actual evolution of z_t is determined by a time-varying coefficient equation in the state variables $\hat{k}_t$ and $\hat{\gamma}_t$, where the coefficients evolve according to (18) and (19). The evolution of z_t depends on $\hat{\omega}_{t-1}$, while at the same time $\hat{\omega}_t$ depends on z_t. Learning induces self-referential behavior. Agents use current prices and capital holdings to make inferences about future macroeconomic conditions, but in equilibrium prices depend on agents' beliefs, and prices in turn affect the evolution of beliefs. This dependence on z_t is related to the fact that outside the rational expectations equilibrium $T_1(\hat{\omega}_{t-1}) \neq \hat{\omega}'_{t-1}$ and similarly for T_2.

Model mis-specification. Agents' beliefs (12) – (14) are mis-specified because they fail to account for how aggregate variables are affected by market participants' learning process. The mis-specification exists because individual agents are 'small' and therefore their individual actions do not have an impact on prices, which are taken as given. Moreover, agents know only their own preferences, technology and information updating technology and have no knowledge of these objects for other agents. To be clear, the agent's model can be reinterpreted in terms of a drifting coefficients model

$$z_t = \omega'_{t-1} q_t + \tilde{e}_t \tag{21}$$

$$\omega_t = \omega_{t-1} + \Psi_t \tag{22}$$

where $\tilde{e}_t$ and Ψ_t are viewed as mean normal shocks with $cov(\tilde{e}_t) = \Sigma_{\tilde{e}}$ and $cov(\Psi_t) << \Sigma_{\tilde{e}}$. Sargent and Williams (2005) show how the steady state Kalman filter estimate of (21) and the constant gain beliefs (15) are closely related.[6] Inspection of (21) shows that the evolution of ω_t depends only on ω_{t-1}: it is assumed to be a random walk. Conversely, the *actual* dynamics of $\hat{\omega}_t$ is stationary[7].[MORE]

[6] See Sargent and Williams (2005) for the derivation details.

[7] The actual dynamic is stationary provided g is sufficiently small and the eigenvalues of T_1 evaluated at

Given mis-specification, the agents' expectations formation scheme is suboptimal. In fact, the 'rational' predictor would use equations (17) — (19). An individual agent adopting this forecasting model would increase her utility. But as this model is adopted by more agents, it becomes a suboptimal model and the recursion leads back to rational expectations – see Sargent (1993). In this paper, we constrain agents to use (21). This represents an assumption of bounded rationality on the part of the agents. The deviation from rationality is regulated by the constant gain g. As $g \to 0$, $T_1(\hat{\omega}_{t-1}) \to \hat{\omega}_{t-1} \to \bar{\omega}$: in other words, the distribution of agents' estimates collapses to the rational expectations coefficients.

4 Calibration

Concerning households's preferences we set the discount rate $\beta = 0.99$. We assume separable preferences between consumption and leisure with log-utility for consumption and close-to-linear utility of leisure.[8] Accordingly, we set $\epsilon_L = -\nu'' L/\nu' = 0.01$ and a steady state level of hours $\bar{H} = 0.2$ so that the Frisch elasticity of labor supply is equal to 0.0025. Firms' technology is calibrated by setting the capital share $\alpha = 0.34$ and the steady state growth rate of labor augmenting technical progress to $\gamma = 1.0053$ consistent with the quarterly mean output growth over the sample.

Two parameters are left to calibrate: the standard deviation of the shock σ_X and the constant gain g. We calibrate these two parameters by minimizing the sum of squared distances between the model implied volatility of HP detrended output and the first autocorrelation coefficient of output growth and the corresponding data moments. To do this, at each iteration in the minimization problem the model is simulated for 20000 periods, and the first 2000 periods of the simulation are discarded. Required statistics are then computed using the remaining observations. This insures that the model reaches its stationary distribution of belief parameters, implying that our calibration and subsequent results do not depend on the initial conditions on the belief parameters. The procedure integrates out initial beliefs and guarantees that our conclusions are not influenced by transitional dynamics resulting from

the rational expectations equilibria have real parts less than 1 –see Evans and Honakapohja (2001). This condition is verified for the parameter values studied in the paper.

[8] This approximates the labor supply properties of Rogerson (1988) model of indivisable labor.

unusual initial conditions.

As illustration of the possible effects of initial beliefs on inference, consider the following example. Suppose that data is generated according to the process $x_t = \bar{x} + \varepsilon_t$ where $\bar{x} > 0$ is a constant and ε_t and i.i.d mean zero disturbance. Consider estimating the mean using all sample observations and assume that the initial condition on the expectation of this mean is $\bar{x}^e > \bar{x}$. Over time beliefs about the mean of x_t will be revised down as realizations of this random variable fluctuate around the true mean. The resulting estimates of the mean would exhibit positive autocorrelation. Yet the true model has zero serial correlation. Our simulation approach ensures inference on various model statistics are not driven by transitional dynamics of this kind.[9]

The procedure gives a gain of 0.0029. To interpret this magnitude, note the gain measures the weight that agents assign to past data. This value of the gain implies that observations that are 50 years old receive a weight of $(1 - 0.0029)^{200} \simeq 0.5$, implying agents do not discount past data too heavily. [10] To gauge the relative magnitude across observations, the weight assigned to the most recent data observation is approximately one.

One concern about the analysis might be that this choice of gain provides excessive freedom to fit the observed data moments. Several points worth making. First, many deviations from benchmark theory involve increased parameterization. For example, this is true in when investment adjustment costs, variable capital utilization, financial market frictions and labor market search — see, for example, Burnside and Eichenbaum (1996), Andolfatto (1996) and Carlstrom and Fuerst (1997). All these model variants engender more highly parameterized models and all seek to match the kinds of properties discussed here. In the same spirit, learning is an example of alternative friction whose implications for model fit is being evaluated.

Second, our calibrated gain is considerably smaller than values found in the literature, which range from $0.01 - 0.05$ — see, for example, Milani (2004), which estimates the gain, and Orphanides and Williams (2005). Branch and Evans (2006) show that a simple VAR with constant gain performs well in forecasting output growth and inflation, with respect to alternative more sophisticated models. The constant gain model is also shown to approximate

[9] This is way in which our work is distinguished from Milani (2006).

[10] For this value of g, agents would give approximately zero weight to observations that are 500 years old.

well the behavior of output growth forecasts in the Survey of Professional Forecasters. The choice of gain that maximizes the fit in their VAR is 0.007, which is also above our calibration.

Finally note that under rational expectations we only have to choose σ_x to match the standard deviation of HP filtered output growth.

5 Central Results

5.1 Statistical Properties

Tables 1 and 2 report summary statistics on the cyclical properties of various U.S. data series and the model under both rational expectations and learning dynamics. The sample is 1955:Q1 to 2004:Q4 and a short description of each series is contained in the Appendix. For each variable the relative standard deviation and correlation with output are reported. Table 1 reports these statistics for HP filtered series, while Table 2 presents the corresponding statistics for the growth rates of each series.

The first row of Table I shows learning dynamics amplify the effects of technology shocks. To match the variance and serial correlation properties of output, the learning model requires a technology disturbance with a standard deviation that is about 20 percent smaller than required under rational expectations. Moreover, the relative volatility of hours and investment are 40 and 25 percent higher respectively, bearing closer resemblance to data implied moments than the rational expectations model. The former represents a significant success, being problematic for standard real business cycle theory — see Hansen (1985).[11] In regards to consumption, wages and labor productivity, the learning model performs less well. Given the high elasticity of the labor supply and the assumption of perfectly competitive markets the model predicts $\hat{C}_t \approx \hat{w}_t \approx \hat{Y}_t - \hat{H}_t$ and it is therefore too stylized to capture the different dynamics of these variables. The source of the discrepancy between the model and the data is in the behavior of consumption, discussed further below. Table 2 shows the same set of

[11] Using different measures of hours and real activity, or different sample sizes affects the specific values of σ_H/σ_Y, but does not alter our conclusions regarding the model's performance. For example, using nonfarm business output as a measure of economic activity yields $\sigma_H/\sigma_Y = 0.83$ for whole sample and $\sigma_H/\sigma_Y = 1.1$ for the sample 1982:Q3-2006:Q1. Using average weekly hours (from the BLS household survey data) and real GDP as a measure of output, gives a relative standard deviation of 0.88, over the same period.

statistics in terms of growth rates, underscoring that the model under learning unambiguously delivers a better fit. In particular, the model does not display the counterfactually large output growth volatility which occurs under rational expectations.[12]

Table 1: HP filtered moments

	Statistic	Data	REE	Learning
Technology:	σ_X	-	1.22	0.98
Output:	σ_Y	1.5383	1.5422	1.5152
Consumption:	σ_C/σ_Y	0.5246	0.5378	0.3807
	$\rho_{Y,C}$	0.6903	0.9727	0.8349
Investment:	σ_I/σ_Y	2.8727	2.4234	3.0583
	$\rho_{Y,I}$	0.8986	0.9886	0.9797
Hours:	σ_H/σ_Y	1.1255	0.4918	0.7118
	$\rho_{Y,H}$	0.8754	0.9674	0.9559
Wages:	σ_w/σ_Y	0.5413	0.5389	0.3819
	$\rho_{Y,w}$	0.1192	0.9729	0.8369
Labor Prod:	σ_{Pr}/σ_Y	0.6785	0.5389	0.3819
	$\rho_{Y,Pr}$	0.5202	0.9729	0.8369

Turning to the correlations between each series and output, all moments are closer to the data than are those under rational expectations. Of particular note are the weaker correlations of consumption, wages and labor productivity with output. To presage later discussion, these improvements in fit arise from learning endogenously generating demand shocks. Revisions to beliefs shift the marginal utility of income. And for a given technology frontier these variations in marginal utility have qualitative similarities to demand or government expenditure shocks. As shown by Christiano and Eichenbaum (1992), the inclusion of such shocks in conjunction with technology disturbances can improve the fit of unconditional moments pertaining to labor market variables.

[12] The rational expectations model over predicts the standard deviation of output growth by some 30 percent in contrast to 10 percent for the learning model.

Table 2: Growth rates

	Statistic	Data	REE	Learning
Output:	σ_{Δ_Y}	0.8763	1.1944	0.9878
Consumption:	$\sigma_{\Delta_C}/\sigma_{\Delta_Y}$	0.5981	0.5187	0.5397
	ρ_{Δ_Y,Δ_C}	0.5119	0.9824	0.8027
Investment:	$\sigma_{\Delta_I}/\sigma_{\Delta_Y}$	2.5397	2.4518	2.8206
	ρ_{Δ_Y,Δ_I}	0.7139	0.9933	0.9426
Hours:	$\sigma_{\Delta_H}/\sigma_{\Delta_Y}$	0.9290	0.4987	0.6502
	ρ_{Δ_Y,Δ_H}	0.6979	0.9810	0.8696
Wages:	$\sigma_{\Delta_w}/\sigma_{\Delta_Y}$	0.6023	0.5198	0.5404
	ρ_{Δ_Y,Δ_w}	0.0791	0.9825	0.8043
Labor Prod:	$\sigma_{\Delta_{Pr}}/\sigma_{\Delta_Y}$	0.9499	0.5198	0.5404
	$\rho_{\Delta_Y,\Delta_{Pr}}$	0.6828	0.9825	0.8043

Since Cogley and Nason (1993, 1995), the internal propagation mechanisms of technology shocks have been a central preoccupation of real business cycle theory. These papers demonstrate that the impulse response functions of model variables are entirely determined by the assumed stochastic properties of technology shocks — the existence of capital as a state variable adds little propagation. Rotemberg and Woodford (1996) show, in a related criticism of real business cycle theory, that predictable variation in model simulated output, hours and consumption data is negligible, despite evidence of substantial forecastable variation of these series in observed data. Moreover, what predictable variation there is in the model is of the wrong kind.

