Financial Vulnerability and Monetary Policy *

Tobias Adrian   Fernando Duarte

This version: March 20, 2017

Abstract

We present a microfounded New Keynesian model that features financial vulnerabilities. Financial intermediaries' occasionally binding value at risk constraints give rise to vulnerabilities that generate time varying downside risk of the output gap. Monetary policy impacts the output gap directly via the IS curve, and indirectly via its impact on the tightness of the value at risk constraint. The optimal monetary policy rule always depends on financial vulnerabilities in addition to output, inflation, and the real rate. We show that a classic Taylor rule exacerbates downside risk of GDP growth relative to an optimal Taylor rule, thus generating welfare losses associated with negative skewness of GDP growth.

Keywords: monetary policy, macro-finance, financial stability
JEL classification: G10, G12, E52

*Adrian is with the International Monetary Fund. Duarte is with the Federal Reserve Bank of New York. The authors thank Ricardo Caballero, John Campbell, Richard Crump, Gauti Eggertson, Bill English, Jordi Gali, Marc Giannoni, James Hamilton, Nellie Liang, Alp Simsek, Lars Svensson, Andrea Tambalotti, Ivan Werning, and seminar and conference participants at the Massachusetts Institute of Technology, the Federal Reserve Bank of New York, National Bureau of Economic Research, the European Central Bank, Yale University, University of Tokyo, Kyoto University, and the Bank of Japan for helpful comments. The views expressed in this paper are those of the authors and do not necessarily represent those of the International Monetary Fund, the Federal Reserve Bank of New York, or the Federal Reserve System.

1 Introduction

FOMC statements mention financial stability and financial conditions with increasing frequency (see Peek, Rosengren, and Tootell (2015)). Additionally, the notion of downside risks to growth has become more prevalent in the speeches of monetary policy makers. In the academic literature, authors increasingly consider the roles of financial conditions and vulnerabilities in monetary policy settings (see Adrian and Shin (2010), Borio and Zhu (2012), Curdia and Woodford (2010), and Gambacorta and Signoretti (2014)).

An important early strand of the literature triggered by Bernanke and Gertler (1989) and Bernanke and Blinder (1992) argues for the credit channel of monetary policy. In the credit channel, financial frictions of borrowers or lenders shift credit demand and supply curves when monetary policy changes, thus giving financial conditions a role in monetary policy via the "external finance premium." However, Bernanke and Gertler (2000) argue that financial vulnerabilities should impact central banks' monetary policy actions only to the extent that vulnerabilities change the forecasts for inflation and real activity, thus promoting a form of the Taylor (1993) and Taylor (1999) rule within an inflation targeting framework. Financial vulnerabilities refer to downside risks to GDP growth caused by risks to asset valuations, the level of leverage in the financial and nonfinancial sectors, and the degree of maturity transformation (see Adrian, Covitz, and Liang (2015) for a framework on measuring financial vulnerability).

More recent literature has argued for a Taylor rule with financial variables. In the setups of Curdia and Woodford (2010) and Gambacorta and Signoretti (2014), the optimal monetary policy rule is augmented to take financial conditions into account. Limited capital in the financial intermediary sector leads to a distortion in aggregate activity that optimal monetary policy takes into account. However, just as the literature on the credit channel focused on financial conditions, and not financial vulnerabilities, the setups of Curdia and Woodford (2010) and Gambacorta and Signoretti (2014) do not feature financial vulnerabilities either. While financial conditions refer to the notion that the pricing of risk impacts aggregate activity, settings with financial vulnerability give rise to downside risks to growth from financial frictions that impact aggregate economic welfare.

In a series of influential of papers, Svensson (2016b,a) explicitly incorporates financial vulnerabilities in the form of financial crises. Svensson develops a cost benefit framework, argues that financial stability considerations should not enter monetary policy decisions if costs exceed benefits, and finds that, for existing empirical estimates, costs exceed benefits by a substantial margin. In Svensson's setup, the costs consist of distorting the dual mandate objectives, while the benefits are measured in terms of the reduction of the likelihood or severity of financial crises. In contrast, Adrian and Liang (2016) present a comprehensive literature review of the role of financial conditions and vulnerabilities in monetary policy and argue that there is compelling empirical evidence that financial vulnerabilities should affect monetary policy considerations.[1]

In this paper, we present a parsimonious macroeconomic framework for incorporating financial vulnerabilities in monetary policy. Our starting point is the standard New Keynesian model of Woodford (2003) and Galí (2008). Households have risk averse utility over differentiated products and supply labor to an intermediate goods producing sector. Intermediate goods have a constant returns technology with exogenous productivity and labor as only input. These intermediate goods producing firms maximize profits subject to a demand curve for differentiated products and Calvo style price stickiness. Their output is sold to the final goods producers in a monopolistically competitive way. The intermediate goods profits are distributed as dividends to shareholders. Without loss of generality, as the Modigliani-Miller theorem holds, these intermediate goods producing firms are fully equity financed. The final good sector is perfectly competitive and uses intermediate goods as only input.

The point of departure from the standard NK model is the existence of banks. Households cannot directly invest in the shares of the intermediate goods producing sector, as it is assumed that all financing is intermediated by the banks. There is a continuum of identical banks that issue riskless deposits that pay out the risk free rate of return. The deposits, as well as the risky equity of banks, are owned by households. Banks maximize profits by investing in all available risky assets in the economy. The banks' portfolio selection problem is subject to a value at risk (VaR) constraint on their net worth (i.e. on bank equity). Banks do not consume. Relative to earlier NK models with banks, the VaR is the main difference (see, for example, Gertler and Karadi (2011) and Curdia and Woodford (2010)).

The only source of risk in the economy are shocks to the time preference rate of banks. These shocks capture differences in beliefs between banks and the other agents in the economy. Because there is only one source of risk, and bank equity and bank deposits trade continuously, markets are complete. Using martingale techniques from financial economics, we can solve for the equilibrium in closed form. Importantly, we

[1] See Svensson (2017) for a response.

solve for the full stochastic equilibrium, which is characterized by conditional means and conditional volatilities as a function of state variables. For analytical tractability, we linearize first and second moments. This is a novel approach relative to standard first order approximations, as it preserves the equilibrium conditions that are imposed on second moments. The model and solution technique thus lends themselves to study the risk return tradeoff of monetary policy.

The linearized solution can be represented as a parsimonious four equation reduced form model. Relative to the standard NK model, the IS curve (which is derived from the Euler equation) features risk premia. Risk premia, in turn, depend on the vulnerability of aggregate economic activity. Vulnerability is defined as the VaR of the output gap. The evolution of vulnerability is the third equation. Finally, there is the stochastic process that determines risk.

Our modeling approach is motivated by the empirical evidence that financial conditions forecast tail risks. Estrella and Hardouvelis (1991) and Estrella and Mishkin (1998) show that the term spread, an indicator of the pricing of interest rate risk, forecasts recessions. Gilchrist and Zakrajšek (2012), López-Salido, Stein, and Zakrajšek (2016), and Krishnamurthy and Muir (2016) find that credit spreads forecast downside risks to GDP growth. More generally, Adrian, Boyarchenko, and Giannone (2016) document that financial conditions are strong forecasters of downside risks to GDP growth. Deteriorating financial conditions give rise to an increase in the conditional volatility of GDP and a decline in the conditional mean of GDP in such a way that upper quantiles of GDP growth are more or less constant, while lower quantiles are varying sharply. Hence the unconditional distribution of GDP is highly skewed to the left as a function of financial conditions.

The four equation reduced form NK model captures these dynamics of the conditional output distribution. The model gives rise to a relationship between the conditional mean and the conditional volatility of output that generates the empirical features of the conditional output distribution, namely, the negative correlations between mean and volatility that Adrian, Boyarchenko, and Giannone (2016) document. As a result, the model features strongly time varying downside risk as a function of financial conditions, while upside risks are more or less constant. Monetary policy impacts not only conditional means, but also conditional volatilities via its impact on the tightness of the VaR constraint of banks.

The central bank is assumed to minimize a standard loss function with the squared output gap and the squared inflation rate entering as arguments. We can solve for the optimal policy rules in closed form using dynamic programming. We consider optimal monetary policy rules with flexible prices (no Phillips curve) and sticky prices.

Our optimal monetary policy rule can be cast in the language of a flexible inflation targeting framework, such as the one in Svensson (1999), Rudebusch and Svensson (1999), Svensson (2002), and Giannoni and Woodford (2012). Relative to the standard New Keynesian model, there are two important differences. First, vulnerability becomes a target variable. Second, the coefficients in the linear optimal targeting criterion rule that trade off deviations in output, inflation and financial vulnerability from their desired levels, depend on the parameters that govern GDP vulnerability.

Optimal monetary policy can also be expressed as an augmented Taylor rule. The nominal interest rate not only depends on inflation and output, but also on financial vulnerability. The optimal coefficients on output and inflation are taking the parameters that govern GDP vulnerability into account.

The NKV model (New Keynesian Vulnerability model) is straightforward to calibrate from GDP, inflation, and financial conditions data. When we simulate the model under the optimal monetary policy rule, and the alternative standard New Keynesian Taylor rule, we find that the latter generates higher equilibrium GDP skewness. Therefore, policy makers that take vulnerability into account mitigate downside movements of output.

Our model captures the intuition that in recent years monetary policy has explicitly taken into account and influenced financial conditions.[2] A deterioration of financial conditions corresponds to an increase in tail risk, as conditional GDP volatility rises, while the conditional growth forecast deteriorates. As a result of such an increase in financial vulnerability, i.e. an increase in the downside risk to GDP growth, monetary policy is relatively easier than under the classic Taylor rule. This results in a lowering of vulnerability, and hence in less severe left skewness of GDP.

We also study an extension with a zero lower bound on nominal interest rates. The zero lower bound implies a flexible inflation targeting rule when interest rates are away from the bound, and a forward guidance rule when the zero lower bound is reached. Therefore, the New Keynesian model with financial vulnerability can be extended to settings with a zero lower bound.

In the setting of our paper, monetary policy *always* takes GDP vulnerability into account, even though the policy maker only cares about output and inflation. Incorporating

[2]Dudley (2015); Yellen (2016).

financial vulnerability in the flexible inflation targeting framework strictly dominates the standard Taylor rules that condition only on output and inflation. This result is in stark contrast to Svensson (2016b,a) who argues that financial stability considerations should not be taken into account if costs exceed benefits. Despite the contrasting conclusions, the framework of analysis that we are using here is similar to the framework that Svensson is using. However, while Svensson focuses on tail risks to GDP growth that only occur very rarely, we focus on vulnerabilities that are present most of the time. Importantly, tail risks can naturally occur within our setup, due to a particular form of nonlinearity involving first and second moments. Hence our setting also captures the extreme tail events that Svensson studies.

The remainder of the paper is organized as follows. Section 2 provides the motivation for our model from the existing empirical and theoretical literature on financial stability in a macroeconomic context. Section 3 presents the model. The solution ot the model is presented in Section 4. Section 5 derives the optimal monetary policy rule in the reduced form. Section 6 concludes.

2 Financial Vulnerability

Financial vulnerability refers to the presence of amplification mechanisms that are caused by leverage, maturity transformation, or asset valuations. When financial vulnerability is large, small shocks can have severe aggregate macroeconomic consequences. Adrian, Covitz, and Liang (2015) present a framework for the monitoring of financial vulnerability. They measure leverage, maturity transformation, and asset valuations across four sectors: asset markets, the banking system, the market based financial system, and the nonfinanical system. Aikman, Kiley, Lee, Palumbo, and Warusawitharana (2015) propose a quantitative indicator for the vulnerabilities in this framework.

In this paper, following Adrian, Boyarchenko, and Giannone (2016), we construct a measure of financial vulnerability by using the National Financial Conditions Index (NFCI) of the Federal Reserve Bank of Chicago. That index aggregates 105 financial market, money market, credit supply, and shadow bank indicators to compute a single index using the filtering methodology of Stock and Watson (1998). Adrian, Boyarchenko, and Giannone (2016) show that the conditional GDP distribution features strong downside risk as a function of financial conditions. We reproduce the main results of Adrian, Boyarchenko, and Giannone (2016) here using a conditionally heteroskedastic model to estimate the conditional first and second moments of GDP gap

$$y_t = \gamma_0^y + \gamma_1^y y_{t-1} + \gamma_2^y \pi_{t-1} + \gamma_3^y x_{t-1} + \sigma_t^y \varepsilon_t^y \tag{1}$$

$$\ln\left(\sigma_t^y\right) = \delta_0^y + \delta_1^y x_{t-1} \tag{2}$$

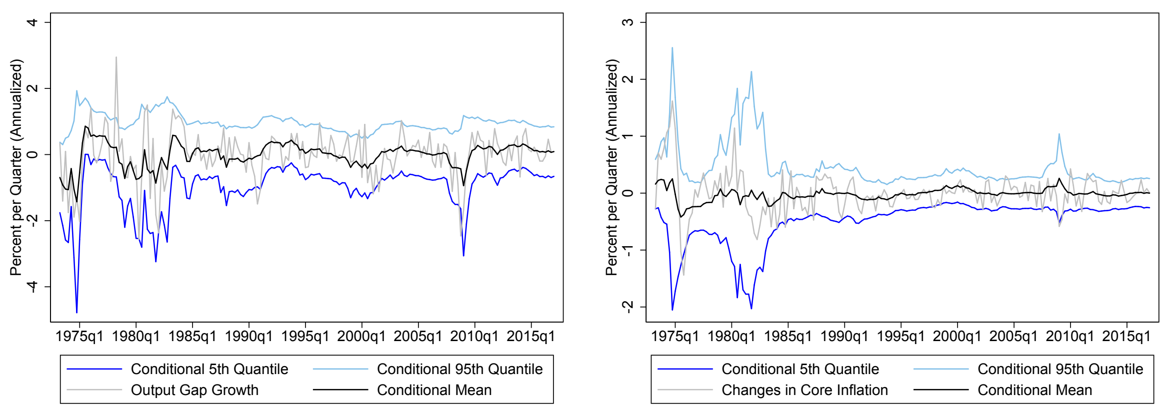
where $\varepsilon_t^y \sim N(0,1)$, x_t denotes the NFCI financial conditions index, and y_t is the GDP gap. Mean GDP gap also depends on the lagged quarterly core PCE inflation rate π and on the lagged GDP gap. In addition to estimating the conditional mean and conditional volatility of the GDP gap, we also estimate an analogous equation for the inflation rate:

$$\pi_t = \gamma_0^\pi + \gamma_1^\pi y_{t-1} + \gamma_2^\pi \pi_{t-1} + \sigma_t \varepsilon_t^\pi \tag{3}$$

$$\ln\left(\sigma_t^\pi\right) = \delta_0^\pi + \delta_1^\pi \pi_{t-1} \tag{4}$$

The model is estimated via maximum likelihood.

Figure 1: **Estimated Conditional Distribution of One Quarter Ahead GDP Gap and PCE Inflation**. The figure reports estimates from equations (1), (2), (3), and (4). Panel (a) shows the actual GDP gap, the conditional mean of GDP gap, and the 5th and 95th quantile. Panel (b) shows the actual PCE inflation, the conditional mean of inflation, and the 5th and 95th quantile.



The estimation results are in Figure 1 and Table 1. In Panel (a) of Figure 1, we present the conditional mean of GDP gap, actual GDP gap, and the 5th and 95th quantiles. The distribution is left skewed as deteriorating financial conditions are associated with an increase in conditional volatility, and at the same time a decline in the conditional mean of GDP gap (see Table 1). Due the negative correlation of mean and volatility the unconditional distribution is negatively skewed, even though the conditional distribution is conditionally Gaussian. For inflation, financial conditions aren't significant for either

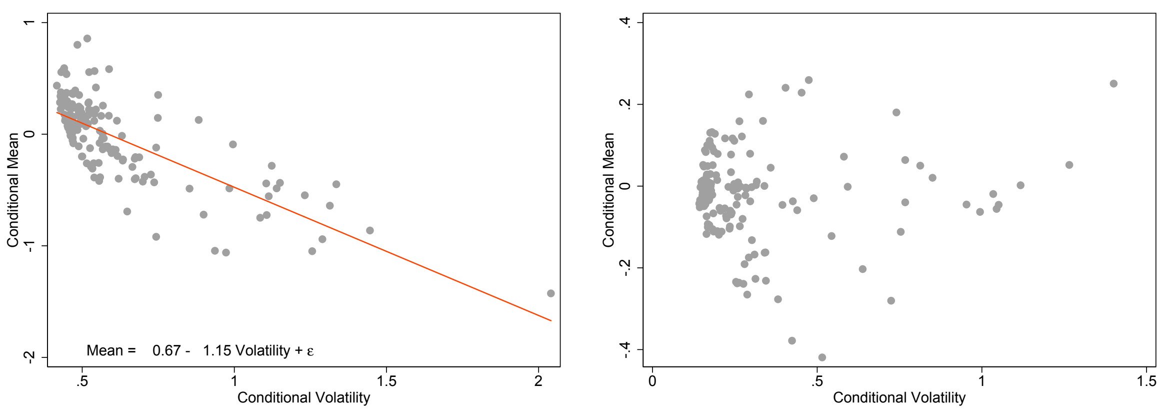
Table 1: **GDP Gap and Inflation Conditional Mean and Volatility Estimates**

VARIABLES	(1) GDP	(2) HET
Financial Conditions (lag)	-1.715***	0.551***
	[-5.096]	[3.765]
GDP Gap (lag)	-0.000356	
	[-1.510]	
Inflation Rate (lag)	0.00277	
	[0.0842]	
Constant	6.213***	1.785***
	[11.02]	[21.69]
Observations	173	173

*** p<0.01, ** p<0.05, * p<0.1

the conditional mean equation or the conditional volatility. However, the volatility of inflation scales in the level of inflation. Hence the conditional mean and the conditional volatility are positively correlated. Importantly, financial conditions play no role for inflation dynamics.

Figure 2: **Estimated Conditional Mean and Conditional Volatility of One Quarter Ahead GDP Gap and PCE Inflation**. The figure reports estimates from equations (1), (2), (3), and (4). Panel (a) plots the GDP gap mean against the GDP gap volatility, panel (b) plots PCE inflation mean against PCE inflation volatility.



The estimates for GDP have the unusual property that the shift in the conditional mean and volatility of GDP offset each other in such a way that the 95th quantile is close to constant. In contrast, the 5th quantile strongly varies as a function of financial conditions. Importantly, this property only arises when the GDP distribution is estimated as a function of financial conditions—real economic indicators do not contain significant information for the tail of the GDP distribution. This is shown more generally by Adrian, Boyarchenko, and Giannone (2016). This property is shown in Figure 2, which scatters the conditional mean against the conditional volatility for the GDP gap and PCE inflation. For the GDP gap, mean and volatility are strongly negatively correlated, but there is no correlatoin for PCE inflation.

These results suggest that GDP vulnerability is related to the time varying left tail of the GDP distribution, as a function of financial conditions. In this paper, we define GDP vulnerability V_t as the value at risk of the GDP gap

$$V_t = \mathcal{N}^{-1}(p)\mathbb{V}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\sqrt{\tau} - \mathbb{E}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\tau. \tag{5}$$

$\mathcal{F}_t$ denotes the filtration generated by the underlying stochastic processes, $\mathbb{V}$ denotes the volatility operator (the square root of the instantaneous variance of dy_t), and $\mathbb{E}$ denotes the expectations operator. The expectation is multiplied by the horizon of the value at risk, τ, while the volatility is multiplied by the square root of τ. $\mathcal{N}^{-1}(p)$ denotes the inverse cumulative Gaussian distribution function with probability p. As vulnerability measures the left tail of the GDP gap distribution, p is small, and therefore $\mathcal{N}^{-1}(p)$ is negative. For example, $\mathbb{N}^{-1}(5\%) = -1.96$. To save notation, we will denote $\alpha = -\mathcal{N}^{-1}(p)$.