Figure 1 plots the autocorrelation function for output growth together with model predictions under both rational expectations and learning. It is immediate that the rational expectations real business cycle model has virtually no propagation — recall that the growth rate of technology is given by an i.i.d. process — having an autocorrelation function that is essentially equal to zero at all horizons. In contrast, the learning model matches the first order serial correlation properties, though generates little persistence beyond that. While matching this feature of the data was part of the calibration, we view it as a success given how well remaining model properties are captured.

Table 3 reports the autocorrelation properties of the growth rate of key model variables. Investment, output and hours growth are remarkably well matched relative to the predictions of the model under rational expectations. That wages, labor productivity and consumption are counterfactually predicted to have negative serial autocorrelation, stems from the well known comovement problem in real business cycle theory. As noted by Barro and King (1984), and explored in detail in the recent literature falling under the rubric *expectations driven business cycles*, this class of model is unable to generate positive comovement in these series. While the impact effect of technology shocks does induce positive comovement, subsequent dynamics under learning are driven by revisions to beliefs. The next section shows that these revisions to beliefs are isomorphic to demand shocks in the sense that for a given production frontier shifts in expectations implies consumption and hours must be negatively correlated from the labor-leisure condition (3). This explains the observed positive serial correlation in hours and concomitant negative serial autocorrelation in consumption. In the final section we introduce a simple extension to the baseline model that resolves these counterfactual predictions.

Table 3: Autocorrelation in growth rates

	Statistic	Data	REE	Learning
Wages:	Δ_w	0.1898	0.1047	-0.1424
Consumption:	Δ_C	0.2545	0.1053	-0.1439
Investment:	Δ_I	0.3386	-0.0280	0.4248
Output	Δ_Y	0.2835	0.0033	0.2768
Labor Prod.	Δ_{Pr}	0.0527	0.1047	-0.1424
Hours	Δ_H	0.5790	-0.0342	0.4446

5.2 Impulse Response Functions

Further insight can be gleaned from impulse response functions to a unit technology shock. The effects of a disturbance depend on the precise beliefs maintained by households at that time. Impulse response functions for the learning model are therefore generated by simulating the model twice for $2000 + T$ periods. The first 2000 periods guarantee convergence to the model stationary distribution and are discarded. The second simulation includes a unit shock in period 2001. The T period impulse response to a unit technology shock is then given by difference between these two trajectories. The simulation is repeated 3000 times.

For stationary variables, the impulse response functions are expressed in percentage deviations from steady state. For non-stationary series, the impulse responses are reported in percentage deviations from the trend growth rate.[13] For these later series, a unit technology shock results in a permanent increase in their level. In each plot the solid lines correspond to the median (pointwise) impulse response function, while the dotted lines provide a 75 percent band — that is, the 12.5 and 87.5 percentiles of the simulated impulse responses. The dashed line gives the corresponding impulse response predicted by a rational expectations analysis of the model.

Figures 2 - 6 report the impulse response functions for output, consumption, investment, hours and the rental rate of capital. For all series the impact effects of a technology shock are almost identical when comparing the median impulse response under learning and the impulse response under rational expectations. This is because agents' beliefs are distributed around the rational expectations prediction function, as shown in the next section. However, in the case of learning, there is variation in the impact effects. The observed amplification of technology shocks in the previous sections is in part sourced to this variation. Depending on the precise beliefs of households and firms at the time of the shock, which along with the capital stock determine the state of the economy, the impact effect of the technology shock could be larger all smaller.

Output, hours and investment display a hump-shaped profile in response to a technology shock. This reflects earlier noted persistence properties induced by learning dynamics. At the time of the shock, belief coefficients are fixed so that the impact effects are on average the same. In subsequent periods, beliefs are revised in response to observed data with a one period lag. This generates persistence in the actual data generating process for all series.

[13] Equivalently, the dynamics are those observed in transition to the new steady state associated with the higher level of technology.

An interesting feature of model concerns dynamics the period after the technology shock dissipates. In a rational expectations equilibrium, all model variables, appropriately normalized, are a linear function of the capital stock and the disturbance to the growth rate of technology. As the disturbance is assumed to be i.i.d., the observed dynamics one period after the shock are entirely determined by adjustment in the capital stock. Under learning, this is not the case. The technology shock leads to revisions in beliefs that commence the period after the disturbance. Subsequent dynamics are largely driven by revisions to beliefs.

In the period after the disturbance, agents revise upwards their beliefs about the returns to investment and downwards their beliefs about wages (in efficiency units) — not just for the next period, but for all future periods in the household's decision horizon. Hence, the present discounted value of capital returns rise and the present discounted labor returns fall relative to the predictions of rational expectations. Figure 7 plots the time series of these sums under each belief structure. Recalling aggregate consumption dynamics given by equation (6), optimism about future returns (a steeper profile) tilts the consumption profile towards greater future consumption. This and the increased pessimism about returns on the labor market (a flatter profile) serve to increase the marginal utility of income relative to rational expectations — leading to larger labor supply and investment effects. Both predictions are, therefore, in part realized in equilibrium outcomes in the period after the shock: the return to capital rises and investment demand surges, while the real wage drops as aggregate labor supply increases. Thus the model generates dynamics that are consistent with those described by Pigou (1926)

Learning amplifies the standard substitution and income effects that operate in real business cycle theory in response to a technology shock. Shifts in expectations are endogenous to technology disturbances giving model greater flexibility in fitting various second order moments. Increased variation in marginal utility of income generates increased volatility in hours worked. Because these variations in hours are caused by variations in the supply of labor for a given production frontier, the model also better matches the various statistics relating to labor market variables — to wit, hours, wages and average labor productivity. Moreover, it simultaneously serves to break the tight correlation between consumption and output.

5.3 Distributions of Beliefs

Because beliefs are central to our story it is useful to study their properties further. Consider the following thought experiment. An econometrician observes an economy with data generated according to the real business model under rational expectations. For each observed sample, the econometrician runs the exact regressions that comprise the beliefs in the learning model — recall equations (12) – (14) — calibrated with a gain equal to $g = 0.0029$. The coefficients are recorded for many simulations.[14]

The dashed line in Figure 8 plots a kernel estimate of the implied distribution of the resulting parameter estimates. Six distributions are reported corresponding to the intercept and slope coefficient in each of the three forecasting equations. What is the interpretation of this distribution? Because the econometrician is outside the model — equivalently, the econometrician is small relative to the population of rational expectation agents — the distribution reflects pure sampling error: there is no feedback of this sampling error on the true data generating process. The distributions are centered on the rational expectations equilibrium, exhibit negligible bias, and have a fairly tight variance. This variance would go to zero as the gain parameter goes to zero, as this would imply that all data are given equal weight. But with the chosen positive gain it is evident that the econometrician has fairly accurate estimates of the parameters characterizing the true data generating process, and would therefore make comparably good forecasts of future returns as the rational agent.

Now imagine a world where all agents modeled by our real business cycle theory actually construct forecasts based on these estimated models. This is precisely the model discussed in this paper. The kernel estimate of the resulting ergodic distribution of beliefs is given by the solid lines. The distribution of the estimated coefficients on capital is not centered on the rational expectations parameters. The distributions are re-centered around the rational expectations coefficients to facilitate comparison with the non-feedback case.[15] However the median impact impulse responses shown in the previous section indicate that agents' median forecast is in line with rational expectations.

[14]In order to compute the distribution of agents's estimates, the model is simulated 2250 times and agents estimates are recorded after discaring the first 2000 observations. The simulation is repeated 7000 times.

[15]The "bias" in the estimates is about 6% for each coefficient.

It is immediate that the variation in possible beliefs that can be held by agents is substantially more dispersed than in the previous thought experiment. This dispersion is what leads to the non-linear impulse response functions and the associated uncertainty of their paths. This in turn generates the increased volatility in the learning model.

The figures show that the bulk of the dispersion in agents' beliefs is *endogenously* determined by the interaction between observed prices and agents' beliefs updating. The dispersion in beliefs reflects that prices are less informative about future macroeconomic conditions. This model feature is further manifestation of shifting expectations as a source of business cycle fluctuations that is very much in the spirit of Pigou and Keynes. Shifting beliefs about the future returns to capital and wages, perhaps due to greater optimism about future investment opportunities, leads to changes in current market clearing prices for labor and capital. In turn, these prices reinforce beliefs.

These dynamics obviously relate to a number of recent papers on news shocks and business cycle dynamics — see for example Beaudry and Portier (2006) and Jaimovich and Rebelo (2008). The present analysis is distinct in the sense that there is only a single source of disturbance — technology shocks. The observed dynamics can be sourced to two kinds of variation: that due to the direct impact of the shock and that due to revisions in beliefs. Because the latter are endogenous to variations to technology they could arguably be termed "endogenous news shocks". Note, however, that the mechanism in each case is somewhat different. In our model, current dynamics are generated by current technology shocks and the endogenous pessimism and optimism reflected in revisions to beliefs. In contrast, these other papers generate shifts in current equilibrium prices in response to signals about productivity as some future date that are exogenous to current technology. Irrespective, learning clearly provided a mechanism through which expectations driven business cycles emerge.

6 Expectations-Driven Business Cycles

Under learning dynamics, real business cycle theory still faces difficulty in matching two key characteristics of the data. The first is the relative volatility of hours and output — and labor market variables more generally. Without an high elasticity of labor supply, the model struggles to replicate the volatility of output. And while learning alleviates the magnitude of the discrepancy between data and model predictions, there remains the question of what other model features would better fit this dimension of the data.[16] The second regards the problem of comovement. Given a shift in expectations, hours and consumption are negatively correlated. Introducing of an alternative belief structure can do little to resolve this model prediction. For a given production frontier, and under the assumption that consumption and leisure are normal goods, shifting beliefs, regardless of how they are modeled, cause variation in the marginal utility of income for which optimal decisions demand negative comovement in these variables — see Barro and King (1984).

An emerging literature under the rubric expectations-driven business cycles studies assumptions on preferences and technology that resolve this comovement problem. The motivating example is typically a news shock about the state of future technology. In the benchmark model under rational expectations it creates an increase in consumption and a decrease in hours and investment.

Beaudry and Portier (2006) explore primitive assumptions on production technology that, in a competitive environment, are consistent with positive comovement in these variables. They show that if production in a multi-sector model displays cost complimentarities in intermediate goods inputs then an otherwise standard real business cycle model will produce expectations-driven business cycles: that is positive comovement between consumption, output, hours and investment in response to an expectations shock. A growing number of papers have proposed alternative resolutions to the comovement problem by considering more complex variants of the standard real business cycle model. Jaimovich and Rebelo (2008) propose modified preferences, variable capital utilization and adjustment costs to investment; Chen and Song (2007) introduce financial frictions; den Haan and Kaltenbrunner (2007) focus on labor market frictions; Floden (2006) considers a model with vintage capital; and Christiano, Motto, and Rostagno (2006) introduce monetary frictions.

[16] Introducing labor market search as in Andolfatto (1996) is one possible remedy, though this friction appears to have more success with persistence properties than as a source of amplification.

6.1 The Model

Motivated by this literature, and the finding in section 2 that consumption and hours growth display, respectively, negative and positive serial correlation, the benchmark model is augmented in the following way. First, a more general assumption on household preferences is considered that remains consistent with a long-run balanced growth path. Second, a production technology with a small degree of increasing returns is introduced along with variable capital utilization. These model features resolve the comovement problem. This modeling choice is dictated by keeping the model as simple as possible (no state variable is added) to provide a meaningful comparison with the benchmark real business cycle framework.

Households are assumed to maximize

$$\hat{E}_t^j \sum_{T=t}^{\infty} \beta^{T-t} \frac{\left(C_T^j v\left(L_t^j\right)\right)^{1-\sigma}}{1-\sigma}$$

subject to

$$C_t^j + K_{t+1}^j = R_t^K \left(U_t K_t^j\right) + W_t H_t^j + \left(1 - \delta\left(U_t\right)\right) K_t^j.$$

The notation remains as before, with the following additions. U_t is the utilization rate of capital in any period t. The function $\delta\left(\cdot\right)$ gives the associated capital depreciation costs attached to a given utilization rate of capital. We choose $\delta\left(U_t\right) = \theta^{-1} U_t^{\theta}$. It is included to address the potential criticism that the benchmark model is designed to minimize amplification and propagation under rational expectations. The results show that even in the presence of this friction learning amplifies volatility relative to rational expectations by a greater magnitude than in the benchmark analysis. The only other change in the household's problem is the more general utility function. The utility function is consistent with constant hours on the balanced growth path, it displays a constant consumption intertemporal elasticity of substitution and has a constant Frisch elasticity of labor supply. Given that we are interested in the first order approximation of the model we need only to specify $\epsilon_L = \nu'' L/L$ and $\psi = \frac{\nu'}{\nu} L$ (additional detail can be found in the Appendix).

The firm's problem is to maximize profits

$$Y_T - W_T H_T - R_T^K \left(U_t K_t\right)$$

by choice of effective capital input $u_t K_t$ and labor input H_t subject to the production technology

$$Y_t = \Psi_t \left(U_t K_t\right)^{\alpha} \left(X_t H_t\right)^{1-\alpha}$$

where

$$\Psi_t = \left[\left(U_t K_t\right)^{\alpha} \left(X_t H_t\right)^{1-\alpha}\right]^{\eta} \ X_t^{-\eta}.$$

The term, Ψ_t, denotes the external effects of aggregate capital, indexed by the constant $\eta \geq 0$. The term $X_t^{-\eta}$ guarantees that a balanced growth path exists in this model. The assumptions $\sigma = 1$, $\eta = 0$ and $U_t = 1$ for all t delivers our benchmark model. Details of the first order conditions; log-linear approximation; and resulting model equations are found in the appendix. To gain intuition about the results, the assumption of increasing returns implies that efficiency wages decrease by less after an outwards shift in the labor supply while the non-separability between consumption and leisure tighten the comovement between hours and consumption by rising the marginal utility of consumption as hours increase.

6.2 Calibration

In this version of the model, we keep the Frisch elasticity of labor supply to the same value as in the simple real business cycle model. From the log-linearized labor supply equation we have

$$\sigma^{-1}\hat{\lambda}_t + \hat{w}_t = \left[\epsilon_L + \sigma^{-1}\psi\left(2\sigma - 1\right)\right] \frac{H}{1-H}\hat{H}_t$$

where ψ depends on the model steady state and $\epsilon_L = -\nu'' L/\nu'$. We then choose ϵ_L such that $\epsilon_L + \sigma^{-1}\psi\left(2\sigma - 1\right) = 0.01$. There are two extra parameters with respect to the simple RBC model. The first parameter, measuring the aggregate externality, is set as $\eta = 0.1$, consistent with the lowest estimate in Baxter and King (1991). The parameter implies a "small" degree of externality which implies a locally determinate equilibrium under rational expectations.[17] The second parameter is the household's intertemporal elasticity of substitution σ. We choose σ so that the ratio of the standard deviations of consumption and output in the model and in the HP filtered data are as close as possible. As for the simple RBC model, we calibrate σ_X and

[17] The parameter implies a downward-sloping demand for labor. For the connection between externality and indeterminacy, see Benhabib and Farmer (1994).

g to match the standard deviation of output in the filtered data and the first autocorrelation of output growth respectively. The calibrated gain is now $g = 0.0015$, half as much as in the simple RBC model. This gain implies that agents put 74% weight on observations that are 50 years old. The Appendix shows the parameter θ is pinned down by the steady state return on capital and the depreciation rate.

6.3 Results

The table 4 reports a subset of earlier presented statistics for the generalized model. It is immediate that the model does well in most dimensions. Moreover, the autocorrelation of consumption is significantly positive. The model continues to perform poorly in terms of wage dynamics, as we should expect, though does better than a rational expectations equilibrium analysis (not reported).

Table 4: Model with increasing returns and non-separable preferences

	Statistic									
	σ_Y	σ_C/σ_Y	σ_I/σ_Y	σ_H/σ_Y	$\rho_{Y,C}$	$\rho_{Y,H}$	Δ_C	Δ_Y	Δ_I	Δ_H
Data	1.54	0.52	2.87	1.13	0.69	0.88	0.25	0.28	0.34	0.58
Model:										
$\sigma = 1.5$	1.50	0.52	2.42	0.70	0.99	0.99	0.14	0.27	0.35	0.39
$\sigma = 1$	1.50	0.33	3.04	0.70	0.92	0.98	0.01	0.22	0.30	0.33

The impulse response functions in figures 9 - 13 confirm that the model can generate expectations-driven business cycles, as consumption, investment and hours rise also *after* the productivity shock has occurred. Assuming a consumption intertemporal elasticity above one achieves a stronger correlation between consumption and hours, which is shown in the positive autocorrelation of consumption. This comes at the cost of slightly lower volatility of investment. The second row shows the performance of the model when σ is equal to 1. This weakens the autocorrelation properties of consumption, which, as before, is noticeably less volatile than output. One last important result is that the extended model improves the overall fit with the data but also increases considerably amplification. The standard deviation of the shock that is required to match the volatility of output is more than 30% lower than the required value under rational expectations.

Interestingly, learning as an endogenous news shock generates greater amplification and propagation compared to other recent models of news-driven business cycles. For example, in Jaimovich and Rebelo (2008), model implied statistics are virtually identical across models with and without the news shock — current technology shocks determine time series properties. This is not the case in our model.

7 Robustness

Modeling learning dynamics introduces one free parameter in the gain coefficient. It might reasonably be asked how sensitive are our results to the choice of gain parameter. Furthermore, our approach maybe criticized on the ground that it is well known that real business cycle models need to be augmented with additional frictions to have any hope of replicating observed data. And that if we permitted the real business cycle model under rational expectations a one parameter deviation from the benchmark model it would provide a similarly good fit as the model under learning dynamics. Or that the presence of such frictions would mitigate the role of learning as an amplification and propagation mechanism. The following exercises allay such concerns, demonstrating that:

- The choice of gain is not unconstrained. Large gain coefficients generate excess volatility in many variables and therefore inferior fit of observed data;

- Introducing other frictions to the benchmark model under rational expectations, such as variable capital utilization or adjustment costs in investment, continue to fail to fit the data as well as our one parameter deviation of learning dynamics; and

- Even when learning is introduced in conjunction with these frictions, it continues to provide significant amplification and propagation relative to the same model under rational expectations.

Table 5 reports a subset of statistics for a number of variants of the benchmark model. The calibration is held fixed at our benchmark values for the model under learning, so that the standard deviation of technology shocks remains unchanged across simulations. Models 1 and 2 show the benchmark results for the rational expectations and learning models. The latter reiterates earlier results for ease of comparison while the former gives the results under rational expectations assuming the same standard deviation of technology shocks as model 2. The improved amplification is again immediate. Models 3 and 4 show the cases of a low elasticity of labor supply for each belief structure. Under both rational expectations and learning, the volatility of output falls for a given standard deviation technology shock. Concomitantly, the relative volatility of investment and hours also decline, while the relative volatility of consumption increases. The serial correlation properties adjust accordingly. These results underscore the centrality of the elasticity of labor supply in generating plausible volatility in real business cycle models.

Model 5 shows the learning model under a higher gain, $g = 0.009$, which is three times as large as our benchmark case. It significantly increases volatility in all series, but tends to overshoot corresponding sample moments. This makes clear that the modeler is not unconstrained in choosing this parameter — larger and larger gains do not translate into increasingly better correspondence with data.

The final row reports statistics for an alternative model of learning. Many recent papers have proposed analyses of learning dynamics in the context of models where agents solve infinite horizon decision problems, but without requiring that agents make forecasts more than one period into the future. In these papers, agents' decisions depend only on forecasts of future variables that appear in Euler equations used to characterize rational expectations equilibrium. Important contributions include Bullard and Mitra (2002) and Evans and Honkapohja (2003).

Table 5: Robustness

	Statistic								
	σ_Y	σ_C/σ_Y	σ_I/σ_Y	σ_H/σ_Y	$\rho_{Y,C}$	$\rho_{Y,H}$	Δ_C	Δ_Y	Δ_I
Data	1.54	0.52	2.8727	1.13	0.69	0.88	0.25	0.28	0.34
Model:									
Baseline RE	1.24	0.54	2.42	0.49	0.97	0.97	0.11	0.00	-0.03
Baseline Learn	1.52	0.38	3.07	0.72	0.83	0.96	-0.14	-0.28	0.42
Low Elast. RE	1.13	0.56	2.35	0.39	0.98	0.97	0.10	0.01	-0.02
Low Elast. Learn	1.28	0.43	2.91	0.55	0.87	0.95	-0.17	0.22	0.41
High Gain	2.30	0.32	4.00	1.04	0.03	0.95	-0.35	0.44	0.26
Euler Equation	1.24	0.54	2.42	0.49	0.97	0.97	0.10	0.00	-0.03

Of particular relevance to the present study are the analyses of Williams (2003) and Carceles-Poveda and Giannitsarou (2007). The former studies precisely the question explored here: can learning be a source of business cycle fluctuations? The latter is similarly motivated, with specific focus on asset pricing implications of real business cycle theory. Both papers make use of models with learning dynamics in which only one period ahead expectations matter to expenditure and production plans of households and firms. Both conclude that learning dynamics are unpromising in generating amplification and propagation.

The final row replicates this kind of analysis in the context of the model developed here. Williams (2003) proceeds assuming that the Euler equations predicted by a rational expectations analysis of the model represent decision rules of agents under learning. The only model equation to change is that for consumption demand. The Euler equation is

$$c_t = E_t c_{t+1} - E_t \left(\beta \bar{R} R^K_{t+1} + \hat{\gamma}_{t+1} \right). \tag{23}$$

The model under learning then assumes household consumption decisions are determined as

$$c_t = \hat{E}_t c_{t+1} - \hat{E}_t \left(\beta \bar{R} R^K_{t+1} + \hat{\gamma}_{t+1} \right) \tag{24}$$

This requires the further assumption that households directly forecast their own future consumption using regressions of the kind specified in section 2. Preston (2005) shows that this decision rule leads to suboptimal decisions — see also Marcet and Sargent (1989).[18] All remaining model equations are unchanged as they do not directly depend on the specification of beliefs.