We will derive a fully microfounded NK model in the next sections. To foreshadow where we will end up, we present the reduced form model here, which consists of the following four equations:

$$dy_t = \gamma^{-1}\left(R_t - r_t + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right)dt + \xi\left(V_t - s_t\right)dZ_t \tag{6}$$

$$V_t = -\alpha\mathbb{V}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\sqrt{\tau} - \mathbb{E}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\tau \tag{7}$$

$$ds_t = \kappa_s\left(\bar{s} - s_t\right)dt + \sigma_s dZ_t \tag{8}$$

$$d\pi_t = \left(\beta\pi_t - \kappa y_t\right)dt \tag{9}$$

Equation (6) is the Euler equation (or IS curve) of a standard NK model augmented with a risk premium. The risk premium has drift $\gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)$ and volatility $\xi\left(V_t - s_t\right)$. Note that both drift and volatility are proportional to $V_t - s_t$, where V_t is the VaR of GDP as defined in (7) and s_t is a state variable defined in (8). Importantly, vulnerability is *endogenous* to the stochastic evolution of GDP as a function of shocks to the risk premium. The shock dZ_t is a standard Brownian motion. We can interpret V_t as conditional volatility of GDP, and s_t as a mean reverting shock to volatility.

Using (6) in (5) and solving for V_t gives vulnerability as a function of the interest rate gives

$$V_t = \frac{-\gamma^{-1}\left(R_t - r_t\right) + \alpha\xi s_t\sqrt{\tau} + \hat{\eta}\xi\left(s_t + \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\tau}{1 + \alpha\xi\sqrt{\tau} + \hat{\eta}\xi\tau} \tag{10}$$

Vulnerability depends on the interest rate in excess of the natural rate $R_t - r_t$ and the process s_t. We can thus interpret s_t as a shock to vulnerability. Higher interest rates make vulnerability more negative.

The sign of the dependence of V_t on the interest rate R_t depends on the sign of $-\left(1 + \alpha\xi\sqrt{\tau} + \hat{\eta}\xi\tau\right)$. The empirical results presented above can help us pin down the sign of these parameters. The mean-variance tradeoff for y_t follows by writing $\mathbb{E}\left[dy_t|\mathcal{F}_t\right]$ and $\mathbb{V}\left[dy_t|\mathcal{F}_t\right]$ as functions of vulnerability V_t and the shock to vulnerability s_t

$$\mathbb{E}\left[dy_t|\mathcal{F}_t\right] = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right) \tag{11}$$

$$\mathbb{V}\left[dy_t|\mathcal{F}_t\right] = \xi\left(V_t - s_t\right) \tag{12}$$

and then eliminating V_t to get

$$\mathbb{E}\left[dy_t|\mathcal{F}_t\right] = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau\xi}\mathbb{V}\left[dy_t|\mathcal{F}_t\right] - \left(\frac{1}{\tau}\right)s_t. \tag{13}$$

Empirically, the slope is negative and the intercept is positive, hence we need

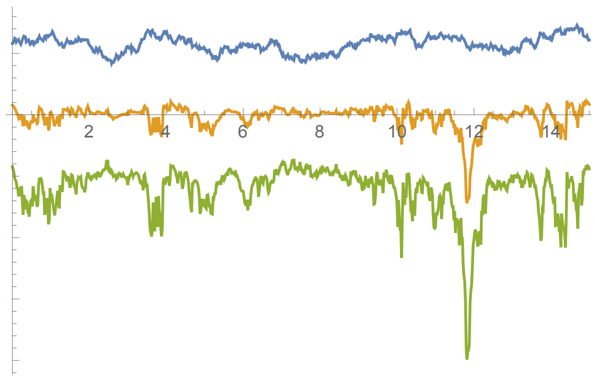
$$-\frac{\alpha\sqrt{\tau}\xi + 1}{\tau\xi} < 0 \tag{14}$$

$$\left(\frac{1}{\tau}\right)\bar{s} > 0 \tag{15}$$

To calibrate the reduced form model, we set $\alpha = -1.645$, which corresponds to a VaR value of 5%. We choose a VaR horizon of one year, $\sqrt{\tau} = 1$. To match the data, we set the slope $-\frac{\alpha\sqrt{\tau}\xi+1}{\tau\xi} = -1.15$ and the intercept $\bar{s} = -0.67\tau$ which gives $\xi = 0.36$ and $\bar{s} = -0.67$. These calibrations imply that GDP vulnerability V_t and interest rates i_t

are negatively correlated. This correlation is consistent with the empirical observation that when financial conditions deteriorate, GDP vulnerability increases, and short-term interest rates decline. Figure 3 shows a simulated path of (6), (7), (8) setting $R - r$ to zero, for simplicity. The simulation clearly features the stylized facts of Figure 1.

Figure 3: **Simulated Conditional Distribution of One Quarter Ahead GDP Growth**. The figure shows simulated conditional mean of GDP, and the 5th and 95th quantile of model (6), (7), (8).



The IS curve augmented with the shocks to risk premia that depend on vulnerability lead to an additional channel for monetary policy. The traditional transmission channel is via the drift of the IS curve: higher interest rates are associated with a higher growth rate of output. This is because a higher interest rate shifts consumption from the present to the future, via increased savings. The additional channel that arises in the current setup is the impact of monetary policy on vulnerability, and hence on the volatility of the risk premium. Hence monetary policy impacts total risk in the economy. This channel is sometimes called the "risk taking channel of monetary policy" (see Adrian and Shin (2010) and Borio and Zhu (2012)). When we study optimal monetary policy in the next section, this tradeoff is going to emerge prominently.

3 The Model

3.1 Physical Environment

Time is continuous. There is a continuum of identical, infinitely lived households who rank consumption streams C_t and labor streams N_t according to

$$E_0 \int_0^\infty e^{-\beta t} \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\xi}}{1+\xi} \right) dt, \tag{16}$$

where $\beta > 0$ is a time-preference parameter, $\gamma > 0$ is the coefficient of relative risk aversion and $\xi > 0$ is the inverse of the Frisch elasticity of labor supply. The variable C_t represents a consumption index given by

$$C_t \equiv \left(\int_0^1 C_t\left(i\right)^{1-\frac{1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{17}$$

where $C_t\left(i\right)$ is the quantity of differentiated good $i \in [0,1]$ consumed by the household at time t and $\varepsilon > 1$ is the constant elasticity of substitution across different goods. The variable N_t is the aggregate labor supplied to all firms and given by

$$N_t \equiv \int_0^1 N_t\left(i\right) di, \tag{18}$$

where $N_t\left(i\right)$ is the amount of labor supplied at time t to the firm that produces goods of type i. Output $Y_t\left(i\right)$ for each good i can be produced by the following constant returns to scale technology

$$Y_t(i) = AN_t\left(i\right), \tag{19}$$

where A is the constant economy-wide level of technology. The final good Y_t is produced with the technology

$$Y_t = \left(\int_0^1 Y_t\left(i\right)^{1-\frac{1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}. \tag{20}$$

There is no government spending and the economy is closed to imports and exports. The resource constraint of the economy is

$$C_t = Y_t. \tag{21}$$

3.2 First Best

The first best is obtained by maximizing the utility of the representative household, given in equation (16), subject only to the structure of the economy's physical environment described by equations (17)-(21). This central planner problem is

$$
\begin{aligned}
V_s^{FB} &= \max_{\{C_t(i),N_t(i),Y_t(i)\}_{t\geq s}} E_s\left\{\int_s^\infty e^{-\beta(t-s)}\left(\frac{C_t^{1-\gamma}}{1-\gamma}-\frac{N_t^{1+\xi}}{1+\xi}\right)dt\right\}, \\
& s.t. \\
C_t &= \left(\int_0^1 C_t\left(i\right)^{1-\frac{1}{\varepsilon}}di\right)^{\frac{\varepsilon}{\varepsilon-1}}, \\
Y_t &= \left(\int_0^1 Y_t\left(i\right)^{1-\frac{1}{\varepsilon}}di\right)^{\frac{\varepsilon}{\varepsilon-1}}, \\
N_t &= \int_0^1 N_t\left(i\right)di, \\
Y_t(i) &= AN_t\left(i\right), \\
C_t &= Y_t.
\end{aligned}
$$

The solution to this problem is

$$
\begin{aligned}
C_t\left(i\right) &= C_t\left(j\right), && (22) \\
Y_t\left(i\right) &= Y_t\left(j\right), && (23) \\
N_t\left(i\right) &= N_t\left(j\right), && (24)
\end{aligned}
$$

for all i and j and

$$
\begin{aligned}
N_t &= A^{\frac{1-\gamma}{\gamma+\xi}}, && (25) \\
C_t &= Y_t = A^{\frac{1-\gamma}{\gamma+\xi}+1}. && (26)
\end{aligned}
$$

3.3 Market Structure

Now we describe the structure of the market economy that we use to find the decentralized equilibrium. The only source of uncertainty is a single standard Brownian motion. There are two types of firms, intermediate good producers and final good producers. There is a continuum of identical households. They are as in the standard New Keynesian model with two differences. First, they cannot invest in shares of the intermediate good producers (but can still invest in all other available financial assets). The reason is that all financing of the intermediate good producers must be intermediated by banks.

Second, they are allowed to deposit money in riskless accounts at banks[3] and hold shares of the banks.

There is a continuum of identical banks. They issue riskless deposits to households that pay out the risk-free rate of return. In addition, the banks issue shares that can be purchased and held by the households or by the banks themselves. Banks are also allowed to trade a riskless bond among themselves and with the households. Banks have a risk averse objective function over total distributions (dividends, plus payouts to depositors, plus net payoffs from positions in the riskless bond) that has preference shocks. The Modigliani-Miller theorem holds for banks[4], so how total distributions are split between its components is irrelevant. We henceforth use the word "dividends" to refer to total bank distributions. Banks maximize their objective by choosing a portfolio of investments in all financial assets available in the economy. The banks' portfolio selection problem is subject to a Value-at-Risk (VaR) constraint on their wealth (net worth). In summary, the liabilities of the banks are deposits and equity while their assets are positions in the bond and stocks.

There is a central bank that sets the nominal interest rate by paying interest on base money in the cashless limit, as in Woodford (2003). There is no fiscal spending, so no need for the government to issue bonds or levy taxes (so we assume it does neither). Fiscal policy is therefore "Ricardian" (in the terminology of Woodford (2001)) or "passive" (in the terminology of Leeper (1991)).

Unlike the standard New Keynesian model, in this setup it becomes important to spell out the financial assets available in the economy. There are two types of stocks in positive net supply: the stocks of banks and the stocks of the intermediate good producers. Because banks receive identical aggregate shocks, we can group them into a single banking sector stock; for the same reason, we can group all intermediate good producer stock into a single intermediate good producer sector stock. We can therefore assume that there are exactly two stocks in positive net supply. The banking sector stock pays the aggregate dividends of all banks and the producer sector stock pays the aggregate dividends of all producers[5].

[3]It does not matter whether they are riskless in real or nominal terms, so we henceforth assume they are real accounts.

[4]That the Modigliani-Miller theorem holds for banks follows at once by using the results in (Merton, 1977).

[5]We could allow for unrestricted trade among households and banks of a complete set of Arrow-Debreu securities in zero net supply that span all risks in the economy and none of our results would change. In our setup, markets will be complete in equilibrium even without introducing these Arrow-Debreu securities, so adding any new securities would be redundant.

We index the financial securities by $j \in \{0, goods, banks\}$ and write their price, $S_{j,t}$, and dividend processes, $D_{j,t}$, in real terms. The first security is a riskless bond with price $S_{0,t}$ that follows

$$dS_{0,t} = S_{0,t} R_t dt, \tag{27}$$

where R_t is the equilibrium real riskless interest rate. The remaining two securities are risky stocks that have equilibrium prices given by

$$\frac{dS_{j,t}}{S_{j,t}} = \alpha_{j,t} dt + \sigma_{j,t} dZ_t, \tag{28}$$

where $\alpha_{j,t}$ is the real expected return (including any dividends[6]) and $\sigma_{j,t}$ is the exposure to the standard Brownian motion Z_t. We can write the stock price processes in vector notation

$$dS_t = diag\left(S_t\right)\left(\alpha_t dt + \sigma_t dZ_t\right), \tag{29}$$

where S_t is a 2×1 vector (that does not include the bond), α_t is a 2×1 vector and σ_t is a 2×1 vector. Because σ_t is endogenous, we do not yet know whether markets are complete in equilibrium (but we show below they will be). If either stock has $\sigma_{j,t} \neq 0$ a.s., markets are complete. Because there is a single source of uncertainty and two stocks and a bond, one of the stocks will be redundant in equilibrium, in that it can be fully replicated by a portfolio of bonds and the other stock. However, although one of stocks can be replicated, it is only the banks that can do so, since households can only trade in the stock of good producers while the banks can trade in both stocks. It is therefore important for our results to have both stocks in the economy.

We define real expected excess returns μ_t as the 2×1 vector

$$\mu_t \equiv \alpha_t - R_t, \tag{30}$$

and the market price of risk (MPR) as the process η_t that satisfies

$$\sigma_t^T \eta_t \equiv \mu_t. \tag{31}$$

where the superscript T denotes the transpose of a vector or a matrix. Equation (31) has a solution iff there is no arbitrage, which is a necessary condition for equilibrium. If

[6]With some abuse of language, we refer to S_t as the "price" of the stock instead of using the sometimes more precise "gain process" terminology.

markets are complete, η_t is unique and given by

$$\eta_t = \left(\sigma_t^T\right)^{-1} \mu_t. \tag{32}$$

The MPR is closely related to the equivalent martingale measure $\mathbb{Q}$. Indeed, $Z_t^{\mathbb{Q}}$ defined by

$$Z_t^{\mathbb{Q}} \equiv Z_t + \eta_t, \tag{33}$$

is a standard Brownian motion under the equivalent martingale measure (the "risk-neutral measure"). Under $\mathbb{Q}$, discounted stock prices are martigales, i.e.

$$E_t^{\mathbb{Q}}\left[d\left(\frac{S_{j,t}}{S_{0,t}}\right)\right] = 0. \tag{34}$$

We also define the real state price density (SPD) Q_t as the solution to

$$dQ_t \equiv -Q_t R_t dt - Q_t \eta_t^T dZ_t, \tag{35}$$

$$Q_0 \equiv 1. \tag{36}$$

and the nominal SPD, $Q_t^{\$}$, by

$$Q_t^{\$} \equiv Q_t P_t.$$

Under the physical measure, stock prices are given by the SPD-deflated stream of dividends

$$Q_t S_{j,t} = E_t\left[\int_t^\infty Q_s D_{j,s} ds\right].$$

In equilibrium, we will find that markets are complete and that $\sigma_{j,t} \neq 0$ for $j = \{goods, banks\}$. No arbitrage then requires

$$\left(\sigma_{goods,t}\right)^{-1} \mu_{goods,t} = \left(\sigma_{banks,t}\right)^{-1} \mu_{banks,t}. \tag{37}$$

3.4 Banks

Bank liabilities consist of wealth (equity capital or net worth) and nominal deposits issued to households. Denote real wealth of the bank by X_t. Bank assets consist of a portfolio of the two traded stocks and bonds. Because banks can replicate one of the stocks by trading on the other stock and the bond, the portfolio choice of the bank only determines the allocation of wealth between the bond and a portfolio of the two stocks.

We therefore solve the portfolio problem of banks assuming for simplicity that there is a single risky asset (a portfolio of the two stocks) instead of two risky assets. By equation (37), this portfolio of stocks can be taken to have any portfolio weights in each of the two stocks without affecting the optimal portfolio choice of banks. Therefore, μ_t and σ_t in the portfolio choice problem of the bank described below should be interpreted as the drfit and volatility of the protfolio of stocks and not as a vector of drifts and volatilities that contain the drifts and volatilities of each stock. The actual equilibrium weights for the portfolio of two stocks that banks invest in will be determined by the market clearing condition that banks must hold the entire supply of good producers' stock.

The bank solves a standard Merton portfolio problem augmented by a Value-at-Risk constraint and preference shocks

$$
\begin{aligned}
& \max_{\{\theta_t,\delta_t\}_{t\geq s}} E_s\left[\int_s^\infty e^{-\beta(t-s)} e^{\zeta_t} \log\left(\delta_t X_t\right) dt\right] && (38)\\
& s.t. \\
\frac{dX_t}{X_t} &= \left(R_t - \delta_t + \theta_t \mu_t\right) dt + \theta_t \sigma_t dZ_t, && (39)\\
VaR_{\tau,\alpha}\left(t,\theta_t,\delta_t,X_t\right) &\leq a_V X_t, && (40)\\
d\zeta_t &= -\frac{1}{2} s_t^2 dt - m_t dZ_t,\ \zeta_0 = 0, && (41)\\
dm_t &= -\kappa m_t + \sigma_m dZ_t, && (42)\\
& X_s \text{ given}, && (43)
\end{aligned}
$$

where ζ_t is a preference (risk aversion) or belief shock, δ_t is the share of wealth distributed to the household, $\delta_t X_t$ are real dividends distributed to the household, θ_t is the share of wealth invested in the portfolio of risky assets, $VaR_{\tau,\alpha}\left(t,\theta_t,\delta_t,X_t\right)$ is the value-at-risk of the portfolio of the bank over the interval $[t, t+\tau]$ at level $\alpha \in (0, 1/2]$ with $\tau > 0$ and $a_V \in (0, 1)$.

The dynamics of ζ_t in equations (41)-(42) imply that e^{ζ_t} is a Radon–Nikodym derivative (a change of measure). Changing to the measure defined by e^{ζ_t}, the bank problem can be restated as

$$\max_{\{\theta_t,\delta_t\}_{t\geq s}} E_s^{bank}\left[\int_s^\infty e^{-\beta(t-s)}\log\left(\delta_t X_t\right)dt\right] \tag{44}$$

$$s.t.$$

$$\frac{dX_t}{X_t} = \left(R_t - \delta_t + \theta_t\left(\mu_t - \sigma_t m_t\right)\right)dt + \theta_t\sigma_t dZ_t^m, \tag{45}$$

$$VaR_{\tau,\alpha}\left(t,\theta_t,\delta_t,X_t\right) \leq a_V X_t, \tag{46}$$

$$X_s \text{ given}, \tag{47}$$

where, under the new measure, Z_t^m is a standard Brownian motion and $E_s^{bank}[\cdot]$ is the expectation operator, with

$$Z_t^m = Z_t + m_t.$$

As is the case in practice, the portfolio manager evaluates its VaR by assuming that the portfolio weights remain constant between t and $t+\tau$. Let

$$Q\left(t,\theta_t,\delta_t\right) \equiv R_t - \delta_t + \theta_t\left(\mu_t - \sigma_t m_t\right) - \frac{1}{2}\left(\theta_t\sigma_t\right)^2$$

be the drift of $d\log X_t$ under the bank measure. Then, the dynamic budget constraint of the bank has a strong solution[7]

$$\begin{aligned} X_t &= X_0\exp\left\{\int_0^t Q\left(t,\theta_s,\delta_s\right)ds + \int_0^t \theta_s\sigma_s dZ_s^m\right\}, \\ & X_0 \text{ given.} \end{aligned}$$

Projected wealth loss between t and $t+\tau$ when keeping the portfolio constant at (θ_t,δ_t) for $t\in[t,t+\tau]$ is

$$X_t - X_{t+\tau} = X_t\left[1-\exp\left\{Q\left(t,\theta_t,\delta_t\right)\tau + \theta_t\sigma_t\left(Z_{t+\tau}^m - Z_t^m\right)\right\}\right]$$

Thus, the α^{th} percentile of the projected wealth loss, $X_t - X_{t+\tau}$, conditional on time-t information is

$$X_t\left[1-\exp\left\{Q\left(t,\theta_t,\delta_t\right)\tau + \mathcal{N}^{-1}(\alpha)\left|\theta_t\sigma_t\right|\sqrt{\tau}\right\}\right]$$

Value-at-risk is then defined by

$$VaR_{\tau,\alpha}\left(t,\theta_t,\delta_t,X_t\right) \equiv X_t\left[1-\exp\left\{Q\left(t,\theta_t,\delta_t\right)\tau + \mathcal{N}^{-1}(\alpha)\left|\theta_t\sigma_t\right|\sqrt{\tau}\right\}\right]$$

[7]In this context, a solution is "strong" if it holds path by path.

Define

$$g_V(t, \theta_t, \delta_t) \equiv -Q(t, \theta_t, \delta_t)\,\tau - \mathcal{N}^{-1}(\alpha)\,|\theta_t \sigma_t|\,\sqrt{\tau}$$

Then

$$VaR_{\tau,\alpha}(t, \theta_t, \delta_t, X_t) \leq X_t a_V \iff g_V(t, \theta_t, \delta_t) \leq \overline{VaR}$$

where

$$\overline{VaR} \equiv \log \frac{1}{1 - a_V}$$

The choice $\alpha \in (0, 1/2]$ guarantees that $\mathcal{N}^{-1}(\alpha) \leq 0$ and that $g_V(t, \theta_t, \delta_t)$ is convex in (θ_t, δ_t).

In Appendix A we show that the banks' problem can be simplified to a non-stochastic one. To maximize

$$E_0^{bank} \int_0^\infty e^{-\beta t} \log(\delta_t X_t)\, dt$$

over the constrained set, it suffices to maximize

$$h(t, \theta_t, \delta_t) \equiv \log(\delta) + \frac{1}{\beta} Q(t, \theta_t, \delta_t)$$

pathwise over the constrained set. For a fixed path and a fixed time t, the bank then solves

$$\begin{aligned} & \max_{\theta_t, f_t} h(t, \theta_t, \delta_t) \\ & s.t. \\ g_V(t, \theta_t, \delta_t) \quad \leq \quad & \overline{VaR} \end{aligned} \tag{48}$$

The function $h(t, \theta_t, \delta_t)$ is concave in (θ_t, δ_t) and maximized over (θ_t, δ_t) by

$$\begin{aligned} \delta_{M,t} &= \beta \\ \theta_t &= \theta_{M,t} \end{aligned}$$

when the VaR constraint is not binding, where we derive $\theta_{M,t}, \delta_{M,t}$ using the FOC

$$\begin{aligned}
[f_t] \quad &: \quad 0 = \frac{\partial}{\partial \delta_t} h\left(t, \theta_t, \delta_t\right) \\
&: \quad 0 = \frac{1}{\delta_{M,t}} - \frac{1}{\beta} \\
&: \quad \delta_{M,t} \equiv \beta \\
[\theta_t] \quad &: \quad 0 = \frac{\partial}{\partial \theta_t} h\left(t, \theta_t, \delta_t\right) \\
&: \quad 0 = \frac{1}{\beta}\left(\mu_t - \sigma_t m_t - \sigma_t^2 \theta_{M,t}\right) \\
&: \quad \theta_{M,t} \equiv \frac{1}{\sigma_t}\left(\frac{\mu_t}{\sigma_t} - m_t\right)
\end{aligned}$$

The solution $(\theta_{M,t}, \delta_{M,t})$ when the VaR constraint is not binding coincides with the standard Merton portfolio solution for an agent that does not face a VaR constraint. Using the definition of the market price of risk η_t in equation (31), we can also write

$$\theta_{M,t} \equiv \left(\sigma_t\right)^{-1}\left(\eta_t - m_t\right)$$

As just derived, if $(\theta_{M,t}, \delta_{M,t})$ satisfy

$$g_V\left(t, \theta_{M,t}, \delta_{M,t}\right) \leq \overline{VaR}$$

then $(\theta_t^*, \delta_t^*) = (\theta_{M,t}, \delta_{M,t})$ is the solution to the bank's problem with the VaR constraint (and the VaR constraint does not bind). Otherwise, because the constraint set is compact and convex, and the objective is continuous, there will be a unique solution (θ_t^*, δ_t^*). Moreover, (θ_t^*, δ_t^*) must be such that the VaR holds with equality.

In Appendix C we derive the optimal portfolio of the bank, which is given by

$$\theta_t = \min\left\{1, \max\left\{0, \varphi_t\right\}\right\} \theta_{M,t} \tag{49}$$

$$\begin{aligned}
\delta_t = \; & u\left(t, \min\left\{1, \varphi_t\right\}\right) f_{M,t} 1_{\{\varphi_t > 0\}} \\
& + \left(R_t + \frac{1}{\tau}\log\frac{1}{1 - a_V}\right) 1_{\{\varphi_t \leq 0\}}
\end{aligned} \tag{50}$$

$$\varphi_t \text{ such that:} \quad g_V\left(t, \varphi_t \theta_{M,t}, u\left(t, \varphi_t\right) f_{M,t}\right) = \overline{VaR} \tag{51}$$

where we omit the asterisks for ease of notation and

$$u(t,z) \equiv 1 + \frac{\sqrt{\tau}\,|\theta_{M,t}\sigma_t|}{\mathcal{N}^{-1}(\alpha)}(1-z)$$

We see that under the bank measure, the VaR constraint does not distort the composition of the portfolio, as θ_t is a multiple of mean-variance efficient portfolio. It does however change the amount invested in the mean-variance efficient portfolio. Instead of $\theta_t = \theta_{M,t}$ as would obtain without the VaR constraint, we now get $\theta_t = \theta_{M,t}/\gamma_t$ where

$$\begin{aligned}
\gamma_t &= \frac{1}{\min\{1, \max\{0, \varphi_t\}\}} \in [1,\infty) \\
&= \begin{cases} \infty & , \text{ if } \varphi_t \in (-\infty, 0] \\ \frac{1}{\varphi_t} & , \text{ if } \varphi_t \in (0,1] \\ 1 & , \text{ if } \varphi_t \in (1,\infty) \end{cases}
\end{aligned}$$

Thus, the VaR constraint makes the agent behave as an agent with time-varying risk aversion γ_t that is higher than its true risk aversion of 1. Under the physical measure, the VaR constraint distorts the conditional composition of the portfolio but not its unconditional composition. In other words, the bank invests in the mean-variance efficient portfolio on average, but not necessarily at any given point in time.

We can find an explicit expression for φ_t, as shown in Appendix D

$$\varphi_t = 1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau}\,|\eta_t - m_t|} \pm \sqrt{2(R_t - \delta_t)\tau + 2\overline{VaR} + |\eta_t - m_t|^2 \tau^2 \left(1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau}\,|\eta_t - m_t|}\right)^2}$$

Evaluating the left-hand side of equation (48) at the optimal policies of the bank in equations (195)-(197) gives

$$\begin{aligned}
g_V(t,\theta_t,\delta_t,\nu) = &-\left(R_t - \delta_t + \theta_t\mu_t - \theta_t\sigma_t m_t - \frac{1}{2}(\theta_t\sigma_t)^2\right)\tau && (52) \\
&-\mathcal{N}^{-1}(\alpha)\,|\theta_t\sigma_t|\sqrt{\tau} && (53) \\
= &-(R_t - \delta_t)\tau && (54) \\
&-\left(\min\{1,\max\{0,\varphi_t\}\} - \frac{1}{2}\min\{1,\max\{0,\varphi_t\}\}^2\right)(\eta_t - m_t)^2\tau && (55) \\
&-\mathcal{N}^{-1}(\alpha)\min\{1,\max\{0,\varphi_t\}\}\,|\eta_t - m_t|\sqrt{\tau} && (56)
\end{aligned}$$

Of course, (53) evaluated at any $\varphi_t \in [0,1]$ gives $g_V(t,\theta_t,\delta_t) = \overline{VaR}$.

Finally, we note that the Lagrange multiplier λ for the VaR constraint is

$$\lambda = \frac{1}{\tau}\left(\frac{1}{\delta_t} - \frac{1}{\beta}\right)$$

so the Lagrange multiplier for the original problem under the bank's probability measure is

$$\begin{aligned}
\lambda_{VaR,t,m} &= \lambda e^{-\beta t} \\
&= \frac{1}{\tau}\left(\frac{1}{\delta_t} - \frac{1}{\beta}\right) e^{-\beta t}
\end{aligned}$$

and under the physical measure is

$$\begin{aligned}
\lambda_{VaR,t} &= \lambda e^{-\beta t} e^{\zeta_t} \\
&= \frac{1}{\tau}\left(\frac{1}{\delta_t} - \frac{1}{\beta}\right) e^{-\beta t} e^{\zeta_t}
\end{aligned}$$

Note that since

$$\delta_t \leq \beta$$

we have

$$\lambda_{VaR,t} \geq 0$$

3.5 Market Completeness and Banks' SPD

So far, we have not used the fact that markets are complete. We can use market completeness to recover the bank's SPD. The bank problem under the physical measure is given by equations (38)-(43). Complete markets imply that the dynamic budget constraints of the bank in equation (39) is equivalent to the static budget constraint

$$X_0 = E_0\left[\int_0^\infty Q_t \delta_t X_t dt\right] \tag{57}$$

where the banks take the SPD Q_t as given. The Lagrangian for the bank's problem is then

$$\mathcal{L} = E_0\left[\int_0^\infty e^{-\beta t} e^{\zeta_t}\log\left(\delta_t X_t\right)dt\right] + \lambda_{bc}\left(X_0 - E_0\left[\int_0^\infty Q_t\delta_t X_t dt\right]\right) + \int_0^\infty \lambda_{VaR,t}\left(g_V\left(t,\theta_t,\delta_t\right) - \overline{VaR}\right)dt$$

where $\lambda_{bc} > 0$ is a number but $\lambda_{VaR,t} > 0$ is a function of time since we have one VaR constraint for each t. The FOC for an interior solution are

$$[F_t] \quad : \quad 0 = \frac{e^{-\beta t}e^{\zeta_t}}{\delta_t X_t} - \lambda_{bc}Q_t + \lambda_{VaR,t}\frac{\tau}{X_t} \tag{58}$$

$$[\theta_t] \quad : \quad 0 = \frac{e^{-\beta t}e^{\zeta_t}}{\beta}\frac{\partial Q\left(t,\theta_t,\delta_t\right)}{\partial\theta_t} + \lambda_{VaR,t}\frac{\partial g_V\left(t,\theta_t,\delta_t\right)}{\partial\theta_t} \tag{59}$$

Re-arranging (58) gives the SPD of the bank

$$\begin{aligned} Q_t &= \frac{e^{-\beta t}e^{\zeta_t}}{\lambda_{bc}\delta_t X_t} + \frac{\lambda_{VaR,t}}{\lambda_{bc}}\frac{\tau}{X_t} \\ &= \frac{e^{-\beta t}e^{\zeta_t}}{\lambda_{bc}}\frac{1}{\delta_t X_t} + \frac{\lambda_{VaR,t}\tau}{\lambda_{bc}}\frac{1}{X_t} && (60) \\ &= \frac{1}{\lambda_{bc}X_t}\left[e^{-\beta t}e^{\zeta_t}\frac{1}{\delta_t} + \lambda_{VaR,t}\tau\right] && (61) \end{aligned}$$

We can interpret the terms

$$\frac{\lambda_{VaR,t}\tau}{\lambda_{bc}}\frac{1}{X_t} : \text{Marginal value of relaxing the } VaR \text{ constraint}$$

$$\frac{e^{-\beta t}e^{\zeta_t}}{\lambda_{bc}}\frac{1}{f_t X_t} : \text{marginal value of issuing dividends}$$

Using $\lambda_{VaR,t}$ from equation (57)

$$\lambda_{VaR,t} = \frac{e^{-\beta t}e^{\zeta_t}}{\tau}\left(\frac{1}{\delta_t} - \frac{1}{\beta}\right)$$

gives

$$Q_t = \frac{e^{-\beta t}e^{\zeta_t}}{\lambda_{bc}X_t}\left(\frac{2}{\delta_t} - \frac{1}{\beta}\right) \tag{62}$$

The multiplier λ_{bc} can be found from noting that we must have $Q_0 = 1$,

$$\lambda_{bc} = \frac{2}{X_0}\left(\frac{1}{\delta_0} - \frac{1}{\beta}\right)$$

or from the budget constraint (57).

3.6 Households

3.6.1 Setup

The representative household maximizes utility subject to its budget constraint, a portfolio constraint on its holdings of stocks of goods producers, and a solvency constraint (transversality condition). For time s, the household problem is

$$\max_{\{C_t(i),N_t,\omega_t\}_{t\geq s}} E_s\left\{\int_s^\infty e^{-\beta(t-s)}\left(\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{N_t^{1+\xi}}{1+\xi}\right)dt\right\}, \tag{63}$$

subject to

$$d\left(P_tF_t\right) \leq W_tN_tdt + \omega_{banks,t}d\left(P_tS_{banks,t}\right) + \omega_{0,t}d\left(P_tS_{0,t}\right) - P_tC_tdt \text{ for all } t \geq s, \tag{64}$$

$$\lim_{t\to\infty} E_s\left[Q_tF_t\right] = 0, \tag{65}$$

$$F_s \text{ given,} \tag{66}$$

and the definition of the aggregator for consumption in equation (17). The household maximizes utility by choosing the path $\{C_t(i), N_t, \omega_t\}_{t\geq s}$ of consumption of good i, $C_t(i)$, supply of labor, N_t, its position (number of shares) $\omega_{banks,t}$ in the stock of banks and its position $\omega_{0,t}$ in bonds. The variable F_t is the household's real financial wealth at time t, W_t is the nominal wage paid for labor supplied to the firms in an integrated competitive market[8], $S_{banks,t}$ is the real price of the stock of banks, $S_{0,t}$ is the real price of riskless bonds, P_t is the aggregate price level and Q_t is the nominal state-price density (SPD).

The dynamic flow budget constraint in equation (64) states that changes in the household's nominal financial wealth must be less than or equal to nominal labor income plus the nominal payoff on financial assets (which can be negative), minus nominal consumption expenditures. We can write equation (64) in real terms and using portfolio

[8]It follows that all firms pay the same wage for homogeneous labor. This means that the household picks $N_t(i) = N_t$ for all i and thus we can simplify the household problem by optimizing directly over N_t instead of over $N_t(i)$.

weights $\tilde{\omega}_t$ instead of number of shares as follows

$$\frac{dF_t}{F_t} = R_t dt + \tilde{\omega}_{banks,t}\mu_{banks,t} - \left(\frac{C_t}{F_t} - \frac{W_t}{P_t}\frac{N_t}{F_t}\right) dt$$

where $\tilde{\omega}_{banks,t}$ is defined by

$$\tilde{\omega}_{banks,t} \equiv \frac{S_{banks,t}\omega_{banks,t}}{F_t} \tag{67}$$

The portfolio weight on the bond, $\tilde{\omega}_{0,t}$, is then

$$\tilde{\omega}_{0,t} = 1 - \tilde{\omega}_{banks,t}$$

since total wealth is

$$F_t = \omega_{banks,t} S_{banks,t} + S_{0,t}\omega_{0,t}$$

The transversality condition in equation (65) is a no-Ponzi condition for the household. In its maximization, the household takes $\{W_t, S_{banks,t}, S_{0,t}, P_t(i), P_t, Q_t\}_{t\geq s}$ and F_s as given. We can also add a zero lower bound (ZLB) constraint on nominal interest rates to the household problem

$$i_t \geq 0 \text{ for all } t \geq s. \tag{68}$$

where i_t is the interest rate on a riskless nominal bond. This condition would emerge as a FOC for the household problem if we had money in the utility function, and money paid no interest while providing the same risk profile as a riskless bond when the riskless rate is at zero[9]. Here we consider the cashless limit of such an economy and thus can, if desired, include what would have been the FOC $i_t \geq 0$ directly as a constraint.

3.6.2 Solving Households' Problem

We solve the problem of the household in two stages. First, we find the optimal allocation $C_t(i)$ across goods i for a given level of consumption expenditures. Second, we solve the consumption/savings problem for the household who picks $C_t(i)$ optimally according to the first stage.

[9] Woodford and Eggertsson (2003).

The Lagrangian for the first stage is

$$\mathcal{L} = \left(\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} \, di \right)^{\frac{\varepsilon}{\varepsilon-1}} - \mu \left(E_t - \overline{E} \right). \tag{69}$$

where

$$E_t \equiv \int_0^1 P_t(i) \, C_t(i) \, di \tag{70}$$

are total nominal consumption expenditures and $\overline{E}$ is a given constant. The associated first-order condition is

$$C_t(i)^{-\frac{1}{\varepsilon}} \, C_t^{\frac{1}{\varepsilon}} = \mu P_t(i) \text{ for all } i. \tag{71}$$

Therefore,

$$C_t(i) = C_t(k) \left(\frac{P_t(i)}{P_t(k)} \right)^{-\varepsilon} \tag{72}$$

for any two goods i and k. Plugging in (72) into (70) gives

$$C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} \frac{Z_t}{P_t} \tag{73}$$

where we have defined the aggregate price level as

$$P_t \equiv \left(\int_0^1 P_t(i)^{1-\varepsilon} \, di \right)^{\frac{1}{1-\varepsilon}}. \tag{74}$$

Multiplying (73) by $P_t(i)$, raising both sides by the power $1 - \frac{1}{\varepsilon}$ and integrating over i gives

$$\int_0^1 P_t(i) \, C_t(i) \, di = P_t C_t, \tag{75}$$

where we have also used the definition of C_t, equation (17). Combining (70), (73) and (75) gives

$$C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t. \tag{76}$$

Now we solve the second stage of the household maximization. We restate the optimization problem of the household in a simplified way by doing three things. First, because utility is increasing in consumption and decreasing in labor, the household's budget constraint (64) holds with equality for all $t \geq s$. Second, we replace (64) and (65) by an equivalent intertemporal budget constraint.Note that to write down the intertemporal budget constraint, it is not necessary that financial markets are complete. Third, we use the solution of the first stage so that the household now chooses $\{C_t\}_{t\geq s}$ instead of $\{C_t(i)\}_{t\geq s}$ and allocates $C_t(i)$ according to equation (76). With these changes, the problem is

$$\max_{\{C_t,N_t\}_{t\geq s}} E_s\left\{\int_s^\infty e^{-\beta(t-s)}\left[\frac{C_t^{1-\gamma}}{1-\gamma}-\frac{N_t^{1+\xi}}{1+\xi}\right]dt\right\} \tag{77}$$

$$\text{subject to}$$

$$Q_sF_s = E_s\left[\int_s^\infty Q_t\left(C_t-\frac{W_t}{P_t}N_t\right)dt\right] \tag{78}$$

$$F_s \text{ given} \tag{79}$$

The Lagrangian for the optimization is

$$\begin{aligned}\mathcal{L} \;=\; & E_s\left\{\int_s^\infty e^{-\beta(t-s)}\left[\frac{C_t^{1-\gamma}}{1-\gamma}-\frac{N_t^{1+\xi}}{1+\xi}\right]dt\right\}+ \\ & -\lambda^{bc}\left[E_s\left[\int_s^\infty Q_t\left(C_t-\frac{W_t}{P_t}N_t\right)dt\right]-Q_sF_s\right]\end{aligned} \tag{80}$$

where λ^{bc} is the Lagrange multiplier associated with the constraint. The first order conditions for the households problem are given by

$$[N_t] \;:\; -e^{-\beta(t-s)}N_t^{\xi}+\lambda^{bc}Q_t\frac{W_t}{P_t}=0 \tag{81}$$

$$[C_t] \;:\; e^{-\beta(t-s)}C_t^{-\gamma}-\lambda^{bc}Q_t=0 \tag{82}$$

Combining (81) and (82) to eliminate λ^{bc} gives the intra-temporal optimality condition, which defines the labor supply curve

$$C_t^{\gamma}N_t^{\xi}=\frac{W_t}{P_t} \tag{83}$$

Using equation (82) for times s and t gives

$$\frac{Q_t}{Q_s}=e^{-\beta(t-s)}\left(\frac{C_t}{C_s}\right)^{-\gamma} \tag{84}$$

We identify the real and nominal state price densities

$$
\begin{aligned}
Q_t &= e^{-\beta t} C_t^{-\gamma}, && (85) \\
Q_t^{\$} &= e^{-\beta t} \frac{C_t^{-\gamma}}{P_t}, && (86)
\end{aligned}
$$

the real stochastic discount factor

$$
SDF_{t,s} \equiv \frac{Q_t}{Q_s} \tag{87}
$$

and the nominal stochastic discount factor $SDF_{t,s}^{\$} \equiv Q_t^{\$}/Q_s^{\$}$.

3.7 Firms

3.7.1 Final Good Sector

Firms in the final good sector produce a homogeneous good, Y_t, using intermediate goods, $Y_t(i)$, of different varieties $i \in [0,1]$. There is continuum of competitive final good producers of measure one. The production functions for all final good producers are identical and given by

$$
Y_t = \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{88}
$$

where $\varepsilon > 1$ is the constant elasticity of substitution for differentiated goods (and taken to be equal to the elasticity of substitution across goods for consumers). The production function has constant returns to scale and diminishing marginal product.