It is immediate that not modeling optimal decisions and assuming consumption decisions are made according to (24) leads to dramatically different conclusions. Learning dynamics fail to generate amplification and propagation. Model implied moments are essentially indistinguishable from a rational expectations analysis of the model. This negative finding has less to do with learning than it does with the assumed nature of economic decisions. In real business cycle theory the only intertemporal decision is the household's consumption and saving decision. To make this decision households must forecast the entire future sequence of wages and real interest rates. These beliefs about future prices determine current market clearing prices, which in turn determine beliefs. A consequence of the model of household behavior given by (24) is the connections between market prices that govern future consumption and investment opportunities and current allocations and prices is severed. The economic structure of the model is completely changed and revealed to matter greatly for implied model dynamics. Only by properly modeling the interactions of households' and firms' beliefs about the economy and the markets in which they operate can we understand the potential of near-rational beliefs to explain observed data.

7.1 Alternative Frictions

[To be added ...]

8 Conclusion

[To be added ...]

[18] That (23) describes optimal decisions under rational expectations and not learning, reflects the property under under rational expectations of equilibirum probability laws embed information about all relevant constraints, including transversality conditions and intertemporal budget constraints. This is not true once beliefs are exogenously specified as in the learning model contemplated here.

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A Appendix

A.1 Data

We use data for the US economy, the sample ranges from 1955:Q1 to 2004:Q4. The variables are constructed as follows (DLX codes in parenthesis). Output is Real Gross Domestic Product (GDPH); nominal consumption is computes as the sum of nondurable goods (CN), services (CS) and government expenditures (G); nominal investment is the sum of private nonresidential investment structures (FNS), Equipment and software (FNE), private residential investment (FR) and consumption durable goods (CD). Consumption and investment are converted in real terms by using the GDP deflator (GDP/GDPH). Hours are measured by non farm business hours (LXNFH). All variables a transformed in per capita terms by using the civilian noninstitutional population above 16 years (LN16N). Productivity is measured as output per hour in the nonfarm business sector (LXNFA). Finally, the hourly wage is measured by compensation per hour in the nonfarm business sector (LXNFC). Real wage is obtained by using the nonfarm output price deflator (LXNFI). We also document the volatility of hours by using (as an alternative measure) the average hours of all persons at work from the household survey (LENCLWHN). For this series, we use the sample 1982:Q3-2006:Q1.

A.2 Model

This section shows the solution of the model that includes capacity utilization, non-separability between consumption and leisure and externalities of production. Consumers choose consumption, leisure and capital in order to maximize

$$E_t \sum_{T=t}^{\infty} \beta^{T-t} u\left(C_T, L_T\right)$$

subject to

$$C_t + K_{t+1} = R_t^K \left(u_t K_t\right) + W_t H_t + \left(1 - \delta\left(U_t\right)\right) K_t$$

$$L_t = 1 - H_t.$$

The first order conditions are

C_t

$$u_c(C_t, L_t) = \Lambda_t$$

K_{t+1}

$$\beta \hat{E}_t \Lambda_{t+1} R^K_{t+1} u_{t+1} - \Lambda_t + \beta \hat{E}_t \left[\Lambda_{t+1} \left(1 - \delta \left(U_{t+1}\right)\right)\right] = 0$$

L_t

$$u_L(C_t, L_t) = \Lambda_t W_t$$

u_t

$$R^K_t = \delta' \left(U_t\right).$$

In the sequel we assume

$$u\left(C_t, L_t\right) = \frac{\left(C_T v\left(L_t\right)\right)^{1-\sigma}}{1-\sigma}$$

where we specify $\psi = \nu'\left(L\right) L/\nu\left(L\right)$, ($L$ is steady state leisure) and $\epsilon_L = -\nu'' L/\nu'$. Also, we assume

$$\delta\left(U_t\right) = \frac{1}{\theta} U_t^\theta.$$

Non-stationary variables (expressed in efficiency units) are denoted in lower case letters. Stationary variables are left unchanged.

A.3 Households

here we transform the variables in stationary form

C_t

$$\begin{aligned} X_t^\sigma \Lambda_t &= X_t^\sigma u_c(C_t, L_t) = X_t^\sigma C_t^{-\sigma} v\left(L_t\right)^{1-\sigma} \\ \lambda_t &= c_t^{-\sigma} v\left(L_t\right)^{1-\sigma} \end{aligned}$$

K_{t+1}

$$\beta \hat{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} R^K_{t+1} U_{t+1} + \frac{\Lambda_{t+1}}{\Lambda_t}\left(1 - \delta\left(U_{t+1}\right)\right)\right] = 1$$

re-arranging

$$\beta \hat{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \frac{X_{t+1}^\sigma}{X_{t+1}^\sigma} \frac{X_t^\sigma}{X_t^\sigma} R_{t+1}^K U_{t+1} + \frac{\Lambda_{t+1}}{\Lambda_t} \frac{X_{t+1}^\sigma}{X_{t+1}^\sigma} \frac{X_t^\sigma}{X_t^\sigma} \left(1 - \delta \left(U_{t+1}\right)\right) \right] = 1$$

$$\beta \hat{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\gamma_{t+1}^\sigma} R_{t+1}^K U_{t+1} + \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\gamma_{t+1}^\sigma} \left(1 - \delta \left(U_{t+1}\right)\right) \right] = 1$$

$$\beta \hat{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\gamma_{t+1}^\sigma} \left(R_{t+1}^K U_{t+1} + \left(1 - \delta \left(U_{t+1}\right)\right) \right) \right] = 1$$

L_t

$$\begin{aligned} X_t^{\sigma-1} \Lambda_t W_t &= X_t^{\sigma-1} u_L(C_t, L_t) = X_t^{\sigma-1} C_t^{1-\sigma} \frac{v'\left(L_t\right)}{v\left(L_t\right)^\sigma} \\ \lambda_t w_t &= c_t^{1-\sigma} \frac{v'\left(L_t\right)}{v\left(L_t\right)^\sigma}. \end{aligned}$$

Log-linearizing the equations we get:

1) Marginal utility of consumption

$$\hat{\lambda}_t = -\sigma \hat{c}_t - \psi \left(1 - \sigma\right) \frac{H}{1-H} \hat{H}_t$$

where in steady state

$$\psi = \frac{L v'\left(L\right)}{v\left(L\right)} = \frac{1-H}{H} \frac{wH}{k} \frac{k}{c}.$$

2) Euler equation

$$\beta \hat{E}_t \left[\beta^{-1} \left(\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma \hat{\gamma}_{t+1} \right) + \left(\beta^{-1} - \frac{(1-\delta)}{\gamma^\sigma} \right) \left(\hat{R}_{t+1}^K + \hat{U}_{t+1} \right) - \frac{\delta}{\gamma^\sigma} \theta \hat{U}_{t+1} \right] = 0$$

which, using

$$\frac{r^K U}{\gamma^\sigma} = \left(\beta^{-1} - \frac{(1-\delta)}{\gamma^\sigma} \right) = \frac{\theta \delta}{\gamma^\sigma}$$

becomes

$$\beta \hat{E}_t \left[\beta^{-1} \left(\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma \hat{\gamma}_{t+1} \right) + \left(\beta^{-1} - \frac{(1-\delta)}{\gamma^\sigma} \right) \hat{R}_{t+1}^K \right] = 0.$$

3) Labor equation

$$\left(1 - \sigma\right) \hat{c}_t - \epsilon_L \hat{L}_t = \hat{\lambda}_t + \hat{w}_t$$

where

$$\epsilon_L = -\frac{\nu''}{\nu'} L$$

and the expression for marginal utility we get

$$\begin{aligned}\hat{\lambda}_t + \hat{w}_t &= [\sigma\epsilon_L + \psi(2\sigma - 1)] \frac{H}{1-H}\hat{H}_t \\ &= \tilde{\epsilon}_L \frac{H}{1-H}\hat{H}_t\end{aligned}$$

We can then set ϵ_L such that the elasticity of the labor supply ($\tilde{\epsilon}_L$) is at the desired value.

4) capacity utilization

$$\hat{U}_t = \frac{1}{(\theta - 1)} \hat{R}_t^K \tag{25}$$

Using the expressions for capital utilization and the marginal utility of consumption, the Euler equation can be expressed in familiar form

$$\begin{aligned}0 &= \beta \hat{E}_t \left[\beta^{-1}\left(\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma\hat{\gamma}_{t+1}\right) + \left(\beta^{-1} - \frac{(1-\delta)}{\gamma^\sigma}\right)\hat{R}_{t+1}^K\right] \\ &= \hat{E}_t \left[\left(\hat{\lambda}_{t+1} - \hat{\lambda}_t - \sigma\hat{\gamma}_{t+1}\right) + \beta\left(\beta^{-1} - \frac{(1-\delta)}{\gamma^\sigma}\right)\hat{R}_{t+1}^K\right] = \\ \hat{\lambda}_t &= \hat{E}_t \left[\left(\hat{\lambda}_{t+1} - \sigma\hat{\gamma}_{t+1}\right) + \beta\bar{R}\hat{R}_{t+1}^K\right] = \\ -\sigma\hat{c}_t - \bar{H}\hat{H}_t &= \hat{E}_t\left[-\sigma\hat{c}_{t+1} - \bar{H}\hat{H}_{t+1}\right] - \sigma\hat{E}_t\hat{\gamma}_{t+1} + \hat{E}_t\beta\bar{R}\hat{R}_{t+1}^K = \\ \hat{Q}_t &= -\hat{E}_t\beta\bar{R}\left(\hat{R}_{t+1}^K\right) + \hat{E}_t\hat{Q}_{t+1} + \sigma\hat{E}_t\hat{\gamma}_{t+1}\end{aligned}$$

where

$$\begin{aligned}\bar{R} &= \left(\beta^{-1} - \frac{(1-\delta)}{\gamma^\sigma}\right), \\ \bar{H} &= \psi(1-\sigma)\frac{H}{1-H}\end{aligned}$$

and

$$\hat{Q}_t = \sigma\hat{c}_t + \bar{H}\hat{H}_t.$$

A.4 Firms

The firms's problem is

$$\max_{C_t, I_t, u_t K_t, L_t} \hat{E}_t \sum_{T=t}^{\infty} \tilde{Q}_{t,T}\left[Y_T - W_T H_T - R_T^K (U_t K_t)\right]$$

sub

$$Y_t = \Psi_t \left(U_t K_t\right)^{\alpha} \left(X_t H_t\right)^{1-\alpha}$$

where

$$\Psi_t = \left[\left(U_t K_t\right)^{\alpha} \left(X_t H_t\right)^{1-\alpha}\right]^{\eta} \; X_t^{-\eta}$$

denotes the external effects of aggregate capital. Finally, X_t denotes labor-augmenting technical progress. The term $X_t^{-\eta}$ guarantees that a balanced growth path with exogenous growth exists in this model. The exponent on the exogenous process becomes

$$Z_t = X_t^{(1-\alpha)\eta-\eta+(1-\alpha)} = X_t^{1-\alpha-\eta\alpha} = X_t^{1-(1+\eta)\alpha}$$

which can be made stationary by the following transformation

$$\Psi_t \gamma_t^{-\alpha} k_t^{\alpha} U_t^{\alpha} H_t^{1-\alpha} = y_t$$

which is log-linearized to

$$\hat{\Psi}_t - \alpha\hat{\gamma}_t + \alpha\hat{k}_t + \alpha\hat{U}_t + (1-\alpha)\hat{H}_t = \hat{y}_t. \tag{26}$$