The representative firm chooses inputs $\{Y_t(i)\}_{i\in[0,1]}$ to maximize real profits

$$
Y_t - \frac{1}{P_t} \int_0^1 P_t(i) Y_t(i) \, di
$$

where $(1/P_t) \int_0^1 P_t(i) Y_t(i) \, di$ are real costs and Y_t is real total revenue. Because final good producers are competitive, they take $P_t(i)$ and P_t as given. Because of constant returns and competition, the size of any one final goods firm is indeterminate. However, their input demand is determined by the following cost minimization problem

$$\begin{aligned} & \min_{Y_t(i)} \int_0^1 P_t(i)\, Y_t(i)\, di \\ & s.t \\ Y_t \;\; \leq \;\; & \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\, di \right)^{\frac{\varepsilon}{\varepsilon-1}} \end{aligned}$$

The cost minimization yields a demand for intermediate good i that is homogeneous of degree one in total final output

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y_t \tag{89}$$

where ε turns out to be the elasticity of demand.

3.7.2 Intermediate Goods Sector

There is continuum of mass one of monopolistically competitive firms owned by the households, indexed by $i \in [0,1]$. Each firm faces a demand curve given by equation (89). Firms use labor $N_t(i)$ to produce output according to the technology

$$Y_t(i) = AN_t(i) \tag{90}$$

Labor is hired in a competitive market with perfect mobility.

Firms set prices according to Calvo staggered pricing. The probability density of receiving the signal to change prices after an amount of time h has elapsed is independent of the last time the firm received the signal and across firms, and given by

$$\delta e^{-\delta h},$$

where $\delta > 0$ is the Calvo parameter. Hence, the probability of not having received a signal between t and τ is

$$\begin{aligned} 1 - \int_t^\tau \delta e^{-\delta(s-t)} ds \;\; &= \;\; e^{-\delta(\tau-t)} \\ &\approx \;\; 1 - \delta(\tau - t) \end{aligned}$$

Firms that are able to adjust the price choose the price optimally. These firms maximize expected real discounted profits subject to their production technology (90), the demand curve (89) and the constraint on the frequency of price adjustment. Firms that cannot change their price adjust output to meet demand at the pre-established price. Both types of firms choose inputs to minimize costs, given output demand.

We characterize first the input choice problem conditional on output. We then characterize the optimal price adjustment and output decisions. We start by deriving input demand and marginal cost. Firm i chooses $N_t(i)$ to minimize total cost, given by

$$\frac{W_t}{P_t} N_t(i) \tag{91}$$

subject to

$$AN_t(i) - Y_t(i) \geq 0 \tag{92}$$

where, as mentioned earlier, W_t/P_t is the real wage. Let MC_t denote the Lagrange multiplier with respect to the constraint. Note that MC_t is the firm's real marginal cost (the derivative of total cost with respect to $Y_t(i)$).

The FOC with respect to $N_t(i)$ is

$$[N_t(i)] : MC_t = \frac{W_t}{AP_t} \tag{93}$$

Since the firm takes W_t/P_t as given, real marginal cost is constant across firms, a result of constant returns to scale and perfect factor mobility. Equation (92) with equality gives labor demand

$$N_t(i) = \frac{Y_t(i)}{A} \tag{94}$$

We next consider optimal price setting. A firm that is allowed to change its price at time t picks $P_t(i)$ to maximize

$$E_t \int_t^{\infty} SDF_{s,t} \left(\delta e^{-\delta(s-t)}\right) \left(\frac{P_t(i)}{P_s} Y_{s|t}(i) - MC_s Y_{s|t}(i)\right) ds \tag{95}$$

subject to

$$Y_{s|t}(i) = \left(\frac{P_t(i)}{P_s}\right)^{-\varepsilon} Y_s \tag{96}$$

where $Y_{s|t}(i)$ is the demand of good i at time s conditional on having changed prices for the last time at time t. In the optimal price setting decision, the firm takes as given the paths of $SDF_{s,t}$, P_s, Y_s and MC_s. Plugging equation (96) into (95) gives

$$E_t \int_t^{\infty} \frac{Q_s}{Q_t} \left(\delta e^{-\delta(s-t)}\right) \left(\frac{Y_s}{P_s^{1-\varepsilon}} P_t(i)^{1-\varepsilon} - \frac{Y_s MC_s}{P_s^{-\varepsilon}} P_t(i)^{-\varepsilon}\right) ds \tag{97}$$

The FOC with respect to $P_t(i)$ is

$$[P_t(i)] : E_t \int_t^{\infty} \frac{Q_s}{Q_t} \left(\delta e^{-\delta(s-t)}\right) \left(\frac{Y_s}{P_s^{1-\varepsilon}} (1-\varepsilon) P_t^*(i)^{-\varepsilon} + \varepsilon \frac{Y_s MC_s}{P_s^{-\varepsilon}} P_t^*(i)^{-\varepsilon-1}\right) ds = 0$$

or, rearranging,

$$\frac{P_t^*(i)}{P_t} = \frac{1}{MC} \frac{E_t \int_t^{\infty} \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{-\varepsilon} \frac{Y_s}{Y_t} MC_s ds}{E_t \int_t^{\infty} \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{1-\varepsilon} \frac{Y_s}{Y_t} ds} \tag{98}$$

where $P_t^*(i)$ is the optimal desired price and where we have defined

$$MC \equiv \left(1 - \frac{1}{\varepsilon}\right),$$

which is the steady-state level of the real marginal cost (the inverse of the steady-state gross markup). We can also write

$$P_t^*(i) = (1+\mu) E_t \int_t^{\infty} \Upsilon_{s,t} MC_s ds$$

where

$$\Upsilon_{s,t} \equiv \frac{Q_s e^{-\delta s} P_s^{-\varepsilon} Y_s}{E_t \int_t^{\infty} Q_s e^{-\delta s} P_s^{\varepsilon-1} Y_s ds}$$

which shows that the optimal price is a weighted average of real marginal costs times the markup (using that the nominal marginal cost $MC_s^n = P_s MC_s$, the price is also a weighted average of *nominal* marginal costs). Defining

$$\begin{aligned} x_{1,t} &\equiv E_t \int_t^{\infty} \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{1-\varepsilon} \frac{Y_s}{Y_t} ds \\ x_{2,t} &\equiv E_t \int_t^{\infty} \frac{Q_s}{Q_t} \delta e^{-\delta(s-t)} \left(\frac{P_t}{P_s}\right)^{-\varepsilon} \frac{Y_s}{Y_t} \frac{MC_s}{MC} ds \end{aligned}$$

and

$$\Pi_t \equiv \frac{P_t^*}{P_t}$$

we have

$$\Pi_t = \frac{x_{2,t}}{x_{1,t}}$$

and

$$\begin{aligned}
dx_{1,t} &= d\left(E_t \int_t^\infty \frac{Q_s}{Q_t}\delta e^{-\delta(s-t)}\left(\frac{P_t}{P_s}\right)^{1-\varepsilon}\frac{Y_s}{Y_t}ds\right) \\
&= d\left(\frac{e^{\delta t}P_t^{1-\varepsilon}}{Q_tY_t}\right)\left(\frac{e^{\delta t}P_t^{1-\varepsilon}}{Q_tY_t}\right)^{-1}x_{1,t}+\delta dt \qquad (99) \\
dx_{2,t} &= d\left(E_t \int_t^\infty \frac{Q_s}{Q_t}\delta e^{-\delta(s-t)}\left(\frac{P_t}{P_s}\right)^{-\varepsilon}\frac{Y_s}{Y_t}\frac{MC_s}{MC}ds\right) \\
&= d\left(\frac{e^{\delta t}P_t^{-\varepsilon}}{Q_tY_t}\right)\left(\frac{e^{\delta t}P_t^{-\varepsilon}}{Q_tY_t}\right)^{-1}x_{2,t}+\frac{\delta MC_t}{MC}dt \qquad (100)
\end{aligned}$$

Note that we dropped the index i from P_t^* (and hence from Π_t) because the optimal price P_t^* depends only on aggregate variables, so all firms that are allowed to change the price pick the same optimal price. Since the price changes are stochastically independent across firms, we have

$$P_t^{1-\varepsilon} = \int_{-\infty}^t \delta e^{-\delta(t-s)}\left(P_s^*\right)^{1-\varepsilon}ds$$

It follows that the price level is a predetermined variable at time t given by the past price quotations. Differentiating with respect to time gives

$$\begin{aligned}
\frac{d\left(P_t^{1-\varepsilon}\right)}{dt} &= \delta\left(P_t^*\right)^{1-\varepsilon} - \delta\int_{-\infty}^t \delta e^{-\delta(t-s)}\left(P_s^*\right)^{1-\varepsilon}ds \\
&= \delta\left[\left(P_t^*\right)^{1-\varepsilon} - P_t^{1-\varepsilon}\right] \qquad (101)
\end{aligned}$$

and

$$\frac{d\left(P_t^{1-\varepsilon}\right)}{dt} = (1-\varepsilon)P_t^{-\varepsilon}\frac{dP_t}{dt} \qquad (102)$$

Combining (101) and (102) gives

$$\frac{dP_t}{P_t} = \frac{\delta}{1-\varepsilon}\left(\Pi_t^{1-\varepsilon}-1\right)dt \qquad (103)$$

Defining inflation as

$$\pi_t \equiv \frac{1}{dt}\frac{dP_t}{P_t}$$

we get

$$\pi_t = \frac{\delta}{1-\varepsilon}\left(\Pi_t^{1-\varepsilon}-1\right)$$

Using that P_t is locally deterministic by equation (103) and Ito's lemma,equations (99) and (100) become

$$\begin{aligned} dx_{1,t} &= \frac{x_{1,t}Q_tY_t}{e^{\delta t}}d\left(\frac{e^{\delta t}}{Q_tY_t}\right)+\delta dt+(1-\varepsilon)\,x_{1,t}\pi_t dt && (104)\\ dx_{2,t} &= \frac{x_{2,t}Q_tY_t}{e^{\delta t}}d\left(\frac{e^{\delta t}}{Q_tY_t}\right)+\frac{\delta MC_t}{MC}dt-\varepsilon x_{2,t}\pi_t dt && (105)\end{aligned}$$

Therefore, the price dynamics are determined by the following four equations

$$\begin{aligned} \pi_t &= \frac{\delta}{1-\varepsilon}\left(\Pi_t^{1-\varepsilon}-1\right) && (106)\\ \Pi_t &= \frac{x_{2,t}}{x_{1,t}} && (107)\\ dx_{1,t} &= (x_{1,t}+1)\,\delta dt+(1-\varepsilon)\,x_{1,t}\pi_t dt+x_{1,t}Q_tY_t d\left(\frac{1}{Q_tY_t}\right) && (108)\\ dx_{2,t} &= \left(x_{2,t}+\frac{MC_t}{MC}\right)\delta dt-\varepsilon x_{2,t}\pi_t dt+x_{2,t}Q_tY_t d\left(\frac{1}{Q_tY_t}\right) && (109)\end{aligned}$$

where π_t , Π_t, $x_{1,t}$ and $x_{2,t}$ are all stationary.

Let us next turn to the determination of profits and dividends. The real profits for the producer of intermediate good producer i is

$$D_{t,goods}(i) = \frac{P_t(i)\,Y_t(i)}{P_t} - MC_tY_t(i) \tag{110}$$

Aggregating across firms gives the aggregate profits for the sector, which are paid out as dividends to shareholders

$$\begin{aligned} D_{t,goods} &= \int_0^1 D_{t,goods}(i)\,di && (111)\\ &= \frac{1}{P_t}\int_0^1 P_t(i)\,Y_t(i)\,di - MC_t\int_0^1 Y_t(i)\,di && (112)\end{aligned}$$

3.7.3 Aggregation

Integrating (89) over i gives

$$\int_0^1 Y_t(i)\,di = v_t Y_t$$

where

$$v_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} di.$$

so

$$Y_t \neq \int_0^1 Y_t(i)\,di$$

unless all prices are identical across firms.

Itegrating (94) over i gives

$$Y_t = \frac{A}{v_t} N_t \tag{113}$$

The term $1/v_t$ gives the aggregate efficiency loss due to price distortions.

Because of Calvo pricing, we have

$$v_t = \int_{-\infty}^t \delta e^{-\delta(t-s)} \left(\frac{P_s^*}{P_t}\right)^{-\varepsilon} ds$$

where recall that P_s^* is the optimal price chosen by firms that can reset their price at time s given that the last time they were able to change their price was at t. Differentiating this expression gives the dynamics of v_t in terms of aggregate variables

$$\begin{aligned}
dv_t &= d\left[\int_{-\infty}^t \delta e^{-\delta(t-s)} \left(\frac{P_s^*}{P_t}\right)^{-\varepsilon} ds\right] \\
&= d\left[e^{-\delta t} P_t^{\varepsilon} \int_{-\infty}^t \delta e^{\delta s} P_s^{*-\varepsilon} ds\right] \\
&= d\left[e^{-\delta t} P_t^{\varepsilon}\right] \int_{-\infty}^t \delta e^{\delta s} P_s^{*-\varepsilon} ds + e^{-\delta t} P_t^{\varepsilon} d\left[\int_{-\infty}^t \delta e^{\delta s} P_s^{*-\varepsilon} ds\right] \\
&= \left(\varepsilon \frac{dP_t}{P_t} - \delta dt\right) \int_{-\infty}^t \delta e^{-\delta(t-s)} \left(\frac{P_s^*}{P_t}\right)^{-\varepsilon} ds + \delta \left(\frac{P_t^*}{P_t}\right)^{-\varepsilon} dt
\end{aligned}$$

so that

$$dv_t = \delta \Pi_t^{-\varepsilon} dt + (\varepsilon \pi_t - \delta) v_t dt \tag{114}$$

We can also express equation (112) in terms of aggregate variables only

$$D_{t,goods} = \frac{1}{P_t}\int_0^1 P_t(i)\, Y_t(i)\, di - MC_t \int_0^1 Y_t(i)\, di \tag{115}$$

$$= (1 - MC_t v_t)\, Y_t. \tag{116}$$

4 Equilibrium

An equilibrium is a collection

$$\mathbb{E} = \{C_t, C_t(i), Y_t, Y_t(i), N_t, F_t, \omega_t, W_t, P_t, Q_t, D_t, \theta_t, X_t, \delta_t, S_t, \eta_t, \sigma_t, R_t\}_{t\geq 0} \tag{117}$$

such that households, firms and banks optimize, and markets for labor, intermediate goods, the final good and all financial assets clear. We now collect the first-order conditions and market clearing conditions that determine an equilibrium, and then combine them to have an explicit characterization of the equilibrium.

Household optimization gives

$$\text{Labor supply: } N_t = \left(\frac{W_t}{P_t} C_t^{-\gamma}\right)^{\frac{1}{\xi}} \tag{118}$$

$$\text{Intertemporal consumption: } Q_t = e^{-\beta t} C_t^{-\gamma} \tag{119}$$

$$\text{Budget constraint: } F_t = E_t\left[\int_t^\infty \frac{Q_s}{Q_t}\left(C_s - \frac{W_s}{P_s} N_s\right) ds\right] \tag{120}$$

$$\text{Demand for financial assets: } \frac{dF_t}{F_t} = \left(R_t + \omega_t \mu_t - \frac{C_t}{F_t} + \frac{W_t}{P_t}\frac{N_t}{F_t}\right) dt + \omega_t \sigma_t dZ_t \tag{121}$$

$$\tag{122}$$

Optimization for the final good produer gives

$$\text{Demand of intermediate goods:} Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t \tag{123}$$

$$\text{Supply of final goods:} Y_t = \left(\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \tag{124}$$

Intermediate good producers' optimization for inputs of prodcution is characterized by

$$\text{Labor demand: } N_t(i) = \frac{Y_t(i)}{A} \tag{125}$$

$$\text{Supply of goods: } Y_t(i) = AN_t(i) \tag{126}$$

$$\text{Profits/Dividends: } D_{goods,t}(i) = \frac{P_t(i)\,Y_t(i)}{P_t} - MC_t Y_t(i) \tag{127}$$

where we recall that $MC_t = W_t/AP_t$. Intermediate good producers' price setting decision is given by

$$\pi_t = \frac{\vartheta}{1-\varepsilon}\left(\Pi_t^{1-\varepsilon} - 1\right) \tag{128}$$

$$\Pi_t = \frac{x_{2,t}}{x_{1,t}} \tag{129}$$

$$dx_{1,t} = (x_{1,t} + 1)\,\vartheta dt + (1-\varepsilon)\,x_{1,t}\pi_t dt + x_{1,t}Q_tY_t d\left(\frac{1}{Q_tY_t}\right) \tag{130}$$

$$dx_{2,t} = \left(x_{2,t} + \frac{MC_t}{MC}\right)\vartheta dt - \varepsilon x_{2,t}\pi_t dt + x_{2,t}Q_tY_t d\left(\frac{1}{Q_tY_t}\right) \tag{131}$$

Optimization for banks is given by

$$\text{Dividends: } \delta_t = u\left(t, \min\{1, \varphi_t\}\right) f_{M,t} 1_{\{\varphi_t > 0\}} + \left(R_t + \frac{1}{\tau}\overline{VaR}\right) 1_{\{\varphi_t \leq 0\}} \tag{132}$$

$$\text{Optimal portfolio: } \theta_t = \min\{1, \max\{0, \varphi_t\}\}\,\theta_{M,t} \tag{133}$$

$$\text{Wealth accumulation: } \frac{dX_t}{X_t} = (R_t - \delta_t + \theta_t\mu_t)\,dt + \theta_t\sigma_t dZ_t \tag{134}$$

where

$$f_{M,t} = \beta \tag{135}$$

$$\theta_{M,t} = (\sigma_t)^{-1}(\eta_t - m_t) \tag{136}$$

are the dividends and portfolio positions of an unconstrained but otherwise identical bank, and φ_t is such that the VaR constraint holds with equality

$$\varphi_t = 1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau}\,|\eta_t - m_t|} \pm \sqrt{2\left(R_t - \delta_t\right)\tau + 2\overline{VaR} + |\eta_t - m_t|^2\,\tau^2\left(1 + \frac{\mathcal{N}^{-1}(\alpha)}{\sqrt{\tau}\,|\eta_t - m_t|}\right)^2} \tag{137}$$

Market clearing for labor and goods are

$$\text{Intermediate goods: } \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t = AN_t(i) \tag{138}$$

$$\text{Final goods: } C_t = Y_t \tag{139}$$

$$\text{Labor: } \left(\frac{W_t}{P_t}C_t^{-\gamma}\right)^{\frac{1}{\xi}} = \frac{Y_t(i)}{A} \tag{140}$$

and market clearing for financial assets are

$$\text{Banks' stock: } \frac{F_t\omega_{banks,t}}{S_{banks,t}} + \frac{X_t\theta_{banks,t}}{S_{banks,t}} = 1 \tag{141}$$

$$\text{Good producers' stock: } \frac{X_t\theta_{goods,t}}{S_{goods,t}} = 1 \tag{142}$$

$$\text{Bond: } \frac{F_t\omega_{0,t}}{S_{0,t}} + \frac{X_t\theta_{0,t}}{S_{0,t}} = 0 \tag{143}$$

Since banks must hold the entire stock of good producers, $\varphi_t > 0$. Aggregation of output gives

$$\int_0^1 Y_t(i)\,di = v_t Y_t = AN_t$$

where

$$dv_t = \vartheta\Pi_t^{-\varepsilon}dt + \left(\varepsilon\pi_t - \vartheta\right)v_t dt \tag{144}$$

Aggregation of dividends across good producers gives

$$D_{goods,t} = \left(1 - MC_t v_t\right)Y_t$$

Finally, the central bank sets nominal interest i_t, which are linked to real interest rates and inflation by $i_t = R_t + \pi_t$.

4.1 Equilibrium characterization

Combining labor demand, labor supply and clearing of labor markets gives real wages and labor

$$
\begin{aligned}
\frac{W_t}{P_t} &= \left(\frac{v_t}{A}\right)^{\xi} Y_t^{\xi+\gamma} && (145)\\
N_t &= \frac{v_t Y_t}{A} && (146)
\end{aligned}
$$

Equations (145) and (146) in turn imply that

$$
\begin{aligned}
MC_t &= \frac{1}{A}\left(\frac{v_t}{A}\right)^{\xi} Y_t^{\xi+\gamma} && (147)\\
D_{goods,t} &= \left(1-\left(\frac{v_t}{A}\right)^{1+\xi} Y_t^{\xi+\gamma}\right) Y_t && (148)
\end{aligned}
$$

Defining the natural rate of output Y_t^n, the natural rate of interes, r_t, and the output gap y_t, as

$$
\begin{aligned}
Y_t^n &\equiv v_t^{-\frac{\xi}{\xi+\gamma}} A^{\frac{1+\xi}{\xi+\gamma}} MC^{\frac{1}{\xi+\gamma}}\\
y_t &\equiv \log Y_t - \log Y_t^n\\
r_t &\equiv \beta - \frac{\varphi\sigma}{\varphi+\sigma}\frac{E_t\left[d\log v_t\right]}{dt}
\end{aligned}
$$

equations (85), (119) and (139) give the dynamic IS equation

$$
dy_t = \frac{1}{\gamma}\left(R_t - r_t + \frac{1}{2}\eta_t^2\right)dt + \frac{\eta_t}{\gamma}dZ_t \tag{149}
$$

Because the household and the bank trade in complete markets, their SPD must agree,

$$
e^{-\beta t}Y_t^{-\gamma} = \frac{e^{-\beta t}e^{\zeta_t}}{\lambda_{bc}X_t}\left(\frac{2}{\delta_t}-\frac{1}{\beta}\right)
$$

Taking derivatives,

$$
-\gamma d\log Y_t = d\zeta_t - d\log X_t + d\log\left(\frac{2}{\delta_t}-\frac{1}{\beta}\right) \tag{150}
$$

Matching drift and stochastic parts of the left and right-hand sides of (150) gives

$$\begin{aligned} d\delta_t &= \left(\delta_t - \beta + \frac{1}{2}\left(\eta_t^2 - m_t^2\right) - \theta_t\mu_t + \frac{1}{2}\left(\theta_t\sigma_t\right)^2\right)\left(\frac{2\left(2\beta - \delta_t\right)}{\beta}\right)dt \\ &\quad + \left(\eta_t - m_t - \theta_t\sigma_t\right)^2\left(\frac{1}{2} - \frac{\delta_t\left(12\beta^2 - \delta_t^2 - 2\beta\delta_t\right)}{16\beta^3}\right)dt \\ &\quad + \left(\eta_t - m_t - \theta_t\sigma_t\right)\left(\frac{2\beta - \delta_t}{2\beta}\right)dZ_t \end{aligned} \tag{151}$$

Using Ito's lemma, equations (132), (137) and (151) give

$$d\varphi_t = G\left(\varphi_t, R_t\right)dt + S\left(\varphi_t, R_t\right)dZ_t \tag{152}$$

for two functions G, S. Any three of the four equations (132), (137), (151), (152) characterize the optimal decision of the banks.

Using the definition of vulnerability in equation (5) and the dynamic IS in equation (156), we obtain

$$V_t = -\frac{1}{\gamma}\left(R_t - r_t + \frac{1}{2}\eta_t^2\right)\tau - \mathcal{N}^{-1}\left(\alpha\right)\frac{\eta_t}{\gamma}\sqrt{\tau} \tag{153}$$

Solving (153) for R_t gives

$$R_t\left(V_t, \eta_t\right) = -\frac{1}{2}\eta_t^2 - \frac{\mathcal{N}^{-1}\left(\alpha\right)}{\sqrt{\tau}}\eta_t - \frac{\gamma V_t}{\tau} + r_t \tag{154}$$

Assuming[10] $\eta_t - m_t > 0$, plugging (154) into (137) and using (132), we can solve for η_t as a function of φ_t, m_t and V_t

$$\eta_t = \eta\left(\varphi_t, m_t, V_t\right) \tag{155}$$

[10]We later linearize the model around a steady-state with $\eta_t - m_t > 0$ so that small perturbations always preserve the sign of $\eta_t - m_t$. With that in mind, we simplify at this point by assuming $\eta_t - m_t > 0$ to simplify exposition. Solving the non-linear version of the model allowing for $\eta_t - m_t < 0$ is straightforward.

where the function η is given by

$$
\begin{aligned}
\eta\left(\varphi_t, m_t, V_t\right) &= -\frac{1}{2A}\left(B-\sqrt{-4AC+B^2}\right) \\
A &= -\frac{1}{2}\tau\left(\varphi_t-1\right)^2 \\
B &= \tau\varphi_t^2 m_t+\left(-2\tau\right)\varphi_t m_t+\left(N\sqrt{\tau}-\frac{1}{N}\beta\tau^{\frac{3}{2}}\right)\varphi_t-\tau\left(\frac{N}{\sqrt{\tau}}-\frac{1}{N}\beta\sqrt{\tau}\right) \\
C &= \overline{VaR}+\left(-\frac{1}{2}\tau\right)\varphi_t^2 m_t^2+\tau\varphi_t m_t^2+\left(\frac{1}{N}\beta\tau^{\frac{3}{2}}-N\sqrt{\tau}\right)\varphi_t m_t \\
&\quad +\left(-\frac{1}{N}\beta\tau^{\frac{3}{2}}\right)m_t+\left(-\gamma\right)V_t+\tau\left(r_t-\beta\right)
\end{aligned}
$$

The characterization of the equilibrium is then given by the following equations:

$$
\begin{aligned}
dy_t &= \frac{1}{\gamma}\left(i_t-\pi_t-r_t+\frac{1}{2}\eta_t^2\right)dt+\frac{\eta_t}{\gamma}dZ_t && (156) \\
\frac{d\pi_t}{dt} &= \left(\left(1-\varepsilon\right)\pi_t+\vartheta\right)\left[\left(\frac{\varepsilon}{\varepsilon-1}\frac{Y_t^{\xi+\gamma}}{A}\left(\frac{v_t}{A}\right)^{\xi}\left(\frac{1-\varepsilon}{\vartheta}\pi_t+1\right)^{\frac{1}{\varepsilon-1}}-1\right)\vartheta e_{1,t}\right] && (157) \\
&\quad -\left(\left(1-\varepsilon\right)\pi_t+\vartheta\right)\pi_t && (158) \\
d\varphi_t &= G_{\varphi}\left(\varphi_t, R_t\right)dt+S_{\varphi}\left(\varphi_t, R_t\right)dZ_t && (159) \\
i_t &= f\left(y_t, \pi_t, \varphi_t\right) && (160) \\
\eta_t &= \eta\left(\varphi_t, m_t, V_t\right) && (161) \\
V_t &= -\alpha\mathbb{V}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\sqrt{\tau}-\mathbb{E}\left[\frac{dy_t}{dt}|\mathcal{F}_t\right]\tau && (162)
\end{aligned}
$$

Equation (156) is the dynamic IS equation, the demand block of the model. Equation (158) gives inflation dynamics, the supply side of the model. The inflation dynamics depend on the present discounted value of nominal output for firms that can reset their price, $x_{1,t}$, with dynamics given by $e_{1,t}\equiv x_{1,t}^{-1}$

$$
\begin{aligned}
\frac{de_{1,t}}{e_{1,t}} &= -\left[\left(1+e_{1,t}\right)\vartheta+\left(1-\varepsilon\right)\pi_t+\beta+\frac{\left(\gamma-1\right)}{2\gamma^2}\eta_t^2+\frac{\left(\gamma-1\right)}{\gamma}\left(R_t-\beta\right)\right]dt && (163) \\
&\quad -\frac{\left(\gamma-1\right)\eta_t}{\gamma}dZ_t && (164)
\end{aligned}
$$

Inflation dynamics also depend on and the output losses due to inefficient price dispersion, v_t , with dynamics

$$dv_t = \vartheta \left(\frac{1-\varepsilon}{\vartheta} \pi_t + 1 \right)^{-\frac{\varepsilon}{1-\varepsilon}} dt + (\varepsilon \pi_t - \vartheta) v_t dt \tag{165}$$

Equation (159) corresponds to the financial sector block of the model and gives the dynamics of the tightness of the VaR constraint of the bank, φ_t. Equation (160) is the monetary policy rule for the central bank. Equation (161) connects the household and bank behavior through the market price of risk, vulnerability and the tightness of the VaR constraint.

4.2 Deterministic Steady State

Variables without their time subscript denote their values in a deterministic steady state. In a deterministic steady state, we have

$$\begin{aligned}
v &= 1 \\
P &= 1 \\
\pi &= 0 \\
MC &= 1 - \frac{1}{\varepsilon} \\
Y &= Y^n = A^{\frac{1+\varphi}{\varphi+\sigma}} \left(1 - \frac{1}{\varepsilon} \right)^{\frac{1}{\varphi+\sigma}} \\
\frac{W}{P} &= A \left(1 - \frac{1}{\varepsilon} \right) \\
N &= \frac{Y}{A} \\
D_{goods} &= \frac{1}{\varepsilon} \left(1 - \frac{1}{\varepsilon} \right)^{\frac{1}{\varphi+\sigma}} A^{\frac{1+\varphi}{\varphi+\sigma}} \\
r &= \beta \\
R &= i = r = \beta \\
s &= 0 \\
\eta &= 0 \\
\delta &= \beta \\
\theta &= 0 \\
g_V(t, \theta, \delta) &= 0
\end{aligned}$$

4.3 Linearized Version

We linearize $\eta(\varphi_t, V_t), \eta(\varphi_t, V_t)^2, G_\varphi(\varphi_t, R_t), S_\varphi(\varphi_t, R_t)$ in

$$\begin{aligned}
dy_t &= \frac{1}{\gamma}\left(R_t - r_t + \frac{1}{2}\eta_t^2\right)dt + \frac{\eta_t}{\gamma}dZ_t \\
d\varphi_t &= G_\varphi(\varphi_t, R_t)\,dt + S_\varphi(\varphi_t, R_t)\,dZ_t \\
\eta_t &= \eta(\varphi_t, V_t) \\
V_t &= V(\eta_t, R_t)
\end{aligned}$$

around the deterministic steady state

$$\begin{aligned}
\eta_t &= \Phi_0 + \Phi_v V_t + \Phi_\varphi \varphi_t + \Phi_m m_t \\
\eta_t^2 &= -\hat{\eta}^2 + 2\hat{\eta}\left(\Phi_0 + \Phi_v V_t + \Phi_\varphi \varphi_t + \Phi_m m_t\right) \\
d\varphi_t &= \left(\Upsilon_0 + \Upsilon_r R_t + \Upsilon_\varphi \varphi_t + \Upsilon_m m_t\right)dt + \left(\Psi_0 + \Psi_r R_t + \Psi_\varphi \varphi_t + \Psi_m m_t\right)dZ_t
\end{aligned}$$

where $\hat{\eta}$ is the point around which $\hat{\eta}$ is linearized, i.e. $\hat{\eta} = \eta\left(\hat{\varphi}, \hat{V}, \hat{m}\right)$ where the Taylor expansion of η was performed on (φ_t, V_t, m_t) around $\left(\hat{\varphi}, \hat{V}, \hat{m}\right)$. Also note that to first order, $v_t = 1$ and thus the natural rate r_t is constant at r.

Using the linearizations, we get

$$\begin{aligned}
dy_t &= \frac{1}{\gamma}\left(R_t - r + \frac{1}{2}\eta_t^2\right)dt + \frac{\eta_t}{\gamma}dZ_t \\
d\varphi_t &= \left(\Upsilon_0 + \Upsilon_r R_t + \Upsilon_\varphi \varphi_t + \Upsilon_m m_t\right)dt \\
&\quad + \left(\Psi_0 + \Psi_r R_t + \Psi_\varphi \varphi_t + \Psi_m m_t\right)dZ_t \\
\eta_t &= \Phi_0 + \Phi_v V_t + \Phi_\varphi \varphi_t + \Phi_m m_t \\
\eta_t^2 &= -\hat{\eta}^2 + 2\hat{\eta}\left(\Phi_0 + \Phi_v V_t + \Phi_\varphi \varphi_t + \Phi_m m_t\right) \\
V_t &= V(\eta_t, R_t)
\end{aligned}$$

Rearrange

$$
\begin{aligned}
dy_t &= \frac{1}{\gamma}\left(R_t - r + \frac{1}{2}\eta_t^2\right)dt + \frac{\eta_t}{\gamma}dZ_t \\
dy_t &= \frac{1}{\gamma}\left(R_t - r + \frac{1}{2}\left(-\hat{\eta}^2 + 2\hat{\eta}\Phi_v\left(V_t + \frac{\Phi_\varphi}{\Phi_v}\varphi_t + \frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t\right)\right)\right)dt \\
&\quad + \frac{1}{\gamma}\Phi_v\left(V_t + \frac{\Phi_\varphi}{\Phi_v}\varphi_t + \frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t\right)dZ_t \\
dy_t &= \frac{1}{\gamma}\left(R_t - r + \hat{\eta}\Phi_v\left(V_t + \frac{\Phi_\varphi}{\Phi_v}\varphi_t + \frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t - \frac{1}{2}\frac{\hat{\eta}^2}{\hat{\eta}\Phi_v}\right)\right)dt \\
&\quad + \frac{1}{\gamma}\Phi_v\left(V_t + \frac{\Phi_\varphi}{\Phi_v}\varphi_t + \frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t\right)dZ_t
\end{aligned}
$$

Define

$$
\begin{aligned}
\varphi_t &\equiv -\frac{\Phi_\varphi}{\Phi_v}\varphi_t \\
s_t &\equiv -\left(\frac{\Phi_0}{\Phi_v} + \frac{\Phi_m}{\Phi_v}m_t\right) \\
\bar{s} &\equiv -\frac{\Phi_0}{\Phi_s} \\
\xi &\equiv \frac{\Phi_v}{\gamma} \\
\alpha &\equiv N^{-1}(\alpha) \\
d(rp_t) &\equiv \hat{\eta}\xi\left(V_t - \chi_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)dt \\
&\quad + \xi(V_t - \chi_t - s_t)\,dZ_t
\end{aligned}
$$

and

$$
\begin{aligned}
\left(-\frac{\Phi_\varphi\Upsilon_0}{\Phi_v}\right) &\equiv a_0, \left(-\frac{\Phi_\varphi\Upsilon_r}{\Phi_v}\right) \equiv a_r, \left(-\frac{\Phi_\varphi\Upsilon_m}{\Phi_v}\right) \equiv a_s, \Upsilon_\varphi \equiv a_\varphi \\
\left(-\frac{\Phi_\varphi\Psi_0}{\Phi_v}\right) &\equiv b_0, \left(-\frac{\Phi_\varphi\Psi_r}{\Phi_v}\right) \equiv b_r, \left(-\frac{\Phi_\varphi\Psi_m}{\Phi_v}\right) \equiv b_s, \Psi_\varphi \equiv a_\varphi
\end{aligned}
$$

Then we get

$$\begin{aligned}
dy_t &= \frac{1}{\gamma}\left(R_t - r\right) dt + d\left(rp_t\right) \\
d\left(rp_t\right) &= \hat{\eta}\xi\left(V_t - \varphi_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right) dt + \xi\left(V_t - \varphi_t - s_t\right) dZ_t \\
V_t &= -\frac{1}{dt}\mathbb{E}\left[dy_t|\mathcal{F}_t\right]\tau - \alpha\mathbb{V}\left[dy_t|\mathcal{F}_t\right]\sqrt{\tau} \\
d\varphi_t &= \left(a_0 + \Upsilon_\varphi\varphi_t + a_r R_t + a_s s_t\right) dt + \left(b_0 + b_\varphi\varphi_t + b_r R_t + b_s s_t\right) dZ_t \\
ds_t &= -\kappa\left(s_t - \overline{s}\right) + \sigma_s dZ_t
\end{aligned}$$

4.4 No Direct Feedback from Monetary Policy to φ_t

We consider the simpler case of $\varphi_t = 0$ and fixed prices. The case with $\varphi_t = 0$ corresponds to monetary policy not affecting the bank's VaR constraint directly, but only through general equilibrium (discount rate) effects. We can analyze the case in which $\varphi_t \neq 0$ and a Phillips curve is present in the same way that we analyze the simpler case; the control problem for monetary policy is still linear-quadratic even in the general case.

Without a Phillips curve and with $\varphi_t = 0$, the linearized equilibrium is characterized by

$$dy_t = \frac{1}{\gamma}\left(R_t - r + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right) dt + \xi\left(V_t - s_t\right) dZ_t \tag{166}$$

$$V_t = -\mathbb{E}_t\left[dy_t\right]\tau - \alpha\mathbb{V}\left[dy_t|\mathcal{F}_t\right]\sqrt{\tau} \tag{167}$$

$$ds_t = -\kappa\left(s_t - \overline{s}\right) + \sigma_s dZ_t \tag{168}$$

so that

$$\begin{aligned}
\mathbb{E}_t\left[dy_t\right] &= \frac{1}{\gamma}\left(R_t - r + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right) \\
\mathbb{V}\left[dy_t|\mathcal{F}_t\right] &= \xi\left(V_t - s_t\right)
\end{aligned}$$

Solve for R_t and V_t in (167) to get

$$R_t = r - \frac{\gamma}{\tau}\left(\xi\sqrt{\tau}\left(\alpha + \sqrt{\tau}\eta\right) + 1\right) V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right) s_t + \frac{1}{2}\eta^2 \tag{169}$$

Plug in (169) into (166) to get

$$dy_t = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)dt + \xi\left(V_t - s_t\right)dZ_t \tag{170}$$

Use

$$\begin{aligned}
\mathbb{E}\left[dy_t|\mathcal{F}_t\right] &= -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right) \\
\mathbb{V}\left[dy_t|\mathcal{F}_t\right] &= \xi\left(V_t - s_t\right)
\end{aligned}$$

and then eliminating V_t to get

$$\mathbb{E}\left[dy_t|\mathcal{F}_t\right] = -\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi}\mathbb{V}\left[dy_t|\mathcal{F}_t\right] - \frac{1}{\tau}s_t \tag{171}$$

We have thus obtained the mean-volatility line of Figure 2. Equation (171) also makes clear that the shocks s_t are shifts to vulnerability that shift the mean-volatility line up and down, while all other changes in the economy involve moving along the mean-volaitlity line. Empirically, slope is negative and intercept is positive on average

$$\begin{aligned}
-\frac{1 + \alpha\sqrt{\tau}\xi}{\tau\xi} &< 0 \\
-\frac{\overline{s}}{\tau} &> 0
\end{aligned}$$

This implies $\overline{s} < 0$ and

$$\begin{aligned}
\xi &> 0 \text{ and } 1 + \alpha\sqrt{\tau} > 0 \\
&\text{or} \\
\xi &< 0 \text{ and } 1 + \alpha\sqrt{\tau} < 0
\end{aligned}$$

To match empirical estimates, we set

$$\begin{aligned}
\alpha &= -1.645 \\
\sqrt{\tau} &= 1
\end{aligned}$$

To match the actual slope and intercept

$$\begin{aligned}
-\frac{1+\alpha\sqrt{\tau}\xi}{\tau\xi} &= -1.15 \\
\overline{s} &= -0.67\tau
\end{aligned}$$

which gives

$$\begin{aligned}
\xi &= 0.36 \\
\overline{s} &= -0.67
\end{aligned}$$

We identify $s_t - \overline{s}$ with the residuals of the regression of $\mathbb{E}\left[dy_t|\mathcal{F}_t\right]$ on $\mathbb{V}\left[dy_t|\mathcal{F}_t\right]$. The standard deviation and $AR(1)$ coefficient of these residuals then identify σ_s and κ, respectively. Since

$$\begin{aligned}
Std\left(-\frac{1}{\tau}\left(s_t-\overline{s}\right)\right) &= 0.62 \\
AR(1) &= 0.12
\end{aligned}$$

we get, converting to annualized values

$$\begin{aligned}
\kappa &= -\log\left(0.12\right) = 2.12 \\
\sigma_s &= 0.31
\end{aligned}$$

5 Monetary Policy

5.1 Optimal Monetary Policy

The central bank is minimizing a quadratic loss function over the output gap and inflation

$$L\left(y_t,\pi_t\right) = \min_{i_t} \mathbb{E}_t \int_t^{\infty} e^{-t\beta}\left(y_t^2+\pi_t^2\right)dt. \tag{172}$$

subject to the dynamics of the economy (166), (167), (168). Minimizing the quadratic loss function is a standard approach in the NK literature, as Rotemberg and Woodford (1997), Rotemberg and Woodford (1999) and Woodford (2003) have shown that aggregate welfare can be approximated by such a loss function.