L_t

$$-W_t + (1-\alpha)\Psi_t \left(U_t K_t\right)^{\alpha} \left(X_t\right)^{1-\alpha} H_t^{-\alpha} = 0$$

becomes

$$(1-\alpha)\Psi_t X_{t-1}^{\alpha} \left(U_t \frac{K_t}{X_{t-1}}\right)^{\alpha} \frac{\left(X_t\right)^{1-\alpha}}{X_t} H_t^{-\alpha} = \frac{W_t}{X_t}$$
$$(1-\alpha)\Psi_t \gamma_t^{-\alpha} \left(U_t k_t\right)^{\alpha} H_t^{-\alpha} = w_t$$

where

$$\begin{aligned}\Psi_t &= \left[\left(U_t K_t\right)^{\alpha} \left(X_t H_t\right)^{1-\alpha}\right]^{\eta} \; X_t^{-\eta} = \left[\left(U_t \frac{K_t}{X_t}\right)^{\alpha} H_t^{1-\alpha}\right]^{\eta} \\ &= \left[\left(\frac{U_t}{\gamma_t} k_t\right)^{\alpha} H_t^{1-\alpha}\right]^{\eta}\end{aligned}$$

which can be log-linearized to

$$\hat{\Psi}_t = \eta\alpha\left(\hat{U}_t + \hat{k}_t - \hat{\gamma}_t\right) + (1-\alpha)\eta\hat{H}_t. \tag{27}$$

Combined with the definition of output gives

$$w_t = (1-\alpha)\frac{y_t}{H_t}$$

which in log-linear form becomes

$$\hat{w}_t = \hat{y}_t - \hat{H}_t. \tag{28}$$

$u_t K_t$

$$-R_t^K + \alpha\Psi_t (U_t K_t)^{\alpha-1} (X_t H_t)^{1-\alpha} = 0$$

$$-R_t^K + \alpha\Psi_t \left(U_t \frac{K_t}{X_t}\right)^{\alpha-1} H_t^{1-\alpha} = 0$$

$$-R_t^K + \alpha\Psi_t \left(\frac{U_t}{\gamma_t} k_t\right)^{\alpha-1} H_t^{1-\alpha} = 0$$

using the definition of output

$$R_t^K = \alpha\gamma_t \frac{y_t}{U_t k_t}$$

in log-linear form

$$\hat{R}_t^K = \hat{\gamma}_t + \hat{y}_t - \hat{U}_t - \hat{k}_t \tag{29}$$

and finally capital

$$\hat{k}_{t+1} = \frac{i}{k}\hat{\imath}_t + \frac{(1-\delta)}{\gamma}\left(\hat{k}_t - \hat{\gamma}_t\right) - \frac{\delta\theta}{\gamma}\hat{U}_t. \tag{30}$$

B Consumption decision rule

In the following we derive the consumption decision rule. Households choose a path for consumption, taking as given their initial capital holdings, capital and labor prices and their expectations about future prices. Let us start by defining the household's intertemporal budget constraint. The flow budget constraint can be expressed as

$$c_t + k_{t+1} = (\gamma_t)^{-1} R_t^K (U_t k_t) + w_t H_t + (1 - \delta(U_t))(\gamma_t)^{-1} k_t.$$

Log-linearization gives

$$\gamma^{\sigma-1}\beta^{-1}\hat{k}_t = \left\{ \begin{array}{c} \frac{c}{k}\hat{c}_t + \hat{k}_{t+1} - \\ \bar{R}\gamma^{\sigma-1}\left(\hat{R}_t^K + \hat{U}_t + \frac{1-\alpha}{\alpha}\hat{w}_t + \frac{1-\alpha}{\alpha}\hat{H}_t - \hat{\gamma}_t\right) - \\ \frac{(1-\delta)}{\gamma}\left[-\hat{\gamma}_t - \frac{\delta}{(1-\delta)}\theta\hat{U}_t\right] \end{array} \right\}$$

where we use

$$\frac{wH}{k} = \frac{1-\alpha}{\alpha}\frac{UR^k}{\gamma} = \frac{1-\alpha}{\alpha}\bar{R}\gamma^{\sigma-1}$$

and

$$\left[\frac{R^k U}{\gamma} + \frac{1-\delta}{\gamma}\right] = \gamma^{\sigma-1}\beta^{-1} = \tilde{\beta}^{-1}$$

and where we can define

$$\begin{aligned} \bar{R}\gamma^{\sigma-1} &= \gamma^{\sigma-1}\left(\beta^{-1} - \frac{(1-\delta)}{\gamma^{\sigma}}\right) \\ &= \tilde{\beta}^{-1} - \frac{(1-\delta)}{\gamma} \\ &= \tilde{R} \end{aligned}$$

from the steady state condition of the Euler equation. The expression above can be further simplified by using $\frac{R^k U}{\gamma} = \frac{\delta\theta}{\gamma}$, which gets

$$\hat{k}_t = \tilde{\beta}\left[\frac{c}{k}\hat{c}_t + \hat{k}_{t+1} + \tilde{\beta}^{-1}\hat{\gamma}_t - \tilde{R}\left(\hat{R}_t^K + \frac{1-\alpha}{\alpha}\hat{w}_t + \frac{1-\alpha}{\alpha}\hat{H}_t\right)\right].$$

Using the labor supply

$$\hat{\lambda}_t + \sigma\hat{w}_t = \tilde{\epsilon}_L\frac{H}{1-H}\hat{H}_t$$

and the definition of marginal utility

$$\begin{aligned} -\sigma\hat{w}_t + \tilde{\epsilon}_L\frac{H}{1-H}\hat{H}_t &= -\sigma\hat{c}_t - \psi\left(1-\sigma\right)\frac{H}{1-H}\hat{H}_t \\ \left[\frac{\tilde{\epsilon}_L}{\sigma} - \psi\frac{(\sigma-1)}{\sigma}\right]\frac{H}{1-H}\hat{H}_t &= -\hat{c}_t + \hat{w}_t \end{aligned}$$

$$\hat{H}_t = \frac{1-H}{H}\left[\frac{\tilde{\epsilon}_L}{\sigma} - \psi\frac{(\sigma-1)}{\sigma}\right]^{-1}\left(-\hat{c}_t + \hat{w}_t\right)$$

We can substitute for labor supply decision $\hat{L}_t$ by using the household first order conditions which gives

$$\hat{k}_t = \tilde{\beta}\left(\epsilon_c \hat{c}_t + \tilde{\beta}^{-1}\hat{\gamma}_t + \hat{k}_{t+1} - \epsilon_w \hat{w}_t - \tilde{R}\hat{R}_t^K\right)$$

where

$$\epsilon_c = \frac{c}{k} + \frac{1-H}{H}\left[\frac{\tilde{\epsilon}_L}{\sigma} - \psi\frac{(\sigma-1)}{\sigma}\right]^{-1}\tilde{R}\frac{1-\alpha}{\alpha}$$

and

$$\epsilon_w = \left(1 + \frac{1-H}{H}\left[\frac{\tilde{\epsilon}_L}{\sigma} - \psi\frac{(\sigma-1)}{\sigma}\right]^{-1}\right)\tilde{R}\frac{1-\alpha}{\alpha}$$

By iterating forward and taking expectations we get

$$\epsilon_c \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\hat{c}_T = \tilde{\beta}^{-1}\hat{k}_t + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t}\bar{y}_T^P$$

where

$$\bar{y}_t^P = -\tilde{\beta}^{-1}\hat{\gamma}_t + \epsilon_w \hat{w}_t + \tilde{R}\hat{R}_t^K$$

denotes a linear combination of the market prices that affect household's permanent income. the expression above defines household expected intertemporal budget constraint, as of time t. Recall the Euler equation

$$\hat{Q}_t = -\hat{E}_t \beta \bar{R}\left(\hat{R}_{t+1}^K\right) + \hat{E}_t \hat{Q}_{t+1} + \sigma \hat{E}_t \hat{\gamma}_{t+1}$$

which can be re-written as

$$\begin{aligned}
\hat{Q}_t &= -\hat{E}_t \beta \gamma^{1-\sigma}\gamma^{\sigma-1}\bar{R}\left(\hat{R}_{t+1}^K\right) + \hat{E}_t \hat{Q}_{t+1} + \sigma \hat{E}_t \hat{\gamma}_{t+1} \\
&= -\hat{E}_t \tilde{\beta}\tilde{R}\hat{R}_{t+1}^K + \hat{E}_t \hat{Q}_{t+1} + \sigma \hat{E}_t \hat{\gamma}_{t+1}
\end{aligned}$$

By solving backward from time T we get

$$\begin{aligned}
\hat{E}_t Q_T &= \hat{Q}_t + \hat{E}_t\left[\sum_{T=t}^{T-1}\left(\tilde{\beta}\tilde{R}\hat{R}_{T+1}^K - \sigma\hat{\gamma}_{T+1}\right)\right] \\
\hat{E}_t\left(\sigma\hat{c}_T + \bar{H}\hat{H}_T\right) &= \sigma\hat{c}_t + \bar{H}\hat{H}_t + \hat{E}_t\left[\sum_{T=t}^{T-1}\left(\tilde{\beta}\tilde{R}\hat{R}_{T+1}^K - \sigma\hat{\gamma}_{T+1}\right)\right],
\end{aligned}$$

where I use

$$\hat{Q}_t = \sigma \hat{c}_t + \bar{H} \hat{H}_t.$$

Substitute labor supply

$$\hat{E}_t \left(\sigma \hat{c}_T + \bar{H} \frac{1-H}{H} \left[\frac{\tilde{\epsilon}_L}{\sigma} - \psi \frac{(\sigma - 1)}{\sigma} \right]^{-1} (-\hat{c}_T + \hat{w}_T) \right) = \sigma \hat{c}_t + \bar{H} \hat{H}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \sigma \hat{\gamma}_{T+1} \right) \right]$$

and using

$$\begin{aligned} \bar{H} \frac{1-H}{H} \left[\frac{\tilde{\epsilon}_L}{\sigma} - \psi \frac{(\sigma - 1)}{\sigma} \right]^{-1} &= \psi (1-\sigma) \left[\frac{\tilde{\epsilon}_L}{\sigma} - \psi \frac{(\sigma - 1)}{\sigma} \right]^{-1} \\ &= \left[\frac{\tilde{\epsilon}_L}{\psi (1-\sigma) \sigma} - \psi \frac{(\sigma - 1)}{\psi (1-\sigma) \sigma} \right]^{-1} \\ &= \sigma \left[\frac{\tilde{\epsilon}_L}{\psi (1-\sigma)} + 1 \right]^{-1} \end{aligned}$$

we get where we use

$$1 - \frac{\psi (1-\sigma)}{\psi (1-\sigma) + \tilde{\epsilon}_L} = 1 - \chi$$

so that

$$\hat{E}_t \left(\left[(1-\chi) \sigma \hat{c}_T + \chi \sigma \hat{w}_T \right] \right) = \sigma \hat{c}_t + \bar{H} \hat{H}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \sigma \hat{\gamma}_{T+1} \right) \right]$$

which implies

$$\hat{E}_t \hat{c}_T = \frac{1}{1-\chi} \left\{ \hat{c}_t + \sigma^{-1} \bar{H} \hat{H}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\sigma^{-1} \tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \hat{\gamma}_{T+1} \right) \right] - \chi \hat{w}_T \right\}.$$

By substituting into the intertemporal budget constraint we get

$$\begin{aligned} & \epsilon_c \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \left[\frac{1}{1-\chi} \left\{ \hat{c}_t + \sigma^{-1} \bar{H} \hat{H}_t + \hat{E}_t \left[\sum_{T=t}^{T-1} \left(\sigma^{-1} \tilde{\beta} \tilde{R} \hat{R}^K_{T+1} - \hat{\gamma}_{T+1} \right) \right] - \chi \hat{w}_T \right\} \right] \\ = \; & \tilde{\beta}^{-1} \hat{k}_t + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \bar{y}^P_T \end{aligned}$$

which can be simplified to

$$\begin{aligned}
\hat{c}_t + \sigma^{-1}\bar{H}\hat{H}_t &= \frac{1-\tilde{\beta}}{\tilde{\epsilon}_c\tilde{\beta}}\hat{k}_t + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \left[\frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c} \left(\bar{y}_T^P + \tilde{\epsilon}_c \chi \hat{w}_T\right) - \tilde{\beta}\left(\sigma^{-1}\tilde{\beta}\tilde{R}\hat{R}_{T+1}^K - \hat{\gamma}_{T+1}\right) \right] \\
&= \frac{1-\tilde{\beta}}{\tilde{\epsilon}_c\tilde{\beta}}\hat{k}_t + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \left[\frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c} \tilde{y}_T^P - \tilde{\beta}\left(\sigma^{-1}\tilde{\beta}\tilde{R}\hat{R}_{T+1}^K - \hat{\gamma}_{T+1}\right) \right]
\end{aligned}$$

where

$$\tilde{\epsilon}_c = \frac{\epsilon_c}{1-\chi}$$

and

$$\tilde{y}_t^P = -\tilde{\beta}^{-1}\hat{\gamma}_t + \left(\epsilon_w + \tilde{\epsilon}_c\chi\right)\hat{w}_t + \tilde{R}\hat{R}_t^K.$$

Finally we can substitute for $\tilde{y}_t^P$ to obtain the consumption decision rule, depending only on forecast of prices that are beyond the control of the household,

$$\begin{aligned}
\hat{c}_t + \sigma^{-1}\bar{H}\hat{H}_t &= \frac{1-\tilde{\beta}}{\tilde{\epsilon}_c\tilde{\beta}}\hat{k}_t + \frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c}\tilde{R}\hat{R}_t^K - \frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c}\tilde{\beta}^{-1}\hat{\gamma}_t + \frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c}\left(\epsilon_w + \tilde{\epsilon}_c\chi\right)\hat{w}_t \\
&\quad + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \left[\frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c} \tilde{\beta}\tilde{y}_{T+1}^P - \tilde{\beta}\left(\sigma^{-1}\tilde{\beta}\tilde{R}\hat{R}_{T+1}^K - \hat{\gamma}_{T+1}\right) \right]
\end{aligned}$$

$$\begin{aligned}
\hat{c}_t + \sigma^{-1}\bar{H}\hat{H}_t &= \frac{1-\tilde{\beta}}{\tilde{\epsilon}_c\tilde{\beta}}\hat{k}_t + \frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c}\tilde{R}\hat{R}_t^K - \frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c}\tilde{\beta}^{-1}\hat{\gamma}_t + \frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c}\left(\epsilon_w + \tilde{\epsilon}_c\chi\right)\hat{w}_t \\
&\quad + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \left[-\frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c} + \tilde{\beta} \right] \hat{\gamma}_{T+1} + \\
&\quad + \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \left[\frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c} - \tilde{\beta}\sigma^{-1} \right] \tilde{\beta}\tilde{R}\hat{R}_{T+1}^K + \\
&\quad \hat{E}_t \sum_{T=t}^{\infty} \tilde{\beta}^{T-t} \frac{\left(1-\tilde{\beta}\right)}{\tilde{\epsilon}_c} \tilde{\beta}\left(\epsilon_w + \tilde{\epsilon}_c\chi\right)\hat{w}_{T+1}.
\end{aligned}$$

Notice that by setting $\sigma = 1$ and $\eta = 0$ we get back to the simple RBC model.

C Steady State

From the Euler equation we get

$$\begin{aligned}\frac{R^k U}{\gamma} &= \gamma^{\sigma-1}\beta^{-1} - \frac{1-\delta}{\gamma} \\ &= \tilde{\beta}^{-1} - \frac{1-\delta}{\gamma}\end{aligned}$$

and from the capacity utilization first order condition

$$\begin{aligned}\frac{R^k U}{\gamma} &= \frac{\delta\theta}{\gamma} \\ &\Longrightarrow \\ \theta &= \frac{R^k U}{\delta}\end{aligned}$$

which defines θ, allowing to determine U and therefore R^k. The ratios

$$\frac{y}{k} = (\alpha)^{-1}\frac{R^k U}{\gamma}$$

$$\frac{i}{k} = 1 - \frac{1-\delta}{\gamma}$$

$$\frac{c}{k} = \frac{y}{k} - \frac{i}{k}$$

$$\frac{c}{y} = \frac{c}{k}/\frac{y}{k}$$

and finally the steady state level ψ, for a given choice of H. Using

$$\begin{aligned}\psi &= \frac{Lv'(L)}{v(L)} = \frac{1-H}{H}\frac{wH}{k}\frac{k}{c} \\ &= \frac{1-H}{H}\frac{1-\alpha}{\alpha}\bar{R}\gamma^{\sigma-1}\frac{k}{c} \\ &= \frac{1-H}{H}\frac{1-\alpha}{\alpha}\tilde{R}\frac{k}{c}.\end{aligned}$$

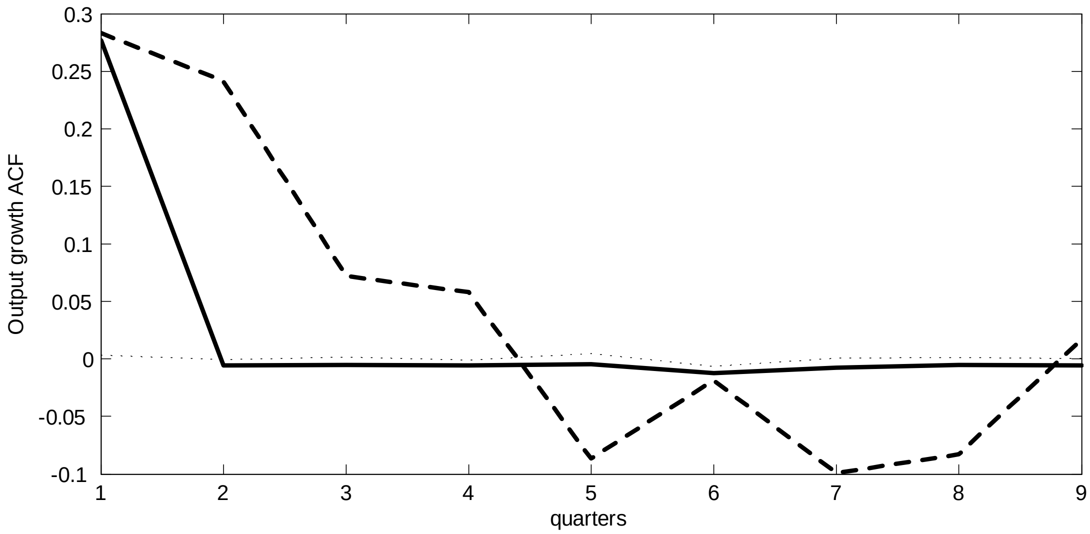

Figure 1: Output autocorrelation function. The thick dashed line denotes US data, the thick solid line denoted the model with learning, while the dotted line denotes the model under rational expectations.

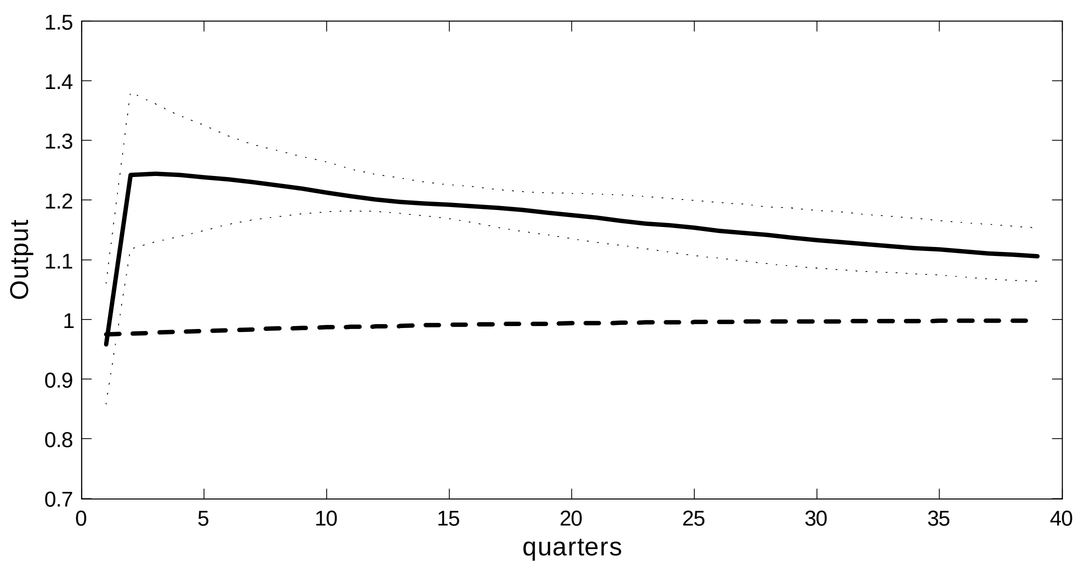

Figure 2: Dotted lines denote the 75% bands, solid line denotes the median impulse response under learning. The dashed line denotes the impulse response under rational expectations.