We focus on the case described in the last section for ease of exposition. The interest rate R_t can be eliminated from the optimization problem, so that the central bank's problem can be written as

$$L(y_t, s_t) = \min_{\{V_s\}_{s=t}^{\infty}} \mathbb{E}_t \int_t^{\infty} e^{-s\beta} y_s^2 ds \tag{173}$$

$$s.t.$$

$$V_t = \frac{-\gamma^{-1}(R_t - r_t) + \alpha\xi s_t\sqrt{\tau} + \hat{\eta}\xi\left(s_t + \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\tau}{1 + \alpha\xi\sqrt{\tau} + \hat{\eta}\xi\tau} \tag{174}$$

$$dy_t = -\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s_t\right)dt + \xi(V_t - s_t)dZ_t \tag{175}$$

$$ds_t = -\kappa(s_t - \overline{s}) + \sigma_s dZ_t \tag{176}$$

The central bank thus effectively picks V_t, which is connected to R_t in a one-to-one fashion by

$$R_t = r - \frac{\gamma}{\tau}\left(\xi\sqrt{\tau}\left(\alpha + \sqrt{\tau}\eta\right) + 1\right)V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}} + \eta\right)s_t + \frac{1}{2}\eta^2$$

The Hamilton-Jacobi-Bellman (HJB) equation for the central banker's optimization is

$$\begin{aligned} 0 \;=\; & \min_V \left\{ y^2 - \beta L - \frac{\partial L}{\partial y}\frac{\alpha\sqrt{\tau}\xi + 1}{\tau}\left(V - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi + 1}s\right) + \frac{1}{2}\frac{\partial^2 L}{\partial y^2}\xi^2(V - s)^2 \right\} \\ & -\kappa(s - \overline{s})\frac{\partial L}{\partial s} + \frac{1}{2}\frac{\partial^2 L}{\partial s^2}\sigma_s^2 \end{aligned}$$

Intuitively, the HJB takes into account the current value of welfare, as well as the change in welfare associated with changes in the state variables y and s.

The first order condition is

$$0 = -\frac{\partial L}{\partial y}\frac{\alpha\sqrt{\tau}\xi + 1}{\tau} + \frac{\partial^2 L}{\partial y^2}\xi^2(V - s) \tag{177}$$

$$V = \frac{\partial L}{\partial y}\frac{\alpha\sqrt{\tau}\xi + 1}{\tau\xi^2}\left(\frac{\partial^2 L}{\partial y^2}\right)^{-1} + s \tag{178}$$

Hence at the optimum, vulnerability is proportional to s, and depends on the first and second derivative of welfare with respect to output. It is also noteworthy that $\frac{\alpha\sqrt{\tau}\xi+1}{\tau\xi^2}$, which defines the slope of output volatility with respect to expected output, appears in the FOC.

We look for a quadratic solution of the form

$$L\left(y,x\right)=c_0+c_1y+c_2y^2+c_3s+c_4s^2+c_5ys$$

where $c.$ are constants.

Plugging into the HJB, and using

$$\begin{aligned}\frac{\partial L}{\partial y} &= c_1+2c_2y+c_5s \\ \frac{\partial^2 L}{\partial y^2} &= 2c_2 \\ \frac{\partial L}{\partial s} &= c_3+2c_4s+c_5y \\ \frac{\partial^2 L}{\partial s^2} &= 2c_4\end{aligned}$$

we get the following system of equations on the coefficients $c_0,...,c_5$

$$\begin{aligned}\left[y^2\right] &: 0=\left(-\beta-\frac{1}{\tau^2\xi^2}\left(\alpha\sqrt{\tau}\xi+1\right)^2\right)c_2+1 \\ \left[ys\right] &: 0=\left(-\frac{2}{\tau}\right)c_2+\left(-\frac{1}{\tau^2\xi^2}\left(2\alpha\sqrt{\tau}\xi+\alpha^2\tau\xi^2+\beta\tau^2\xi^2+1\right)\right)c_5 \\ \left[y\right] &: 0=-\frac{1}{\tau^2\xi^2}c_1\left(2\alpha\sqrt{\tau}\xi+\alpha^2\tau\xi^2+\beta\tau^2\xi^2+1\right) \\ \left[s^2\right] &: 0=-\frac{1}{4\tau^2\xi^2c_2}\left(c_5^2\left(2\alpha\sqrt{\tau}\xi+\alpha^2\tau\xi^2+1\right)+4\tau\xi^2c_2c_5+4\beta\tau^2\xi^2c_2c_4\right) \\ \left[s\right] &: 0=-\frac{1}{2\tau^2\xi^2c_2}\left(c_1c_5\left(2\alpha\sqrt{\tau}\xi+\alpha^2\tau\xi^2+1\right)+2\tau\xi^2c_1c_2+2\beta\tau^2\xi^2c_2c_3\right) \\ \left[const\right] &: 0=-\frac{1}{4\tau^2\xi^2c_2}\left(c_1^2\left(2\alpha\sqrt{\tau}\xi+\alpha^2\tau\xi^2+1\right)+4\beta\tau^2\xi^2c_0c_2-4\tau^2\xi^2\sigma_s^2c_2c_4\right)\end{aligned}$$

with solution

$$\begin{aligned}
c_0 &= \frac{\tau^2\xi^4\sigma_s^2\left(\left(\alpha\sqrt{\tau}\xi+1\right)^2+2\beta\tau^2\xi^2\right)}{\beta^2\left(\left(\alpha\sqrt{\tau}\xi+1\right)^2+\beta\tau^2\xi^2\right)^3} > 0 \\
c_1 &= 0 \\
c_2 &= \frac{\tau^2\xi^2}{\tau^2\xi^2\beta+\left(\alpha\sqrt{\tau}\xi+1\right)^2} > 0 \\
c_3 &= 0 \\
c_4 &= \frac{\xi^4\tau^2\left(\left(\alpha\sqrt{\tau}\xi+1\right)^2+2\beta\tau^2\xi^2\right)}{\beta\left(\left(\alpha\sqrt{\tau}\xi+1\right)^2+\beta\tau^2\xi^2\right)^3} > 0 \\
c_5 &= -\frac{2\tau^3\xi^4}{\left(\left(\alpha\sqrt{\tau}\xi+1\right)^2+\beta\tau^2\xi^2\right)^2} < 0
\end{aligned}$$

To pick the optimal initial conditions, we minimize L with respect to y_0 taking s_0 as given

$$\begin{aligned}
L\left(y_0, s_0\right) &= c_0 + c_1 y_0 + c_2 y_0^2 + c_3 s_0 + c_4 s_0^2 + c_5 y_0 s_0 \\
FOC &: \frac{\partial L}{\partial y_0} = 0 \\
SOC &: \frac{\partial^2 L}{\partial y_0^2} > 0
\end{aligned}$$

The FOC and SOC can be solved to get

$$\begin{aligned}
y_0^* &= -\left(\frac{c_1}{2c_2} + \frac{c_5}{2c_2} s_0\right) \\
&= \frac{\tau\xi^2 s_0}{\left(\alpha\sqrt{\tau}\xi+1\right)^2+\beta\tau^2\xi^2} \\
c_2 &> 0
\end{aligned}$$

The optimal policy in terms of V_t is given by plugging in the optimal solution into the FOC in equation (177):

$$V = \frac{\left(\alpha\sqrt{\tau}\xi+1\right)}{\tau\xi^2} y + \left(1 - \frac{\left(\alpha\sqrt{\tau}\xi+1\right)}{\left(\alpha\sqrt{\tau}\xi+1\right)^2+\beta\tau^2\xi^2}\right) s \tag{179}$$

This can be viewed as a "flexible inflation targeting rule" (see Svensson (1999), Svensson (2002) and Rudebusch and Svensson (1999)) or, more generally, as a linear optimal targeting criterion (Giannoni and Woodford (2012)). Even though vulnerability and its shocks, V_t and s_t, are not target variables, i.e., they do not appear in the loss function equation (172), they still enter the inflation targeting rule, the first-order condition given by equation (179). There are no independent target values for V_t and s_t that the central bank hopes to achieve. The reason V_t and s_t enter the targeting rule is that they forecast the conditional mean and variance of y_t even after controlling for the information already contained in the mean of y_t itself (more generally, in the means of y_t and π_t when a Phillips Curve is included). This is consistent with the empirical results in Table 1 and with the findings in Adrian, Boyarchenko, and Giannone (2016), who show that financial conditions are excellent predictors of the tail of the GDP distribution in a way that non-financial variables are not. Alternatively, equation (179) can be interpreted as a traditional flexible inflation targeting rule in which the targets for inflation and/or output are time-varying and depend on V_t and s_t. It also important to note that even if a central bank decided not to condition its actions on V_t and s_t, the tradeoff between inflation and output –reflected in the coefficients of the rule in equation (179)– now depends on γ and ξ, the parameters that dictate the strength of the mean-variance tradeoff of output.

Using the optimal solution in the process for the output gap in equation (175), we then find that

$$\begin{aligned} dy_t &= -\frac{\alpha\sqrt{\tau}\xi+1}{\tau}\left(V_t - \frac{\alpha\sqrt{\tau}\xi}{\alpha\sqrt{\tau}\xi+1}s_t\right)dt + \xi\left(V_t - s_t\right)dZ_t \\ &= -\left(\frac{\left(\alpha\sqrt{\tau}\xi+1\right)^2}{\tau^2\xi^2}y_t + \frac{\beta\tau\xi^2}{\left(\alpha\sqrt{\tau}\xi+1\right)^2+\beta\tau^2\xi^2}s_t\right)dt \\ &\quad + \left(\frac{\left(\alpha\sqrt{\tau}\xi+1\right)}{\tau\xi}y_t - \frac{\xi\left(\alpha\sqrt{\tau}\xi+1\right)}{\left(\alpha\sqrt{\tau}\xi+1\right)^2+\beta\tau^2\xi^2}s_t\right)dZ_t \end{aligned}$$

Recalling that

$$\mathbb{E}\left[dy_t|\mathcal{F}_t\right] = -\frac{1+\alpha\sqrt{\tau}\xi}{\tau\xi}\mathbb{V}\left[dy_t|\mathcal{F}_t\right] - \frac{1}{\tau}s_t \tag{180}$$

And defining the slope as

$$M \equiv -\frac{1+\alpha\sqrt{\tau}\xi}{\tau\xi}$$

we get

$$V = -\frac{M}{\xi}y + \left(1 + \frac{M}{\tau\xi\left(M^2+\beta\right)}\right)s$$

and

$$dy_t = -\left(M^2 \times y_t + \frac{\beta/\tau}{M^2+\beta} \times s_t\right)dt - \left(M \times y_t - \frac{M/\tau}{M^2+\beta} \times s_t\right)dZ_t \tag{181}$$

The last equation makes clear that the magnitude of the tradeoff between stabilizing the mean and variance of the output gap is given by the slope M of the mean-volatility line in Figure 2.

We can also express monetary policy as an interest rate rule. Using the FOC for V, the optimal interest rate is

$$\begin{aligned}
R_t &= r - \frac{\gamma}{\tau}\left(\xi\sqrt{\tau}\left(\alpha+\sqrt{\tau}\eta\right)+1\right)V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}}+\eta\right)s_t + \frac{1}{2}\eta^2 \\
&= r + \frac{1}{2}\eta^2 - \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}}+\frac{1}{\tau}+\eta\right)V_t + \gamma\xi\left(\frac{\alpha}{\sqrt{\tau}}+\eta\right)s_t \\
&= \gamma M\left(\eta+\frac{\alpha}{\sqrt{\tau}}\right)\frac{M^2+\beta}{\beta+M^2+M/\tau\xi}y_t - \gamma\xi\left(\left(\eta+\frac{\alpha}{\sqrt{\tau}}\right)\frac{M/\tau\xi}{\beta+M^2+M/\tau\xi}+\frac{1}{\tau}\right)V_t \\
&\quad + \left(\frac{1}{2}\eta^2 + r\right)
\end{aligned}$$

The optimal interest rule can thus be viewed as an augmented Taylor rule. In addition to the output gap y and the equilibrium rate of interest r (and inflation π_t in the more general case), the level of vulnerability V enters the optimal rule. As before, the coefficients on y (and π in the more general case) depend on the parameters that define vulnerability ξ and γ and thus monetary policy is different from the typical NK model without vulnerabilities not only because vulnerability enters the augmented Taylor rule directly, but also because the presence of vulnerabilities alter the optimal response of interest rates to changes in output and inflation.

5.2 Alternative Monetary Policy Rules

In general, the central bank might follow other monetary policy rules. We consider alternative linear rules that do not explicitly condition on vulnerability or its shocks:

$$i_t = \psi_0 + \psi_y y_t \tag{182}$$

We show that even after picking the coefficients ψ_0, ψ_y in an optimal way, the rule in equation (182) implies quantitatively large welfare losses compared to the optimal monetary policy found in the last section. To find the coefficients ψ_0, ψ_y that minimize welfare losses, we solve

$$\min_{(\psi_0,\psi_y)} L\left(y_0, s_0\right) \tag{183}$$

$$s.t.$$

$$dy_t = \frac{1}{\gamma}\left(i_t - r + \gamma\hat{\eta}\xi\left(V_t - s_t - \frac{1}{2}\frac{\hat{\eta}}{\xi\gamma}\right)\right)dt + \xi\left(V_t - s_t\right)dZ_t \tag{184}$$

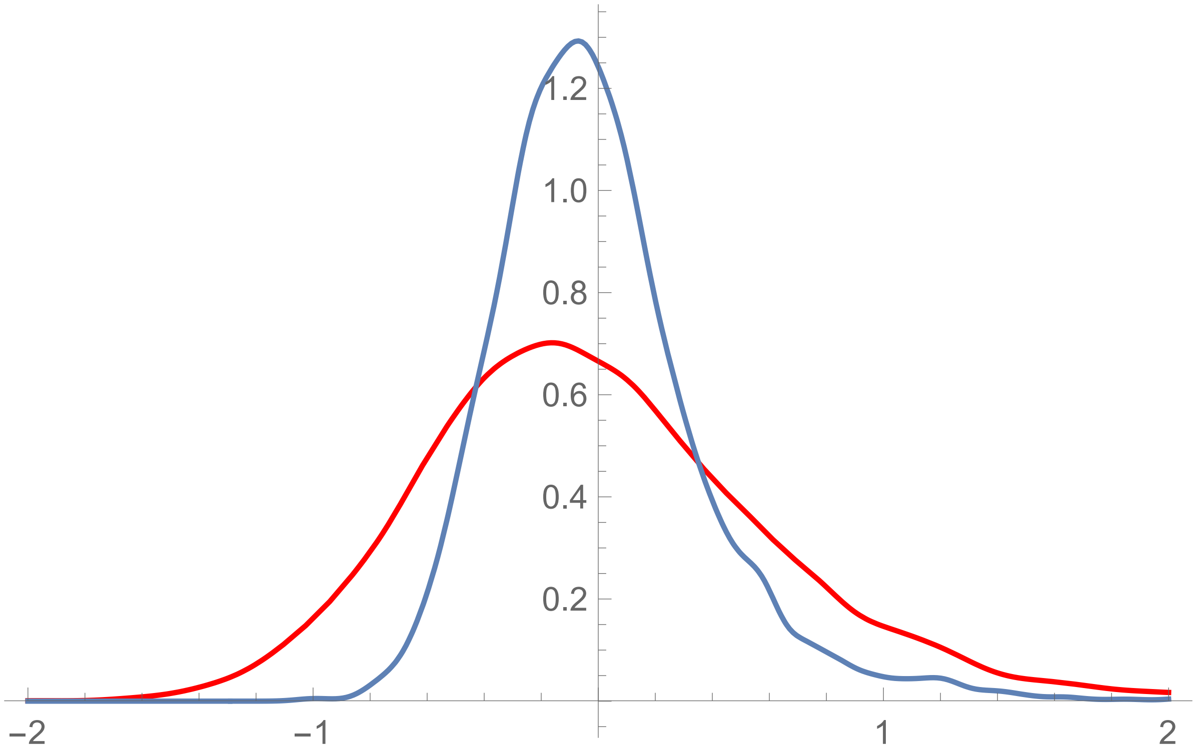
$$i_t = \psi_0 + \psi_y y_t \tag{185}$$

$$V_t = -\mathbb{E}_t\left[dy_t\right]\tau - \alpha\mathbb{V}\left[dy_t|\mathcal{F}_t\right]\sqrt{\tau} \tag{186}$$

$$ds_t = -\kappa\left(s_t - \overline{s}\right) + \sigma_s dZ_t \tag{187}$$

Figure 4 shows the steady-state distribution of the output gap y_t using the optimal policy rule that explicitly takes vulnerability into account (using equation (181)), and the Taylor-type rule that does not condition on vulnerability V_t, given by equation (182) with coefficients found by solving (183)-(187). Intuitively, shocks to vulnerability s contain information about the conditional distribution of the output gap that the policy maker should take into account in setting optimal policy. For a given level of the output gap, a higher vulnerability –a larger VaR of output– calls for higher interest rates. Higher interest rates induce the private sector to save more and consume less, thus shifting the conditional future distribution of y_t upwards by shifting its conditional mean upwards. Given the link between the expected mean and the expected volatility of output induced by the presence of vulnerability, a higher conditional mean induces a lower volatility of y_t. Together, higher mean and lower volatility mean lower vulnerability – lower VaR for output. For the suboptimal Taylor rule that ignores vulnerability, interest rates remain unchanged when, for a given level of y_t, V_t changes. Compared to the optimal rule, when V_t increases but i_t remains unchanged, the conditional mean of output is lower and its conditional volatility is higher. Over time, more frequent visit to states of lower mean and higher volatility create an unconditional distribution that is more negatively skewed. When instead V_t decreases, the optimal rule and the suboptimal Taylor rule produce similar right tails for the unconditional distribution of output. The reason is that lower V_t induces both higher mean and lower volatility of output. Therefore, even though the changes in mean and volatility of y_t are different for the two different rules, the actual differences in outcomes for y_t are small because the lower volatility minimizes all fluctuations.

Figure 4: **Probability Density Functions of the Output under the Optimal Policy Rule and a Standard Taylor Rule**. The figure shows the PDFs using the optimal policy rule and the standard Taylor rule. The standard Taylor rule coefficients are calculated for the economy assuming that the policy maker is ignoring the presence of financial vulnerability.



6 Conclusion

The degree to which financial stability considerations should be incorporated in the conduct of monetary policy has long been debated, see Adrian and Liang (2016) for an overview. In this paper, we extend the basic, two equation New Keynesian model to incorporate a notion of financial vulnerability. Shocks to risk premia impact aggregate demand via the Euler equation. The shocks to risk premia are assumed to impact the volatility of output, which is motivated from the empirical observation by Adrian, Boyarchenko, and Giannone (2016) that financial conditions forecast both the mean and the volatility of output. Importantly, our framework reproduces the stylized fact that the conditional mean and the conditional volatility of output are strongly negatively correlated, giving rise to a sharply negatively skewed unconditional output distribution. Vulnerability thus captures movements in the conditional GDP distribution that correspond to the downside risk of growth.

We further assume that the central bank minimizes the expected discounted sum of squared output gaps and squared inflation, which is standard in the literature. This is therefore a central bank that is subject to a dual mandate, without an independent financial stability objective. Despite that narrow objective function, the optimal flexible inflation targeting rule conditions on the level of vulnerability. Intuitively, all variables that provide information about the conditional distribution of GDP should be taken into account in setting optimal monetary policy. This translates into an augmented Taylor rule, where financial vulnerability—as measured by output gap tailrisk as a function of financial variables—is an input into the Taylor rule. Furthermore, the magnitude of the Taylor rule coefficients on output gap and inflation depend on the parameters that determine vulnerability.

The striking result from our setup is that the central bank should **always** condition monetary policy on financial vulnerability. Relative to earlier literature that has made similar arguments (e.g. Curdia and Woodford (2010), Cúrdia and Woodford (2016) and Gambacorta and Signoretti (2014), our modeling approach is deeply rooted in empirical observations which capture macoreconomic shocks of the 2008 crisis very well. Through the negative correlation between conditional mean and conditional variance, our setup captures nonlinearity in macro dynamics in a tractable linear-quadratic setting. The implications of our results for the conduct of monetary policy are in line with the arguments or Adrian and Shin (2010) and Borio and Zhu (2012).

References

Adrian, T., N. Boyarchenko, and D. Giannone (2016): "Vulnerable Growth," *Federal Reserve Bank of New York Staff Reports*, 794.

Adrian, T., D. Covitz, and N. Liang (2015): "Financial Stability Monitoring," *Annual Review of Financial Economics*, 7, 357–395.

Adrian, T., and N. Liang (2016): "Monetary Policy, Financial Conditions, and Financial Stability," *International Journal of Central Banking*, forthcoming.

Adrian, T., and H. Shin (2010): "Financial Intermediaries and Monetary Economics," *Handbook of Monetary Economics*, 3, 601–650.

Aikman, D., M. T. Kiley, S. J. Lee, M. G. Palumbo, and M. N. Warusawitharana (2015): "Mapping Heat in the US Financial System," *Federal Reserve Board Finance and Economics Discussion Series*, 59.

Bernanke, B., and A. Blinder (1992): "The Federal Funds Rate and the Channels of Monetary Transmission," *American Economic Review*, pp. 901–921.

Bernanke, B., and M. Gertler (1989): "Agency Costs, Net Worth, and Business Fluctuations," *American Economic Review*, pp. 14–31.

——— (2000): "Monetary Policy and Asset Price Volatility," *National Bureau of Economic Research*, 7559.

Borio, C., and H. Zhu (2012): "Capital Regulation, Risk-taking and Monetary Policy: A Missing Link in the Transmission Mechanism?," *Journal of Financial Stability*, 8(4), 236–251.

Curdia, V., and M. Woodford (2010): "Credit Spreads and Monetary Policy," *Journal of Money, Credit and Banking*, 42(1), 3–35.

Cúrdia, V., and M. Woodford (2016): "Credit Frictions and Optimal Monetary Policy," *Journal of Monetary Economics*, 84, 30–65.

Dudley, W. C. (2015): "Remarks at The Fed at a crossroads: Where to go next?, Brookings Institution, Washington, D.C.," `https://www.newyorkfed.org/newsevents/speeches/2015/dud151015`.

Estrella, A., and G. A. Hardouvelis (1991): "The Term Structure as a Predictor of Real Economic Activity," *Journal of Finance*, 46(2), 555–576.

Estrella, A., and F. S. Mishkin (1998): "Predicting US Recessions: Financial Variables as Leading Indicators," *Review of Economics and Statistics*, 80(1), 45–61.

Galí, J. (2008): *Monetary Policy, Inflation, and the Business Cycle: an Introduction to the New Keynesian Framework and its Applications*. Princeton University Press.

Gambacorta, L., and F. M. Signoretti (2014): "Should Monetary Policy Lean against the Wind?: An Analysis based on a DSGE Model with Banking," *Journal of Economic Dynamics and Control*, 43, 146–174.

Gertler, M., and P. Karadi (2011): "A Model of Unconventional Monetary Policy," *Journal of Monetary Economics*, 58(1), 17–34.

Giannoni, M., and M. Woodford (2012): "Optimal target criteria for stabilization policy," Staff Reports 535, Federal Reserve Bank of New York.

Gilchrist, S., and E. Zakrajšek (2012): "Credit Spreads and Business Cycle Fluctuations," *American Economic Review*, 102(4), 1692–1720.

Krishnamurthy, A., and T. Muir (2016): "How Credit Cycles across a Financial Crisis," *Unpublished paper, Stanford University (June)*.

Leeper, E. M. (1991): "Equilibria under 'active' and 'passive' monetary and fiscal policies," *Journal of Monetary Economics*, 27(1), 129–147.

López-Salido, D., J. C. Stein, and E. Zakrajšek (2016): "Credit-Market Sentiment and the Business Cycle," *National Bureau of Economic Research*, 21879.

Merton, R. C. (1977): "On the pricing of contingent claims and the Modigliani-Miller theorem," *Journal of Financial Economics*, 5(2), 241–249.

Peek, J., E. S. Rosengren, and G. M. Tootell (2015): "Should US Monetary Policy Have a Ternary Mandate?," *Federal Reserve Bank of Boston 59 th Economic Conference*, Macroprudential and Monetary Policy.

Rotemberg, J., and M. Woodford (1997): "An Optimization-Based Econometric Framework for the Evaluation of Monetary Policy," *NBER Macroeconomics Annual 1997, Volume 12*, pp. 297–361.

Rotemberg, J. J., and M. Woodford (1999): "Interest Rate Rules in an Estimated Sticky Price Model," *Monetary Policy Rules*, pp. 57–126.

Rudebusch, G., and L. E. Svensson (1999): "Policy Rules for Inflation Targeting," *Monetary Policy Rules*, pp. 203–262.

Stock, J. H., and M. W. Watson (1998): "Diffusion Indexes," *National Bureau of Economic Research*, 6702.

Svensson, L. E. (1999): "Inflation Targeting as a Monetary Policy Rule," *Journal of Monetary Economics*, 43(3), 607–654.

——— (2002): "Inflation Targeting: Should It be Modeled as an Instrument Rule or a Targeting Rule?," *European Economic Review*, 46(4), 771–780.

——— (2016a): "Cost-Benefit Analysis of Leaning Against the Wind: Are Costs Larger Also with Less Effective Macroprudential Policy?," *National Bureau of Economic Research Working Paper*, 21902.

——— (2016b): "Inflation Targeting and "Leaning Against the Wind"," *Israel Economic Review*, 13(1).

——— (2017): "How Robust Is the Result That the Cost of "Leaning Against the Wind" Exceeds the Benefit? Response to Adrian and Liang," Discussion paper, CEPR Discussion Papers.

Taylor, J. B. (1993): "Discretion Versus Policy Rules in Practice," *Carnegie-Rochester Conference Series on Public Policy*, 39, 195–214.

——— (1999): "A Historical Analysis of Monetary Policy Rules," *Monetary Policy Rules*, pp. 319–348.

Woodford, M. (2001): "Fiscal Requirements for Price Stability," *Journal of Money, Credit and Banking*, pp. 669–728.

——— (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

Yellen, J. (2016): "The Federal Reserve's Monetary Policy Toolkit: Past, Present, and Future, speech at "Designing Resilient Monetary Policy Frameworks for the Future," a symposium sponsored by the Federal Reserve Bank of Kansas City, Jackson Hole, Wyoming," `https://www.federalreserve.gov/newsevents/speech/yellen20160826a.htm`.

A Reformulating the Bank's Problem

Log utility allows us to transform the bank's optimization problem into a non-stochastic problem. Indeed,

$$\log X_t = \log X_0 + \int_0^t Q\left(s, \theta_s, f_s, \nu_s\right) ds + \int_0^t \theta_s^T \sigma_s dB_s^\nu$$

Consider the following

$$\begin{aligned}
& \int_0^\infty e^{-\beta t} \log\left(f_t X_t\right) dt && (188)\\
= & \int_0^\infty e^{-\beta t} \log\left(X_t\right) dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt \\
= & \int_0^\infty e^{-\beta t} \log\left(X_0\right) dt + \int_0^\infty e^{-\beta t} \left\{ \int_0^t Q\left(s, \theta_s, f_s, \nu_s\right) ds + \int_0^t \theta_s^T \sigma_s dB_s^\nu \right\} dt \\
& + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt \\
= & \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \int_0^\infty \int_0^t e^{-\beta t} Q\left(s, \theta_s, f_s, \nu_s\right) ds dt \\
& + \int_0^\infty \int_0^t e^{-\beta t} \theta_s^T \sigma_s dB_s^\nu dt && (189)\\
= & \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \int_0^\infty \int_s^\infty e^{-\beta t} Q\left(s, \theta_s, f_s, \nu_s\right) dt ds \\
& + \int_0^\infty \int_0^t e^{-\beta t} \theta_s^T \sigma_s dB_s^\nu dt && (190)\\
= & \log\left(X_0\right) \int_0^\infty e^{-\beta t} dt + \int_0^\infty e^{-\beta t} \log\left(f_t\right) dt + \int_0^\infty Q\left(s, \theta_s, f_s, \nu_s\right) \left[\int_s^\infty e^{-\beta t} dt \right] ds \\
& + \int_0^\infty \int_0^t e^{-\beta t} \theta_s^T \sigma_s dB_s^\nu dt && (191)
\end{aligned}$$

where the change in the order of integration follows from Fubini's theorem. We assume all the usual regularity conditions. In particular, we will need that

$$\int_0^\infty \left\| \sigma_t^{-1} \mu_t \right\|^2 dt < \infty \qquad (192)$$

Under the regularity condition in equation the stochastic part of the bank's objective function is a martingale and not just a local martingale, so

$$E_0^{bank} \int_0^\infty \int_0^t e^{-\beta t} \theta_s^T \sigma_s dB_s^\nu dt = 0$$

Therefore, taking expectations in (191) gives

$$\begin{aligned} E_0^{bank}\int_0^\infty e^{-\beta t}\log\left(f_tX_t\right)dt &= \log\left(X_0\right)\int_0^\infty e^{-\beta t}dt + E_0^{bank}\int_0^\infty e^{-\beta t}\log\left(f_t\right)dt \\ &\quad + E_0^{bank}\int_0^\infty Q\left(s,\theta_s,f_s,\nu_s\right)\left[\int_s^\infty e^{-\beta t}dt\right]ds \end{aligned}$$

$$\begin{aligned} E_0^{bank}\int_0^\infty e^{-\beta t}\log\left(f_tX_t\right)dt &= \frac{\log\left(X_0\right)}{\beta} + E_0\int_0^\infty e^{-\beta t}\log\left(f_t\right)dt \\ &\quad + \frac{1}{\beta}E_0\int_0^\infty Q\left(s,\theta_s,f_s,\nu_s\right)e^{-\beta s}ds \\ &= \frac{\log\left(X_0\right)}{\beta} + E_0\int_0^\infty e^{-\beta t}\left(\log\left(f_t\right)+\frac{1}{\beta}Q\left(t,\theta_t,f_t,\nu_t\right)\right)dt \end{aligned}$$

B Appendix: Finding the Lagrange Multiplier for the VaR

We first compute some derivatives

$$\begin{aligned} \theta_t^T\mu_t &= \sum_j \theta_{j,t}\mu_{j,t} \\ \frac{\partial\left(\theta_t^T\mu_t\right)}{\partial\theta_{j,t}} &= \frac{\partial}{\partial\theta_{j,t}}\sum_{k=1}^M\theta_{k,t}\mu_{k,t} = \mu_{j,t} \\ \theta_t^T\sigma_t &= \left[\begin{array}{cccc} \sum_{k=1}^M\theta_{k,t}\sigma_{k1,t} & \sum_{k=1}^M\theta_{k,t}\sigma_{k2,t} & \ldots & \sum_{k=1}^M\theta_{k,t}\sigma_{kN,t}\end{array}\right] \\ \left\|\theta_t^T\sigma_t\right\|^2 &= \sum_{n=1}^N\left(\sum_{k=1}^M\theta_{k,t}\sigma_{kn,t}\right)^2 \\ \frac{\partial}{\partial\theta_{j,t}}\left\|\theta_t^T\sigma_t\right\|^2 &= 2\theta_t^T\sigma_t\sigma_t^{(j)} = \left[\begin{array}{cccc} \sum_{k=1}^M\theta_{k,t}\sigma_{k1,t} & \sum_{k=1}^M\theta_{k,t}\sigma_{k2,t} & \ldots & \sum_k\theta_{k,t}\sigma_{kN,t}\end{array}\right]\left[\begin{array}{c}\sigma_{j1,t}\\ \sigma_{j2,t}\\ \vdots \\ \sigma_{jN,t}\end{array}\right] \end{aligned}$$

where $\sigma_t^{(j)}$ is the j^{th} column of σ_t^T. Putting all the vectors together, we can write

$$\begin{aligned} \frac{\partial\left(\theta_t^T\mu_t\right)}{\partial\theta_t} &= \mu_t \\ \frac{\partial}{\partial\theta_t}\left\|\theta_t^T\sigma_t\right\|^2 &= 2\sigma_t\sigma_t^T\theta_t \end{aligned}$$

The derivative of $\left\|\theta_t^T \sigma_t\right\|$ now follows from

$$
\begin{aligned}
\frac{\partial}{\partial \theta_t}\left\|\theta_t^T \sigma_t\right\|^2 &= 2\left\|\theta_t^T \sigma_t\right\|\left(\frac{\partial\left\|\theta_t^T \sigma_t\right\|}{\partial \theta_t}\right)=2 \sigma_t \sigma_t^T \theta_t \\
\Longrightarrow \quad \frac{\partial\left\|\theta_t^T \sigma_t\right\|}{\partial \theta_t} &= \frac{\sigma_t \sigma_t^T \theta_t}{\left\|\theta_t^T \sigma_t\right\|}
\end{aligned}
$$

Using the above computations and the definitions

$$
\begin{aligned}
Q\left(t, \theta_t, f_t\right) & \equiv R_t-f_t+\theta_t^T \mu_t-\frac{1}{2}\left\|\theta_t^T \sigma_t\right\|^2 \\
g_V\left(t, \theta_t, f_t\right) & \equiv -Q\left(t, \theta_t, f_t\right) \tau-N^{-1}(\alpha)\left\|\theta_t^T \sigma_t\right\| \sqrt{\tau} \\
h\left(t, \theta_t, f_t, \zeta_t\right) & \equiv e^{-\beta t} e^{\zeta_t} \log \left(f_t\right)+\left[\int_t^{\infty} e^{-\beta s} E_t\left[e^{\zeta_s}\right] d s\right] Q\left(t, \theta_t, f_t\right) \\
& = e^{-\beta t} e^{\zeta_t} \log \left(f_t\right)+\frac{e^{-\beta t} e^{\zeta_t}}{\beta} Q\left(t, \theta_t, f_t\right)
\end{aligned}
$$

we get

$$
\begin{aligned}
\frac{\partial}{\partial \theta_t} Q\left(t, \theta_t, f_t\right) &= \mu_t-\sigma_t \sigma_t^T \theta_t \\
\frac{\partial}{\partial f_t} Q\left(t, \theta_t, f_t\right) &= -1
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{f_t} g_V\left(t, \theta_t, f_t\right) &= \tau \\
\nabla_{\theta_t} g_V\left(t, \theta_t, f_t\right) &= -\left(\mu_t-\sigma_t \sigma_t^T \theta_t\right) \tau-N^{-1}(\alpha) \frac{\sigma_t \sigma_t^T \theta_t}{\left\|\theta_t^T \sigma_t\right\|} \sqrt{\tau}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\theta_t} h\left(t, \theta_t, f_t, \zeta_t\right) &= \frac{e^{-\beta t} e^{\zeta_t}}{\beta}\left(\mu_t-\sigma_t \sigma_t^T \theta_t\right) \\
\nabla_{f_t} h\left(t, \theta_t, f_t, \zeta_t\right) &= \frac{e^{-\beta t} e^{\zeta_t}}{f_t}-\frac{e^{-\beta t} e^{\zeta_t}}{\beta}
\end{aligned}
$$

The FOC is

$$
\nabla h\left(t, \theta_t, f_t, \zeta_t\right)=\lambda_{VaR} \nabla g_V\left(t, \theta_t, f_t\right)
$$

i.e.

$$
\begin{aligned}
\nabla_{\theta_t} h\left(t, \theta_t, f_t, \zeta_t\right) &= \lambda_{VaR} \nabla_{\theta_t} g_V\left(t, \theta_t, f_t\right) \\
\nabla_{f_t} h\left(t, \theta_t, f_t, \zeta_t\right) &= \lambda_{VaR} \nabla_{f_t} g_V\left(t, \theta_t, f_t\right)
\end{aligned}
$$

Using the computations above,

$$\begin{aligned}
\nabla_{\theta_t} h(t,\theta_t,f_t,\zeta_t) &= \lambda_{VaR}\nabla_{\theta_t} g_V(t,\theta_t,f_t) \\
\frac{e^{-\beta t}e^{\zeta_t}}{\beta}\left(\mu_t-\sigma_t\sigma_t^T\theta_t\right) &= \lambda_{VaR}\left(-\left(\mu_t-\sigma_t\sigma_t^T\theta_t\right)\tau - N^{-1}(\alpha)\frac{\sigma_t\sigma_t^T\theta_t}{\left\|\theta_t^T\sigma_t\right\|}\sqrt{\tau}\right)
\end{aligned}$$

$$\begin{aligned}
\nabla_{f_t} h(t,\theta_t,f_t,\zeta_t) &= \lambda_{VaR}\nabla_{f_t} g_V(t,\theta_t,f_t) \\
\frac{e^{-\beta t}e^{\zeta_t}}{f_t}-\frac{e^{-\beta t}e^{\zeta_t}}{\beta} &= \tau\lambda_{VaR}
\end{aligned}$$

So we have

$$\lambda_{VaR}=\frac{e^{-\beta t}e^{\zeta_t}}{\tau}\left(\frac{1}{f_t}-\frac{1}{\beta}\right)$$

Now we solve

$$\begin{aligned}
& \min l(\lambda_1,\lambda_2) \\
& s.t. \\
g_V(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M}) &= \log\frac{1}{1-a_V} \\
l(\lambda_1,\lambda_2) &\equiv h(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M},\zeta_t)
\end{aligned}$$

$$\begin{aligned}
\nabla_{\lambda_1} l(\lambda_1,\lambda_1) &= \gamma\nabla_{\lambda_1} g_V(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M}) \\
\nabla_{\lambda_2} l(\lambda_1,\lambda_1) &= \gamma\nabla_{\lambda_2} g_V(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M})
\end{aligned}$$

$$\begin{aligned}
\nabla_{\lambda_1} l(\lambda_1,\lambda_2) &= \frac{\partial}{\partial\lambda_1} h(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M},\zeta_t) \\
&= \frac{\partial}{\partial\lambda_1}\left(e^{-\beta t}e^{\zeta_t}\log(\lambda_2 f_{t,M}) + \left[\int_t^\infty e^{-\beta s}E_t\left[e^{\zeta_s}\right]ds\right]Q(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M})\right) \\
&= \frac{e^{-\beta t}e^{\zeta_t}}{\beta}\frac{\partial}{\partial\lambda_1}Q(t,\lambda_1\theta_{t,M},\lambda_2 f_{t,M}) \\
&= \frac{e^{-\beta t}e^{\zeta_t}}{\beta}\frac{\partial}{\partial\lambda_1}\left(R_t-\lambda_2 f_{t,M}+(\lambda_1\theta_{t,M})^T\mu_t-\frac{1}{2}\left\|(\lambda_1\theta_{t,M})^T\sigma_t\right\|^2\right) \\
&= \frac{e^{-\beta t}e^{\zeta_t}}{\beta}\left((\theta_{t,M})^T\mu_t-\lambda_1\left\|(\theta_{t,M})^T\sigma_t\right\|^2\right)
\end{aligned}$$