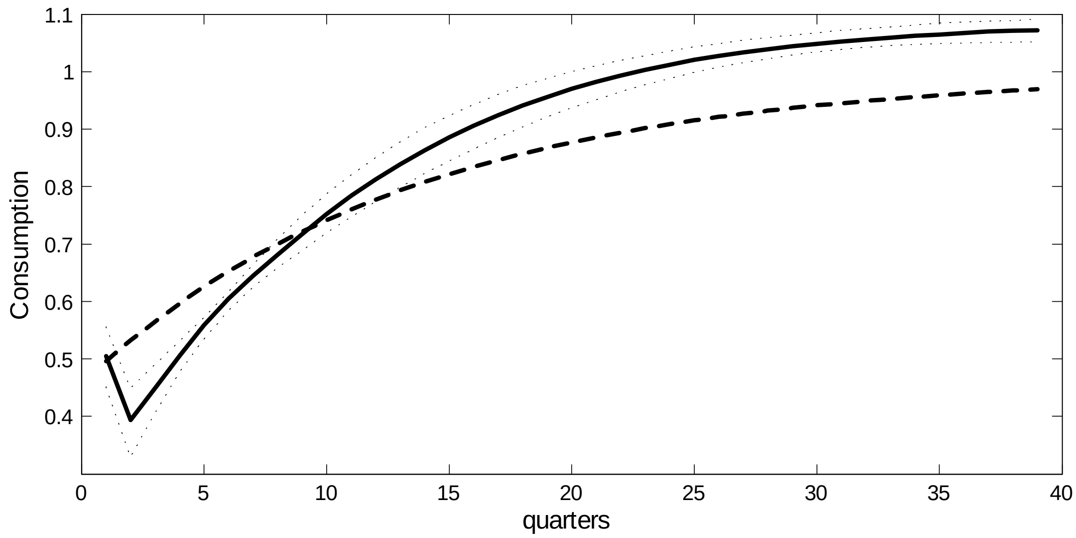

Figure 3: Dotted lines denote the 75% bands, solid line denotes the median impulse response under learning. The dashed line denotes the impulse response under rational expectations.

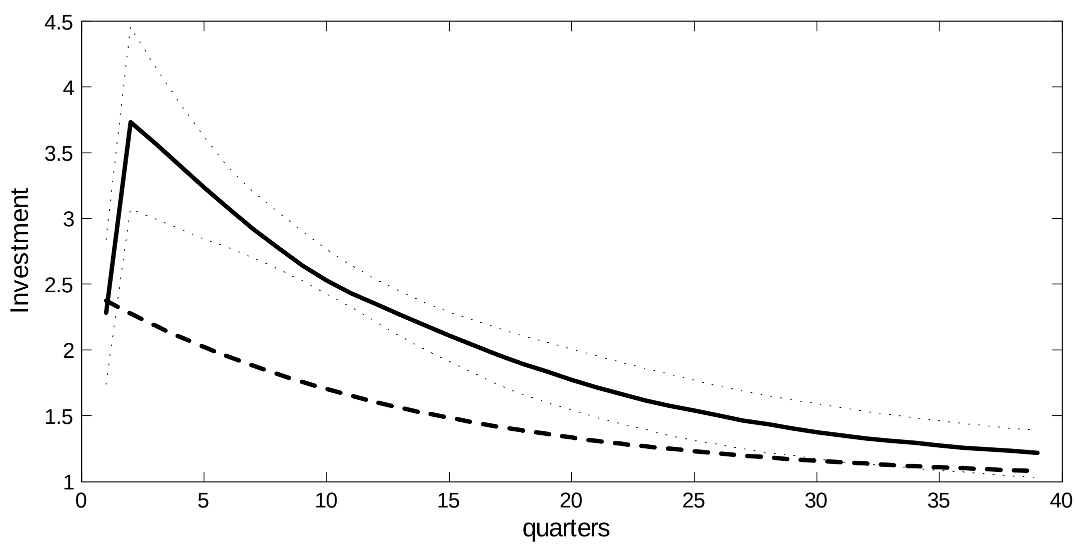

Figure 4: Dotted lines denote the 75% bands, solid line denotes the median impulse response under learning. The dashed line denotes the impulse response under rational expectations.

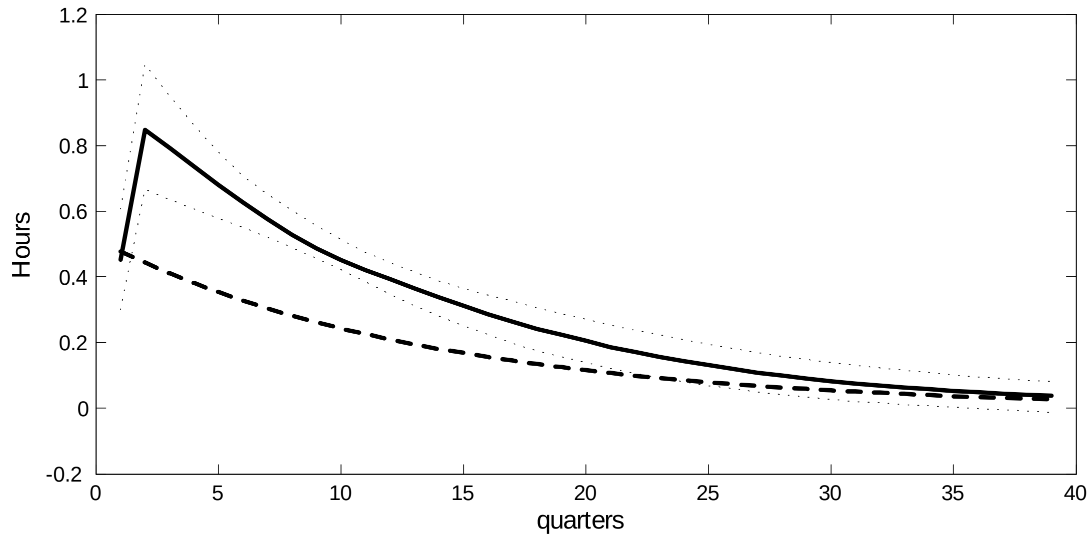

Figure 5: Dotted lines denote the 75% bands, solid line denotes the median impulse response under learning. The dashed line denotes the impulse response under rational expectations.

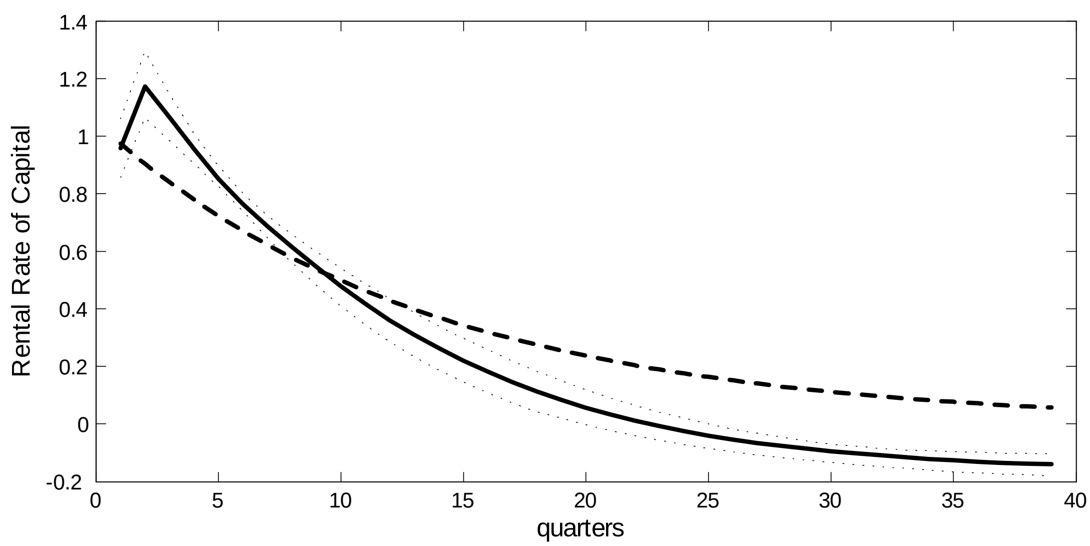

Figure 6: Dotted lines denote the 75% bands, solid line denotes the median impulse response under learning. The dashed line denotes the impulse response under rational expectations.

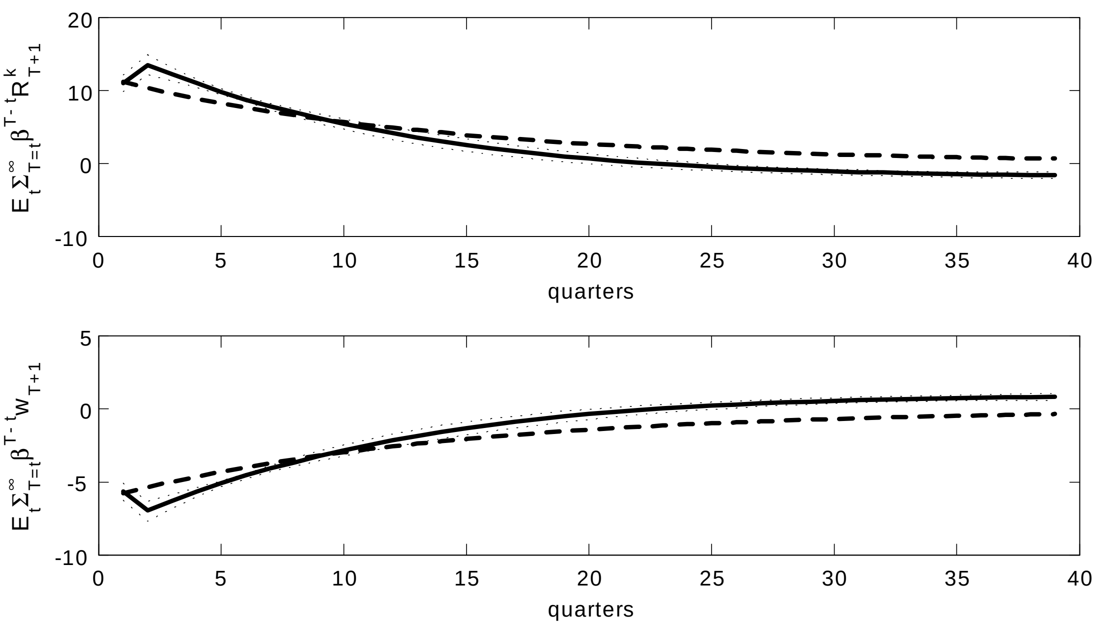

Figure 7: Dotted lines denote the 75% bands, solid line denotes the median impulse response under learning. The dashed line denotes the impulse response under rational expectations. The top panel is the present discounted value of returns to capital and the bottom panel the corresponding value for wages.

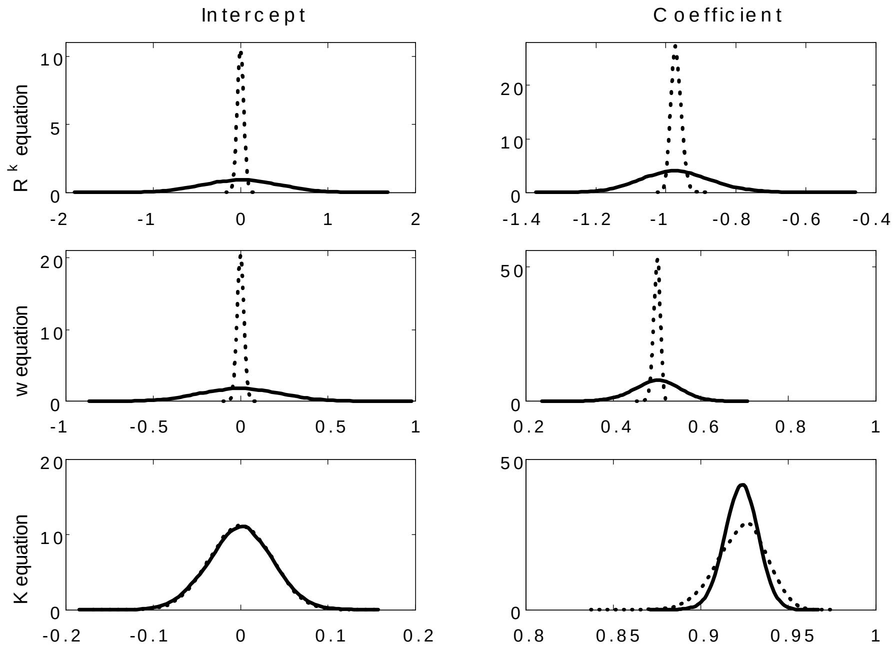

Figure 8: Solid line: model with feedback. Dotted line, model without feedback.

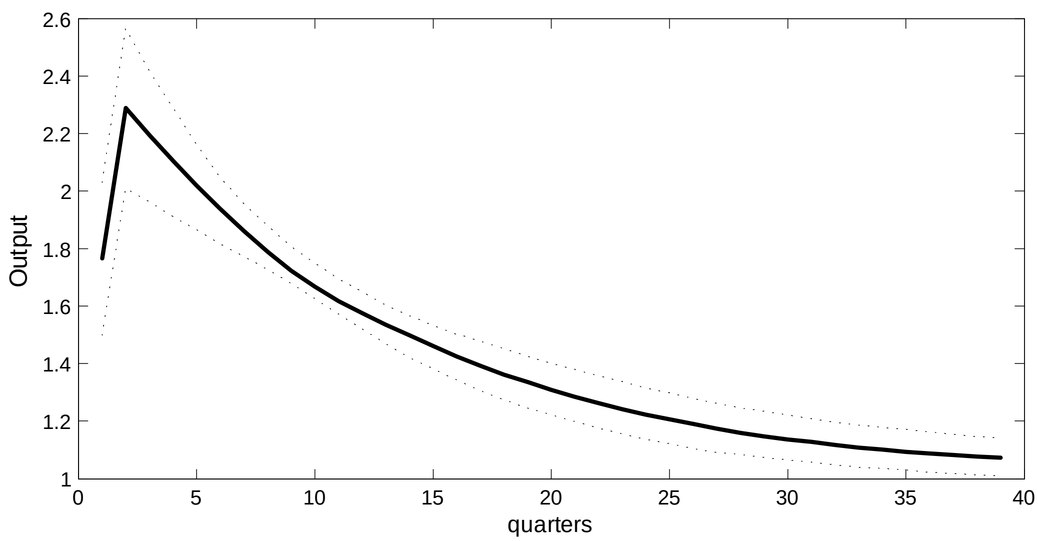

Figure 9: Output dynamics in response to a technology shock with increasing returns and non-separable preferences. The thick solid line denoted the model with learning, with the dotted lines showing the 12.5 and 87.5 percentiles.

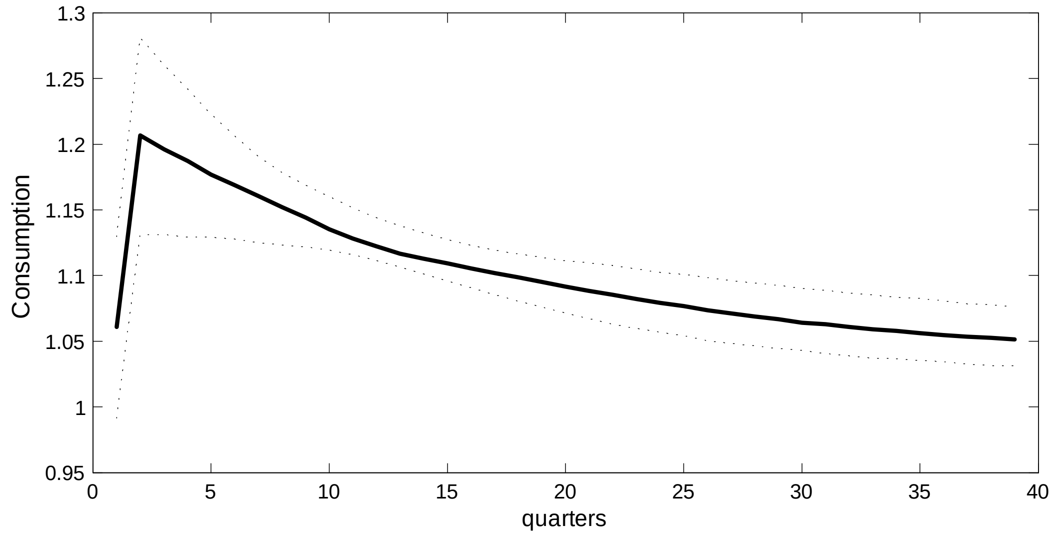

Figure 10: Consumption dynamics in response to a technology shock with increasing returns and non-separable preferences. The thick solid line denoted the model with learning, with the dotted lines showing the 12.5 and 87.5 percentiles.

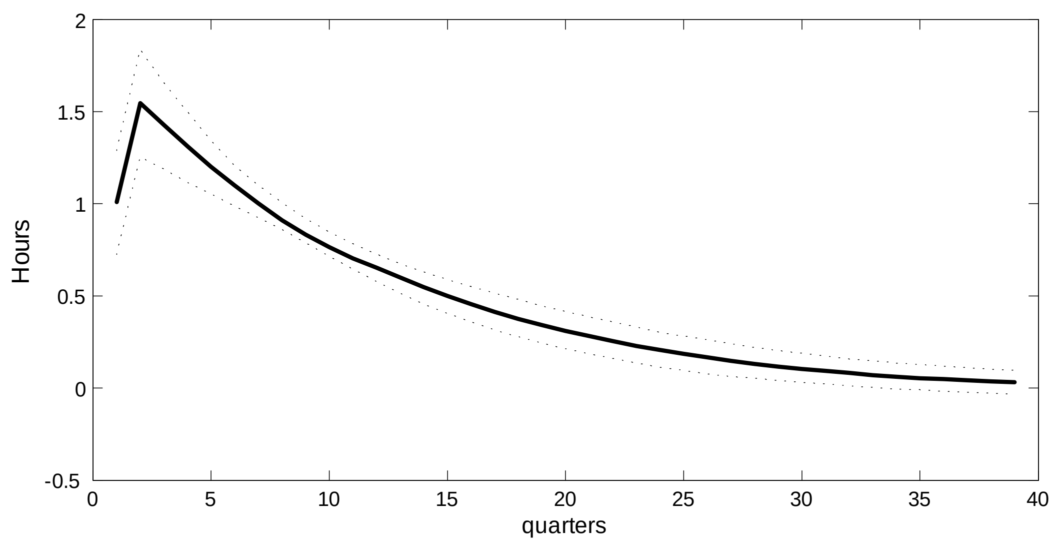

Figure 11: Hours dynamics in response to a technology shock with increasing returns and non-separable preferences. The thick solid line denoted the model with learning, with the dotted lines showing the 12.5 and 87.5 percentiles.

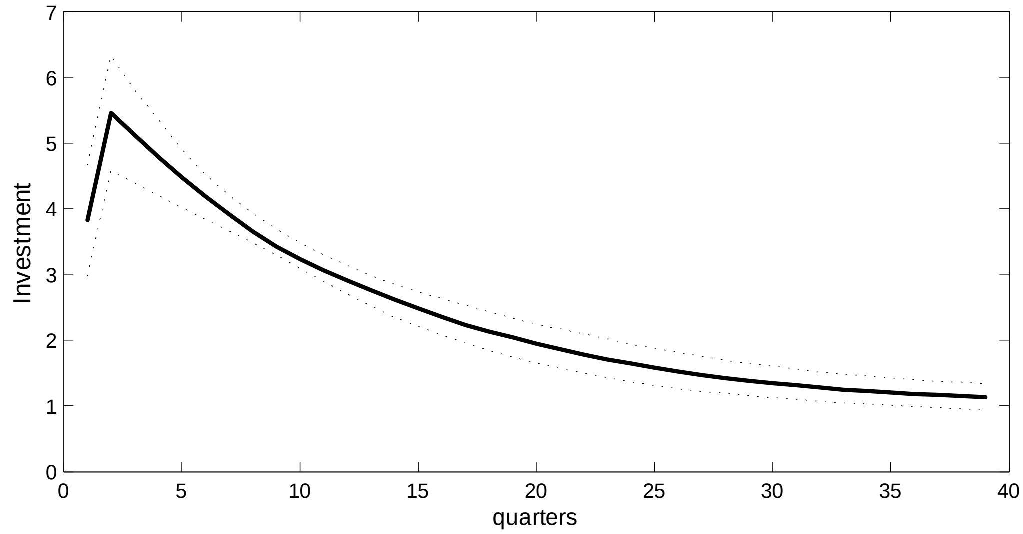

Figure 12: Investment dynamics in response to a technology shock with increasing returns and non-separable preferences. The thick solid line denoted the model with learning, with the dotted lines showing the 12.5 and 87.5 percentiles.

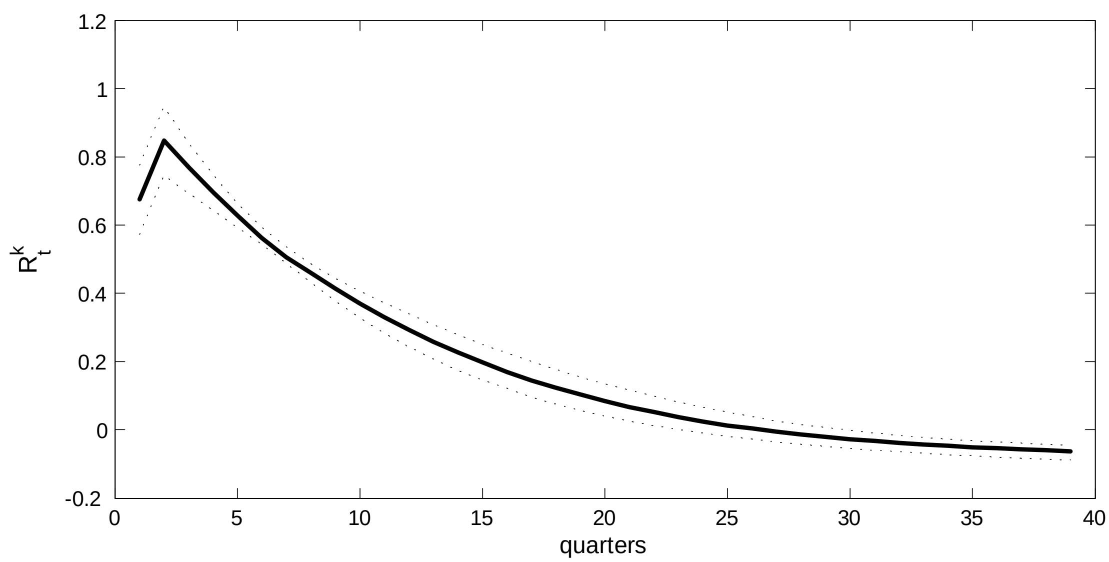

Figure 13: Rental rate dynamics in response to a technology shock with increasing returns and non-separable preferences. The thick solid line denoted the model with learning, with the dotted lines showing the 12.5 and 87.5 percentiles.