$$
\begin{aligned}
\nabla_{\lambda_2} l\left(\lambda_1, \lambda_2\right) &= \frac{\partial}{\partial \lambda_2} h\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}, \zeta_t\right) \\
&= \frac{\partial}{\partial \lambda_2}\left(e^{-\beta t} e^{\zeta_t} \log\left(\lambda_2 f_{t,M}\right) + \left[\int_t^{\infty} e^{-\beta s} E_t\left[e^{\zeta_s}\right] ds\right] Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right)\right) \\
&= e^{-\beta t} e^{\zeta_t} \frac{1}{\lambda_2} + \frac{e^{-\beta t} e^{\zeta_t}}{\beta} \frac{\partial}{\partial \lambda_2} Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right) \\
&= e^{-\beta t} e^{\zeta_t} \frac{1}{\lambda_2} - \frac{e^{-\beta t} e^{\zeta_t}}{\beta} f_{t,M}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\lambda_1} g_V\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right) &= \frac{\partial}{\partial \lambda_1}\left(-Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right) \tau - N^{-1}(\alpha) \left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\| \sqrt{\tau}\right) \\
&= \frac{\partial}{\partial \lambda_1}\left(\begin{array}{c} -\left(R_t - \lambda_2 f_{t,M} + \left(\lambda_1 \theta_{t,M}\right)^T \mu_t - \frac{1}{2}\left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\|^2\right) \tau \\ -N^{-1}(\alpha) \left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\| \sqrt{\tau} \end{array}\right) \\
&= -\left(\left(\theta_{t,M}\right)^T \mu_t - \lambda_1 \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|^2\right) \tau - N^{-1}(\alpha) \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\| \sqrt{\tau}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\lambda_2} g_V\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right) &= \frac{\partial}{\partial \lambda_2}\left(-Q\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right) \tau - N^{-1}(\alpha) \left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\| \sqrt{\tau}\right) \\
&= \frac{\partial}{\partial \lambda_2}\left(\begin{array}{c} -\left(R_t - \lambda_2 f_{t,M} + \left(\lambda_1 \theta_{t,M}\right)^T \mu_t - \frac{1}{2}\left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\|^2\right) \tau \\ -N^{-1}(\alpha) \left\|\left(\lambda_1 \theta_{t,M}\right)^T \sigma_t\right\| \sqrt{\tau} \end{array}\right) \\
&= f_{t,M} \tau - N^{-1}(\alpha) \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\| \sqrt{\tau}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{\lambda_1} l\left(\lambda_1, \lambda_1\right) &= \gamma \nabla_{\lambda_1} g_V\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right) \\
\nabla_{\lambda_2} l\left(\lambda_1, \lambda_1\right) &= \gamma \nabla_{\lambda_2} g_V\left(t, \lambda_1 \theta_{t,M}, \lambda_2 f_{t,M}\right) \\
& \text{become} \\
\frac{e^{-\beta t} e^{\zeta_t}}{\beta}\left(\left(\theta_{t,M}\right)^T \mu_t - \lambda_1 \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|^2\right) &= -\gamma\left(\begin{array}{c} \left(\left(\theta_{t,M}\right)^T \mu_t - \lambda_1 \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\|^2\right) \tau \\ +N^{-1}(\alpha) \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\| \sqrt{\tau} \end{array}\right) \\
e^{-\beta t} e^{\zeta_t} \frac{1}{\lambda_2} - \frac{e^{-\beta t} e^{\zeta_t}}{\beta} f_{t,M} &= \gamma\left(f_{t,M} \tau - N^{-1}(\alpha) \left\|\left(\theta_{t,M}\right)^T \sigma_t\right\| \sqrt{\tau}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\left(f_{t,M}\tau - N^{-1}(\alpha)\left\|(\theta_{t,M})^T\sigma_t\right\|\sqrt{\tau}\right)}{\beta}\left((\theta_{t,M})^T\mu_t - \lambda_1\left\|(\theta_{t,M})^T\sigma_t\right\|^2\right) \\
= & \left(\frac{f_{t,M}}{\beta} - \frac{1}{\lambda_2}\right)\left(\left((\theta_{t,M})^T\mu_t - \lambda_1\left\|(\theta_{t,M})^T\sigma_t\right\|^2\right)\tau + N^{-1}(\alpha)\left\|(\theta_{t,M})^T\sigma_t\right\|\sqrt{\tau}\right)
\end{aligned}
$$

$$
\gamma = \frac{e^{-\beta t}e^{\zeta_t}}{\left(f_{t,M}\tau - N^{-1}(\alpha)\left\|(\theta_{t,M})^T\sigma_t\right\|\sqrt{\tau}\right)}\left(\frac{1}{\lambda_2} - \frac{f_{t,M}}{\beta}\right)
$$

C Appendix: Solving the Banks' Problem

First, assume that $\theta_t \neq \mathbf{0}$ so that $g_V(t, \theta_t, f_t, \nu_t)$ is differentiable. Set up the Lagrangian

$$
\mathcal{L} = h(t, \theta_t, f_t, \nu_t) - \lambda\left(g_V(t, \theta_t, f_t, \nu_t) - \log\frac{1}{1 - a_V}\right)
$$

Direct computation (Appendix B) shows that $\nabla g_V(t, \theta_t, f_t, \nu_t) \neq \mathbf{0}$. Thus, $\lambda \neq 0$ and the FOC is

$$
\nabla h(t, \theta_t, f_t, \nu_t) = \lambda\nabla g_V(t, \theta_t, f_t, \nu_t) \tag{193}
$$

We compute

$$
\begin{aligned}
\nabla_{\theta_t} h &= \frac{1}{\beta}\left(\mu_t - \sigma_t\nu_t - \sigma_t\sigma_t^T\theta_t\right) \\
\nabla_{\theta_t} g_V &= -\left(\mu_t - \sigma_t\nu_t - \sigma_t\sigma_t^T\theta_t\right)\tau - N^{-1}(\alpha)\frac{\sigma_t\sigma_t^T\theta_t}{\left\|\theta_t^T\sigma_t\right\|}\sqrt{\tau} \\
\nabla_{f_t} h &= \frac{1}{f_t} - \frac{1}{\beta} \\
\nabla_{f_t} g_V &= \tau
\end{aligned}
$$

so that the FOC become

$$
\begin{aligned}
\nabla_{\theta_t} h(t, \theta_t, f_t, \nu_t) &= \lambda\nabla_{\theta_t} g_V(t, \theta_t, f_t, \nu_t) \\
\frac{1}{\beta}\left(\mu_t - \sigma_t\nu_t - \sigma_t\sigma_t^T\theta_t\right) &= \lambda\begin{pmatrix} -\left(\mu_t - \sigma_t\nu_t - \sigma_t\sigma_t^T\theta_t\right)\tau \\ -N^{-1}(\alpha)\frac{\sigma_t\sigma_t^T\theta_t}{\left\|\theta_t^T\sigma_t\right\|}\sqrt{\tau} \end{pmatrix} \\
(1 + \beta\tau\lambda_{VaR})(\mu_t - \sigma_t\nu_t) &= \left(1 + \left(\tau - \frac{\sqrt{\tau}N^{-1}(\alpha)}{\left\|\theta_t^T\sigma_t\right\|}\right)\beta\lambda_{VaR}\right)\sigma_t\sigma_t^T\theta_t
\end{aligned}
$$

and

$$\begin{aligned}
\nabla_{f_t} h\left(t, \theta_t, f_t, \nu_t\right) &= \lambda \nabla_{f_t} g_V\left(t, \theta_t, f_t, \nu_t\right) \\
\frac{1}{f_t} - \frac{1}{\beta} &= \lambda \tau \\
f_t &= \frac{\beta}{\beta \lambda \tau + 1}
\end{aligned}$$

Writing

$$\theta_{M,t} = \left(\sigma_t^T\right)^{-1} \sigma_t^{-1} \left(\mu_t - \sigma_t \nu_t\right)$$

we see that θ_t is parallel to $\theta_{M,t}$ so all we need is to find λ_1, λ_2 that solve

$$\begin{aligned}
& \max_{\lambda_1, \lambda_2} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) \\
& s.t. \\
g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) &\leq \log \frac{1}{1 - a_V}
\end{aligned}$$

Again, it can be checked that the constraint holds with equality. The Lagrangian is

$$\mathcal{L} = h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) - \gamma \left(g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) - \log \frac{1}{1 - a_V} \right)$$

The FOC are

$$\begin{aligned}
\frac{\partial}{\partial \lambda_1} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) &= \gamma \frac{\partial}{\partial \lambda_1} g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) \\
\frac{\partial}{\partial \lambda_2} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) &= \gamma \frac{\partial}{\partial \lambda_2} g_V\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right)
\end{aligned}$$

Computing the derivatives gives

$$\begin{aligned}
\frac{\partial}{\partial \lambda_1} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) &= \gamma \frac{\partial}{\partial \lambda_1} g_V\left(t, \theta, f, \nu\right) \\
& \quad \frac{1}{\beta} \left(\theta_{M,t}^T \left(\mu_t - \sigma_t \nu_t\right) - \lambda_1 \left\| \theta_{M,t}^T \sigma_t \right\|^2 \right) \\
&= \gamma \left(- \left(\theta_{M,t}^T \left(\mu_t - \sigma_t \nu_t\right) - \lambda_1 \left\| \theta_{M,t}^T \sigma_t \right\|^2 \right) \tau - N^{-1}\left(\alpha\right) \left\| \theta_{M,t}^T \sigma_t \right\| \sqrt{\tau} \right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \lambda_2} h\left(t, \lambda_1 \theta_{M,t}, \lambda_2 f_{M,t}, \nu_t\right) &= \gamma \frac{\partial}{\partial \lambda_2} g_V\left(t, \theta, f, \nu\right) \\
\frac{1}{\lambda_2} - \frac{f_{M,t}}{\beta} &= \gamma f_{M,t} \tau
\end{aligned}$$

Eliminating γ and using

$$\begin{aligned}
\theta_{M,t}^{T}\sigma_t &= \left(\left(\sigma_t^{-1}\right)\left(\mu_t - \sigma_t\nu_t\right)\right)^T \\
\left\|\theta_{M,t}^{T}\sigma_t\right\| &= \left\|\left(\sigma_t^{-1}\right)\left(\mu_t - \sigma_t\nu_t\right)\right\| \\
\theta_{M,t}\left(\mu_t - \sigma_t\nu_t\right) &= \left\|\left(\sigma_t^{-1}\right)\left(\mu_t - \sigma_t\nu_t\right)\right\|^2 = \left\|\theta_{M,t}^{T}\sigma_t\right\|^2
\end{aligned}$$

we get

$$\begin{aligned}
\lambda_2 &= u\left(t,\lambda_1\right) f_{M,t} \\
u\left(t,z\right) &\equiv \left[1 + \frac{\sqrt{\tau}\left\|\theta_{M,t}^{T}\sigma_t\right\|}{N^{-1}\left(\alpha\right)}\left(1-z\right)\right]
\end{aligned} \tag{194}$$

and λ_1 is the unique number that makes the VaR hold with equality

$$g_V\left(t, \lambda_1\theta_{M,t}, u\left(t,\lambda_1\right) f_{M,t}, \nu_t\right) = \log\frac{1}{1-a_V}$$

If $\lambda_1 \leq 0$, then there is no investment in the risky asset and

$$\begin{aligned}
\theta_t &= \mathbf{0} \\
f_t &= R_t + \frac{1}{\tau}\log\frac{1}{1-a_V}
\end{aligned}$$

Putting everything together, the optimal portfolio is then characterized by

$$\theta_t = \min\left\{1, \max\left\{0, \varphi_t\right\}\right\}\theta_{M,t} \tag{195}$$

$$\begin{aligned}
f_t &= u\left(t, \min\left\{1,\varphi_t\right\}\right) f_{M,t} 1_{\{\varphi_t > 0\}} \\
&\quad + \left(R_t + \frac{1}{\tau}\log\frac{1}{1-a_V}\right) 1_{\{\varphi_t \leq 0\}}
\end{aligned} \tag{196}$$

$$\varphi_t \text{ such that: } \quad g_V\left(t, \varphi_t\theta_{M,t}, u\left(t,\varphi_t\right) f_{M,t}\right) = \log\frac{1}{1-a_V} \tag{197}$$

$$\theta_{M,t} = \left(\sigma_t^T\right)^{-1}\left(\sigma_t^{-1}\mu_t - \nu_t\right) \tag{198}$$

$$f_{M,t} = \beta \tag{199}$$

$$\lambda = \frac{1}{\tau}\left(\frac{1}{f_t} - \frac{1}{\beta}\right) \tag{200}$$

D Appendix: Solving for φ

Solving for φ_t in (197) gives an explicit definition for φ_t. First, we express

$$\begin{aligned}
\left\|\theta_{M,t}^T\sigma_t\right\| &= \left\|\left(\left(\sigma_t^T\right)^{-1}\sigma_t^{-1}\left(\mu_t-\sigma_t\nu_t\right)\right)^T\sigma_t\right\| \\
&= \left\|\left(\sigma_t^{-1}\right)\left(\mu_t-\sigma_t\nu_t\right)\right\| \\
&= \left\|\eta_t-\nu_t\right\|
\end{aligned}$$

$$\begin{aligned}
\theta_{M,t}^T\left(\mu_t-\sigma_t\nu_t\right) &= \left(\left(\sigma_t^T\right)^{-1}\sigma_t^{-1}\left(\mu_t-\sigma_t\nu_t\right)\right)^T\left(\mu_t-\sigma_t\nu_t\right) \\
&= \left[\sigma_t^{-1}\left(\mu_t-\sigma_t\nu_t\right)\right]^T\left(\sigma_t^{-1}\right)\left(\mu_t-\sigma_t\nu_t\right) \\
&= \left\|\left(\sigma_t^{-1}\right)\left(\mu_t-\sigma_t\nu_t\right)\right\|^2 \\
&= \left\|\eta_t-\nu_t\right\|^2
\end{aligned}$$

Then

$$\begin{aligned}
-\log\left(1-a_V\right) &= g_V\left(t,\varphi_t\theta_{M,t},u\left(t,\varphi_t\right)f_{M,t}\right) \\
&= -\left(R_t-f_t+\varphi_t\left\|\eta_t-\nu_t\right\|^2-\frac{1}{2}\varphi_t^2\left\|\eta_t-\nu_t\right\|^2\right)\tau-N^{-1}\left(\alpha\right)\varphi_t\left\|\eta_t-\nu_t\right\|\sqrt{\tau} \\
&= -\left(R_t-f_t+\left\|\eta_t-\nu_t\right\|^2\left(\varphi_t-\frac{1}{2}\varphi_t^2\right)\right)\tau-N^{-1}\left(\alpha\right)\varphi_t\left\|\eta_t-\nu_t\right\|\sqrt{\tau}
\end{aligned}$$

and

$$\begin{aligned}
0 &= -\log\left(1-a_V\right)+\left(R_t-f_t\right)\tau+\tau\left\|\eta_t-\nu_t\right\|^2\varphi_t+N^{-1}\left(\alpha\right)\sqrt{\tau}\left\|\eta_t-\nu_t\right\|\varphi_t-\frac{1}{2}\tau\left\|\eta_t-\nu_t\right\|^2\varphi_t^2 \\
\varphi_t &\equiv 1+\frac{N^{-1}\left(\alpha\right)}{\sqrt{\tau}\left\|\eta_t-\nu_t\right\|}\pm\sqrt{2\left(R_t-f_t\right)\tau-2\log\left(1-a_V\right)+\left\|\eta_t-\nu_t\right\|^2\tau^2\left(1+\frac{N^{-1}\left(\alpha\right)}{\sqrt{\tau}\left\|\eta_t-\nu_t\right\|}\right)^2}
\end{aligned}$$

E Appendix: Solving Stock Market Clearing

We solve for the case of two stocks (*goods*, *bank*) and a single shock from the point of view of the bank, and one stock (*bank*) from the point of view of the household. The supply of goods stocks is 1 share, i.e.

$$\frac{X_t\theta_t}{S_t}=1 \tag{201}$$

The demand comes from the bank's problem

$$\theta_t=\min\left\{1,\max\left\{0,\varphi_t\right\}\right\}\theta_{M,t} \tag{202}$$

We will use the following

$$\begin{aligned}
& f_{M,t} \text{ exogenous} && (203)\\
\theta_{M,t} &= \sigma_t^{-2}\mu_t && (204)\\
\eta_t &= \sigma_t^{-1}\mu_t && (205)\\
dS_t &= S_t\left(\mu_t + R_t\right)dt + S_t\sigma_t dB_t && (206)\\
dX_t &= X_t\left(R_t - f_t + \theta_t\mu_t\right)dt + X_t\theta_t\sigma_t dB_t && (207)\\
f_t &= u\left(t, \min\left\{1, \varphi_t\right\}\right) f_{M,t} && (208)
\end{aligned}$$

Consider first the case $\varphi_t > 1$. Then (202),(205),(207),(208) give

$$\begin{aligned}
\theta_t &= \sigma_t^{-2}\mu_t && (209)\\
f_t &= f_{M,t} && (210)\\
dX_t &= X_t\left(R_t - f_{M,t} + \sigma_t^{-2}\mu_t^2\right)dt + X_t\sigma_t^{-1}\mu_t dB_t && (211)
\end{aligned}$$

and (201), (206), (211) give

$$\begin{aligned}
S_t &= X_t\theta_t && (212)\\
dS_t &= \theta_t dX_t + X_t d\theta_t + dX_t d\theta_t && (213)\\
S_t\left(\mu_t + R_t\right)dt + S_t\sigma_t dB_t &= \theta_t\left(X_t\left(R_t - f_{M,t} + \sigma_t^{-2}\mu_t^2\right)dt + X_t\sigma_t^{-1}\mu_t dB_t\right) && (214)\\
&\quad + X_t d\theta_t + X_t\sigma_t^{-1}\mu_t\left(d\theta_t dB_t\right) && (215)
\end{aligned}$$

Matching the drift and stochastic parts of the left and hand side of equation (214) gives

$$\begin{aligned}
S_t\left(\mu_t + R_t\right) &= \theta_t X_t\left(R_t - f_{M,t} + \sigma_t^{-2}\mu_t^2\right) + X_t\frac{1}{dt}E_t\left[d\theta_t\right] && (216)\\
&\quad + X_t\sigma_t^{-1}\mu_t\left(\frac{1}{dt}d\theta_t dB_t\right) && (217)\\
S_t\sigma_t &= \theta_t X_t\sigma_t^{-1}\mu_t + X_t \, stoch\left(d\theta_t\right) && (218)
\end{aligned}$$

Using (201) and (209) in (218) gives

$$stoch\left(d\theta_t\right) = \mu_t\sigma_t^{-3}\left(\sigma_t^2 - \mu_t\right) \quad (219)$$

Using (201), (209) and (219) in (216) gives

$$\frac{1}{dt}E_t\left[d\theta_t\right] = \sigma_t^{-2}\mu_t f_{M,t} \quad (220)$$

Equations (219) and (220) mean that

$$d\theta_t = \theta_t f_{M,t}dt + \eta_t\left(1 - \theta_t\right)dB_t \quad (221)$$

Consider now the case of $0 < \varphi \leq 1$. Equations (202),(205),(207),(208) give

$$\begin{aligned}
\theta_t &= \varphi_t \sigma_t^{-2} \mu_t && (222)\\
f_t &= u(t, \varphi_t) f_{M,t} && (223)\\
dX_t &= X_t \left(R_t - u(t, \varphi_t) f_{M,t} + \varphi_t \eta_t^2 \right) dt + X_t \varphi_t \eta_t dB_t && (224)
\end{aligned}$$

Equations (201), (206), (224) give

$$\begin{aligned}
S_t &= X_t \theta_t && (225)\\
dS_t &= \theta_t dX_t + X_t d\theta_t + dX_t d\theta_t && (226)\\
S_t (\mu_t + R_t) dt + S_t \sigma_t dB_t &= \theta_t X_t \left(R_t - u(t, \varphi_t) f_{M,t} + \varphi_t \eta_t^2 \right) dt && (227)\\
&\quad + \theta_t X_t \varphi_t \eta_t dB_t + X_t d\theta_t + X_t \varphi_t \eta_t (dB_t d\theta_t) && (228)
\end{aligned}$$

Matching the drift and stochastic parts of the left and hand side of equation (227) gives

$$\begin{aligned}
S_t (\mu_t + R_t) &= \theta_t X_t \left(R_t - u(t, \varphi_t) f_{M,t} + \varphi_t \eta_t^2 \right) + X_t \frac{1}{dt} E_t [d\theta_t] && (229)\\
&\quad + X_t \varphi_t \eta_t \frac{1}{dt} (dB_t d\theta_t) && (230)\\
S_t \sigma_t dB_t &= \theta_t X_t \varphi_t \eta_t dB_t + X_t stoch(d\theta_t) && (231)
\end{aligned}$$

Using (201), (205) and (222) in (231) gives

$$stoch(d\theta_t) = \eta_t \varphi_t (1 - \theta_t \varphi_t) \quad (232)$$

Using (201), (205), (222) and (232) in (229) gives

$$\frac{1}{dt} E_t [d\theta_t] = \theta_t u(t, \varphi_t) f_{M,t} + \eta_t^2 \varphi_t (\varphi_t - 1)(\theta_t + \theta_t \varphi_t - 1) \quad (233)$$

Equations (232) and (233) mean that

$$d\theta_t = \left(\eta_t^2 \varphi_t (\varphi_t - 1)(\theta_t (\varphi_t + 1) - 1) + \theta_t \varphi_t f_{M,t} \right) dt + \eta_t \varphi_t (1 - \theta_t \varphi_t) dB_t \quad (234)$$

Note that (221) and (234) match when $\varphi_t = 1$.